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AN IMPROVED BOX-COUNTING METHOD TO ESTIMATE FRACTAL DIMENSION OF IMAGES

Jundong Yan[1], Yuanyuan Sun[1,†], Shanshan Cai[1], and Xiaopeng Hu[1]

Abstract Fractal dimension (FD) reflects the intrinsic self-similarity of an image and can be used in image classification, image segmentation and texture analysis. The differential box-counting (DBC) method is a common approach to calculating the FD values. This paper proposes an improved DBC-based approach to optimizing the performance of the method in the following ways: reducing fitting errors by decreasing step lengths, considering under-counting boxes on the border of two neighboring box-blocks and making better use of all the pixels in the blocks while not neglecting the middle parts. The experimental results show that the fitting error of the new method can be decreased to 0.012879. The average distance of the FD values is decreased by 16.0% in the divided images and the average variance of the FD values is decreased by 30% in the scaled images, compared with other modified methods. The results show that the new method has a better performance in the recognition of the same type of images and the scaled images.

Keywords Ddifferential box-counting (DBC) method, fractal dimension, improved DBC method.

MSC(2010) 68U10, 28A80.

1. Introduction

The self-similarity plays an important role in describing the textures of images, which is too complicated to be described by the traditional Euclidean geometry [4, 6, 14, 19]. Mandelbrot proposed a new mathematical model of fractal dimension (FD) to solve the problem [13]. The method has comprehensive applications in image segmentation, texture analysis, graphic classification and so on [1, 5]. There are three kinds of methods for calculating the FD according to the classification by Lopes et al. [12]: the differential box-counting (DBC) method, the fractional Brownian motion (fBm) method and the area measurement method. Sarkar et al. proposed the DBC method which is frequently used in computing FD due to its conciseness and automatic computability [16–18]. Many research reports show that the method leads to better presentation in the calculation of FD for gray images compared with other methods [3, 7, 8, 15]. However, the DBC method shows some disadvantages. The prime drawbacks of the DBC method lies in three aspects [9, 10, 17]: the biggish fitting errors, under-counting boxes at the border of two neighboring box blocks and neglecting the pixels of middle part of a graph. Some improved methods were proposed afterwards. Wang et al. proposed an improved

[†]the corresponding author. Email address:syuan@dlut.edu.cn(Y. Sun)
[1]School of Computer Science and Technology, Dalian University of Technology, Dalian, China

DBC method by considering all the pixels of a box [20]. Li et al. optimized the DBC method in counting boxes [10]. Liu et al. proposed an improved DBC method by considering the border of two neighboring box blocks [11]. In this paper, we propose an improved DBC method by considering the major drawbacks of the original DBC method.

The paper is organized as follows. In section 2, the fundamental analysis of DBC method is discussed. In section 3, an improved method is proposed. In Section 4, some real texture images are used to discuss and evaluate the new approach. Section 5 presents a conclusion.

2. Related work

2.1. The DBC method

The basic principle of DBC method is derived from the self-similarity of the texture of the images. In the classic method of DBC, the $(x,\ y)$ is the pixel coordinates of the image and the z is the gray value of the image. The plane of the image is partitioned into non-overlapping blocks of size $s \times s$, where $2 \leq s \leq M/2$ and M is the size of picture. The scale is $r = M/s$. In each block, there are a column of boxes of size $s \times s \times s'$. The s' represents the height of each box, which is computed by the formula $G/s' = M/s$. The G is the gray level of the image and it is 255 for gray image. Assign the numbers 1, 2... to the boxes as shown in Figure 1. The

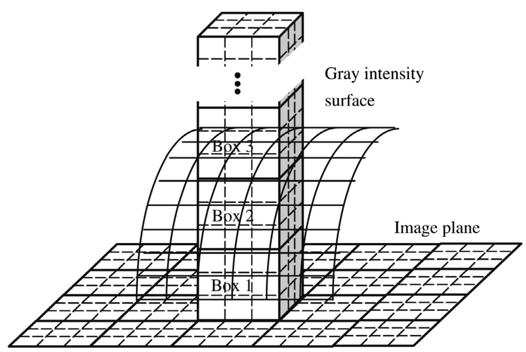


Figure 1. Sketch of determination of the number of boxes (n_r) by DBC method (quoted from Li et al. [10]).

