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TOPOLOGICAL LINEARIZATION OF DEPCAGS WITH UNBOUNDED NONLINEAR TERMS*

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Abstract In this paper, we study the global topological linearization of a differential equation with piecewise constant argument of generalized type (DEPCAG) when the nonlinear term is unbounded. Some sufficient conditions are established for the topological conjugacy between a nonlinear system and its linear system. Our work generalizes the main result of Pinto and Robledo in [25].

Keywords Piecewise constant argument, global topological linearization, unbounded nonlinear term, exponential dichotomy.

MSC(2010) 34A30, 34D09, 34K34.

1. Introduction

The classical topological linearization theorem of autonomous differential equations was given by Hartman and Grobman [12, 13]. Palmer generalized the Hartmann-Grobman theorem to the nonautonomous case in [20, 21], where he obtained a version of the global topological linearization theorem.

Later, many scholars obtained a series of results on topological linearization. Barreira and Valls [5,6], Jiang [15,16], Shi and Xiong [28], Shi [29] obtained various results about ordinary differential equations. Castañeda and Robledo [8], Kurzweil and Papaschinopoulos [17], Papaschinopoulos [22] considered topological linearization of difference equations. Topological linearization of impulsive equations and time-scale systems were studied in [18, 31, 33, 34] and [26, 32], respectively. In 1996, Papaschinopoulos [23] generalized the topological linearization theorem to a differential equation with piecewise constant argument (DEPCA).

Nineteen years later, Pinto and Robledo [25] generalized the work of Papaschinopoulos to a differential equation with piecewise constant argument of generalized type (DEPCAG). They studied the following system

$$z'(t) = M(t)z(t) + M_0(t)z(\gamma(t)) + h(t, z(t), z(\gamma(t))), \tag{1.1}$$

where $t \in \mathbb{R}, z(t) \in \mathbb{R}^n$, $M(t)$ and $M_0(t)$ are $n \times n$ matrices, $h : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $\gamma(t) : \mathbb{R} \to \mathbb{R}$.

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Under suitable conditions, they proved that the above nonlinear system is topologically conjugated to its linear system

$$z'(t) = M(t)z(t) + M_0(t)z(\gamma(t)). \tag{1.2}$$

If the function $\gamma(t) = [t]$, where $[t]$ is the integer part of t, then system (1.1) is a DEPCA, which has been well studied by Papaschinopoulos [23]. DEPCAs and DEPCAGs have been extensively studied, readers could refer to [2, 7, 10, 11, 14, 19, 27, 35] for more details.

However, the results in [23] and [25] require that there exists a constant $\mu > 0$ such that $|h(t, z(t), z(\gamma(t)))| \leqslant \mu$. That is the boundness of the nonlinear term $h(t, z(t), z(\gamma(t)))$. So a natural question is what happens when $h(t, z(t), z(\gamma(t)))$ is unbounded?

In this paper, we prove that even if $h(t, z(t), z(\gamma(t)))$ is unbounded, system (1.1) can also be topologically conjugated to system (1.2) as long as it has a proper structure.

The rest of this paper is organized as follows: In Section 2, we give some definitions, notations and preliminary lemmas. Our main result is stated in Section 3. Proofs are given in Section 4 and 5.

2. Preliminaries

2.1. General assumptions

For convenience, in this paper, we assume that system (1.1) has the following form:

$$\begin{cases} x'(t) = A(t)x(t) + A_0(t)x(\gamma(t)) + f(t, x(t), y(t), x(\gamma(t)), y(\gamma(t))), \\ y'(t) = B(t)y(t) + A_0(t)y(\gamma(t)) + g(t, x(t), y(t), x(\gamma(t)), y(\gamma(t))). \end{cases} \tag{2.1}$$

Thus, system (1.2) can be rewritten as:

$$\begin{cases} x'(t) = A(t)x(t) + A_0(t)x(\gamma(t)), \\ y'(t) = B(t)y(t) + A_0(t)y(\gamma(t)). \end{cases} \tag{2.2}$$

Corresponding to systems (1.1) and (1.2),

$$z(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad M(t) = \begin{pmatrix} A(t) & 0 \\ 0 & B(t) \end{pmatrix}, \quad M_0(t) = \begin{pmatrix} A_0(t) & 0 \\ 0 & B_0(t) \end{pmatrix},$$
$$h(t, z(t), z(\gamma(t))) = \begin{pmatrix} f(t, x(t), y(t), x(\gamma(t)), y(\gamma(t))) \\ g(t, x(t), y(t), x(\gamma(t)), y(\gamma(t))) \end{pmatrix} \triangleq \begin{pmatrix} \hat{f}(t) \\ \hat{g}(t) \end{pmatrix},$$

where $t \in \mathbb{R}, x(t) \in \mathbb{R}^{n_1}, y(t) \in \mathbb{R}^{n_2}$, $n_1 + n_2 = n$, $A(t)$, $A_0(t)$ are $n_1 \times n_1$ matrices, $B(t)$, $B_0(t)$ are $n_2 \times n_2$ matrices, $f, g : \mathbb{R} \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^n$.

