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MULTI-PEAKON SOLUTIONS TO A FOUR-COMPONENT CAMASSA-HOLM TYPE SYSTEM*

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Abstract A four-component Camassa-Holm type system with cubic nonlinearity is investigated. It allows an arbitrary function $\Gamma(x,t)$ to be involved in to include some existing integrable peakon equations as special reductions. We obtain N-peakon solutions of the four-component Camassa-Holm type system with two special cases of $\Gamma(x,t)$. In particular, we give one- and two-peakon solutions in an explicit formula and are graphically plotted. Further, some interesting peaked solutions are found: some peakon waves possessing positive and negative amplitudes while others decaying and growing amplitudes.

Keywords Peakon wave, four-component Camassa-Holm type system, integrable system.

MSC(2010) 35D30, 35C08, 35C07, 37K10.

1. Introduction

In 1993, Camassa and Holm (CH) derived a completely integrable dispersive shallow water equation [2], which has been studied quite extensively in the past two decades. A significant property of this equation is that the CH equation admits peaked soliton (peakon) and multi-peakon wave solutions. As an integrable equation more diverse studies on the CH equation have been remarkably developed in the literatures [1, 10, 17]. Recently, more integrable equations with peakon properties attract much attention, including the Degasperis-Procesi (DP) equation [3, 11], the Fokas-Olver-Rosenau-Qiao(FORQ) equation [4, 5, 9, 12–14], the Novikov equation [7, 15], and other CH type equations [6, 8, 16, 18]. The CH and the DP equations are completely integrable peakon systems with quadratical nonlinearity, and the FORQ and the Novikov equations are typical integrable peakon systems with cubic nonlinearity.

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Recently, Li, Liu and Popowicz studied the following 3×3 spectral problem [8]

$$\Phi_x = U\Phi,\ U = \begin{pmatrix} 0 & \lambda m_1 & 1 \\ \lambda n_1 & 0 & \lambda m_2 \\ 1 & \lambda n_2 & 0 \end{pmatrix}, \tag{1.1}$$

where $m_i = m_i(x,t)$, $n_i = n_i(x,t)$, $i = 1,2$. Actually, this spectral problem is a special case of the multi-component problem studied in [17]. The spectral problem (1.1) is interesting because it could cover some 3×3 spectral problems for CH type equations as special cases, such as the three-component CH system proposed by Geng and Xue [6], one and two-component Novikov equations [7,15], and one and two-component Qiao equations [12,16].

Based on the spectral problem (1.1), the authors [8] gave the following four-component CH type system

$$\begin{cases} m_{1t} + (\Gamma m_1)_x + n_2(g_1 g_2 - \Gamma) + m_1(f_2 g_2 + 2 f_1 g_1) = 0, \\ m_{2t} + (\Gamma m_2)_x - n_1(g_1 g_2 - \Gamma) - m_2(f_1 g_1 + 2 f_2 g_2) = 0, \\ n_{1t} + (\Gamma n_1)_x - m_2(f_1 f_2 - \Gamma) - n_1(f_2 g_2 + 2 f_1 g_1) = 0, \\ n_{2t} + (\Gamma n_2)_x + m_1(f_1 f_2 - \Gamma) + n_2(f_1 g_1 + 2 f_2 g_2) = 0, \end{cases} \tag{1.2}$$

where

$$\begin{aligned} & f_1 = u_2 - v_{1x},\ f_2 = u_1 + v_{2x},\ g_1 = v_2 + u_{1x},\ g_2 = v_1 - u_{2x}, \\ & m_i = u_i - u_{ixx},\ n_i = v_i - v_{ixx},\ i = 1,2, \end{aligned} \tag{1.3}$$

and $\Gamma = \Gamma(x,t)$ is an arbitrary function. The system (1.2) is integrable in the sense of Lax pair associated with the spectral problem (1.1) and the following auxiliary spectral problem

$$\Phi_t = V\Phi,\ V = \begin{pmatrix} -f_1 g_1 & \lambda^{-1} g_1 - \lambda\Gamma m_1 & -g_1 g_2 \\ \lambda^{-1} f_1 - \lambda\Gamma n_1 & -\lambda^{-2} + f_1 g_1 + f_2 g_2 & \lambda^{-1} g_2 - \lambda\Gamma m_2 \\ -f_1 f_2 & \lambda^{-1} f_2 - \lambda\Gamma n_2 & -f_2 g_2 \end{pmatrix}. \tag{1.4}$$

We notice that the system (1.2) contains an arbitrary function Γ, which amazingly leads (1.2) to some different CH type equations through certain choices of Γ. For instance, some cubic systems could be reduced (see [8]).