pixels of the minimum and maximum gray-level (written as $I_{\min}$ and $I_{\max}$) in the $(i,\ j)$ block fall in the boxes numbered k and l, respectively. The number of boxes covering this block are calculated as follows:

$$n_r(i,j) = l - k + 1. \tag{2.1}$$

And N_r is counted for different values of r as

$$N_r = \sum n_r(i,j). \tag{2.2}$$

The formula of computing the FD of DBC method is as follows:

$$D = \lim_{r \to 0} \frac{\log(N_r)}{\log(1/r)}, \tag{2.3}$$

where N_r denotes the least number of boxes for covering the region with radius r. The FD of DBC method is a certain value for a rigorous self-similarity image. But the objects in the real world do not have strict self-similarity. Therefore it is difficult to compute an accurate value of FD. However those nature objects still possess the self-similarity in a major extent. And the DBC method is still helpful in exploring and recognizing the self-similarities of various objects in reality.

2.2. The discussion of the DBC method

Although the DBC method is sound in theoretical support, it still has three inherent shortcomings. In the following we discuss in detail.

(i) Lee et al. proposed that the classical method of DBC has major fitting errors in the process of the least square method [9]. The result of the least square method of a sample image is illustrated in Figure 2. The picture is stemmed from the texture image of communal atlas of Brodatz[2]. The x axis represents $\log(1/r)$ and the y axis represents $\log(N_r)$. We can see that the both ends of points have sharp drops compared with other points. It causes a substantial increase of variance summation of the fitting results.

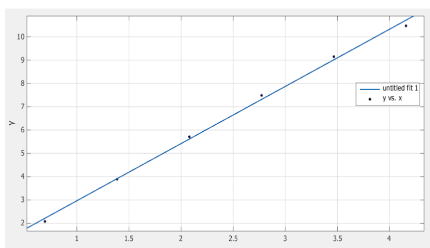
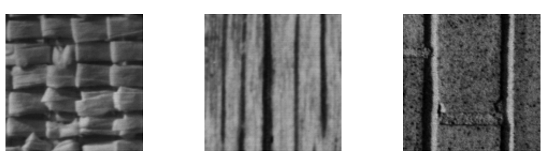


Figure 2. The fitting result of the least square method

(ii) Sarkar et al. presented that the DBC method has under-counting box number at the border of two neighboring box blocks [17]. Under-counting the number of boxes may happen when the picture has a dramatic gray-scaled variation at the border of two neighboring box blocks. Figures 3 show an interpretation of this phenomenon.



(a) Wall (b) Board (c) Brick

Figure 3. Images of dramatic gray-scaled variation at the border

(iii) The boxes for covering $s \times s$ block are counted in the number $l - k + 1$. It results in the pixels of middle part not being considered. As shown in Eq. (2.1), The n_r is calculated by the maximum-value pixel l and the minimum-value pixel k. The pixels of middle part are neglected in this method. Wang et al. proposed that the neglecting of pixels in middle part results in a biggish fitting error [20].

Many improved DBC methods are proposed to optimize the performance of classical DBC method. Wang et al. proposed a DBC method counting the box number by the maximum pixel and minimum pixel in the block [20]. It neglects the other pixels. So they optimize the DBC method by considering all the pixels in the block. Li et al. improved the performance of DBC method by changing the part in counting box [10]. In fact, they optimized the DBC method by considering the border of two neighboring block in the horizontal direction. In the comparison experiments, the last rows and columns pixels are ignored because of the transforming of the counting box means. Liu et al. modified the DBC method by considering the border of two neighboring blocks in the horizontal and vertical directions, which manifests that the one line is suitable for the block [11].

2.3. An improved box-counting method

According to the analysis of the DBC method, there are still much to be improved. In this paper three modifications are proposed in the improved DBC method.