We suppose that $A(t)$, $A_0(t)$, $B(t)$, $B_0(t)$, f and g satisfy condition **(A)** as follows:

(A1) There exist constants $\beta > 0$ and $\beta_0 > 0$ such that

$$\sup_{t\in\mathbb{R}} |A(t)| \leqslant \beta, \quad \sup_{t\in\mathbb{R}} |B(t)| \leqslant \beta,$$

$$\sup_{t\in\mathbb{R}}|A_0(t)|\leqslant\beta_0,\quad \sup_{t\in\mathbb{R}}|B_0(t)|\leqslant\beta_0,$$

where $|\cdot|$ denotes a matrix norm.
(A2) There exist constants $\mu>0$ and $\ell>0$ such that for any $(t,x(t),y(t),x(\gamma(t)),y(\gamma(t)))\in\mathbb{R}\times\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$,

$$\begin{aligned}&|f(t,x(t),y(t),x(\gamma(t)),y(\gamma(t)))|\leqslant\mu,\\&|g(t,x(t),y(t),x(\gamma(t)),y(\gamma(t))|\leqslant\ell(|x(t)|+|x(\gamma(t))|)+\mu,\end{aligned}$$

where $|\cdot|$ denotes a vector norm.
(A3) $\forall(t,x_1(t),y_1(t),x_1(\gamma(t)),y_1(\gamma(t))),\ (t,x_2(t),y_2(t),x_2(\gamma(t)),y_2(\gamma(t)))\in\mathbb{R}\times\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$, for the above ℓ,

$$\begin{aligned}&|f(t,x_1(t),y_1(t),x_1(\gamma(t)),y_1(\gamma(t)))-f(t,x_2(t),y_2(t),x_2(\gamma(t)),y_2(\gamma(t)))|\\\leqslant&\ell\Big(|x_1(t)-x_2(t)|+|y_1(t)-y_2(t)|+|x_1(\gamma(t))-x_2(\gamma(t))|+|y_1(\gamma(t))-y_2(\gamma(t))|\Big),\\&|g(t,x_1(t),y_1(t),x_1(\gamma(t)),y_1(\gamma(t)))-g(t,x_2(t),y_2(t),x_2(\gamma(t)),y_2(\gamma(t)))|\\\leqslant&\ell\Big(|x_1(t)-x_2(t)|+|y_1(t)-y_2(t)|+|x_1(\gamma(t))-x_2(\gamma(t))|+|y_1(\gamma(t))-y_2(\gamma(t))|\Big).\end{aligned}$$

$\gamma:\mathbb{R}\to\mathbb{R}$ satisfies condition **(B)**: there exist two sequences $\{t_i\}_{i\in\mathbb{Z}}$ and $\{\zeta_i\}_{i\in\mathbb{Z}}$ satisfying
(B1) $t_i<t_{i+1}$ and $t_i\leqslant\zeta_i\leqslant t_{i+1}$, $\forall i\in\mathbb{Z}$,
(B2) $t_i\to\pm\infty$ as $i\to\pm\infty$,
(B3) $\gamma(t)=\zeta_i$ for $t\in[t_i,t_{i+1})$,
(B4) there exists a constant $\theta>0$ such that $t_{i+1}-t_i\leqslant\theta,\forall i\in\mathbb{Z}$.

For any $i\in\mathbb{Z}$, let $I_i=[t_i,t_{i+1})$ and $\bar{I}_i=[t_i,t_{i+1}]$.

For any $m\times m$ matrix $Q(t)$ ($m=n_1,n_2$ or n), define

$$\rho_i^+(Q)=\exp(\int_{t_i}^{\zeta_i}|Q(s)|ds)\quad\text{and}\quad\rho_i^-(Q)=\exp(\int_{\zeta_i}^{t_{i+1}}|Q(s)|ds).$$

Now we introduce the following condition **(C)**.

Condition **(C)**: There exist $\nu^+>0$, $\nu^->0$ such that matrices $A(t)$, $A_0(t)$, $B(t)$ and $B_0(t)$ satisfy the following properties:

$$\begin{aligned}&\sup_{i\in\mathbb{Z}}\rho_i^+(A)\ln\rho_i^+(A_0)\leqslant\nu^+<1,\quad \sup_{i\in\mathbb{Z}}\rho_i^-(A)\ln\rho_i^-(A_0)\leqslant\nu^-<1,\\&\sup_{i\in\mathbb{Z}}\rho_i^+(B)\ln\rho_i^+(B_0)\leqslant\nu^+<1,\quad \sup_{i\in\mathbb{Z}}\rho_i^-(B)\ln\rho_i^-(B_0)\leqslant\nu^-<1.\end{aligned}$$

Note that **(A1)** and **(B4)** imply that

$$1\leqslant\rho(A)\triangleq\sup_{i\in\mathbb{Z}}\rho_i^+(A)\rho_i^-(A)<+\infty\quad\text{and}\quad1\leqslant\rho(B)\triangleq\sup_{i\in\mathbb{Z}}\rho_i^+(B)\rho_i^-(B)<+\infty.\tag{2.3}$$

Thus,

$$\alpha_0(A)\triangleq\rho(A)^2(\frac{1+\nu^+}{1-\nu^-})>1\quad\text{and}\quad\alpha_0(B)\triangleq\rho(B)^2(\frac{1+\nu^+}{1-\nu^-})>1.\tag{2.4}$$

Throughout the rest of the paper, we assume that conditions **(A), (B)** and **(C)** hold.