In particular, if $\Gamma = 0$, we have

$$\begin{cases} m_{1t} + n_2 g_1 g_2 + m_1(f_2 g_2 + 2 f_1 g_1) = 0, \\ m_{2t} - n_1 g_1 g_2 - m_2(f_1 g_1 + 2 f_2 g_2) = 0, \\ n_{1t} - m_2 f_1 f_2 - n_1(f_2 g_2 + 2 f_1 g_1) = 0, \\ n_{2t} + m_1 f_1 f_2 + n_2(f_1 g_1 + 2 f_2 g_2) = 0, \end{cases} \tag{1.5}$$

where f_i, g_i, m_i and n_i, $i = 1,2$ are given in (1.3). The system (1.5) can be rewritten in the form of the following bi-Hamiltonian structure

$$\begin{pmatrix} m_1 \\ m_2 \\ n_1 \\ n_2 \end{pmatrix}_t = \mathscr{K} \begin{pmatrix} \frac{\delta H_0}{\delta m_1} \\ \frac{\delta H_0}{\delta m_2} \\ \frac{\delta H_0}{\delta n_1} \\ \frac{\delta H_0}{\delta n_2} \end{pmatrix} = (\mathscr{J} + \mathscr{F}) \begin{pmatrix} \frac{\delta H_1}{\delta m_1} \\ \frac{\delta H_1}{\delta m_2} \\ \frac{\delta H_1}{\delta n_1} \\ \frac{\delta H_1}{\delta n_2} \end{pmatrix}, \tag{1.6}$$

where

$$\mathscr{K} = \begin{pmatrix} 0 & -1 & \partial & 0 \\ 1 & 0 & 0 & \partial \\ \partial & 0 & 0 & 1 \\ 0 & \partial & -1 & 0 \end{pmatrix}, \mathscr{J} = \begin{pmatrix} 2m_1\partial^{-1}m_1 & -m_1\partial^{-1}m_2 & \mathscr{J}_{13} & \mathscr{J}_{14} \\ -m_2\partial^{-1}m_1 & 2m_2\partial^{-1}m_2 & \mathscr{J}_{23} & \mathscr{J}_{24} \\ -\mathscr{J}_{13}^* & -\mathscr{J}_{23}^* & 2n_1\partial^{-1}n_1 & -n_1\partial^{-1}n_2 \\ -\mathscr{J}_{14}^* & -\mathscr{J}_{24}^* & -n_2\partial^{-1}n_1 & 2n_2\partial^{-1}n_2 \end{pmatrix},$$

$$\mathscr{F} = (2\mathcal{P} + \mathcal{S}\partial)(\partial^3 - 4\partial)^{-1}\mathcal{P}^T - (2\mathcal{S} + \mathcal{P}\partial)(\partial^3 - 4\partial)^{-1}\mathcal{S}^T,$$

$$\mathscr{J}_{13} = -2m_1\partial^{-1}n_1 - n_2\partial^{-1}m_2, \mathscr{J}_{14} = m_1\partial^{-1}n_2 + n_2\partial^{-1}m_1,$$

$$\mathscr{J}_{23} = m_2\partial^{-1}n_1 + n_1\partial^{-1}m_2, \mathscr{J}_{24} = -2m_2\partial^{-1}n_2 - n_1\partial^{-1}m_1,$$

$$\mathcal{P} = (m_1, m_2, -n_1, -n_2)^T, \mathcal{S} = (-n_2, n_1, -m_2, m_1)^T,$$

$$H_0 = \int (f_1g_1 + f_2g_2)(m_2f_2 + n_1g_1)dx, H_1 = \int (m_2f_2 + n_1g_1)dx.$$

The aim of this paper is to construct multi-peakon solutions for the four-component CH type system (1.2) with a special Γ. In the case of $\Gamma = 0$, we solve the system (1.5) and obtain its multi-peakon solutions, which are not in the traveling wave type. In the case of $\Gamma = \rho$ (ρ is a non-zero constant), we find the four-component CH type system (1.2) possesses the traveling wave type multi-peakon solutions.

2. Multi-peakon solutions

In the following, we will derive multi-peakon solutions to the four-component CH type system (1.2) with $\Gamma = \rho$, where ρ is a constant.

Case 1. Let us suppose that one-peakon solution of the four-component CH type system (1.2) with $\Gamma = \rho$ is of the following form

$$u_1 = p_1e^{-|x-q_1|}, u_2 = r_1e^{-|x-q_1|}, v_1 = s_1e^{-|x-q_1|}, v_2 = \tau_1e^{-|x-q_1|}, \tag{2.1}$$

where p_1, r_1, s_1, τ_1 and q_1 are functions of t to be determined. With the help of distribution theory, we are able to write out u_{1x}, u_{2x}, v_{1x}, v_{2x}, m_1, m_2, n_1 and n_2 as follows

$$\begin{aligned} u_{1x} &= -p_1\mathrm{sgn}(x-q_1)e^{-|x-q_1|}, \ \ m_1 = 2p_1\delta(x-q_1), \\ u_{2x} &= -r_1\mathrm{sgn}(x-q_1)e^{-|x-q_1|}, \ \ m_2 = 2r_1\delta(x-q_1), \\ v_{1x} &= -s_1\mathrm{sgn}(x-q_1)e^{-|x-q_1|}, \ \ n_1 = 2s_1\delta(x-q_1), \\ v_{2x} &= -\tau_1\mathrm{sgn}(x-q_1)e^{-|x-q_1|}, \ \ n_2 = 2\tau_1\delta(x-q_1). \end{aligned} \tag{2.2}$$

