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PERIODIC ORBITS IN TWO CLASSES OF PIECEWISE SMOOTH MAPS WITH POSITIVE NONLINEAR PARTS*

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Abstract In this paper we consider two classes of one dimensional piecewise smooth continuous maps that have been derived as normal forms for grazing bifurcations of piecewise smooth dynamical systems. These maps are linear on one side of the phase space and nonlinear on the other side. The case of nonlinear parts with negative coefficients has been studied previously and it is proved that period-adding scenarios are generic in this case. In contrast to this result, in our analytical and numerical results, the period-adding scenarios are not observed when the nonlinear parts have positive coefficients. Furthermore, our results suggest that the typical bifurcation scenario is period doubling cascade leading to chaos in this case, which is similar to that of the smooth logistic map.

Keywords Piecewise smooth map, period doubling bifurcation, grazing bifurcation, chaos.

MSC(2010) 37E05, 37G10.

1. Introduction

Piecewise smooth (PWS) dynamical systems have become very popular recently due to their capability of modeling many real world problems involving collision, friction and devices with switching components, as well as their mathematical interest. See, for example, [4, 8, 13, 16, 19, 22–27, 33] and the references therein. It is well known from these works that PWS systems often exhibit very complicated dynamics. Besides the occurrence of all types of traditional bifurcations, the non-smoothness also leads to many new discontinuity induced phenomena, such as grazing, sticking, sliding and chattering. Among them, grazing bifurcation is one of the most interesting one and has been investigated by many authors. To mention only a few of them, see [6, 7, 9–12, 14, 29, 30, 34–37]. More recently, grazing induced bifurcation phenomena, such as "invisible grazing", have been studied by Wiercigroch's group in [3, 20, 21, 32].

A powerful tool to analyze grazing bifurcation is the discontinuity-mapping technique. It was originally introduced by Nordmark in [29] for impacting systems and then was extended to general PWS systems by Dankowicz and Nordmark in [10]. Using this method, the Poincaré map at a grazing bifurcation for a two dimensional PWS system, after some transformations, can be written to leading-order in the

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form

$$\mathbf{x} \to \begin{cases} N\mathbf{x} + M\mu, & \text{if } \ C^T\mathbf{x} \le 0, \\ N\mathbf{x} + M\mu + E(C^T\mathbf{x})^\gamma, & \text{if } \ C^T\mathbf{x} > 0, \end{cases} \tag{1.1}$$

where $\mathbf{x} \in \mathbb{R}^2$, $\gamma, \mu \in \mathbb{R}$, N is a 2×2 matrix, C, E and M are two dimensional column vectors. It has been shown that the value of γ depends heavily on the smoothness of the vector field across the discontinuity set. For example, for impact systems and discontinuous Filippov systems, $\gamma = \frac{1}{2}$ and the corresponding two dimensional square-root map (1.1) is often referred to as the *Nordmark map*; for continuous but non-differentiable PWS systems, $\gamma = \frac{3}{2}$, see, for example, [13, 21]. It is worth mentioning that the normal form maps of sliding bifurcations also have the form (1.1) with $\gamma = 2$ [13].

In order to unfold the near-grazing dynamics of a PWS system, it is very important to understand the bifurcations of map (1.1). The Nordmark map that corresponds to (1.1) with $\gamma = \frac{1}{2}$ has been systematically studied by Chin et al. in [7]. Their results have been used to investigate grazing bifurcations of periodic orbits and quasiperiodic orbits by many authors and the results were found to be in excellent agreement with direct simulations of the original systems [9–11, 14, 30]. However, to the best of our knowledge, there are no results on the two-dimensional map (1.1) with $\gamma > 1$. An important progress was made by Halse, Homer and di Bernardo in [18], in which they considered one-dimensional maps of the following two forms derived as normal form maps of grazing and sliding bifurcations in planar PWS systems:

$$x \to \begin{cases} \alpha x - \mu, & \text{if } x \le 0, \\ \beta x^\gamma - \mu, & \text{if } x > 0, \end{cases} \tag{1.2}$$

and

$$x \to \begin{cases} \alpha x - \mu, & \text{if } x \le 0, \\ \alpha x + \beta x^\gamma - \mu, & \text{if } x > 0, \end{cases} \tag{1.3}$$

where $\alpha \in \mathbb{R}$ and $\gamma > 1$. Maps (1.2) and (1.3) are the one-dimensional counterparts of (1.1). Note that since the magnitude of β can be scaled out, only its sign is important. Thus we further assume that $\beta \in \{1, -1\}$.

Let $m \ge 2$ be an integer. We label a stable period-m orbit of map (1.2) or map (1.3) which has one iteration on the nonlinear side and $m-1$ iterations on the linear side as $A^{m-1}B$ and label an unstable orbit of this type as $a^{m-1}b$. In [18], Halse, Homer and di Bernardo extended Feigin's classification methods (see [15]) for piecewise linear maps to be applicable to maps (1.2) and (1.3) and studied border collision bifurcations of fixed points of these maps. An important feature often observed in piecewise smooth maps is the period-incrementing or period-adding cascade, i.e. windows of stable periodic orbits are observed whose periodicities form an arithmetic sequence as the bifurcation parameter varies [2, 28]. For (1.2) and (1.3), Halse, Homer and di Bernardo [18] obtained analytical conditions for the existence of stable period-m orbits of the form $A^{m-1}B$ under the assumptions that $\beta = -1$, $0 < \alpha < 1$, $\mu < 0$ and $\gamma > 1$, implying that period-adding scenarios are generic in maps of this form with the given conditions.

It is then natural to ask what is the dynamic behaviors of maps (1.2) and (1.3) with positive nonlinear parts (i.e. when $\beta = 1$)? In particular, does period-adding bifurcations occur in maps of this form when $\beta = 1$? This question is not only important to the study of grazing bifurcations for PWS systems as mentioned above, but also important in its own because piecewise smooth maps have been studied for many years and they played very important roles in the development of nonlinear dynamics, see, for example, [1,5,31]. As far as we know, this problem has not been studied.

Stimulated by the work of Halse, Homer and di Bernardo [18], in this paper we aim to study the existence and stability of periodic orbits of type $A^{m-1}B/a^{m-1}b$ of maps (1.2) and (1.3) with $\beta = 1$. In contrast to the case of $\beta = -1$, we proved that when $\beta = 1$ and $m \geq 3$, even though a period-m orbit of type $A^{m-1}B/a^{m-1}b$ of map (1.2) or (1.3) may exist, it can not be stable, that is, a stable period-m orbit of type $A^{m-1}B$ does not exist. In fact, in our analytical and numerical results, period-adding scenarios are not observed in maps (1.2) and (1.3) when $\beta = 1$. The results suggest that the typical bifurcation scenario is period doubling cascade leading to chaos, which is very similar to that of the smooth logistic map. We believe that our results are useful for further understanding grazing bifurcations of PWS systems.

Our presentation is organized as follows. The main results are presented in Section 2. In Section 3 we prove the main results. Numerical simulations are given in Section 4.