2.2. Topological conjugacy

The notion of topological equivalence and topological conjugacy can be found in [20, 21, 25, 29, 34].

Definition 2.1 (Topological conjugacy)**.** A continuous function $H : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is topological equivalent between system (1.1) and (1.2) if the following conditions hold:

(i) for each $t \in \mathbb{R}$, $H(t, u)$ is a homeomorphism of $\mathbb{R}^n$,

(ii) $H(t, u) - u$ is bounded in $\mathbb{R} \times \mathbb{R}^n$,

(iii) if $z(t)$ is a solution of system (1.1), then $H(t, z(t))$ is a solution of system (1.2).

In addition, the function $L(t, u) = H^{-1}(t, u)$ has properties (i)–(iii) also.

If such a map H exists, then systems (1.1) and (1.2) are called topologically conjugated.

2.3. Notation of solutions for a DEPCAG

The notion of solutions for a DEPCAG was introduced in [1, 4, 9, 10, 30].

Definition 2.2 (Solutions of a DEPCAG)**.** A function $z(t)$ is a solution of system (1.1) or system (1.2) on $\mathbb{R}$ if:

(i) The derivative $z'(t)$ exists at each point $t \in \mathbb{R}$ with the possible exception of points $t_i, i \in \mathbb{Z}$, where the one side derivatives exist;

(ii) The equation is satisfied for $z(t)$ on each interval (t_i, t_{i+1}) and it holds for the right derivative of $z(t)$ at t_i.

2.4. Transition matrices of systems

In this subsection, we introduce some notations associated with solutions of a DEPCAG.

For convenience, we consider the following subsystems of system (2.1):

$$x'(t) = A(t)x(t) + A_0(t)x(\gamma(t)) + f(t, x(t), y(t), x(\gamma(t)), y(\gamma(t))), \tag{2.5}$$
$$y'(t) = B(t)y(t) + B_0(t)y(\gamma(t)) + g(t, x(t), y(t), x(\gamma(t)), y(\gamma(t))), \tag{2.6}$$

and subsystems of system (2.2):

$$x'(t) = A(t)x(t) + A_0(t)x(\gamma(t)), \tag{2.7}$$
$$y'(t) = B(t)y(t) + B_0(t)y(\gamma(t)). \tag{2.8}$$

Let $\Phi_1(t)$ be the fundamental matrix of system $x' = A(t)x$ with $\Phi_1(0) = I$, and $\Phi_2(t)$ be the fundamental matrix of system $y' = B(t)y$ with $\Phi_2(0) = I$.

For any $t \in I_j$, $\tau \in I_i$, $s \in \mathbb{R}$, we introduce the following notation [10, 24, 25]:

$$\begin{aligned}
&\Phi_k(t,s) = \Phi_k(t)\Phi_k^{-1}(s), \quad k = 1,2,\\
&J_1(t,\tau) = I + \int_\tau^t \Phi_1(\tau,s)A_0(s)ds,\\
&J_2(t,\tau) = I + \int_\tau^t \Phi_2(\tau,s)B_0(s)ds,\\
&E_1(t,\tau) = \Phi_1(t,\tau) + \int_\tau^t \Phi_1(t,s)A_0(s)ds = \Phi_1(t,\tau)J_1(t,\tau),
\end{aligned}$$

and

$$E_2(t,\tau) = \Phi_2(t,\tau) + \int_\tau^t \Phi_2(t,s)B_0(s)ds = \Phi_2(t,\tau)J_2(t,\tau).$$

We define backward and forward products of a set of $n \times n$ ($n_1 \times n_1$ or $n_2 \times n_2$) matrices $\mathcal{Q}_i (i = 1, \ldots, m)$ as follows:

$$\prod_{i=1}^{\leftarrow m} \mathcal{Q}_i = \begin{cases} \mathcal{Q}_m \cdots \mathcal{Q}_2\mathcal{Q}_1, & \text{if} \quad m \geqslant 1,\\ I, & \text{if} \quad m < 1,\end{cases}$$

and

$$\prod_{i=1}^{\rightarrow m} \mathcal{Q}_i = \begin{cases} \mathcal{Q}_1\mathcal{Q}_2 \cdots \mathcal{Q}_m, & \text{if} \quad m \geqslant 1,\\ I, & \text{if} \quad m < 1.\end{cases}$$