Substituting (2.1) and (2.2) into (1.2) with $\Gamma = \rho$, we arrive at the following one-peakon dynamical system

$$\begin{aligned}
&q_{1t} = \rho,\\
&p_{1t} = -\frac{1}{3}\Delta_{11}p_1 + (\rho - \theta_{11})\tau_1,\\
&\tau_{1t} = (\rho - \theta_{11})p_1 - \frac{1}{3}\Delta_{11}\tau_1,\\
&r_{1t} = \frac{1}{3}\Delta_{11}r_1 - (\rho - \theta_{11})s_1,\\
&s_{1t} = -(\rho - \theta_{11})r_1 + \frac{1}{3}\Delta_{11}s_1,
\end{aligned} \tag{2.3}$$

where $\Delta_{11} = p_1 s_1 + r_1\tau_1$, $\theta_{11} = p_1 r_1 + s_1\tau_1$. Δ_{11} and θ_{11} taking derivative with respect to t, and using Eqs. (2.3), we have the following relations

$$\Delta_{11t} = 0,\ \ \theta_{11t} = 0. \tag{2.4}$$

We get

$$\Delta_{11} = A_1,\ \ \theta_{11} = B_1, \tag{2.5}$$

where A_1 and B_1 are arbitrary integration constants. Therefore, Eqs. (2.3) becomes

$$\begin{aligned}
&q_{1t} = \rho,\\
&\begin{pmatrix} p_1 \\ \tau_1 \end{pmatrix}_t = \begin{pmatrix} -\dfrac{1}{3} & A_1\rho - B_1 \\ \rho - B_1 & -\dfrac{1}{3}A_1 \end{pmatrix} \begin{pmatrix} p_1 \\ \tau_1 \end{pmatrix},\\
&\begin{pmatrix} r_1 \\ s_1 \end{pmatrix}_t = \begin{pmatrix} \dfrac{1}{3}A_1 & -(\rho - B_1) \\ -(\rho - B_1) & \dfrac{1}{3}A_1 \end{pmatrix} \begin{pmatrix} r_1 \\ s_1 \end{pmatrix}.
\end{aligned} \tag{2.6}$$

We arrive at the following general solution of Eq. (2.6)

$$\begin{aligned}
&q_1 = \rho t + \omega_1,\\
&p_1 = C_1 e^{\lambda_1^{(1)}t} + C_2 e^{\lambda_2^{(1)}t},\ \tau_1 = C_1 e^{\lambda_1^{(1)}t} - C_2 e^{\lambda_2^{(1)}t},\\
&r_1 = C_3 e^{\lambda_3^{(1)}t} + C_4 e^{\lambda_4^{(1)}t},\ s_1 = -C_3 e^{\lambda_3^{(1)}t} + C_4 e^{\lambda_4^{(1)}t},
\end{aligned} \tag{2.7}$$

where

$$\begin{aligned}
&\lambda_1^{(1)} = \frac{1}{3}(3\rho - A_1 - 3B_1),\ \ \lambda_2^{(1)} = \frac{1}{3}(-3\rho - A_1 + 3B_1),\\
&\lambda_3^{(1)} = \frac{1}{3}(3\rho + A_1 - 3B_1),\ \ \lambda_4^{(1)} = \frac{1}{3}(-3\rho + A_1 + 3B_1),
\end{aligned} \tag{2.8}$$

$A_1 = -2(C_2C_3 - C_1C_4)$, $B_1 = 2(C_2C_3 + C_1C_4)$ and C_1, C_2, C_3, C_4, ω_1 are arbitrary integration constants. From (2.1) and (2.7), we obtain one-peakon solution of (1.2) with $\Gamma = \rho$:

$$\begin{aligned}
&u_1 = (C_1 e^{\lambda_1^{(1)}t} + C_2 e^{\lambda_2^{(1)}t})e^{-|\xi_1|},\ u_2 = (C_3 e^{\lambda_3^{(1)}t} + C_4 e^{\lambda_4^{(1)}t})e^{-|\xi_1|},\\
&v_1 = (-C_3 e^{\lambda_3^{(1)}t} + C_4 e^{\lambda_4^{(1)}t})e^{-|\xi_1|},\ v_2 = (C_1 e^{\lambda_1^{(1)}t} - C_2 e^{\lambda_2^{(1)}t})e^{-|\xi_1|},
\end{aligned} \tag{2.9}$$

where $\xi_1 = x-\rho t-\omega_1$ and $\lambda_i^{(1)}$ $(i = 1, 2, 3, 4)$ are given in (2.8). See Figs. 1-2 for the profile of the one-peakon dynamics for the potentials u_i and v_i $(i = 1, 2)$ in (2.9). In Fig. 1, (a),(d) and (b),(c) show that the one-peakon with amplitudes exponentially decaying and growing with time t, respectively. And an interesting phenomenon is shown in Fig. 2: the amplitude of u_1 (or u_2) is changed from positive to negative (or negative to positive) while v_1 (or v_2) has positive amplitude which is changed from decaying to growing along the t axis.

