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ON CONVOLUTIONS OF HARMONIC UNIVALENT MAPPINGS CONVEX IN THE DIRECTION OF THE REAL AXIS*

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Abstract In this paper, we show that convolutions of some planar harmonic mappings which convex in the direction of the real axis are also convex in the same direction. Furthermore, by means of the *Mathematica* software, we present an example to illuminate the main result.

Keywords Harmonic univalent mapping, convolution, shear construction.

MSC(2010) 58E20, 30C45.

1. Introduction

A complex-valued harmonic function f in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : \ |z| < 1\}$ is given by $f = h + \overline{g}$, where h and g are analytic functions in $\mathbb{D}$. As usual, h is called the analytic part of f, and g is called the co-analytic part of f. The Jacobian of the mapping $f = h + \overline{g}$ is given by $J_f = |h'|^2 - |g'|^2$. A necessary and sufficient condition for f to be locally univalent and sense-preserving in $\mathbb{D}$ is that $|g'| < |h'|$, or equivalently, the dilatation $\omega = g'/h'$ has the property $|\omega| < 1$ in $\mathbb{D}$ for $h' \neq 0$ (see [3] or [11]).

We denote $\mathcal{S}_H$ by the class of harmonic, sense-preserving and univalent mappings in $\mathbb{D}$, normalized by the conditions $f(0) = 0$ and $f_z(z) = 1$. Thus, a harmonic mapping in the class $\mathcal{S}_H$ can be expressed as $f = h + \overline{g}$, where

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_n z^n.$$

Let $\mathcal{S}_H^0$ be the subclass of $\mathcal{S}_H$ whose members satisfy the additional condition $f_{\overline{z}}(0) = 0$. Also, let $\mathcal{K}_H^0$ and $\mathcal{C}_H^0$ be the subclasses of $\mathcal{S}_H^0$ whose image domains are convex and close-to-convex, respectively.

A domain $\Omega \subset \mathbb{C}$ is said to be convex in the direction γ, if for all $a \in \mathbb{C}$, the set $\Omega \cap \{a + te^{i\gamma} : t \in \mathbb{R}\}$ is either connected or empty. In particular, a domain is convex in the horizontal direction (CHD), if every line parallel to the real axis has a connected intersection with the domain. The shear construction is essential to the present work as it allows one to study harmonic functions through their related analytic functions, it produces a univalent harmonic function that maps $\mathbb{D}$ to a

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region which is CHD. This construction relies on the following result due to Clunie and Sheil-Small [3].

Theorem A. *A harmonic function $f = h + \overline{g}$ locally univalent in $\mathbb{D}$ is a univalent mapping of $\mathbb{D}$ onto a domain convex in the direction of the real axis if and only if $h - g$ is a conformal univalent mapping of $\mathbb{D}$ onto a domain convex in the direction of the real axis.*

A function $f = h + \overline{g} \in \mathcal{S}_H^0$ is called a CHD mapping if f maps $\mathbb{D}$ onto a CHD domain. We denote all such CHD mappings by $\mathcal{S}_{CHD}^0$. Clearly, we know that $\mathcal{S}_{CHD}^0 \subset \mathcal{C}_H^0$.

For two harmonic functions given by

$$f(z) = h(z) + \overline{g(z)} = z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=2}^{\infty} \overline{b_n}\,\overline{z}^n$$

and

$$F(z) = H(z) + \overline{G(z)} = z + \sum_{n=2}^{\infty} A_n z^n + \sum_{n=2}^{\infty} \overline{B_n}\,\overline{z}^n,$$

their convolution $f * F$ is defined by

$$f * F = h * H + \overline{g * G} = z + \sum_{n=2}^{\infty} a_n A_n z^n + \sum_{n=2}^{\infty} \overline{b_n B_n}\,\overline{z}^n.$$

For some recent investigations involving harmonic mappings and related topics, one can refer to [1, 2, 4, 5, 7–9, 12–24, 26, 29–32]. In particular, Dorff [8] and Dorff *et al.* [9] derived convolutions involving right half-plane mappings, and obtained the following results, respectively.