If $J_k(t,s)$ $(k = 1,2)$ are nonsingular, we could define the transition matrices $Z_1(t,s)$ of subsystem (2.7) and $Z_2(t,s)$ of subsystem (2.8) as follows:
if $t > \tau$,

$$\begin{aligned}
Z_k(t,\tau) =& E_k(t,\zeta_j)E_k(t_j,\zeta_j)^{-1} \prod_{r=i+2}^{\leftarrow j} \Big(E_k(t_r,\gamma(t_{r-1}))E_k(t_{r-1},\gamma(t_{r-1}))^{-1}\Big)\\
&E_k(t_{i+1},\gamma(\tau))E_k(\tau,\gamma(\tau))^{-1},
\end{aligned}$$

if $t < \tau$,

$$\begin{aligned}
Z_k(t,\tau) =& E_k(t,\zeta_j)E_k(t_{j+1},\zeta_j)^{-1} \prod_{r=j+1}^{\rightarrow i-1} \Big(E_k(t_r,\gamma(t_r))E_k(t_{r+1},\gamma(t_r))^{-1}\Big)\\
&E_k(t_i,\gamma(\tau))E_k(\tau,\gamma(\tau))^{-1},
\end{aligned}$$

where $k = 1,2$.

Through simple calculations, we obtain $Z_k(t,\tau)Z_k(\tau,s) = Z_k(t,s)$ and $Z_k(t,s) = Z_k(s,t)^{-1}$ $(k = 1,2)$. Since $E_1(\tau,\tau) = I$ and $\frac{\partial E_1}{\partial t}(t,\tau) = A(t)E_1(t,\tau) + A_0(t)$, we have

$$\frac{\partial Z_1}{\partial t}(t,\tau) = A(t)Z_1(t,\tau) + A_0(t)Z_1(\gamma(t),\tau).$$

Thus, $Z_1(t,\tau)$ is a solution of subsystem (2.7). Similarly, $Z_2(t,\tau)$ is a solution of subsystem (2.8).

2.5. Formulas of solutions to DEPCAGs

To introduce the formulas of solutions, we first state the following important lemma.

Lemma 2.1 (Lemma 4.3, [24]). *Assume that conditions **(A)**,**(B)**,**(C)** are fulfilled, then $J_k(t,s)(k=1,2)$ is nonsingular for any $t,s\in \bar{I}_r$ and the matrices $Z_k(t,s)(k=1,2)$ and $Z_k(t,s)^{-1}(k=1,2)$ are well defined for any $t,s\in\mathbb{R}$. If $t,s\in\bar{I}_r$, then*

$$\begin{aligned}&|\Phi_1(t,s)|\leqslant\rho(A),\quad |\Phi_2(t,s)|\leqslant\rho(B),\\ &|Z_1(t,s)|\leqslant\alpha_0(A),\quad |Z_2(t,s)|\leqslant\alpha_0(B),\end{aligned}$$

where $\rho(\cdot)$ is defined in (2.3) and $\alpha_0(\cdot)$ is defined in (2.4).

We remark that Lemma 2.1 ensures the continuity of solutions of system (2.1) in $(-\infty,+\infty)$. We introduce the following formulas for DEPCAGs.

Lemma 2.2 (p.239, [24]). *For any $t\in I_j$, $\tau\in I_i$, the solution of subsystem (2.7) with $x(\tau)=\xi$ is defined on $\mathbb{R}$ and is given by*

$$x(t)=Z_1(t,\tau)\xi. \tag{2.9}$$

From Lemma 2.2, the solution of (2.8) with $y(\tau)=\eta$ can be represented as

$$y(t)=Z_2(t,\tau)\eta. \tag{2.10}$$

Lemma 2.3 (Theorem 3.3, [24]). *For $t\in I_j$, $\tau\in I_i$ and $t>\tau$, the solution of subsystem (2.5) with $x(\tau)=\xi$ is defined on $\mathbb{R}$ and is given by*

$$\begin{aligned}x(t)=&Z_1(t,\tau)\xi+\int_\tau^{\zeta_i}Z_1(t,\tau)\Phi_1(\tau,s)\hat{f}(s)ds+\sum_{r=i+1}^{j}\int_{t_r}^{\zeta_r}Z_1(t,t_r)\Phi_1(t_r,s)\hat{f}(s)ds\\ &+\sum_{r=i}^{j-1}\int_{\zeta_r}^{t_{r+1}}Z_1(t,t_{r+1})\Phi_1(t_{r+1},s)\hat{f}(s)ds+\int_{\zeta_j}^{t}\Phi_1(t,s)\hat{f}(s)ds\\ \triangleq&\, Z_1(t,\tau)\eta+\int_\tau^t G_1(t,s)\hat{f}(s)ds,\end{aligned} \tag{2.11}$$

where,

$$G_1(t,s)=\begin{cases}Z_1(t,\tau)\Phi_1(\tau,s), & \textit{if}\ \ s\in[\tau,\zeta_i]\ \textit{or}\ s\in[\zeta_i,\tau],\\ Z_1(t,t_r)\Phi_1(t_r,s), & \textit{if}\ \ s\in[t_r,\zeta_r)\ \textit{for}\ r=i+1,\cdots,j,\\ Z_1(t,t_{r+1})\Phi_1(t_{r+1},s), & \textit{if}\ \ s\in[\zeta_r,t_{r+1})\ \textit{for}\ r=i,\cdots,j-1,\\ \Phi_1(t,s), & \textit{if}\ \ s\in[\zeta_j,t]\ \textit{or}\ s\in[t,\zeta_j].\end{cases}$$