(i) Average gray level (AGL), in the block of size ss, is calculated. Then it is compared with gray value of every pixel in the block. If the value is above AGL, the count of max (CMA) is accumulated; if the value is inferior to AGL, count of min (CMI) is accumulated. The equation is as follows:

$$n_r(i,j) = \frac{2 \times CMA \times pa - 2 \times CMI \times pi}{s \times s}, \tag{2.4}$$

where pa represents the maximal pixel value of block, and pi represents the minimum pixel value of block.

(ii) Figure 2 shows that the DBC has a better performance in the middle points and both ends of points have sharp drops. So the step lengths of new method are decreased. The step lengths of classical DBC method are s by which M can be divisible. The steps length of s, less than 14, are added in the improved method to add more points in the middle part. When M is not divisible by s, the average pixel value is calculated by the border of block and is inserted into the block as rest pixels. This situation is presented as the dashed part in Figure 4. When s is greater than 14, the relative value of $\log(1/r)$ has the same value. So the value of s less than 14 is chosen to improve the accuracy of FD.

(iii) Shift the block in the (x, y) plane with pixels of θ to overcome the shortcoming of under-counting boxes at the border of two neighboring box blocks. The shift block of $(i,j)(i \in x, j \in y)$ is as follows:

$$\text{shiftblock} \begin{cases} s(i+\theta, j+\theta), & i,j \text{ is not the border of the image,} \\ (i-\theta, j+\theta), & i \text{ is the border of the image,} \\ (i+\theta, j-\theta), & j \text{ is the border of the image,} \\ (i-\theta, j-\theta), & i,j \text{ are both borders of the image.} \end{cases}$$

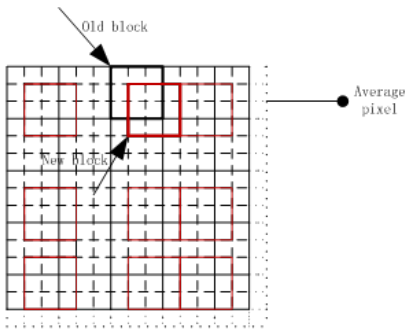


Figure 4. The shift block in method 1

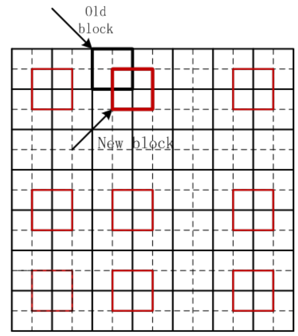


Figure 5. The shift block in method 2

The shift block is to catch the border of two neighboring boxes, so θ taking 1 is the best choice. The shift block is demonstrated in Fig. 4. If M is the exact division by r, the specific circumstance of shift is shown in Fig. 5. The number of the shift block boxes, marked as n_{r_shift}, is the value of the new block with the Eq. (2.4) And n_{r_old} is the value of the original block boxes. The final value of n_r is obtained by

$$n_r(i, j) = \max(n_{r\_old}, n_{r\_shift}). \tag{2.5}$$

Table 1. The procedure of new method

	Operations	Annotations
	Start	
	Load image	// input the image
	$M = image.height$,$N = image.width$;	
Step1	$s = 2$;	//the orgin size of box
	While($s \leq M/2$)	
	If ($s < 13 \| \| M/s == 0$)	
	$r = s/M$;	//define the r
Step2	For($i < M/s$;$j < N/s$)	
	$n_r(i, j) = (CMA \times pa - CMI \times pi)/s \times s$	//use Eq. (2.4)
	Shift block in (x, y) plane with σ pixels	
	$n_r(i, j) = \max(n_{r\_old}, n_{r\_shift})$	//use Eq. (2.5)
	End For	
	End If	
	$N_{r\_old} = \sum(n_r)$;	
	$s++$;	
	End While	
Step3	Fit($\log N_r, \log(1/r)$)	//the least square method
	Obtain FD	
	End	

The procedure of the improved method is listed in Table 1. It was implemented on core (TM) i5 (2.40GHZ). It includes three major steps.

(1) Cover the image with different size boxes;

(2) Count the number of the boxes covering the image completely;

(3) Use the least-squares linear fit of $\log(N_r)$ against $\log(1/r)$ to compute the FD of the image.