2. Main results

When $\beta = 1$, maps (1.2) and (1.3) take the following forms respectively:

$$x \to \begin{cases} \alpha x - \mu, \text{ if } x \leq 0, \\ x^\gamma - \mu, \text{ if } x > 0, \end{cases} \tag{2.1}$$

and

$$x \to \begin{cases} \alpha x - \mu, & \text{if } x \leq 0, \\ \alpha x + x^\gamma - \mu, & \text{if } x > 0. \end{cases} \tag{2.2}$$

As in [18], we choose μ as the bifurcation parameter. In this paper we consider the existence and stability of period-m orbits of map (2.1) or map (2.2) of type $A^{m-1}B/a^{m-1}b$. For map (2.1), we have the following result:

Proposition 2.1. *Suppose that $\gamma > 1$. If $m \geq 3$, then map (2.1) has exactly one period-m orbit of type $A^{m-1}B/a^{m-1}b$ if and only if $\alpha < -1$ and*

$$\mu > -\frac{1}{1+\alpha}\left[\frac{\alpha^2 - \alpha^m}{\alpha^{m-1}(\alpha^2 - 1)}\right]^{\frac{1}{\gamma-1}}.$$

The orbit is always unstable. Map (2.1) has exactly one stable period-2 orbit of type AB if and only if $\alpha < -1$ and $0 < \mu < \mu_{cr1,\gamma}$, where

$$\mu_{cr1,\gamma} = -\frac{1+\gamma}{\gamma(1+\alpha)}(-\gamma\alpha)^{-\frac{1}{\gamma-1}}.$$

The periodic orbit loses stability at $\mu = \mu_{cr1,\gamma}$ and at which period doubling bifurcation occurs.

For map (2.2), we have the following result:

Proposition 2.2. *Suppose that $\gamma > 1$. If $m \geq 3$, then map (2.2) has exactly one period-m orbit of type $A^{m-1}B/a^{m-1}b$ if and only if $\alpha < -1$ and*

$$\mu > -\frac{1}{1+\alpha}\left[\frac{\alpha^2(1-\alpha^m)}{\alpha^{m-1}(\alpha^2-1)}\right]^{\frac{1}{\gamma-1}} > 0.$$

The orbit is always unstable. Map (2.2) has exactly one stable period-2 orbit of type AB if and only if $\alpha < -1$ and $\mu^-_{cr2,\gamma} < \mu < \mu^+_{cr2,\gamma}$, where

$$\mu^-_{cr2,\gamma} = (\alpha-1)\left(1-\frac{1}{\gamma}\right)\left(\frac{1-\alpha^2}{\gamma\alpha}\right)^{\frac{1}{\gamma-1}},$$
$$\mu^+_{cr2,\gamma} = \left(-\frac{1+\alpha^2}{\gamma\alpha}\right)^{\frac{1}{\gamma-1}}\left[\alpha-1-\frac{1+\alpha^2}{\gamma(1+\alpha)}\right].$$

The periodic orbit loses stability at $\mu = \mu^+_{cr2,\gamma}$ and at which period doubling bifurcation occurs.

Remark 2.1. Typical examples for maps (2.1) and (2.2) are the cases of $\gamma = \frac{3}{2}$ and $\gamma = 2$. For map (2.1), the value $\mu_{cr1,2}$ for $\gamma = 2$ and the value $\mu_{cr1,3/2}$ for $\gamma = \frac{3}{2}$ are respectively given by

$$\mu_{cr1,2} = \frac{3}{4\alpha(1+\alpha)}, \quad \mu_{cr1,3/2} = -\frac{20}{27\alpha^2(1+\alpha)}.$$

For map (2.2), the values $\mu^{\pm}_{cr2,2}$ for $\gamma = 2$ and the values $\mu^{\pm}_{cr2,3/2}$ for $\gamma = \frac{3}{2}$ are respectively given by

$$\mu^-_{cr2,2} = \frac{(\alpha-1)(1-\alpha^2)}{4\alpha}, \qquad \mu^+_{cr2,2} = \frac{(1+\alpha^2)(3-\alpha^2)}{4\alpha(1+\alpha)},$$
$$\mu^-_{cr2,3/2} = \frac{4(\alpha-1)(1-\alpha^2)^2}{27\alpha^2}, \qquad \mu^+_{cr2,3/2} = -\frac{4(1+\alpha^2)^2(5-\alpha^2)}{27\alpha^2(1+\alpha)}.$$

Based on Propositions 2.1 and 2.2, we are particularly interested in the case of $\alpha < -1$. It is clear from the expressions (2.1) and (2.2) that neither of the two maps has stable period-1 orbits of type A if $\alpha < -1$. Furthermore, we have the following result:

Proposition 2.3. *Suppose that $\gamma > 1$ and $\alpha < -1$. Then map (2.1) has exactly one stable period-1 orbit of type B if and only if*

$$\mu_{*1,\gamma} := -(\gamma-1)\gamma^{-\frac{\gamma}{\gamma-1}} < \mu < 0.$$

The orbit loses stability at $\mu = \mu_{*1,\gamma}$ and at which saddle-node bifurcation occurs. Map (2.2) has exactly one stable period-1 orbit of type B if and only if $\mu^-_{*2,\gamma} < \mu < \mu^+_{*2,\gamma} < 0$, where

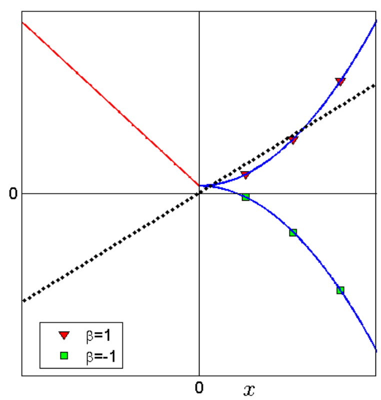
$$\mu^-_{*2,\gamma} := -(\gamma-1)\left(\frac{1-\alpha}{\gamma}\right)^{\frac{\gamma}{\gamma-1}}, \quad \mu^+_{*2,\gamma} := -\left(-\frac{1+\alpha}{\gamma}\right)^{\frac{1}{\gamma-1}}\left[1+\frac{1}{\gamma}-\alpha\left(1-\frac{1}{\gamma}\right)\right].$$

Saddle-node bifurcation for the orbit occurs at $\mu = \mu^-_{*2,\gamma}$ *and period-doubling bifurcation occurs at* $\mu = \mu^+_{*2,\gamma}$.