From Lemma 2.3, if $t>\tau$, the solution of subsystem (2.6) with $y(\tau)=\eta$ can be represented as

$$\begin{aligned}y(t)=&Z_2(t,\tau)\eta+\int_\tau^{\zeta_i}Z_2(t,\tau)\Phi_2(\tau,s)\hat{g}(s)ds+\sum_{r=i+1}^{j}\int_{t_r}^{\zeta_r}Z_2(t,t_r)\Phi_2(t_r,s)\hat{g}(s)ds\\ &+\sum_{r=i}^{j-1}\int_{\zeta_r}^{t_{r+1}}Z_2(t,t_{r+1})\Phi_2(t_{r+1},s)\hat{g}(s)ds+\int_{\zeta_j}^{t}\Phi_2(t,s)\hat{g}(s)ds\\ =&\,Z_2(t,\tau)\eta+\int_\tau^t G_2(t,s)\hat{g}(s)ds,\end{aligned} \tag{2.12}$$

where $G_2(t,s)$ can be defined in the same way as $G_1(t,s)$.

Remark 2.1. If $t<\tau$, one could obtain the solution formulas by replacing $\sum\limits_{r=i+1}^{j}$ and $\sum\limits_{r=i}^{j-1}$ with $\sum\limits_{r=j+1}^{i}$ and $\sum\limits_{r=j}^{i-1}$, respectively.

At the same time, we could obtain $G_k(t,s)$ $(k=1,2)$ for $t<\tau$ by replacing $r=i+1,\cdots,j$, and $r=i,\cdots,j-1$, with $r=j+1,\cdots,i$, and $r=j,\cdots,i-1$, in the definitions of $G_k(t,s)$ $(t>s, k=1,2)$, respectively.

2.6. exponential dichotomy for a DEPCAG

Now we introduce the definition of exponential dichotomy for a DEPCAG. In this paper, we adopt the following definition from Akhmet [3,4].

Definition 2.3 (α-exponential dichotomy for a DEPCAG)**.** The linear system (1.2) admits an α-exponential dichotomy on $(-\infty,\infty)$ if there exist a projection P, positive constants $K\geqslant 1$ and $\alpha>0$ such that the transition matrix $Z(t,s)$ of system (1.2) satisfies

$$|Z_P(t,s)|\leqslant Ke^{-\alpha|t-s|},$$

where $Z_P(t,s)$ is defined by

$$Z_P(t,s)=\begin{cases} Z(t,0)PZ(0,s), & t\geqslant s,\\ -Z(t,0)(I-P)Z(0,s), & s>t.\end{cases}$$

In this paper, we assume that the following condition **(D)** holds.
Condition **(D)**: There exist positive constants $K\geqslant 1$ and $\alpha>0$ such that

$$|Z_1(t,s)|\leqslant Ke^{-\alpha(t-s)},\quad t\geqslant s\quad\text{and}\quad |Z_2(t,s)|\leqslant Ke^{\alpha(t-s)},\quad s>t.$$

2.7. Some lemmas

Lemma 2.4. *If condition **(D)** holds, for $t\in\mathbb{R}$ and $s\in\mathbb{R}$, then*

$$|G_1(t,s)|\leqslant K\tilde{\rho}(A)e^{-\alpha(t-s)},\quad t\geqslant s,\qquad |G_2(t,s)|\leqslant K\tilde{\rho}(B)e^{\alpha(t-s)},\quad t<s,$$

where $\tilde{\rho}(\cdot)=\max(\rho(\cdot)\alpha_0(\cdot),\rho(\cdot)e^{\alpha\theta})$, $\rho(\cdot)$ is defined in (2.3)*, $\alpha_0(\cdot)$ is defined in* (2.4)*, α is defined in **(D)** and θ is defined in **(B4)**.*

Proof. We just prove the first inequality.

Suppose that $t\in I_j$, $\tau\in I_i$ and $t\geqslant s$.

Case 1. $t\geqslant\tau$.

Without loss of generality, we assume that $t_i\leqslant\tau\leqslant\zeta_i\leqslant t_{i+1}\leqslant\cdots t_j\leqslant\zeta_j\leqslant t$.

If $s\in[\tau,\zeta_i]$, due to **(B4)**, we have $s-\tau\leqslant\theta$. It follows from **(D)** and Lemma 2.1 that

$$|G_1(t,s)|=|Z_1(t,\tau)\Phi_1(\tau,s)|\leqslant Ke^{-\alpha(t-\tau)}\rho(A)\leqslant Ke^{-\alpha(t-s)}e^{\alpha\theta}\rho(A).$$

If $s\in[t_r,\zeta_r]$ $(r=i+1,\cdots,j)$, then $s-t_r\leqslant\theta$. In view of **(D)** and 2.1, we have

$$|G_1(t,s)|=|Z_1(t,t_r)\Phi_1(t_r,s)|\leqslant Ke^{-\alpha(t-t_r)}\rho(A)\leqslant Ke^{-\alpha(t-s)}e^{\alpha\theta}\rho(A).$$

If $s \in [\zeta_r, t_{r+1}]$ $(r = i, \cdots, j-1)$, similar to the above inequality, we have the same conclusion.