Theorem B. *Let $f_1 = h_1 + \overline{g_1} \in \mathcal{K}_H^0$, $f_2 = h_2 + \overline{g_2} \in \mathcal{K}_H^0$ with $h_k + g_k = z/(1-z)$ for $k = 1, 2$. If $f_1 * f_2$ is locally univalent and sense-preserving, then $f_1 * f_2 \in \mathcal{S}_{CHD}^0$.*

Theorem C. *Let $f = h + \overline{g} \in \mathcal{K}_H^0$ with $h + g = z/(1-z)$ and $\omega = g'/h' = e^{i\theta} z^n$ $(n \in \mathbb{Z}^+;\ \theta \in \mathbb{R})$. If $n = 1, 2$, then $f_0 * f \in \mathcal{S}_{CHD}^0$, where*

$$f_0 = h_0 + \overline{g_0} = \frac{z - \frac{1}{2}z^2}{(1-z)^2} + \overline{\frac{-\frac{1}{2}z^2}{(1-z)^2}}.$$

We remark that

$$f_0(z) = \frac{z - \frac{1}{2}z^2}{(1-z)^2} + \overline{\frac{-\frac{1}{2}z^2}{(1-z)^2}} = \Re\left(\frac{z}{1-z}\right) + i\Im\left(\frac{z}{(1-z)^2}\right)$$

is the well-known right half-plane mapping.

In this paper, we consider the harmonic mapping $f_c(z) = h_c + \overline{g_c}$ satisfies the condition

$$h_c - g_c = \frac{z}{1-z}$$

with the dilatation $\omega = z$, by applying the shearing technique, we have

$$h_c = \frac{z - \frac{1}{2}z^2}{(1-z)^2} = \frac{1}{2}\left(\frac{z}{(1-z)^2} + \frac{z}{1-z}\right),$$

and

$$g_c = \frac{\frac{1}{2}z^2}{(1-z)^2} = \frac{1}{2}\left(\frac{z}{(1-z)^2} - \frac{z}{1-z}\right),$$

that is,

$$f_c(z) = \Re\left(\frac{z}{(1-z)^2}\right) + i\Im\left(\frac{z}{1-z}\right). \tag{1.1}$$

If a function F is analytic in $\mathbb{D}$ with $F(0) = 0$, then

$$h_c(z) * F(z) = \frac{1}{2}\left(zF'(z) + F(z)\right),$$

and

$$g_c(z) * F(z) = \frac{1}{2}\left(zF'(z) - F(z)\right).$$

In what follows, we consider the image domain $f_c(\mathbb{D})$. For convenience, we write

$$f_c(z) = \Re\left(\frac{z}{(1-z)^2}\right) + i\Im\left(\frac{z}{1-z}\right) = u + iv.$$

Let $z = e^{i\theta} \in \partial\mathbb{D}$. Then

$$u = \Re\left(\frac{z}{(1-z)^2}\right) = \Re\left(\frac{e^{i\theta}}{(1-e^{i\theta})^2}\right) = \frac{2\cos\theta - 2}{(2-2\cos\theta)^2} = -\frac{1}{2-2\cos\theta},$$

and

$$v = \Im\left(\frac{z}{1-z}\right) = \Im\left(\frac{e^{i\theta}}{1-e^{i\theta}}\right) = \frac{\sin\theta}{2-2\cos\theta}.$$

Thus, we get

$$v^2 = -\left(u + \frac{1}{4}\right).$$

Since the point $z = 0$ is mapped into $f_c(0) = 0$, we find that

$$f_c(\mathbb{D}) = \left\{u + iv : \ v^2 > -\left(u + \frac{1}{4}\right)\right\}.$$

The images of concentric circles inside $\mathbb{D}$ under the harmonic mapping $f_c(z)$ are shown in Figure 1, which implies that $f_c(z)$ is a CHD mapping and not a right half-plane mapping.

Recently, Nagpal and Ravichandran [25, Theorems 2.2 and 2.3] gave the radii of convexity and starlikeness of $f_c(z)$ are $2 - \sqrt{3}$ and $\frac{1}{3}\sqrt{\frac{1}{3}(37 - 8\sqrt{10})}$, respectively. In this paper, we aim at deriving convolutions of $f_c(z)$ and some special harmonic mappings in the class $\mathcal{S}^0_{CHD}$ are also belonging to the class $\mathcal{S}^0_{CHD}$. Furthermore, we present an example to illustrate the result with the aid of *Mathematica* software.