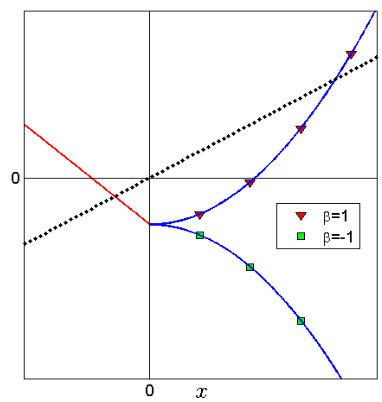
Now we are able to describe the border-collision bifurcations for maps (2.1) and (2.2) when $\alpha < -1$. The results are summarized in the following Theorem:

Theorem 2.1. (1) *Let* $\alpha < -1$. *For map* (2.1), *as* μ *increases from* $\mu < 0$ *to* $\mu > 0$, *a stable period-*1 *orbit of type* B *disappears at* $\mu = 0$ *and at which a stable period-*2 *orbit of type* AB *appears, then it loses stability at* $\mu = \mu_{cr1,\gamma}$ *and at which period doubling bifurcation occurs.*

(2) *Let* $\alpha < -1$. *For* (2.2), *as* μ *increases from* $\mu = \mu^-_{*2,\gamma} < 0$, *a stable period-*1 *orbit of type* B *loses stability at* $\mu = \mu^-_{*2,\gamma}$ *and period-doubling bifurcation occurs, resulting in a stable period-*2 *orbit which is on the nonlinear side* (*i.e. of type* B^2). *Then as* μ *continues to increase, it changes to a stable period-*2 *orbit of type* AB *at* $\mu = \mu^-_{cr2,\gamma} < 0$. *Then the orbit loses stability at* $\mu = \mu^+_{cr2,\gamma}$ *and at which period doubling bifurcation occurs.*



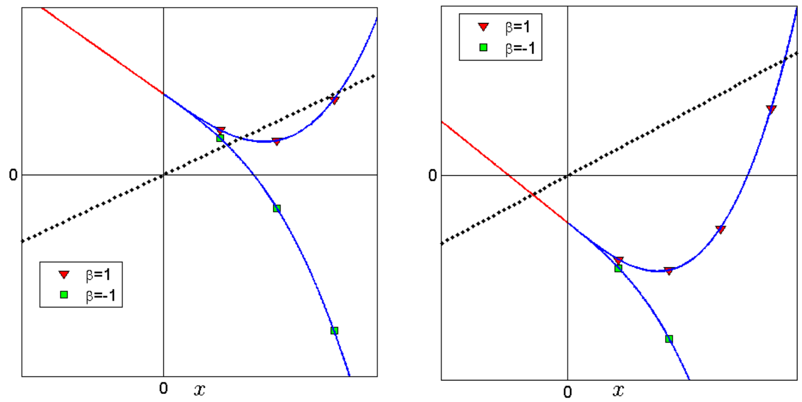
(a) $\mu_{*1,\gamma} < \mu < 0$.



(b) $\mu > 0$.

Figure 1. Typical shapes of system function of map (1.2) with $\alpha < -1$.

In Figs. 1 and 2 we plot typical shapes of system functions of maps (1.2) and (1.3) with $\alpha < -1$. As can be seen from these figures that the case for $\beta = 1$ is quite different from that for $\beta = -1$. Take Fig. 1 as an example. From Fig. 1(a), it is clear that when $\mu_{*1,\gamma} < \mu < 0$ and $\beta = 1$, map (1.2) has two period-1 orbits of type B/b. One is stable and another one is unstable. When μ decreases and passes $\mu = \mu_{*1,\gamma}$, the two orbits coalesce and disappear. All orbits diverge when $\mu < \mu_{*1,\gamma}$. If $\beta = -1$, then (1.2) has a unique period-1 orbit of type B/b. When $|\mu|$ is small enough, the orbit is stable and it attracts all orbits of map (1.2). As μ decreases, the orbit loses stability and all orbits of (2.1) diverge. From Fig. 1(b), we see that when $\mu > 0$ and $\beta = 1$, (1.2) has a unique period-1 orbit of type b and a unique period-1 orbit of type a . It has a point of minimum at the border $x = 0$, which is non-smooth. The absorbing interval is given by $[-\mu, -(1+\alpha)\mu]$. When an orbit enters this interval, it will never escape. If $\beta = -1$, then (1.2) has a unique period-1 orbit, which is of type a. The system function is always strictly decreasing. All orbits diverge. The situation shown in Fig. 2 is a little different in that when



(a) $\mu^-_{*2,\gamma} < \mu < \mu^+_{*2,\gamma} < 0$. (b) $\mu > 0$.

Figure 2. Typical shapes of system function of map (1.3) with $\alpha < -1$.

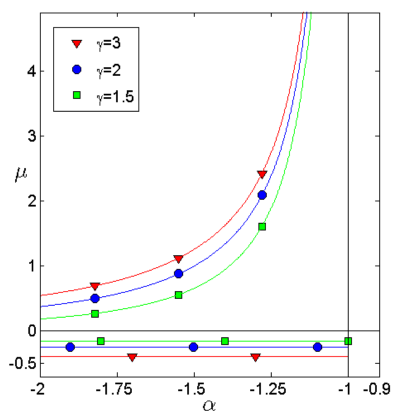
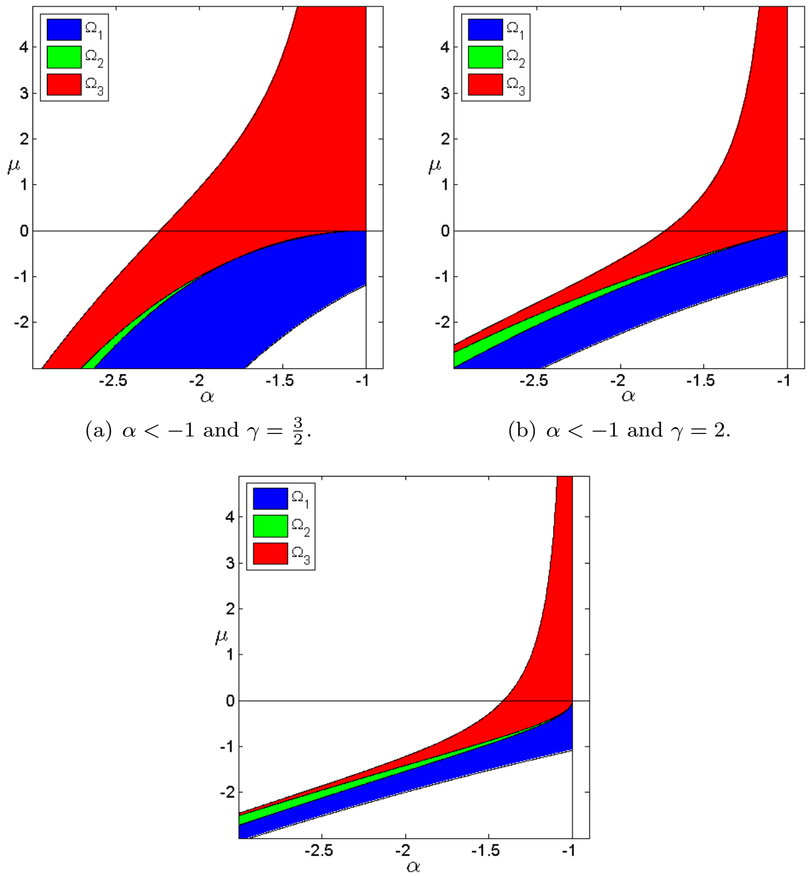


Figure 3. Bifurcation curves $\mu = \mu_{*1,\gamma}$ (below the horizontal axis) and $\mu = \mu_{cr1,\gamma}$ (above the horizontal axis) in (α, μ) space for map (2.1) with $\alpha < -1$ and $\gamma = 3$, $\gamma = 2$, $\gamma = \frac{3}{2}$ respectively.

$\beta = 1$, map (1.3) has a smooth minimum on the $x > 0$ side, thus as μ increases to $\mu < \mu^+_{*2,\gamma} < 0$, the stable period-1 orbit of type B undergoes period-doubling bifurcation.