If $s \in [\zeta_j, t]$, owing to **(B4)**, we have $t - s \leqslant \theta$. It follows from Lemma 2.1 and $K \geqslant 1$ that

$$|G_1(t,s)| = |\Phi_1(t,s)| \leqslant \rho(A) \leqslant Ke^{-\alpha(t-s)}e^{\alpha\theta}\rho(A). \tag{2.13}$$

Case 2. $t \leqslant \tau$.

By definition of $G_1(t,s)$ we have $s \in [\min(t,\zeta_j), \max(\tau,\zeta_i)]$.

If $t \leqslant \zeta_j$, then $t < s$ which contradicts to our assumption that $t \geqslant s$. Thus, we just consider the case that $\zeta_j \leqslant t$. We divide the discussion into two subcases.

Subcase 2.1. $\zeta_j \leqslant t \leqslant t_{j+1} \leqslant \tau$.

For $t \geqslant s$, the only possibility is that $s \in [\zeta_j, t]$. Similar to (2.13), we have

$$|G_1(t,s)| \leqslant Ke^{-\alpha(t-s)}e^{\alpha\theta}\rho(A).$$

Subcase 2.2. $\zeta_j \leqslant t \leqslant \tau \leqslant t_{j+1}$.

If $t \geqslant s$, then $s \in [\zeta_j, t]$ or $s \in [\zeta_j, \tau]$.

When $s \in [\zeta_j, t]$, similar to (2.13), we get

$$|G_1(t,s)| \leqslant Ke^{-\alpha(t-s)}e^{\alpha\theta}\rho(A).$$

When $s \in [\zeta_j, \tau]$, we have $s \in \bar{I}_j$. Since $t \geqslant s$, following **(D)** and Lemma 2.1, we obtain

$$|G_1(t,s)| = |Z_1(t,\tau)\Phi_1(\tau,s)| = |Z_1(t,s)Z_1(s,\tau)\Phi_1(\tau,s)| \leqslant Ke^{-\alpha(t-s)}\alpha_0(A)\rho(A).$$

Note that $\tilde{\rho}(A) = \max(\rho(A)\alpha_0(A), \rho(A)e^{\alpha\theta})$, we complete the proof. □

Lemma 2.5. *Assume that condition **(D)** holds, then*

$$\lim_{t\to-\infty}|Z_1(t,\tau)| = +\infty, \quad \lim_{t\to+\infty}|Z_2(t,\tau)| = +\infty, \quad \forall \tau \in \mathbb{R}.$$

Moreover, the unique bounded solution in $(-\infty,+\infty)$ *of subsystem* (2.7) *((2.8)) is trival.*

Proof. The proof is similar to that of Lemma 2.3 in [25] and so it is omitted. □

3. Main result

Now we are in a position to state our main result as follows.

Theorem 3.1. *If condition **(D)** holds, further assume that*

$$8K\tilde{\rho}(A)\ell < \alpha, \quad 8K\tilde{\rho}(B)\ell < \alpha, \tag{3.1}$$

$$F(\ell,\theta)(\beta_0 + \ell)\theta = \upsilon < 1, \tag{3.2}$$

where $F(\ell,\theta) = \frac{e^{(\beta+\ell)\theta}-1}{(\beta+\ell)\theta}$, $\tilde{\rho}(\cdot)$ *is defined in Lemma 2.4,* ℓ, β *and* β_0 *are defined in **(A)**,* θ *is defined in **(B4)**,* K *and* α *are defined in **(D)**. Then system* (2.1) *is topologically conjugated to system* (2.2).

Remark 3.1. When the solution $z(t)$ of system (2.2) is unbounded, the nonlinear term $h(t,z(t),z(\gamma(t)))$ is possible unbounded. For example, $g(t,x(t),y(t),x(\gamma(t)),y(\gamma(t)))$ in $h(t,z(t),z(\gamma(t)))$ can be a polynomial of order one about $x(t)\sin(y(t))$. In this case, $h(t,z(t),z(\gamma(t)))$ is unbounded, however the topological linearization can be realized.

4. Topological equivalent functions H and L

The main aim in this section is to establish the topological equivalent functions H and L between solutions of system (2.1) and (2.2). We first give some lemmas.