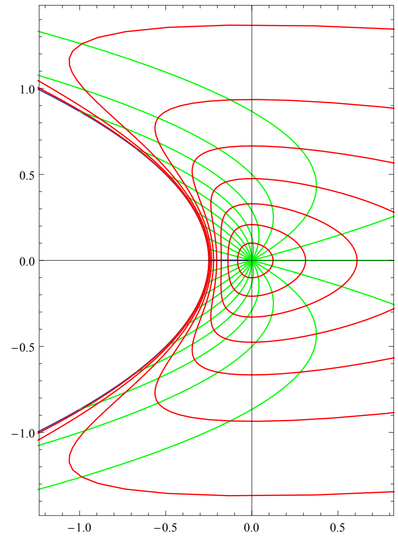

Figure 1. Image of f_c

2. Preliminaries

The following lemmas will be required in the proof of our main result.

Lemma 2.1. (see [27]) *Let f be an analytic function in $\mathbb{D}$ with $f(0) = 0$ and $f'(0) \neq 0$, and let*

$$\varphi(z) = \frac{z}{(1 + ze^{i\theta_1})(1 + ze^{i\theta_2})}, \tag{2.1}$$

where θ_1, $\theta_2 \in \mathbb{R}$. If

$$\Re\left(\frac{zf'(z)}{\varphi(z)}\right) > 0 \quad (z \in \mathbb{D}),$$

then f is convex in the direction of the real axis.

Lemma 2.2. (see [6, Cohn's Rule]) *Given a polynomial*

$$f(z) = a_0 + a_1 z + \cdots + a_n z^n$$

of degree n, let

$$f^*(z) = z^n \overline{f(1/\overline{z})} = \overline{a_n} + \overline{a_{n-1}} z + \cdots + \overline{a_0} z^n.$$

Denote by p and s the number of zeros of f inside the unit circle and on it, respectively. If $|a_0| < |a_n|$, then

$$f_1(z) = \frac{\overline{a_n} f(z) - a_0 f^*(z)}{z}$$

is of degree $n - 1$ with $p_1 = p - 1$ and $s_1 = s$ the number of zeros of f_1 inside the unit circle and on it, respectively.

Lemma 2.3. (see [10]) *Let φ and G be analytic in $\mathbb{D}$ with $\varphi'(0) = G(0) = 0$. If φ is convex and G is starlike, then for each function F analytic in $\mathbb{D}$ and satisfying $\Re(F(z)) > 0$, we have*

$$\Re\left(\frac{(\varphi * FG)(z)}{(\varphi * G)(z)}\right) > 0 \quad (z \in \mathbb{D}).$$

Lemma 2.4. *Let $f_1 = h_1 + \overline{g_1} \in \mathcal{S}^0_{CHD}$ with $h_1 - g_1 = z/(1-z)$, $f_2 = h_2 + \overline{g_2} \in \mathcal{S}^0_{CHD}$ with*

$$h_2 - g_2 = \frac{1}{2i\sin\alpha}\log\left(\frac{1+ze^{i\alpha}}{1+ze^{-i\alpha}}\right)$$

for $\pi/2 \le \alpha < \pi$. If $f_1 * f_2$ is locally univalent, then $f_1 * f_2 \in \mathcal{S}^0_{CHD}$.

Proof. Let

$$F_1 = (h_1 - g_1) * (h_2 + g_2) = h_1 * h_2 + h_1 * g_2 - h_2 * g_1 - g_1 * g_2,$$

and

$$F_2 = (h_1 + g_1) * (h_2 - g_2) = h_1 * h_2 - h_1 * g_2 + h_2 * g_1 - g_1 * g_2.$$

Thus, we have

$$h_1 * h_2 - g_1 * g_2 = \frac{1}{2}(F_1 + F_2). \tag{2.2}$$

Next, we shall prove that $\frac{1}{2}(F_1 + F_2)$ is CHD. Since

$$h_2 - g_2 = \frac{1}{2i\sin\alpha}\log\left(\frac{1+ze^{i\alpha}}{1+ze^{-i\alpha}}\right),$$

we know that

$$\begin{aligned} zF_1' &= (h_1 - g_1) * [z(h_2' + g_2')] = (h_1 - g_1) * \left[z(h_2' - g_2')\left(\frac{h_2' + g_2'}{h_2' - g_2'}\right)\right] \\ &= \frac{z}{1-z} * \frac{z}{(1+ze^{i\alpha})(1+ze^{-i\alpha})}\left(\frac{1+\omega_2}{1-\omega_2}\right) = \frac{zp_2(z)}{(1+ze^{i\alpha})(1+ze^{-i\alpha})}, \end{aligned}$$

where

$$p_2(z) = \frac{1+\omega_2}{1-\omega_2}$$

satisfies the condition $\Re(p_2(z)) > 0$. Thus, we get

$$\Re\left(\frac{zF_1'}{\frac{z}{(1+ze^{i\alpha})(1+ze^{-i\alpha})}}\right) = \Re\left(p_2(z)\right) > 0. \tag{2.3}$$