The bifurcation curves $\mu = \mu_{*1,\gamma}$ and $\mu = \mu_{cr1,\gamma}$ in (α, μ) space for map (2.1) with $\alpha < -1$ and $\gamma = 3$, $\gamma = 2$, $\gamma = \frac{3}{2}$ respectively are shown in Fig. 3. It is easy to show that for fixed $\alpha < -1$, $\mu_{*1,\gamma}$ is strictly increasing in $\gamma > 1$ and $\mu_{cr1,\gamma}$ is strictly decreasing in $\gamma > 1$. In Figs. 4(a)–4(c), we plot the bifurcation sets in (α, μ) space for map (2.2) with $\alpha < -1$ and $\gamma = \frac{3}{2}$, $\gamma = 2$, $\gamma = 3$ respectively. From these figures, we see that as $\gamma > 1$ increases, the regions for stable period-1 orbit of type B (i.e. Region Ω_1) and stable period-2 orbit of type AB (i.e. Region Ω_3) get smaller, while the region for stable period-2 orbit of type B^2 (i.e. Region Ω_2) gets larger. Furthermore, the width of Ω_2 is determined by $\mu = \mu^-_{cr1,\gamma} - \mu^+_{*2,\gamma} > 0$, which is strictly increasing in $\alpha < -1$ for fixed $\gamma > 1$ and approaches zero as $\alpha \to -1-$. It is easy to see that the border curves of Ω_2 start from the point $(-1, 0)$. So the regions Ω_1 and Ω_3 do not have a common border, although in Fig. 4, as $\alpha \to -1-$

the region Ω_2 is too narrow to be seen.



(a) $\alpha < -1$ and $\gamma = \frac{3}{2}$.

(b) $\alpha < -1$ and $\gamma = 2$.

(c) $\alpha < -1$ and $\gamma = 3$.

Figure 4. Bifurcation sets in (α, μ) space for map (2.2). Region Ω_1 is for stable period-1 orbit of type B, Ω_2 is for stable period-2 orbit of type B^2 and Ω_3 is for stable period-2 orbit of type AB.

As can be seen from Theorem 2.1, the bifurcation scenarios of maps (1.2) and (1.3) for $\beta = 1$ are quite different from that for $\beta = -1$. Our analytical and numerical results suggest that the typical bifurcation scenario when $\beta = 1$ is period doubling cascade leading to chaos, which is similar to that of the smooth logistic map. In particular, period-adding scenarios do not exist when $\beta = 1$.

3. Proofs of the main results

In this section we prove Propositions 2.1–2.3. For any positive integer n, consider the following function with parameter α:

$$\varphi_n(x, \alpha) = x^\gamma - \frac{x}{\alpha^{n-1}}.$$

By elementary calculus, the following result is obvious:

Lemma 3.1. *Let $\gamma > 1$, $\alpha < 0$ and $n > 0$ be an odd number. Then $\varphi_n(x,\alpha)$ is strictly decreasing for $x \in \left(0, \left(\gamma\alpha^{n-1}\right)^{-\frac{1}{\gamma-1}}\right)$ and strictly increasing for $x \in \left(\left(\gamma\alpha^{n-1}\right)^{-\frac{1}{\gamma-1}}, \infty\right)$. For $x \in \left(0, \alpha^{-\frac{n-1}{\gamma-1}}\right)$, $\varphi_n(x,\alpha) < 0$ and for $x \in \left(\alpha^{-\frac{n-1}{\gamma-1}}, \infty\right)$, $\varphi_n(x,\alpha) > 0$.*

Lemma 3.2. *For any $\gamma > 1$, we have*

$$\frac{2}{e} < \phi(\gamma) := \gamma^{-\frac{\gamma}{\gamma-1}}(1+\gamma) < 1. \tag{3.1}$$

Proof. Let $\psi(\gamma) = \ln\phi(\gamma)$. Then

$$\psi'(\gamma) = \frac{h(\gamma)}{(\gamma-1)^2(\gamma+1)},$$

where $h(\gamma) = (1+\gamma)\ln\gamma + 2(1-\gamma)$. Clearly $h'(\gamma) = \ln\gamma + \frac{1}{\gamma} - 1$ and $h''(\gamma) = \frac{1}{\gamma} - \frac{1}{\gamma^2}$. Since $\gamma > 1$, we always have $h''(\gamma) > 0$, implying that $h'(\gamma) > h'(1) = 0$ for $\gamma > 1$. Therefore $h(\gamma) > h(1) = 0$. Thus $\psi'(\gamma) > 0$ for $\gamma > 1$, implying that $\phi(\gamma)$ is strictly increasing. On the other hand, it is easy to prove that

$$\lim_{\gamma\to 1}\phi(\gamma) = \frac{2}{e}, \quad \lim_{\gamma\to\infty}\phi(\gamma) = 1.$$

Hence the inequality (3.1) is true. The proof is complete. □

Proof of Proposition 2.1. We first prove that map (2.1) does not have stable period-m orbits of type $A^{m-1}B$ for $m \geq 3$.

Let $m \geq 3$ be an integer. We consider the existence of a period-m orbit of the type $A^{m-1}B/a^{m-1}b$ of map (2.1). Without loss of generality, assuming that the first iterate $x_1 > 0$. Then the existence of such kind of orbit is equivalent to the existence of a sequence of iterates of (2.1) with

$$x_1 > 0, \quad x_2 < 0, \quad \cdots, \quad x_m < 0, \tag{3.2}$$

such that

$$x_2 = x_1^\gamma - \mu, \tag{3.3}$$

$$x_j = \alpha x_{j-1} - \mu, \quad j = 3, 4, \cdots, m, \tag{3.4}$$

$$x_1 = \alpha x_m - \mu. \tag{3.5}$$

By (3.2) and (3.3), we have $\mu > 0$. From $x_1 > 0$, $x_m < 0$, $\mu > 0$ and (3.3), we have $\alpha < 0$. Thus a necessary condition for the existence of such type of orbit is $\mu > 0$ and $\alpha < 0$. Under this condition, from (3.2) and (3.3)–(3.5) we have

$$-\mu < x_2 < 0, \quad -\mu < x_3 < 0, \quad \cdots, \quad -\mu < x_m < 0. \tag{3.6}$$

We divide our discussion into the following three cases.

(1) $\alpha = -1$. In this case, for any $x_1 > 0$, if $x_2 = x_1^\gamma - \mu < 0$, then by (3.3)–(3.5) we have $x_3 = -x_1^\gamma < 0$, $x_4 = x_2$. Thus for any initial value $x_1 > 0$ such that $x_2 = x_1^\gamma - \mu < 0$, the iterative sequence will end up with a period-two orbit which is on the linear side. Thus map (2.1) does not have period-p orbits of type $A^{p-1}B/a^{p-1}b$ for $p \geq 2$.