3. Experimental results and discussions

In this section, the experimental results of the new method are compared with other four methods in three experiments. The test database is the Brodatz texture image library [2]. The images can be downloaded in the web of *http : //www.cipr.rpi.edu/resource/stills/brodatz.html*. For each picture in the image library, the gray value of the image ranges from 0 to 255, namely gray level is 256 totally.

3.1. The fitting errors for the library

The fitting error is one of the most important index for evaluating the method of FD. It is usually used to acquire the degree of accuracy. And the FD is obtained by the least squares linear fitting straight line of a set of point pairs $[\log(1/r), \log(N_r)]$. There are 14 Brodatz texture images with size of 640 $\times$ 640 as shown in Fig. 6. The low fitting error means that the method has a good result.

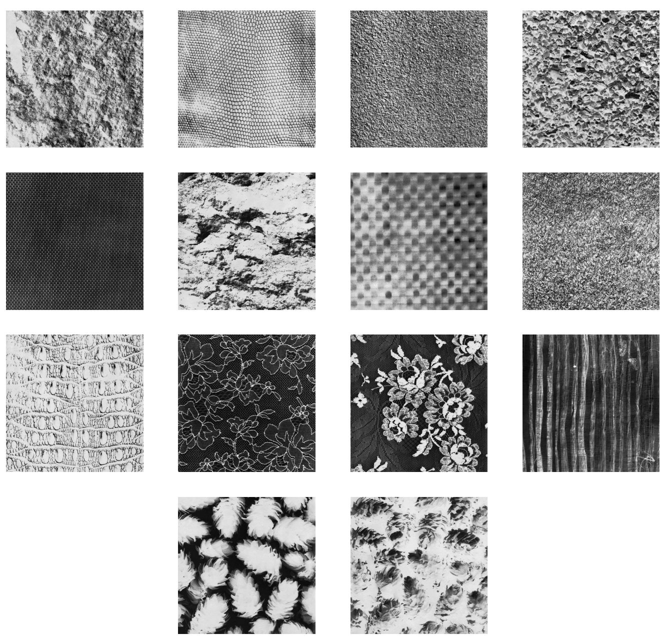


Figure 6. Brodatz texture images

The $y = kx + b$ is the straight fitting line. The y represents the $\log(N_r)$ and the x represents the $\log(1/r)$. The k denotes the result of FD. The DBC and improved DBC methods take a different number of points pairs. So the mean value of fitting errors for point pairs are obtained to evaluate the methods. The equation of fitting error (FE) is as Eq. (3.1):

$$FE = \frac{1}{n}\sum_{i=1}^{n}(kx_i + d - y_i)^2, \tag{3.1}$$

where n is the number of point pairs. The specific results are shown in Table 2. The vlues marking boldface are the best results of five methods.

Table 2. The results of experiment 1

Num of Image	DBC		Liu's method		Wang's method		Li's method		New method	
	FD	Fitting error	FD	Fitting error	FD	Fitting error	FD	Fitting error	FD	Fitting error
1	2.68	0.0387	2.71	0.0697	2.73	0.0972	2.72	0.0272	2.82	**0.0105**
2	2.78	0.0900	2.82	0.0527	2.82	0.1126	2.82	0.0358	2.90	**0.0170**
3	2.78	0.0809	2.81	0.0502	2.82	0.0702	2.82	0.0185	2.89	**0.0106**
4	2.67	0.1393	2.71	0.0977	2.72	0.1257	2.71	0.0438	2.82	**0.0169**
5	2.68	0.1629	2.73	0.1066	2.73	0.0896	2.73	0.0609	2.81	**0.0216**
6	2.64	0.1179	2.68	0.0777	2.70	0.1203	2.67	0.0420	2.82	**0.0115**
7	2.50	0.0631	2.53	0.0446	2.54	0.1018	2.54	0.0787	2.72	**0.0075**
8	2.81	0.014	2.84	0.0348	2.84	0.0521	2.84	0.0197	2.90	**0.0057**
9	2.69	0.1295	2.73	0.0821	2.71	0.1982	2.74	0.0386	2.83	**0.0234**
10	2.72	0.0951	2.76	0.0580	2.81	0.0447	2.75	0.0270	2.87	**0.0063**
11	2.69	0.0666	2.73	0.0364	2.83	0.1015	2.73	0.0187	2.90	**0.0182**
12	2.57	0.0844	2.60	0.0573	2.61	0.0781	2.61	0.0776	2.70	**0.0087**
13	2.44	0.1384	2.48	0.1026	2.48	0.1654	2.49	0.0194	2.68	**0.0105**
14	2.52	0.1312	2.56	0.0922	2.55	0.1943	2.57	0.0477	2.73	**0.0119**