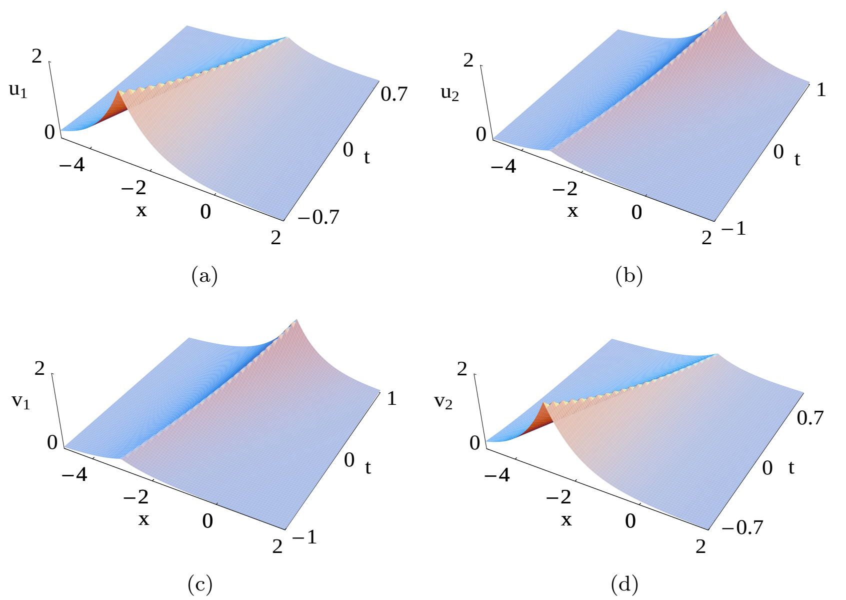

Figure 1. The one-peakon wave (2.9) with $\rho = 1$, $C_1 = 1$, $C_2 = C_3 = 0$, $C_4 = \frac{2}{3}$ and $\omega_1 = -2$.

Case 2. A two-peakon solution is given in the form of

$$\begin{aligned} &u_1 = p_1 e^{-|x-q_1|} + p_2 e^{-|x-q_2|},\ u_2 = r_1 e^{-|x-q_1|} + r_2 e^{-|x-q_2|},\\ &v_1 = s_1 e^{-|x-q_1|} + s_2 e^{-|x-q_2|},\ v_2 = \tau_1 e^{-|x-q_1|} + \tau_2 e^{-|x-q_2|}, \end{aligned} \tag{2.10}$$

where p_i, r_i, s_i, τ_i and q_i $(i = 1, 2)$ are functions of t to be determined. In a similar process as case 1, we can find the two-peakon dynamical system, which consists of ten equations. Let us start from the first two equations: $q_{1t} = \rho$ and $q_{2t} = \rho$, which yield

$$q_1 = \rho t + \omega_1,\ \ q_2 = \rho t + \omega_2, \tag{2.11}$$

where ω_1 and ω_2 are constants. Without loss of generality, we suppose $\omega_2 > \omega_1$. With the help of (2.11), the two-peakon dynamical system can be rewritten as

$$\begin{aligned} p_{it} &= \rho\tau_i + \frac{2}{3}\Delta_{ii} p_i - E_1^{(i)} E_2^{(i)} \tau_i - (E_3^{(i)} E_2^{(i)} + 2E_1^{(i)} E_4^{(i)}) p_i,\\ \tau_{it} &= \rho p_i + \frac{2}{3}\Delta_{ii}\tau_i - E_3^{(i)} E_4^{(i)} p_i - (E_1^{(i)} E_4^{(i)} + 2E_3^{(i)} E_2^{(i)})\tau_i,\\ r_{it} &= -\rho s_i - \frac{2}{3}\Delta_{ii} r_i + E_1^{(1)} E_2^{(1)} s_i + (E_1^{(i)} E_4^{(i)} + 2E_3^{(i)} E_2^{(1)}) r_i,\\ s_{it} &= -\rho r_i - \frac{2}{3}\Delta_{ii} s_i + E_3^{(i)} E_4^{(i)} r_i + (E_3^{(i)} E_2^{(i)} + 2E_1^{(i)} E_4^{(i)}) s_i,\ \ (i = 1, 2). \end{aligned} \tag{2.12}$$