(2) $-1 < \alpha < 0$. We assume that a period-m orbit of the type $A^{m-1}B/a^{m-1}b$ of map (2.1) with $m \geq 3$ exists. From (3.6) and (3.4) we get

$$-\mu < x_2 = \frac{x_3 + \mu}{\alpha} < 0. \tag{3.7}$$

Since $\alpha < 0$, from (3.7) we obtain

$$-\mu < x_3 < -\mu(1+\alpha). \tag{3.8}$$

If $m \geq 4$, then from (3.8) and (3.4) we get

$$-\mu < x_3 = \frac{x_4 + \mu}{\alpha} < -\mu(1+\alpha). \tag{3.9}$$

Since $\alpha < 0$, from (3.9) we obtain

$$-\mu(1+\alpha+\alpha^2) < x_4 < -\mu(1+\alpha).$$

We can continue this process for x_m. If $m \geq 3$ is an odd number, we have

$$-\mu\left(1+\alpha+\cdots+\alpha^{m-3}\right) < x_m < -\mu\left(1+\alpha+\cdots+\alpha^{m-2}\right). \tag{3.10}$$

From (3.5), (3.10) and $\alpha < 0$, we get

$$-\mu\left(1+\alpha+\cdots+\alpha^{m-1}\right) < x_1 < -\mu\left(1+\alpha+\cdots+\alpha^{m-2}\right). \tag{3.11}$$

If $m \geq 3$ is an even number, we have

$$-\mu\left(1+\alpha+\cdots+\alpha^{m-2}\right) < x_m < -\mu\left(1+\alpha+\cdots+\alpha^{m-3}\right). \tag{3.12}$$

From (3.5), (3.12) and $\alpha < 0$, we get

$$-\mu\left(1+\alpha+\cdots+\alpha^{m-2}\right) < x_1 < -\mu\left(1+\alpha+\cdots+\alpha^{m-1}\right). \tag{3.13}$$

But since $-1 < \alpha < 0$, for any nonnegative integer ℓ, we have $1+\alpha+\cdots+\alpha^{\ell} > 0$. Thus either (3.11) or (3.13) implies that $x_1 < 0$ because $\mu > 0$, which contradicts to our assumption that $x_1 > 0$. Hence a period-m orbit of the type $A^{m-1}B/a^{m-1}b$ of map (2.1) with $m \geq 3$ does not exist if $-1 < \alpha < 0$.

(3) $\alpha < -1$. Again we assume that a period-m orbit of the type $A^{m-1}B/a^{m-1}b$ of map (2.1) with $m \geq 3$ exists. We first claim that in this case, in order that (3.6) is satisfied, it is sufficient that $-\mu < x_m < 0$ is satisfied. In fact, if $-\mu < x_m < 0$, then by (3.4), since $\frac{x_m}{\alpha} > 0$, $\alpha < -1$ and $\mu > 0$, it is easy to see that

$$-\mu < x_{m-1} = \frac{x_m}{\alpha} + \frac{\mu}{\alpha} < 0.$$

Then by induction we can prove that (3.6) holds. Moreover, by (3.5), $-\mu < x_m < 0$ if and only if $0 < x_1 < -\mu(1+\alpha)$. From (3.3)–(3.5) we get

$$x_1 = \alpha^{m-1}x_1^{\gamma} - \mu\left(1+\alpha+\cdots+\alpha^{m-1}\right), \tag{3.14}$$

which is equivalent to

$$\frac{\mu\left(1-\alpha^m\right)}{\alpha^{m-1}(1-\alpha)} = x_1^{\gamma} - \frac{x_1}{\alpha^{m-1}} = \varphi_m(x_1,\alpha). \tag{3.15}$$

Clearly, from the above discussion, the existence of a period-m orbit of the type $A^{m-1}B/a^{m-1}b$ of map (2.1) with $m \geq 3$ is equivalent to the existence of a solution of (3.15) with $0 < x_1 < -\mu(1+\alpha)$, $\mu > 0$ and $\alpha < -1$.

Case 1: $m \geq 3$ is an odd number.

In this case $m-1 \geq 2$ is an even number. Thus the left side of (3.15) is positive. By Lemma 3.1, we require that

$$0 < \left(\gamma\alpha^{m-1}\right)^{-\frac{1}{\gamma-1}} < \alpha^{-\frac{m-1}{\gamma-1}} < x_1 < -\mu(1+\alpha). \tag{3.16}$$

By Lemma 3.1, $\varphi_m(x_1, \alpha)$ is increasing for x_1 satisfies (3.16). Thus by (3.15) we get

$$0 < \frac{\mu\,(1-\alpha^m)}{\alpha^{m-1}(1-\alpha)} < \varphi_m(-\mu(1+\alpha), \alpha),$$

implying that a period-m orbit of the type $A^{m-1}B/a^{m-1}b$ of map (2.1) with $m \geq 3$ and m odd exists if and only if

$$\mu > -\frac{1}{1+\alpha}\left[\frac{\alpha^2-\alpha^m}{\alpha^{m-1}(\alpha^2-1)}\right]^{\frac{1}{\gamma-1}}. \tag{3.17}$$

Now we discuss the stability of this orbit. By (3.14), the eigenvalue of the orbit associated to x_1 is given by $\lambda = \gamma\alpha^{m-1}x_1^{\gamma-1}$. Since $m-1$ is even and $x_1 > 0$, thus $\lambda > 0$. For stability, we then require that $\lambda < 1$, implying that $x_1 < \left(\gamma\alpha^{m-1}\right)^{-\frac{1}{\gamma-1}}$, which contradicts the existence condition (3.16). Thus the orbit is certainly unstable.

Case 2: $m \geq 3$ is an even number.

In this case $m-1 \geq 2$ is an odd number. Since $\alpha < -1$, $\varphi_m(x_1, \alpha)$ is always positive and strictly increasing. Therefore similar to the case that $m \geq 3$ is an odd number, we get that a period-m orbit of the type $A^{m-1}B/a^{m-1}b$ of map (2.1) with $m \geq 3$ and m even exists if and only if (3.17) is satisfied. The eigenvalue of the orbit associated to x_1 is again given by $\lambda = \gamma\alpha^{m-1}x_1^{\gamma-1}$. Since $m-1$ is odd and $x_1 > 0$, thus $\lambda < 0$. The stability condition is then given by $\lambda > -1$, implying that $x_1 < \left(-\gamma\alpha^{m-1}\right)^{-\frac{1}{\gamma-1}}$. Thus by (3.15), the stability condition is reduced to

$$0 < \frac{\mu\,(1-\alpha^m)}{\alpha^{m-1}(1-\alpha)} < \varphi_m\left(\left(-\gamma\alpha^{m-1}\right)^{-\frac{1}{\gamma-1}}, \alpha\right),$$

which is equivalent to

$$\mu < \frac{1-\alpha}{\alpha^m-1}\,(-\alpha)^{-\frac{m-1}{\gamma-1}}\,\phi(\gamma), \tag{3.18}$$

where $\phi(\gamma)$ is given in Lemma 3.2. For such an orbit to exist and be stable, conditions (3.17) and (3.18) must be satisfied simultaneously. This requires that

$$-\frac{1}{1+\alpha}\left[\frac{\alpha^2-\alpha^m}{\alpha^{m-1}(\alpha^2-1)}\right]^{\frac{1}{\gamma-1}} < \frac{1-\alpha}{\alpha^m-1}\,(-\alpha)^{-\frac{m-1}{\gamma-1}}\,\phi(\gamma). \tag{3.19}$$

Since $m \geq 3$ is even and $\alpha < -1$, we have $m \geq 4$ and (3.19) is simplified to

$$\phi(\gamma) > \frac{\alpha^m-1}{\alpha^2-1}\left[\frac{\alpha^m-\alpha^2}{\alpha^2-1}\right]^{\frac{1}{\gamma-1}} > 1.$$

But this contradicts to Lemma 3.2.