In what follows, we consider

$$\begin{aligned} zF_2' &= [z(h_1' + g_1')] * (h_2 - g_2) = \left[z(h_1' - g_1')\left(\frac{h_1' + g_1'}{h_1' - g_1'}\right)\right] * (h_2 - g_2) \\ &= \left[z(h_1' - g_1')\left(\frac{1+\omega_1}{1-\omega_1}\right)\right] * (h_2 - g_2) = \frac{zp_1(z)}{(1-z)^2} * (h_2 - g_2), \end{aligned}$$

where

$$p_1(z) = \frac{1+\omega_1}{1-\omega_1}$$

satisfies the condition $\Re(p_1(z)) > 0$. Using the fact that

$$\Psi(z) * \frac{z}{(1-z)^2} = z\Psi'(z)$$

and $h_2 - g_2$ is convex, by Lemma 2.3, we have

$$\begin{aligned} \Re\left(\frac{zF_2'}{\frac{z}{(1+ze^{i\alpha})(1+ze^{-i\alpha})}}\right) &= \Re\left(\frac{(h_2-g_2) * p_1(z)\frac{z}{(1-z)^2}}{z(h_2'-g_2')}\right) \\ &= \Re\left(\frac{(h_2-g_2) * p_1(z)\frac{z}{(1-z)^2}}{(h_2-g_2) * \frac{z}{(1-z)^2}}\right) > 0. \end{aligned} \tag{2.4}$$

Combining (2.3) with (2.4), we get

$$\Re\left(\frac{z(F_1+F_2)'}{\frac{z}{(1+ze^{i\alpha})(1+ze^{-i\alpha})}}\right) > 0,$$

by Lemma 2.1, we know that $F_1 + F_2$ is convex in the direction of the real axis. Thus, by Theorem A, we get the desired result of Lemma 2.4. □

3. Main result

We now give the main result below.

Theorem 3.1. *Let $f_n = h_n + \overline{g_n} \in \mathcal{S}^0_{CHD}$ with*

$$h_n - g_n = \frac{1}{2i\sin\alpha}\log\left(\frac{1+ze^{i\alpha}}{1+ze^{-i\alpha}}\right)$$

for $\pi/2 \le \alpha < \pi$ and $\omega_n = e^{i\theta}z^n$. If $n = 1, 2$, then $f_c * f_n \in \mathcal{S}^0_{CHD}$, where f_c is given by (1.1).

Proof. By Lemma 2.4, we only need to prove that $f_c * f_n = H_n + \overline{G_n}$ are locally univalent. By noting that

$$h_n - g_n = \frac{1}{2i\sin\alpha}\log\left(\frac{1+ze^{i\alpha}}{1+ze^{-i\alpha}}\right)$$

and $g_n' = \omega_n h_n'$, we have $g_n'' = \omega_n h_n'' + \omega_n' h_n'$. Therefore, we know that

$$\widetilde{\omega}(z) = \frac{(g_c * g_n)'}{(h_c * h_n)'} = \frac{(zg_n' - g_n)'}{(zh_n' + h_n)'} = \frac{zg_n''}{2h_n' + zh_n''} = z\frac{\omega_n h_n'' + \omega_n' h_n'}{2h_n' + zh_n''}. \tag{3.1}$$

Moreover, we observe that

$$h_n' = \frac{1}{(1-\omega_n)(1+ze^{i\alpha})(1+ze^{-i\alpha})}, \tag{3.2}$$

and

$$h_n'' = -\frac{2(\cos\alpha + z)(1-\omega_n) - \omega_n'(1+2z\cos\alpha + z^2)}{(1-\omega_n)^2(1+ze^{i\alpha})^2(1+ze^{-i\alpha})^2}. \tag{3.3}$$

By substituting (3.2) and (3.3) into (3.1), yields

$$\widetilde{\omega}(z) = z\frac{\omega_n^2 - \left(\omega_n - \frac{1}{2}\omega_n' z\right) + \frac{1}{2}\omega_n' \frac{1+z\cos\alpha}{\cos\alpha + z}}{\frac{1+z\cos\alpha}{\cos\alpha+z} - \left(\omega_n - \frac{1}{2}\omega_n' z\right)\frac{1+z\cos\alpha}{\cos\alpha+z} + \frac{1}{2}\omega_n' z^2}. \tag{3.4}$$