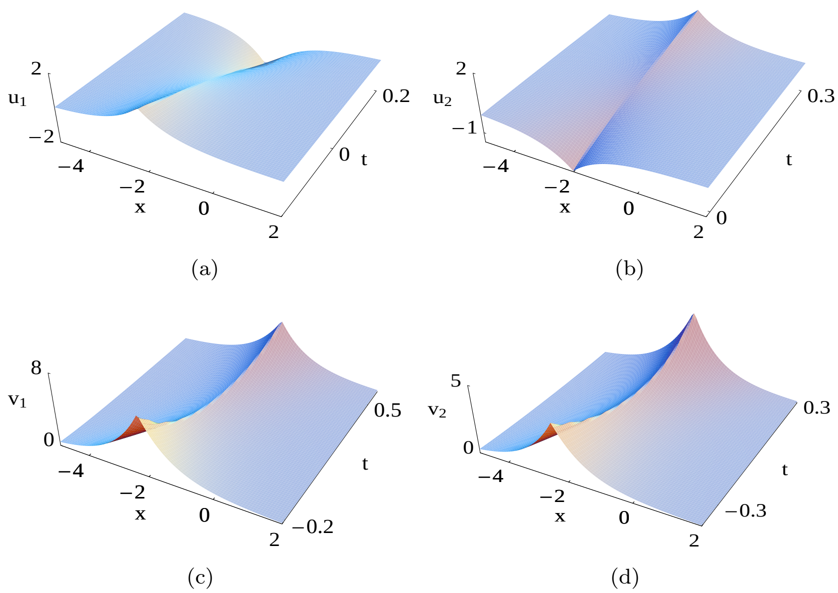

Figure 2. The one-peakon wave (2.9) with $\rho=-4$, $C_1=-C_2=1$, $C_3=-2$, $C_4=\frac{2}{3}$ and $\omega_1=-2$.

where

$$
\begin{aligned}
E_1^{(1)} &= \tau_1+(\tau_2+p_2)\Omega_{12}, \;\; E_2^{(1)}=s_1-(r_2-s_2)\Omega_{12},\\
E_3^{(1)} &= p_1+(\tau_2+p_2)\Omega_{12}, \;\; E_4^{(1)}=r_1+(r_2-s_2)\Omega_{12},\\
E_1^{(2)} &= \tau_2+(\tau_1-p_1)\Omega_{12}, \;\; E_2^{(2)}=s_2+(r_1+s_1)\Omega_{12},\\
E_3^{(2)} &= p_2-(\tau_1-p_1)\Omega_{12}, \;\; E_4^{(2)}=r_2+(r_1+s_1)\Omega_{12},
\end{aligned}
$$

and $\Omega_{12}=e^{\omega_1-\omega_2}$. Apparently, (2.12) implies the following relations

$$\Delta_{iit}=(p_is_i+r_i\tau_i)_t=0,\;\;(i=1,2). \tag{2.13}$$

Therefore, we obtain

$$\Delta_{ii}=A_i,\;\;(i=1,2), \tag{2.14}$$

where A_i $(i=1,2)$ are the integration constants.

In particular, as $\tau_1=p_1$ and $\tau_2=-p_2$, (2.12) is reduced to

$$
\begin{aligned}
&\begin{pmatrix} p_1\\ p_2\end{pmatrix}_t=\begin{pmatrix}\frac{1}{3}(-4A_1+3\rho) & 0\\ 0 & -\frac{1}{3}(4A_2+3\rho)\end{pmatrix}\begin{pmatrix} p_1\\ p_2\end{pmatrix},\\
&\begin{pmatrix} r_1\\ r_2\\ s_1\\ s_2\end{pmatrix}_t=\begin{pmatrix}\frac{1}{3}A_1 & -A_1\Omega_{12} & A_1-\rho & A_1\Omega_{12}\\ -A_2\Omega_{12} & \frac{1}{3}A_2 & -A_2\Omega_{12} & -A_2-\rho\\ A_1-\rho & A_1\Omega_{12} & \frac{1}{3}A_1 & -A_1\Omega_{12}\\ -A_2\Omega_{12} & -A_2-\rho & -A_2\Omega_{12} & \frac{1}{3}A_2\end{pmatrix}\begin{pmatrix} r_1\\ r_2\\ s_1\\ s_2\end{pmatrix},
\end{aligned}
\tag{2.15}
$$