Thus we have proved that map (2.1) does not have stable period-m orbits of type $A^{m-1}B$ for $m \geq 3$.

Now we fix $m = 2$ and consider period-2 orbits of type AB/ab. As can be seen, we have proved that a period-2 orbit of type AB/ab does not exist if $\alpha = -1$. If $-1 < \alpha < 0$, then from (3.3)–(3.5), we have $x_1 = \alpha x_2 - \mu = \alpha x_1^\gamma - \mu(1+\alpha) > 0$. But this is impossible, because $\alpha x_1^\gamma < 0$ and $\mu(1+\alpha) > 0$. Consequently a period-2 orbit of type AB/ab does not exist if $-1 < \alpha < 0$. It remains to consider the case for $\alpha < -1$.

If $\alpha < -1$, then from (3.3)–(3.5), x_1 is a positive solution of the equation $g(x) = 0$, where

$$g(x) = x^\gamma - \frac{x}{\alpha} - \frac{\mu(1+\alpha)}{\alpha}.$$

Since $\alpha < -1$ and $\mu > 0$, $g(x)$ is strictly increasing on $(0, \infty)$. Furthermore

$$g(0) = -\frac{\mu(1+\alpha)}{\alpha} < 0, \qquad \lim_{x\to\infty} g(x) = \infty.$$

Thus $g(x) = 0$ has a unique positive solution x_1 for each $\mu > 0$. For this $x_1 > 0$, we have $x_2 = x_1^\gamma - \mu = (x_1 + \mu)/\alpha < 0$ by (3.3)–(3.5), implying that for each $\mu > 0$, the map has exactly one period-2 orbit of type AB/ab. The eigenvalue of the orbit associated to x_1 is given by $\lambda = \gamma\alpha x_1^{\gamma-1} < 0$. The stability condition is then given by $\lambda > -1$, implying that $x_1 < (-\gamma\alpha)^{-\frac{1}{\gamma-1}}$. Thus from $g(x_1) = 0$, we can easily obtain the existence and stability condition as following

$$0 < \mu < \mu_{cr1,\gamma} := -\frac{1+\gamma}{\gamma(1+\alpha)}\,(-\gamma\alpha)^{-\frac{1}{\gamma-1}}.$$

The orbit loses stability at $\mu = \mu_{cr1,\gamma}$ and at which also $\lambda = -1$. Note that x_1 is the fixed point of the function $\rho(x,\mu) := \alpha x^\gamma - (1+\alpha)\mu$ and $\alpha < -1$, $\gamma > 1$. At the critical values $\mu = \mu_{cr1,\gamma}$ and $x = x_1 = (-\gamma\alpha)^{-\frac{1}{\gamma-1}}$, we have

$$\begin{aligned}
\frac{\partial\rho}{\partial\mu}\frac{\partial^2\rho}{\partial x^2} + 2\frac{\partial^2\rho}{\partial x\partial\mu} &= (1+\alpha)(\gamma-1)\,(-\gamma\alpha)^{\frac{1}{\gamma-1}} < 0,\\
\frac{1}{2}\left(\frac{\partial^2\rho}{\partial x^2}\right)^3 + \frac{1}{3}\frac{\partial^3\rho}{\partial x^3} &= \frac{1}{6}(\gamma-1)(\gamma+1)\,(-\gamma\alpha)^{\frac{2}{\gamma-1}} > 0.
\end{aligned}$$

Thus by Theorem 3.5.1 of [17, p. 158], the period-2 orbit of type AB undergoes supercritical period-doubling bifurcation at $\mu = \mu_{cr1,\gamma}$, resulting in a stable period-4 orbit of type A^2B^2 as μ increases. The proof is complete. □

Proof of Proposition 2.2. Let $m \geq 2$ be an integer. We consider the existence of a period-m orbit of the type $A^{m-1}B/a^{m-1}b$ of map (2.2). Without loss of generality, assuming that the first iterate $x_1 > 0$. Then the second iterate $x_2 = \alpha x_1 + x_1^\gamma - \mu < 0$, implying that $\alpha x_1 - \mu < 0$. Thus if $\alpha \geq 0$, then $\mu > 0$. But from $x_1 = \alpha x_m - \mu > 0$ and $x_m < 0$, we have $\mu < 0$. This is a contradiction. Thus a necessary condition for the existence of such type of orbit is $\alpha < 0$. By exactly the same method as in the proof of Theorem 2.1, we can show that map (2.2) does not have period-m orbits of type $A^{m-1}B/a^{m-1}b$ for $m \geq 2$. So in the following we only consider the cases of $-1 < \alpha < 0$ and $\alpha < -1$.

The proof of the existence and stability of a period-2 orbit of type AB/ab for map (2.2) is very similar to that for (2.1). Thus we omit it for brevity. In the following we only prove that map (2.2) does not have stable period-m orbits of type $A^{m-1}B$ for $m \geq 3$.

The existence of a period-m orbit of the type $A^{m-1}B/a^{m-1}b$ of map (2.2) with $m \geq 3$ is equivalent to the existence of a sequence of iterates of (2.2) with

$$x_1 > 0, \quad x_2 < 0, \quad \cdots, \quad x_m < 0, \tag{3.20}$$

such that

$$x_2 = \alpha x_1 + x_1^\gamma - \mu, \tag{3.21}$$
$$x_j = \alpha x_{j-1} - \mu, \quad j = 3, 4, \cdots, m, \tag{3.22}$$
$$x_1 = \alpha x_m - \mu. \tag{3.23}$$

By (3.22), $x_3 = \alpha x_2 - \mu < 0$. Then from $\alpha < 0$ and $x_2 < 0$, we have $\mu > 0$. By (3.22) we get

$$-\mu < x_3 < 0, \quad \cdots, \quad -\mu < x_m < 0. \tag{3.24}$$

If $-1 < \alpha < 0$, then from (3.23) we get

$$-\mu < x_m = \frac{x_1 + \mu}{\alpha} < 0,$$

implying that $0 < x_1 < -(1+\alpha)\mu$ because $\mu > 0$. Then from (3.22) we have

$$0 > x_2 = \alpha x_1 + x_1^\gamma - \mu > \alpha x_1 - \mu > -\mu(1 + \alpha + \alpha^2).$$

By the same method as in the proof of Theorem 2.1, if $m \geq 3$ is an odd number, we have

$$-\mu\left(1 + \alpha + \cdots + \alpha^{m-3}\right) < x_m < -\mu\left(1 + \alpha + \cdots + \alpha^m\right);$$

if $m \geq 3$ is an even number, we have

$$-\mu\left(1 + \alpha + \cdots + \alpha^m\right) < x_m < -\mu\left(1 + \alpha + \cdots + \alpha^{m-3}\right).$$

In either case we have $x_1 < 0$, which contradicts to our assumption that $x_1 > 0$. Hence a period-m orbit of the type $A^{m-1}B/a^{m-1}b$ of map (2.2) with $m \geq 3$ does not exist if $-1 < \alpha < 0$.