Now, we consider the case $\omega_1 = e^{i\theta}z$, then

$$\begin{aligned}
\widetilde{\omega}(z) &= z\frac{e^{2i\theta}z^2 - \left(e^{i\theta}z - \frac{1}{2}e^{i\theta}z\right) + \frac{1}{2}e^{i\theta}z\frac{1+z\cos\alpha}{\cos\alpha+z}}{\frac{1+z\cos\alpha}{\cos\alpha+z} - \left(e^{i\theta}z - \frac{1}{2}e^{i\theta}z\right)\frac{1+z\cos\alpha}{\cos\alpha+z} + \frac{1}{2}e^{i\theta}z^2} \\
&= ze^{2i\theta}\frac{z^3 + \left(\cos\alpha - \frac{1}{2}e^{-i\theta}\right)z^2 + \frac{1}{2}e^{-i\theta}}{1 + \left(\cos\alpha - \frac{1}{2}e^{i\theta}\right)z + \frac{1}{2}e^{i\theta}z^3} \\
&= ze^{2i\theta}\frac{p(z)}{p^*(z)}.
\end{aligned}$$

Note that $p^*(z) = z^3\overline{p(1/\overline{z})}$, if z_0 is one zero of $p(z)$, then $\frac{1}{\overline{z_0}}$ is one zero of $p^*(z)$. Thus, we have

$$\widetilde{\omega}(z) = ze^{2i\theta}\frac{(z+A)(z+B)(z+C)}{(1+\overline{A}z)(1+\overline{B}z)(1+\overline{C}z)},$$

where $\alpha \in [\pi/2, \pi)$ and $\theta \in [-\pi, \pi]$.

It is sufficient to show that $A, B, C \in \overline{\mathbb{D}}$. We apply *Cohn's Rule* to

$$p(z) = z^3 + \left(\cos\alpha - \frac{1}{2}e^{-i\theta}\right)z^2 + \frac{1}{2}e^{-i\theta}.$$

By noting that $\frac{1}{2}|e^{-i\theta}| = \frac{1}{2} < 1$, we obtain

$$p_1(z) = \frac{\overline{a_3}p(z) - a_0 p^*(z)}{z} = \frac{3}{4}z^2 + \left(\cos\alpha - \frac{1}{2}e^{-i\theta}\right)z - \frac{1}{2}e^{-i\theta}\left(\cos\alpha - \frac{1}{2}e^{i\theta}\right).$$

Since

$$\left|-\frac{1}{2}e^{-i\theta}\left(\cos\alpha - \frac{1}{2}e^{i\theta}\right)\right| \le \frac{1}{4} + \frac{1}{2}|\cos\alpha| < \frac{3}{4},$$

we use *Cohn's Rule* on $p_1(z)$ again, then

$$\begin{aligned}
p_2(z) &= \frac{\frac{3}{4}p_1(z) + \frac{1}{2}e^{-i\theta}\left(\cos\alpha - \frac{1}{2}e^{i\theta}\right)p_1^*(z)}{z} \\
&= \frac{\frac{3}{4}\left[\frac{3}{4}z^2 + \left(\cos\alpha - \frac{1}{2}e^{-i\theta}\right)z - \frac{1}{2}e^{-i\theta}\left(\cos\alpha - \frac{1}{2}e^{i\theta}\right)\right]}{z} \\
&\quad + \frac{\frac{1}{2}e^{-i\theta}\left(\cos\alpha - \frac{1}{2}e^{i\theta}\right)\left[\frac{3}{4} + (\cos\alpha - \frac{1}{2}e^{-i\theta})z - \frac{1}{2}e^{-i\theta}(\cos\alpha - \frac{1}{2}e^{i\theta}z^2)\right]}{z} \\
&= \left(\frac{9}{16} - \frac{1}{4}|\cos\alpha - \frac{1}{2}e^{i\theta}|^2\right)z + \frac{3}{4}\left(\cos\alpha - \frac{1}{2}e^{-i\theta}\right) + \frac{1}{2}e^{-i\theta}\left(\cos\alpha - \frac{1}{2}e^{i\theta}\right)^2
\end{aligned}$$