For any $\tau \in I_i$, $t \in \mathbb{R}$, $\xi \in \mathbb{R}^{n_1}$, $\eta \in \mathbb{R}^{n_2}$, assume that

(i) $(x(t,(\tau,\xi,\eta)), y(t,(\tau,\xi,\eta)))^T$ is the solution of system (2.2) satisfying

$$x(\tau,(\tau,\xi,\eta)) = \xi \text{ and } y(\tau,(\tau,\xi,\eta)) = \eta.$$

(ii) $(X(t,(\tau,\xi,\eta)), Y(t,(\tau,\xi,\eta)))^T$ is the solution of system (2.1) satisfying

$$X(\tau,(\tau,\xi,\eta)) = \xi \text{ and } Y(\tau,(\tau,\xi,\eta)) = \eta.$$

Let

$$\begin{aligned}
q_1(\xi) &= \max(K|\xi|, \alpha_0(A)|\xi|),\\
q_2(\xi) &= \max(K|\xi| + K\tilde{\rho}(A)\alpha^{-1}\mu, \alpha_0(A)|\xi| + \alpha_0(A)\rho(A)\mu\theta).
\end{aligned}$$

Lemma 4.1. *If $t \geqslant \tau$, we have*

$$\begin{aligned}
&|x(t,(\tau,\xi,\eta))| \leqslant q_1(\xi), \quad |x(\gamma(t),(\tau,\xi,\eta))| \leqslant q_1(\xi),\\
&|X(t,(\tau,\xi,\eta))| \leqslant q_2(\xi), \quad |X(\gamma(t),(\tau,\xi,\eta))| \leqslant q_2(\xi).
\end{aligned}$$

Proof. If $t \geqslant \tau$, by (2.9) and **(D)** we have

$$|x(t,(\tau,\xi,\eta))| = |Z_1(t,\tau)\xi| \leqslant Ke^{-\alpha(t-\tau)}|\xi| \leqslant K|\xi| \leqslant q_1(\xi). \tag{4.1}$$

If $\gamma(t) \geqslant \tau$, then

$$|x(\gamma(t),(\tau,\xi,\eta))| \leqslant q_1(\xi).$$

If $\gamma(t) \leqslant \tau \leqslant t$, then $\gamma(t) = \gamma(\tau) = \zeta_i \in \bar{I}_i$. Using (2.9), **(D)** and Lemma 2.1, we get

$$|x(\gamma(t),(\tau,\xi,\eta))| = |Z_1(\gamma(t),\tau)\xi| \leqslant \alpha_0(A)|\xi| \leqslant q_1(\xi). \tag{4.2}$$

From (2.11), we have

$$\begin{aligned}
&X(t,(\tau,\xi,\eta))\\
=&Z_1(t,\tau)\xi + \int_\tau^t G_1(t,s)\cdot f\Big(s, X(s,(\tau,\xi,\eta)), Y(s,(\tau,\xi,\eta)), X(\gamma(s),(\tau,\xi,\eta)),\\
&Y(\gamma(s),(\tau,\xi,\eta))\Big)ds.
\end{aligned}$$

If $t \geqslant \tau$, due to (4.1), **(A2)** and Lemma 2.4, we obtain

$$|X(t,(\tau,\xi,\eta))| \leqslant K|\xi| + \int_\tau^t K\tilde{\rho}(A)e^{-\alpha(t-s)}\mu ds \leqslant K|\xi| + K\tilde{\rho}(A)\alpha^{-1}\mu \leqslant q_2(\xi).$$

If $\gamma(t) \geqslant \tau$, from the above inequality, we have

$$|X(\gamma(t),(\tau,\xi,\eta))| \leqslant q_2(\xi).$$

If $\gamma(t) < \tau \leqslant t$, we have $\gamma(t) = \gamma(\tau) = \zeta_i \in \bar{I}_i$. It follows from (2.11) that

$$\begin{aligned}
&X(\gamma(t),(\tau,\xi,\eta))\\
=&Z_1(\gamma(t),\tau)\xi + \int_\tau^{\gamma(t)} Z_1(\gamma(t),\tau)\Phi_1(\tau,s)\cdot f\Big(s, X(s,(\tau,\xi,\eta)), Y(s,(\tau,\xi,\eta)),\\
&X(\gamma(s),(\tau,\xi,\eta)), Y(\gamma(s),(\tau,\xi,\eta))\Big)ds.
\end{aligned}$$

In view of **(B4)**, we have $|\gamma(t)-\tau| \leqslant \theta$, together with (4.2), **(A2)** and Lemma 2.1, we obtain that

$$|X(\gamma(t),(\tau,\xi,\eta))| \leqslant \alpha_0(A)|\xi| + \alpha_0(A)\rho(A)\mu\theta \leqslant q_2(\xi).$$

□

Lemma 4.2. *If $\tau \in I_i$, then*

$$\begin{aligned}
I \triangleq \sum_{r=i}^{+\infty}\int_{\zeta_r}^{t_{r+1}} & Z_2(\tau,t_{r+1})\Phi_2(t_{r+1},s)\cdot g\Big(s, x(s,(\tau,\xi,\eta)), y(s,(\tau,\xi,\eta)),\\
& x(\gamma(s),(\tau,\xi,\eta)), y(\gamma(s),(\tau,\xi,\eta))\Big)ds
\end{aligned}$$

is convergent.