If $\alpha < -1$, then similar to the proof of Proposition 2.1, we can prove that (3.24) is satisfied if and only if $0 < x_1 < -\mu(1+\alpha)$. From (3.21)–(3.23) we get

$$x_1 = \alpha^m x_1 + \alpha^{m-1} x_1^\gamma - \mu\left(1 + \alpha + \cdots + \alpha^{m-1}\right), \tag{3.25}$$

which is equivalent to

$$\frac{\mu\left(1 - \alpha^m\right)}{\alpha^{m-1}(1-\alpha)} = x_1^\gamma - \frac{1 - \alpha^m}{\alpha^{m-1}} x_1 := \bar{\varphi}_m(x_1, \alpha). \tag{3.26}$$

The existence of a period-m orbit of the type $A^{m-1}B/a^{m-1}b$ of map (2.2) with $m \geq 3$ is equivalent to the existence of a solution of (3.26) with $0 < x_1 < -\mu(1+\alpha)$,

$\mu > 0$ and $\alpha < -1$. Since $\alpha < -1$ and we require that $\mu > 0$, it is easy to see that the left hand side of (3.26) is always positive, the coefficient of x_1 of the second term of $\bar{\varphi}_m(x_1, \alpha)$ is always negative. Thus we require that

$$0 < \left[\frac{(1-\alpha^m)}{\alpha^{m-1}}\right]^{\frac{1}{\gamma-1}} < x_1 < -\mu(1+\alpha).$$

Because $\bar{\varphi}_m(x_1, \alpha)$ is strictly increasing, a period-m orbit of type $A^{m-1}B/a^{m-1}b$ of map (2.2) with $m \geq 3$ exists if and only if

$$\mu > -\frac{1}{1+\alpha}\left[\frac{\alpha^2(1-\alpha^m)}{\alpha^{m-1}(\alpha^2-1)}\right]^{\frac{1}{\gamma-1}} > 0. \tag{3.27}$$

By (3.25), the eigenvalue of the orbit associated to x_1 is given by $\lambda = \alpha^m + \gamma\alpha^{m-1}x_1^{\gamma-1}$. For stability, we require that $-1 < \lambda < 1$. If m is odd, then we have

$$-\frac{1+\alpha^m}{\gamma\alpha^{m-1}} < x_1^{\gamma-1} < \frac{1-\alpha^m}{\gamma\alpha^{m-1}},$$

implying that $\mu < 0$ by (3.26) and $\gamma > 0$. Thus the orbit is unstable if $m \geq 3$ is odd. If m is even, then we have

$$\frac{1-\alpha^m}{\gamma\alpha^{m-1}} < x_1^{\gamma-1} < -\frac{1+\alpha^m}{\gamma\alpha^{m-1}}.$$

By (3.26), to satisfy the condition that $\mu > 0$, we require that

$$-\frac{1+\alpha^m}{\gamma\alpha^{m-1}} > \frac{1-\alpha^m}{\alpha^{m-1}},$$

or equivalently, $|\alpha| < \sqrt[m]{(\gamma+1)/(\gamma-1)}$. Under this condition and (3.26) we obtain the stability condition as following:

$$\mu < -\frac{1-\alpha}{1-\alpha^m}\left[-\frac{1+\alpha^m}{\gamma\alpha^{m-1}}\right]^{\frac{1}{\gamma-1}}\left[\left(1+\frac{1}{\gamma}\right)-\left(1-\frac{1}{\gamma}\right)\alpha^m\right]. \tag{3.28}$$

To simultaneously satisfy (3.27) and (3.28) we must require that

$$\frac{1-\alpha^m}{1-\alpha^2}\left[\frac{\alpha^2(\alpha^m-1)}{(\alpha^m+1)(\alpha^2-1)}\right]^{\frac{1}{\gamma-1}} < \gamma^{-\frac{\gamma}{\gamma-1}}(\gamma-1)\left[\frac{\gamma+1}{\gamma-1}-\alpha^m\right]. \tag{3.29}$$

Since $\alpha < -1$ and $m \geq 3$ is even, it is easy to show that the left hand side of (3.29) is greater than 1. But since $\alpha^m > 1$, we have

$$\gamma^{-\frac{\gamma}{\gamma-1}}(\gamma-1)\left[\frac{\gamma+1}{\gamma-1}-\alpha^m\right] < 2\gamma^{-\frac{\gamma}{\gamma-1}} < \gamma^{-\frac{\gamma}{\gamma-1}}(1+\gamma) < 1,$$

by Lemma 3.2. Thus the condition (3.29) can not hold, implying that the orbit is unstable if $m \geq 3$ is even.

Thus we have proved that map (2.2) does not have stable period-m orbits of type $A^{m-1}B$ for $m \geq 3$. The proof for Proposition 2.2 is complete. □

Proof of Proposition 2.3. The period-1 orbit of type B/b of (2.1) is determined by the positive solutions of the equation $g(x, \mu) := x^\gamma - x - \mu = 0$. Let $\gamma > 1$

and $\mu \in (\mu_{*1,\gamma}, 0)$. Then $g(0,\mu) = g(1,\mu) = -\mu > 0$. It is easy to show that for fixed μ, $g(x,\mu)$ reaches its minimum $g(\gamma^{-\frac{1}{\gamma-1}}, \mu) = \mu_{*1,\gamma} - \mu < 0$ at $x = \gamma^{-\frac{1}{\gamma-1}}$. Thus by the Intermediate Value Theorem, $g(x,\mu) = 0$ has exactly two positive solutions $\bar{x}_1$ and $\bar{x}_2$. Obviously $\bar{x}_1 < \gamma^{-\frac{1}{\gamma-1}} < \bar{x}_2$, implying that (2.1) has exactly two period-1 orbits of type B/b. Let $\lambda_{\bar{x}_j}$ $(j = 1, 2)$ be the eigenvalue associate to the orbit corresponding to $\bar{x}_j$. It is easy to prove that $0 < \lambda_{\bar{x}_1} < 1$ and $\lambda_{\bar{x}_2} > 1$ when $\mu \in (\mu_{*1,\gamma}, 0)$. The equation $g(x,\mu) = 0$ has exactly one positive solution when $\mu = \mu_{*1,\gamma}$ and has no positive solution when $\mu < \mu_{*1,\gamma}$. The eigenvalue associate with the unique positive solution when $\mu = \mu_{*1,\gamma}$ is precisely one. Hence as μ decreases from 0 to $\mu_{*1,\gamma}$, map (2.1) has exactly two period-1 orbits of type B/b. One is stable and another one is unstable. As μ continues to decrease, the two orbits coalesce and disappear at $\mu = \mu_{*1,\gamma}$. Hence map (2.1) has exactly one stable period-1 orbit of type B if $\mu \in (\mu_{*1,\gamma}, 0)$ and the orbit loses stability at $\mu = \mu_{*1,\gamma}$ and at which saddle-node bifurcation occurs. Similarly, we can prove that (2.1) has no stable period-1 orbit of type B if $\mu > 0$. Hence the statements for map (2.1) are true.