has one zero at

$$\begin{aligned} z_0 &= \frac{-\frac{3}{4}\left(\cos\alpha - \frac{1}{2}e^{-i\theta}\right) - \frac{1}{2}e^{-i\theta}\left(\cos\alpha - \frac{1}{2}e^{i\theta}\right)^2}{\frac{9}{16} - \frac{1}{4}|\cos\alpha - \frac{1}{2}e^{i\theta}|^2} \\ &= \frac{-\frac{3}{4}\cos\alpha + \frac{3}{8}e^{-i\theta} - \frac{1}{2}e^{-i\theta}\cos^2\alpha + \frac{1}{2}\cos\alpha - \frac{1}{8}e^{i\theta}}{\frac{9}{16} - \frac{1}{4}\left(\cos\alpha - \frac{1}{2}e^{i\theta}\right)\left(\cos\alpha - \frac{1}{2}e^{-i\theta}\right)} \\ &= \frac{-\cos\alpha - 2e^{-i\theta}\cos^2\alpha + \frac{3}{2}e^{-i\theta} - \frac{1}{2}e^{i\theta}}{2 - \cos^2\alpha + \cos\alpha\cos\theta}. \end{aligned}$$

By noting that

$$\begin{aligned} &\left|2 - \cos^2\alpha + \cos\alpha\cos\theta\right|^2 - \left|\cos\alpha + 2e^{-i\theta}\cos^2\alpha - \frac{3}{2}e^{-i\theta} + \frac{1}{2}e^{i\theta}\right|^2 \\ =&4 - 4\cos^2\alpha + \cos^4\alpha + 4\cos\alpha\cos\theta - 2\cos^3\alpha\cos\theta + \cos^2\alpha\cos^2\theta \\ &- \left(\cos\alpha + 2e^{-i\theta}\cos^2\alpha - \frac{3}{2}e^{-i\theta} + \frac{1}{2}e^{i\theta}\right)\left(\cos\alpha + 2e^{i\theta}\cos^2\alpha - \frac{3}{2}e^{i\theta} + \frac{1}{2}e^{-i\theta}\right) \\ =&4 - 4\cos^2\alpha + \cos^4\alpha + 4\cos\alpha\cos\theta - 2\cos^3\alpha\cos\theta + \cos^2\alpha\cos^2\theta \\ &- 4\left(1 + \cos^4\alpha + \cos^3\alpha\cos\theta - \frac{7}{4}\cos^2\alpha + \cos^2\alpha\cos^2\theta - \frac{1}{2}\cos\alpha\cos\theta - \frac{3}{4}\cos^2\theta\right) \\ =&3\left(-\cos^4\alpha - 2\cos^3\alpha\cos\theta - \cos^2\alpha\cos^2\theta + \cos^2\alpha + \cos\alpha\cos\theta + \cos^2\theta\right) \\ =&3\left(1 - \cos^2\alpha)(\cos\alpha + \cos\theta\right)^2 \geq 0, \end{aligned}$$

which shows that $|z_0| \leq 1$. Thus, by *Cohn's Rule*, we know that f exists three zeros in $\overline{\mathbb{D}}$, that is $A, B, C \in \overline{\mathbb{D}}$, and so $|\widetilde{\omega}(z)| < 1$ for $z \in \mathbb{D}$.

Finally, we consider the case $\omega_2 = e^{i\theta}z^2$, by substituting $\omega_2 = e^{i\theta}z^2$ into (3.4), we have

$$\widetilde{\omega}(z) = z^2 e^{i\theta}\left(\frac{e^{i\theta}z^3 + \frac{1+z\cos\alpha}{\cos\alpha+z}}{\frac{1+z\cos\alpha}{\cos\alpha+z} + e^{i\theta}z^3}\right) = z^2 e^{i\theta},$$

which implies that $|\widetilde{\omega}(z)| < 1$. The proof of Theorem 3.1 is completed. □

Remark 3.1. The range of the dilatation function $\widetilde{\omega}(z)$ in Theorem 3.1 is not contained in $\mathbb{D}$ for $n \geq 3$. To check this, we choose $\omega_n(z) = z^n$ and substitute it into (3.4), yields

$$\widetilde{\omega}(z) = z^n \frac{z^{n+1} + (\frac{n}{2} - 1)z + \frac{n}{2}\frac{1+z\cos\alpha}{\cos\alpha+z}}{\frac{1+z\cos\alpha}{\cos\alpha+z} - (\frac{n}{2} - 1)z^n\frac{1+z\cos\alpha}{\cos\alpha+z} + \frac{n}{2}z^{n+1}} = z^n R(z).$$