The average fitting errors of DBC is 0.096571 and other three modified D-BC methods, Liu's method, Wang's method, Li's method are 0.068757, 0.110836, 0.039686, respectively. The mean fitting error of new method is 0.012879. The standard deviations of five methods, DBC, Liu's method, Wang's method, Li's method and new method, are 0.040829, 0.023689, 0.045703, 0.019874 and 0.005404, respectively. The results illustrate that the new method has a sharp drop in fitting error compared with other improved DBC methods and it is steady in the aspect of fitting error. Figure 7 shows a visualized effect for the comparison of fitting error.

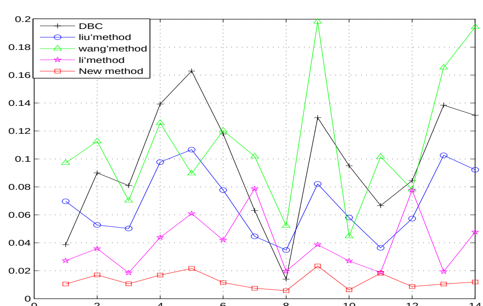


Figure 7. The comparison of fitting errors

3.2. Tests on Brodatz texture image II

Another Brodatz has 26 pictures of size with 512×512. It has two images in the same type of objects, namely it has 13 groups of images in picture library as shown in Figure 8. The FD values of the same class images should be close to each other. The less distance in one group denotes the better accuracy in FD. Each image is divided into 16 same size pictures of 128×128 for better confirming the stability of the method. The average FD of the 16 divided pictures are used to present the FD of the origin picture.The specific distances of FD in groups are presented in Table 3.

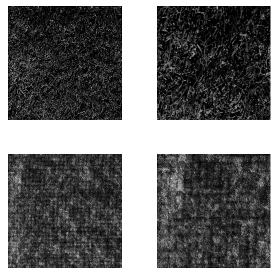


Figure 8. The sample of Brodatz texture image

Table 3. The results of experiment 2

Num of Image	Distance of DBC	Distance of Liu's method	Distance of Wang's method	Distance of Li's method	Distance of New method
1	0.151843	0.155604	0.12887	**0.1139**	0.114188
2	0.093774	0.098036	0.118218	**0.0897**	0.091924
3	0.071703	0.074569	0.075336	0.1007	**0.060466**
4	0.076414	0.075468	0.052885	0.0425	**0.042106**
5	0.057246	0.060133	0.067739	0.0688	**0.039661**
6	0.06895	0.069316	0.049395	**0.0077**	0.035297
7	0.092138	0.095552	0.073189	0.0603	**0.040109**
8	0.156938	0.160444	0.118951	**0.0897**	0.110483
9	0.157617	0.164806	0.16099	0.1293	**0.107626**
10	0.166894	0.170603	0.159434	0.1014	**0.101147**
11	0.12937	0.135551	0.088273	0.0702	**0.005824**
12	0.128676	0.128322	0.093669	0.1107	**0.064489**
13	0.12917	0.135687	0.124886	0.0947	**0.092973**