where $\Omega_{12} = e^{\omega_1 - \omega_2}$. Solving (2.15), we obtain

$$\begin{aligned} p_1 &= \tau_1 = C_1 e^{\lambda_1^{(2)} t}, \ \ p_2 = -\tau_2 = C_2 e^{\lambda_2^{(2)} t}, \\ r_1 &= -\frac{(2A_1 + A_2)C_3}{3A_2\Omega_{12}} e^{\lambda_3^{(2)} t} - C_5 e^{\lambda_5^{(2)} t} + \frac{3A_1\Omega_{12}C_6}{A_1 + 2A_2} e^{\lambda_6^{(2)} t}, \\ r_2 &= C_3 e^{\lambda_3^{(2)} t} + C_4 e^{\lambda_4^{(2)} t} - C_6 e^{\lambda_6^{(2)} t}, \\ s_1 &= -\frac{(2A_1 + A_2)C_3}{3A_2\Omega_{12}} e^{\lambda_3^{(2)} t} + C_5 e^{\lambda_5^{(2)} t} - \frac{3A_1\Omega_{12}C_6}{A_1 + 2A_2} e^{\lambda_6^{(2)} t}, \\ s_2 &= C_3 e^{\lambda_3^{(2)} t} + C_4 e^{\lambda_4^{(2)} t} + C_6 e^{\lambda_6^{(2)} t}, \end{aligned} \tag{2.16}$$

where

$$\begin{aligned} &\lambda_1^{(2)} = \frac{1}{3}(-4A_1 + 3\rho), \ \lambda_2^{(2)} = -\frac{1}{3}(4A_2 + 3\rho), \ \lambda_3^{(2)} = \frac{1}{3}(4A_1 - 3\rho), \\ &\lambda_4^{(2)} = -\frac{1}{3}(2A_2 + 3\rho), \ \lambda_5^{(2)} = \frac{1}{3}(-2A_1 + 3\rho), \ \lambda_6^{(2)} = \frac{1}{3}(4A_2 + 3\rho), \end{aligned} \tag{2.17}$$

with $\Omega_{12} = e^{\omega_1 - \omega_2}$, $A_1 = -\frac{2C_1C_2C_3C_6}{2C_1C_3 + 3C_2C_6\Omega_{12}}$, $A_2 = 2C_2C_6$, and ω_k $(k = 1, 2)$, C_i $(i = 1, 2, 3, 4, 5, 6)$ are the integration constants.

Substituting (2.16) into (2.10), we obtain the two-peakon solution of (1.2)

$$\begin{aligned} u_1 =& C_1 e^{\lambda_1^{(2)} t} e^{-|\xi_1|} + C_2 e^{\lambda_2^{(2)} t} e^{-|\xi_2|}, \\ u_2 =& \left(-\frac{(2A_1 + A_2)C_3}{3A_2\Omega_{12}} e^{\lambda_3^{(2)} t} - C_5 e^{\lambda_5^{(2)} t} + \frac{3A_1\Omega_{12}C_6}{A_1 + 2A_2} e^{\lambda_6^{(2)} t} \right) e^{-|\xi_1|} \\ &+ \left(C_3 e^{\lambda_3^{(2)} t} + C_4 e^{\lambda_4^{(2)} t} - C_6 e^{\lambda_6^{(2)} t} \right) e^{-|\xi_2|}, \\ v_1 =& \left(-\frac{(2A_1 + A_2)C_3}{3A_2\Omega_{12}} e^{\lambda_3^{(2)} t} + C_5 e^{\lambda_5^{(2)} t} - \frac{3A_1\Omega_{12}C_6}{A_1 + 2A_2} e^{\lambda_6^{(2)} t} \right) e^{-|\xi_1|} \\ &+ \left(C_3 e^{\lambda_3^{(2)} t} + C_4 e^{\lambda_4^{(2)} t} + C_6 e^{\lambda_6^{(2)} t} \right) e^{-|\xi_2|}, \\ v_2 =& C_1 e^{\lambda_1^{(2)} t} e^{-|\xi_1|} - C_2 e^{\lambda_2^{(2)} t} e^{-|\xi_2|}, \end{aligned} \tag{2.18}$$

where $\xi_j = x - \rho t - \omega_j$ $(j = 1, 2)$ and $\lambda_i^{(2)}$ $(i = 1, 2, 3, 4, 5, 6)$ are given in (2.17). See Figs. 3-4 for the graph of the two-peakon solution (2.18), which are of traveling wave type. Fig. 3 and Fig. 4 show the right-traveling and left-traveling waves, respectively. The amplitudes of the peakons to equation (2.18) grow/decay exponentially with time t. All two-peakon waves have the same velocity ρ. Namely, the collision between the two-peakon waves will never happen.