Let $x_* > 0$ be the point corresponding to a period-1 orbit of type B of (2.2), then

$$x_* = \alpha x_* + x_*^\gamma - \mu. \tag{3.30}$$

Thus we have

$$\mu = x_*^\gamma - (1 - \alpha)x_*. \tag{3.31}$$

From (3.30) we know that the eigenvalue associated to x_* is $\lambda = \alpha + \gamma x_*^{\gamma-1}$. Thus x_* is stable if and only if $-1 < \alpha + \gamma x_*^{\gamma-1} < 1$, or equivalently

$$-\frac{(1+\alpha)}{\gamma} < x_*^{\gamma-1} < \frac{(1-\alpha)}{\gamma}. \tag{3.32}$$

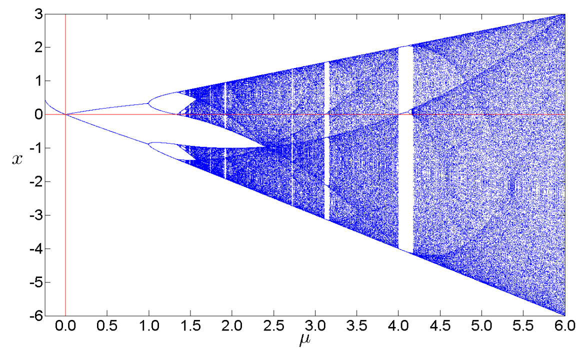
Furthermore, when (3.32) is satisfied, $h(x_*) = x_*^\gamma - (1-\alpha)x_*$ is strictly decreasing because $1 - \alpha > 0$. Thus we can easily obtain the bound for μ and the bifurcation results stated in Proposition 2.3 from (3.31). The proof is complete. □

4. Numerical simulations

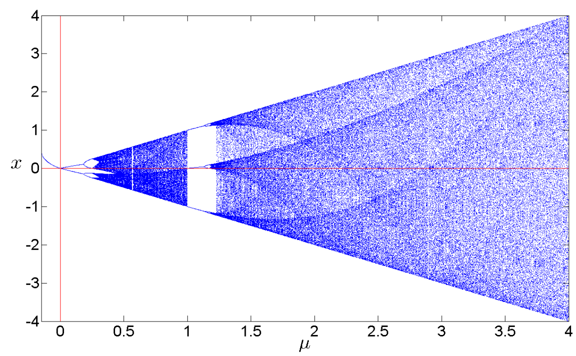
In this section we present numerical simulations for maps (2.1) and (2.2) to validate the theoretical results from Sections 2 and 3. We focus only on two cases when $\gamma = 2$ and $\gamma = \frac{3}{2}$ because they are closely related to the normal form maps of grazing and sliding bifurcations of PWS systems [13, 21].

Fig. 5(a) is the bifurcation diagram for (2.1) with $\gamma = 2$ and $\alpha = -1.5$. By Proposition 2.1, map (2.1) has exactly one stable period-2 orbit of type AB for $\mu \in (0, 1)$. The orbit loses stability at $\mu = 1$ and at which period doubling bifurcation occurs. This is confirmed in Fig. 5(a) and from which we see that the period doubling cascade leads to chaotic orbits as μ increases.

Fig. 5(b) is the bifurcation diagram for (2.1) with $\gamma = \frac{3}{2}$ and $\alpha = -2$, which shows that map (2.1) has exactly one stable period-2 orbit of type AB for $\mu \in \left(0, \frac{5}{27}\right) \approx (0, 0.1852)$ and the orbit loses stability at $\mu \approx 0.1852$ resulting in period



(a) $\gamma = 2$ and $\alpha = -1.5$.



(b) $\gamma = \frac{3}{2}$ and $\alpha = -2$.

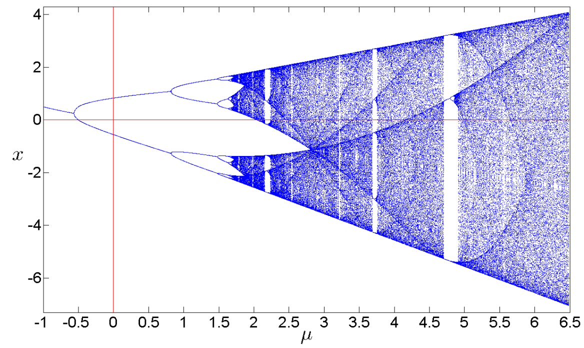
Figure 5. Bifurcation diagrams for (2.1).

doubling bifurcation as predicted by Proposition 2.1. Fig. 5(b) suggests that the period doubling bifurcation also leads to chaotic orbits as μ increases.

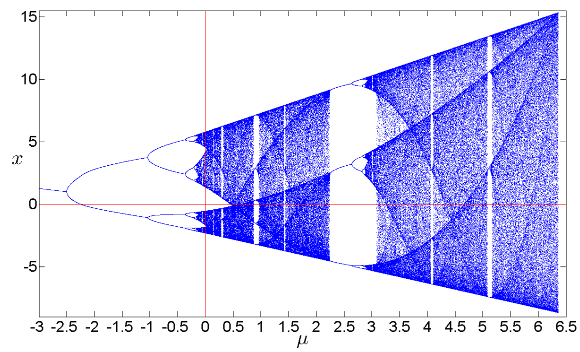
Fig. 6(a) is the bifurcation diagram for (2.2) with $\gamma = 2$ and $\alpha = -1.5$. By Propositions 2.2 and 2.3, a stable period-1 orbit of type B loses stability at $\mu = -\frac{9}{16} = -0.5625$ and at which period-doubling bifurcation occurs, resulting in a stable period-2 orbit of type B^2. As μ continues to increase, it changes to a stable period-2 orbit of type AB at $\mu = -\frac{25}{48} \approx -0.5208 < 0$. Then the orbit loses stability at $\mu = 0.8125$ and at which period doubling bifurcation occurs. Fig. 6(a) also suggests that the period doubling cascade leads to chaos. The numerical results shown in Fig. 6(a) are in good agreement with the theoretical results.

In Fig. 6(b), we take $\gamma = \frac{3}{2}$ and $\alpha = -2.5$ and plot the bifurcation diagram for (2.2). By Propositions 2.2 and 2.3, a stable period-1 orbit of type B loses stability at $\mu = -2.5$ and at which period-doubling bifurcation occurs, resulting in a stable period-2 orbit of type B^2. As μ continues to increase, it changes to a stable period-2 orbit of type AB at $\mu = -\frac{343}{150} \approx -2.2867 < 0$. Then the orbit loses stability at $\mu = -\frac{841}{810} \approx -1.0383$ and at which period doubling bifurcation occurs. Fig. 6(b) also suggests that the period doubling cascade leads to chaos. Again, our numerical results shown in Fig. 6(b) confirm the theoretical results.

We have experimented on some other choices of γ and α and get qualitatively the same results as shown in Figs. 5–6. These bifurcation diagrams suggest that the typical bifurcation scenario for maps (2.1) and (2.2) is period doubling cascade



(a) $\gamma = 2$ and $\alpha = -1.5$.



(b) $\gamma = \frac{3}{2}$ and $\alpha = -2.5$.

Figure 6. Bifurcation diagrams for (2.2).

leading to chaos, which is similar to that of the smooth logistic map. In particular, period adding bifurcations of these maps are not observed.

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