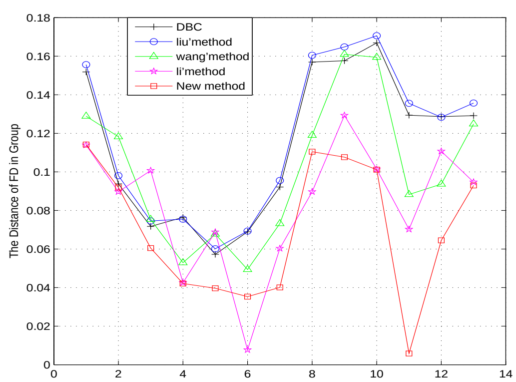


Figure 9. The comparison diagram of distances of FD in every group

The bold part represents the best result of DBC and improved DBC methods. And the comparison diagram is shown in Figure 9. The average distances in groups of DBC, Liu's method, Wang's method, Li's method are 0.1139, 0.1172, 0.1009, 0.0830. And the mean distance in groups of new method is 0.0697. The variances of four methods are 0.0387, 0.0373, 0.0400, 0.0354. The results illustrate that the new method have a better behavior in the recognition the same type of images compared with other three modified methods.

3.3. Experiments in the Shrunken image

In this experiment, the sizes of sixty-seven 640×640 images from Brodatz texture library are reduced to the size of 600×600, 512×512, 450×450 and 400×400 to observe the variances of FD. The origin image is abandoned for fair.

The less variance in one group denotes better accuracy since the reduced images are from the same image and have the similar textures. The variance results of FD in groups are presented in Appendix. The comparison diagram is shown in Figure 10. The number of the new method variance in groups which are inferior to the other method are 53, which means 79.1% of the results are superior to the other methods. And the results of the DBC, Liu's method, Wang's method, Li's method are 0.0147, 0.0130, 0.0158. 0.0148. And the mean variance of FD of the new method in reduced image is 0.0091. The results indicate that the new method have a better performance in identifying the same class of images with different scales compared with other improved DBC methods.

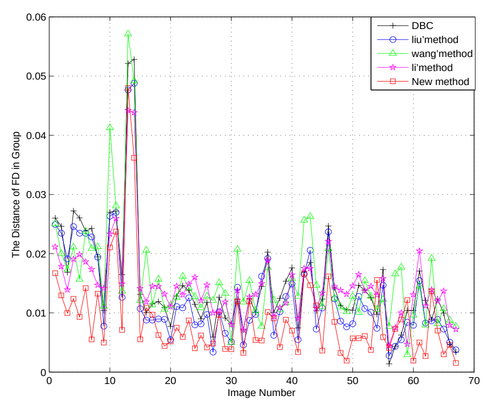


Figure 10. The comparison diagram of variances of FD in the scaled image

4. Conclusions

In this study, we propose a modified DBC method, which is improved in the following aspects: changing the means of the number of counting boxes in the block, shifting the block in (x, y) plane and decreasing the step length. Three experiments were carried out to test the new method and the experimental results are compared with other improved DBC methods. The results of the first experiment shows that the new method has a good fitting and it is steady in the fitting error, which is decreased to 0.012879. The average variance of FD values, compared with the best result of other improved methods, 0.083046 of Li's method, is decreased by 16.0%

in the second experiment. The result illustrates that the new method has a better performance in the recognition of the same type of images. The third experiment shows the new method has a better performance in identifying the scaled images. And the average variance of FD values in each group is decreased by 30% compared with the Lius method.