Case N. Following the procedure in cases 1 and 2, the N-peakon solutions of the four-component Camassa-Holm type system (1.2) are just linear superpositions

$$u_1 = \sum_{j=1}^{N} p_j e^{-|x - q_j|}, u_2 = \sum_{j=1}^{N} r_j e^{-|x - q_j|}, v_1 = \sum_{j=1}^{N} s_j e^{-|x - q_j|}, v_2 = \sum_{j=1}^{N} \tau_j e^{-|x - q_j|}, \tag{2.19}$$

where p_j, r_j, s_j and τ_j $(j = 1, 2, \ldots, N)$ are N amplitudes of the potentials u_1, u_2, v_1 and v_2, respectively, and q_j $(j = 1, 2, \ldots, N)$ are N-peak positions. Functions

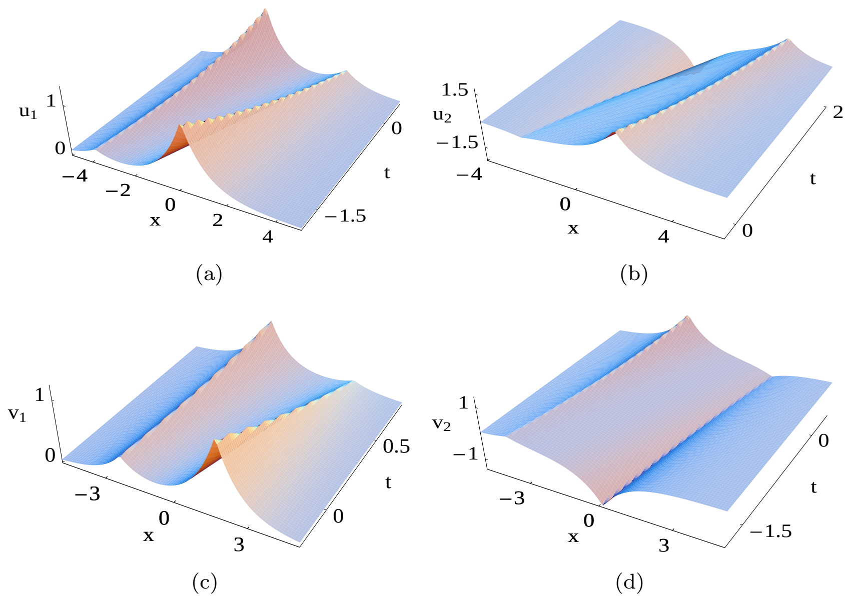

Figure 3. The two-peakon wave (2.18) with $\rho = 1$, $C_1 = 1$, $C_2 = \frac{1}{2}$, $C_3 = \frac{2}{3}$, $C_4 = \frac{3}{5}$, $C_5 = \frac{2}{5}$, $C_6 = -\frac{1}{3}$, $\omega_1 = -2$ and $\omega_2 = 2$.

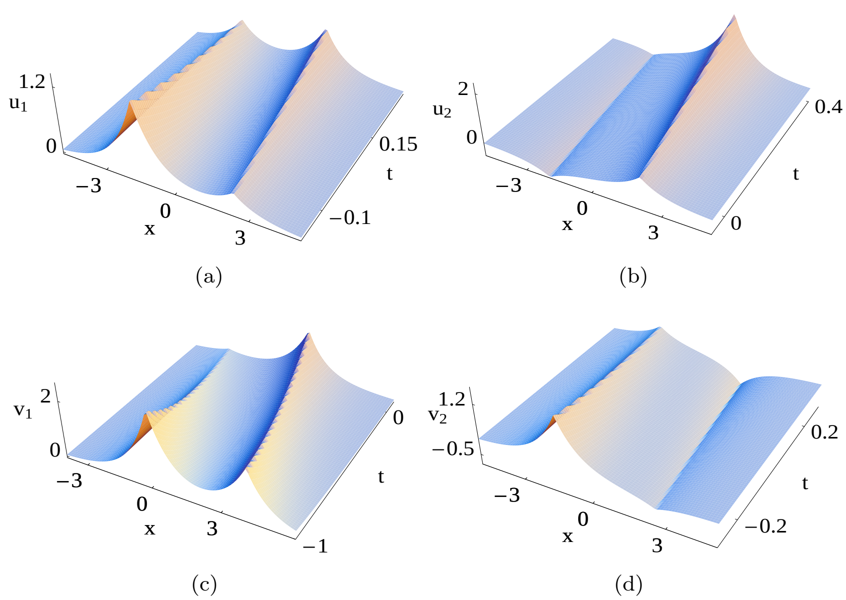

Figure 4. The two-peakon wave (2.18) with $\rho = -2$, $C_1 = 1$, $C_2 = \frac{1}{2}$, $C_3 = \frac{2}{3}$, $C_4 = \frac{3}{5}$,$C_5 = \frac{2}{5}$, $C_6 = \frac{1}{3}$, $\omega_1 = -2$ and $\omega_2 = 2$.