It is a simple calculation to see that $R(e^{i\alpha}) = 1$ and $1/\overline{R(1/\overline{z})} = R(z)$. So $R(z)$ maps the closed unit disk $|z| \leq 1$ onto itself. Hence R can be written as a finite Blaschke product of order $n + 1$. However, $n/2$ is product of the module of zeros of R in the unit disk $\mathbb{D}$. This means that there exists a point $z_0 \in \mathbb{D}$ such that $|\widetilde{\omega}(z_0)| > 1$ for $n \geq 3$. Thus, the restriction of $n = 1, 2$ in Theorem 3.1 becomes necessary for our result.

Finally, we give an example to illuminate the main result.

Example 3.1. In Theorem 3.1, by setting $f_1 = h_1 + \overline{g_1}$ with

$$h_1 - g_1 = \frac{1}{2i} \log \left(\frac{1+iz}{1-iz} \right)$$

and $\omega_1 = z$, we get

$$h_1 = -\frac{1}{2} \log(1-z) + \frac{1-i}{4} \log(1+iz) + \frac{1+i}{4} \log(1-iz),$$

and

$$g_1 = -\frac{1}{2} \log(1-z) + \frac{1+i}{4} \log(1+iz) + \frac{1-i}{4} \log(1-iz).$$

Consider the function

$$F_1 = f_c * f_1 = H_1 + \overline{G_1},$$

we have

$$\begin{aligned} H_1 = h_c * h_1 &= \frac{1}{2}(zh_1' + h_1) \\ &= \frac{1}{2} \left(\frac{z}{(1+z^2)(1-z)} - \frac{1}{2} \log(1-z) + \frac{1-i}{4} \log(1+iz) + \frac{1+i}{4} \log(1-iz) \right), \end{aligned}$$

and

$$\begin{aligned} G_1 = g_c * g_1 &= \frac{1}{2}(zg_1' - g_1) \\ &= \frac{1}{2} \left(\frac{z^2}{(1+z^2)(1-z)} + \frac{1}{2} \log(1-z) - \frac{1+i}{4} \log(1+iz) - \frac{1-i}{4} \log(1-iz) \right). \end{aligned}$$

Thus, we obtain

$$\begin{aligned} F_1 =& \Re(H_1 + G_1) + i\Im(H_1 - G_1) \\ =& \Re \left(\frac{z+z^2}{2(1+z^2)(1-z)} + \frac{1}{4i} \log \frac{1+iz}{1-iz} \right) \\ &+ i\Im \left(\frac{z}{2(1+z^2)} - \frac{1}{2} \log(1-z) + \frac{1}{4} \log\left(1+z^2\right) \right). \end{aligned}$$

The images of concentric circles inside $\mathbb{D}$ under the harmonic mappings f_c and f_1 are shown in Figure 1 and Figure 2, respectively. The images of these concentric circles under the convolution map $f_c * f_1 = F_1$ are shown in Figure 3.

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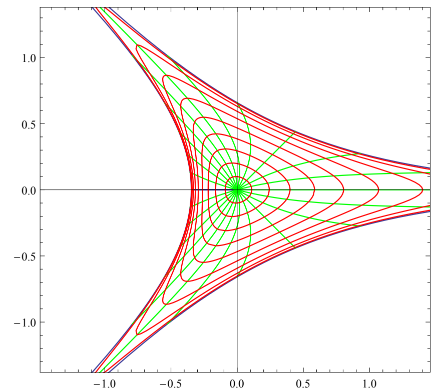

Figure 2. Image of f_1

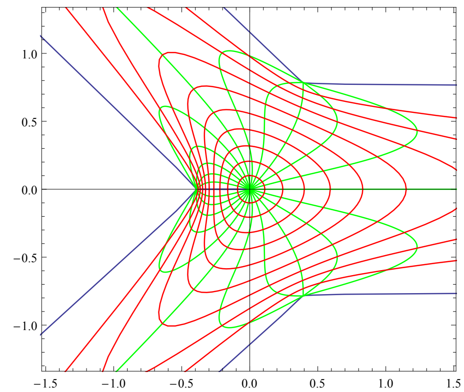

Figure 3. Image of $f_c * f_1$

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