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Appendix

The variances of FD in 67 images

Num of Image	Distance of DBC	Distance of Liu's method	Distance of Wang's method	Distance of Li's method	Distance of New method
1	0.026015	0.024941	0.025035	0.021145	**0.016697**
2	0.024636	0.02347	0.019996	0.017829	**0.01295**
3	0.016838	0.019123	0.017364	0.013903	**0.00998**
4	0.027226	0.024578	0.021121	0.019078	**0.01238**
5	0.026041	0.023458	0.015683	0.019845	**0.009328**
6	0.023815	0.023447	0.023797	0.01862	**0.014184**
7	0.02427	0.022824	0.020878	0.017346	**0.00551**
8	0.019458	0.019419	0.021168	0.014705	**0.013252**
9	0.010392	0.007774	0.011133	0.014158	**0.004981**
10	0.027002	0.026338	0.041266	0.023328	**0.021053**
11	0.027189	0.026961	0.028058	0.02595	**0.02374**
12	0.01646	0.012629	0.013664	0.014914	**0.007123**
13	0.052136	0.047642	0.057117	**0.044211**	0.047975
14	0.052804	0.048762	0.049194	0.043828	**0.036193**
15	0.013143	0.010703	0.012103	0.014115	**0.005534**
16	0.01004	**0.008786**	0.020586	0.011871	0.010857
17	0.011531	**0.008794**	0.011348	0.014524	0.009716
18	0.011903	0.008986	0.015693	0.014436	**0.006252**
19	0.010928	0.008898	0.010595	0.013174	**0.004415**
20	0.007731	0.005441	0.011292	0.01105	**0.005185**
21	0.012839	0.011072	0.012629	0.014518	**0.007507**
22	0.014608	0.010862	0.016174	0.013052	**0.00593**
23	0.013789	0.012548	0.01427	0.014866	**0.008714**
24	0.011449	0.007979	0.012745	0.016017	**0.004025**
25	0.009	0.008139	0.010853	0.01206	**0.006183**
26	0.011782	0.009735	0.012836	0.014715	**0.004172**
27	0.005937	**0.003371**	0.012092	0.010002	0.004832
28	0.012647	0.010207	0.015083	0.010229	**0.009674**
29	0.009132	0.006549	0.01327	0.01183	**0.003918**
30	0.008146	0.005125	0.004967	0.00803	**0.003868**
31	0.012306	0.01433	0.020731	0.013793	**0.011941**
32	0.007093	0.004599	0.011837	0.007116	**0.003247**
33	0.012433	**0.008753**	0.015442	0.012691	0.011945
34	0.010084	0.009717	0.010416	0.013117	**0.005409**
35	0.014457	0.016183	0.007752	0.014943	**0.005331**
36	0.020265	0.019191	0.017689	0.018943	**0.01014**
37	0.010073	**0.00622**	0.012163	0.00941	0.009063

38	0.01246	0.010181	0.010577	0.010889	**0.004225**
39	0.015364	0.012774	0.011706	0.011664	**0.008844**
40	0.017601	0.014882	0.01581	0.0016417	**0.007**
41	0.007456	0.00547	0.012819	0.009085	**0.003373**
42	0.016836	**0.01652**	0.025646	0.017533	0.016635
43	0.018509	0.020582	0.026277	0.017488	**0.014663**
44	0.010341	**0.007271**	0.014662	0.0010759	0.011279
45	0.012337	0.010832	0.011045	0.013364	**0.003631**
46	0.02467	0.023688	0.020855	0.022071	**0.016177**
47	0.014253	0.012367	0.012455	0.014415	**0.008498**
48	0.011263	0.00861	0.010999	0.013816	**0.003236**
49	0.010541	0.007647	0.010217	0.01315	**0.001938**
50	0.010465	0.008167	0.012799	0.014544	**0.005703**
51	0.014667	0.012784	0.010105	0.016473	**0.005732**
52	0.013831	0.010745	0.015423	0.013738	**0.006094**
53	0.012588	0.010101	0.01281	0.014506	**0.003743**
54	0.009751	**0.007401**	0.01057	0.0013024	0.015549
55	0.017322	0.014668	0.012253	0.015814	**0.005891**
56	**0.001385**	0.002779	0.007703	0.0004458	0.004104
57	0.004357	**0.00435**	0.016594	0.007497	0.007273
58	0.006255	**0.005455**	0.017705	0.010003	0.008892
59	0.010427	0.00841	**0.002944**	0.004719	0.012165
60	0.010396	0.007906	0.009637	0.01308	**0.001938**
61	0.017068	0.015453	0.014543	0.020468	**0.005003**
62	0.012118	0.008282	0.007937	0.0112	**0.002708**
63	0.00885	**0.008658**	0.019226	0.013923	0.013646
64	0.012348	0.008907	0.00817	0.011988	**0.007003**
65	0.009966	0.007267	0.01072	0.0013698	**0.002999**
66	0.004721	0.005057	0.008771	0.007918	**0.004611**
67	0.00338	0.003763	0.007744	0.00722	**0.00155**

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