p_j, r_j, s_j, τ_j and q_j $(j = 1, 2, \ldots, N)$ evolve according to the following system:

$$\begin{aligned}
q_{jt} &= \rho,\\
p_{jt} &= \rho\tau_j + \tfrac{2}{3}\Delta_{jj}p_j - \textstyle\sum_{l,k=1}^{N}[\tau_j(\tau_l - p_l\varrho_{jl})(s_k + r_k\varrho_{jk}) + p_j(p_l - \tau_l\varrho_{jl})(s_k + r_k\varrho_{jk})\\
&\quad +2p_j(r_l + s_l\varrho_{jl})(\tau_k - p_k\varrho_{jk})]\Lambda,\\
r_{jt} &= -\rho s_j - \tfrac{2}{3}\Delta_{jj}r_j - \textstyle\sum_{l,k=1}^{N}[-s_j(\tau_l - p_l\varrho_{jl})(s_k + r_k\varrho_{jk}) - r_j(r_l + s_l\varrho_{jl})\\
&\quad \times(\tau_k - p_k\varrho_{jk}) - 2r_j(p_l - \tau_l\varrho_{jl})(s_k + r_k\varrho_{jk})]\Lambda,\\
s_{jt} &= -\rho r_j - \tfrac{2}{3}\Delta_{jj}s_j - \textstyle\sum_{l,k=1}^{N}[-r_j(r_l + s_l\varrho_{jl})(p_k - \tau_k\varrho_{jk})\\
&\quad -s_j(p_l - \tau_l\varrho_{jl})(s_k + r_k\varrho_{jk}) - 2s_j(r_l + s_l\varrho_{jl})(\tau_k - p_k\varrho_{jk})]\Lambda,\\
\tau_{jt} &= \rho p_j + \tfrac{2}{3}\Delta_{jj}\tau_j - \textstyle\sum_{l,k=1}^{N}[p_j(r_l + s_l\varrho_{jl})(p_k - \tau_k\varrho_{jk}) + k_j(r_l + s_l\varrho_{jl})(\tau_k - p_k\varrho_{jk})\\
&\quad +2\tau_j(p_l - \tau_l\varrho_{jl})(s_k + r_k\varrho_{jk})]\Lambda,
\end{aligned} \tag{2.20}$$

where $\varrho_{jl} = \text{sgn}(q_j - q_l)$, $\varrho_{jk} = \text{sgn}(q_j - q_k)$, $\Lambda = e^{-|q_j-q_l|-|q_j-q_k|}$, $\Delta_{jj} = p_js_j + r_j\tau_j$, $(1 \le j; k; l \le N)$. In the above formula, $q_{jt} = \rho$ $(\rho \neq 0)$ implies that N-peakon waves move at the same velocity ρ in the traveling wave type whereas $\rho = 0$ implies that all peak positions do not change along with the time t.

3. Conclusions

In this paper, we study a generalized four-component CH system (1.2) with an arbitrary function $\Gamma(x, t)$. This model provides a large class of peakon dynamical systems and covers several well-known integrable peakon equations associated with 3×3 spectral problems. We obtain two kinds of multi-peakon solutions to the system (1.2) with $\Gamma = \rho$: 1) for $\rho = 0$, the multi-peakon solutions are not in the traveling wave type, and 2) if $\rho \neq 0$, the multi-peakon solutions are in the traveling wave type. Furthermore, the peakon solutions (2.9) and (2.18) can be reduced to the solutions of the model (1.5) if $\rho = 0$.

We believe that some generalizations and reduction of the model (1.2) deserve a further investigation. For example, we can get the one-peakon solution to (1.2) with $\Gamma(x, t) = \rho + \alpha(u_1v_1 + u_2v_2) + \beta(u_1u_2 + v_1v_2)$:

$$\begin{aligned}
&u_1 = (C_1e^{\lambda_1 t} + C_2e^{\lambda_2 t})e^{-|\zeta_1|},\ u_2 = (C_3e^{\lambda_3 t} + C_4e^{\lambda_4 t})e^{-|\zeta_1|},\\
&v_1 = (C_3e^{\lambda_3 t} - C_4e^{\lambda_4 t})e^{-|\zeta_1|},\ v_2 = (-C_1e^{\lambda_1 t} + C_2e^{\lambda_2 t})e^{-|\zeta_1|},
\end{aligned}$$

where $\zeta_1 = x - (\rho + \alpha A_1 + \beta B_1)t - \omega_1$, $\lambda_1 = \frac{1}{3}(-3\rho - A_1 - 3\alpha A_1 + 3B_1 - 3\beta B_1)$, $\lambda_2 = \frac{1}{3}(3\rho - A_1 + 3\alpha A_1 - 3B_1 + 3\beta B_1)$, $\lambda_3 = \frac{1}{3}(-3\rho + A_1 - 3\alpha A_1 + 3B_1 - 3\beta B_1)$, $\lambda_4 = \frac{1}{3}(3\rho + A_1 + 3\alpha A_1 - 3B_1 + 3\beta B_1)$, $A_1 = 2(C_2C_3 - C_1C_4)$, $B_1 = 2(C_2C_3 + C_1C_4)$ and α, β, ρ, ω_1, and C_i $(i = 1, 2, 3, 4)$ are constants. The question arises: how to construct multi-peakon (for $N \geq 2$) solutions to (1.2) with $\Gamma(x, t) = \rho + \alpha(u_1v_1 + u_2v_2) + \beta(u_1u_2 + v_1v_2)$? This question is still under investigation.

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