Proof. Due to $\tau \in I_i$, we have $\tau < t_{r+1}$ for all $r \geqslant i$. It follows from **(D)** that

$$|Z_2(\tau,t_{r+1})| \leqslant Ke^{\alpha(\tau-t_{r+1})}.$$

If $s \in [\zeta_r, t_{r+1}]$, by **(B4)** we have $e^{\alpha(\tau-t_{r+1})} \leqslant e^{\alpha(\tau-s)}e^{\alpha\theta}$. It follows from Lemma 2.1 that

$$|Z_2(\tau,t_{r+1})\Phi_2(t_{r+1},s)| \leqslant Ke^{\alpha(\tau-s)}e^{\alpha\theta}\rho(B) \leqslant Ke^{\alpha(\tau-s)}\tilde{\rho}(B).$$

From **(A2)**, we obtain

$$|I| \leqslant \int_{\zeta_i}^{+\infty} Ke^{\alpha(\tau-s)}\tilde{\rho}(B)\{\ell(|x(s,(\tau,\xi,\eta))| + |x(\gamma(s),(\tau,\xi,\eta))|) + \mu\}ds.$$

Since $s \in [\zeta_i, +\infty)$ and $\tau \in I_i$, similar to Lemma 4.1, we get

$$|x(s,(\tau,\xi,\eta))| \leqslant q_1(\xi), \quad |x(\gamma(s),(\tau,\xi,\eta))| \leqslant q_1(\xi).$$

Due to (3.1), we have

$$|I| \leqslant \frac{1}{4}e^{\alpha(\tau-\zeta_i)}q_1(\xi) + K\tilde{\rho}(B)\alpha^{-1}e^{\alpha(\tau-\zeta_i)}\mu.$$

□

Denote

$$\Omega = \{\varphi(t)|\varphi \in C(R,R^{n_1}), |\varphi(t)| \leqslant K\tilde{\rho}(A)\alpha^{-1}\mu\}.$$

Lemma 4.3. *For any $\varphi \in \Omega$, the system*

$$\begin{aligned}
v'(t) =& B(t)v(t) + B_0(t)v(\gamma(t)) + g\Big(t, x(t,(\tau,\xi,\eta)) + \varphi(t), y(t,(\tau,\xi,\eta)) + v(t),\\
& x(\gamma(t),(\tau,\xi,\eta)) + \varphi(\gamma(t)), y(\gamma(t),(\tau,\xi,\eta)) + v(\gamma(t))\Big) \qquad (4.3)
\end{aligned}$$

has a unique solution $v_\varphi(t,(\tau,\xi,\eta))$ which is bounded for $t \geqslant \tau$ and $\gamma(t) \geqslant \tau$. Moreover, if $t \in I_j$, then it has the following form

$$\begin{aligned}
&v_\varphi(t,(\tau,\xi,\eta))\\
=&-\sum_{r=j+1}^{+\infty}\int_{t_r}^{\zeta_r} Z_2(t,t_r)\Phi_2(t_r,s)\cdot g\Big(s, x(s,(\tau,\xi,\eta))+\varphi(s), y(s,(\tau,\xi,\eta))+v_\varphi\\
&(s,(\tau,\xi,\eta)), x(\gamma(s),(\tau,\xi,\eta))+\varphi(\gamma(s)), y(\gamma(s),(\tau,\xi,\eta))+v_\varphi(\gamma(s),(\tau,\xi,\eta))\Big)ds\\
&-\sum_{r=j}^{+\infty}\int_{\zeta_r}^{t_{r+1}} Z_2(t,t_r)\Phi_2(t_r,s)\cdot g\Big(s, x(s,(\tau,\xi,\eta))+\varphi(s), y(s,(\tau,\xi,\eta))+v_\varphi\\
&(s,(\tau,\xi,\eta)), x(\gamma(s),(\tau,\xi,\eta))+\varphi(\gamma(s)), y(\gamma(s),(\tau,\xi,\eta))+v_\varphi(\gamma(s),(\tau,\xi,\eta))\Big)ds\\
&-\int_t^{\zeta_j}\Phi_2(t,s)\cdot g\Big(s, x(s,(\tau,\xi,\eta))+\varphi(s), y(s,(\tau,\xi,\eta))+v_\varphi(s,(\tau,\xi,\eta)),\\
&x(\gamma(s),(\tau,\xi,\eta))+\varphi(\gamma(s)), y(\gamma(s),(\tau,\xi,\eta))+v_\varphi(\gamma(s),(\tau,\xi,\eta))\Big)ds\\
=&-\int_t^{+\infty} G_2(t,s)\cdot g\Big(s, x(s,(\tau,\xi,\eta))+\varphi(s), y(s,(\tau,\xi,\eta))+v_\varphi(s,(\tau,\xi,\eta)),\\
&x(\gamma(s),(\tau,\xi,\eta))+\varphi(\gamma(s)), y(\gamma(s),(\tau,\xi,\eta))+v_\varphi(\gamma(s),(\tau,\xi,\eta))\Big)ds.
\end{aligned}$$

For $v_\varphi(t,(\tau,\xi,\eta))$, the following system

$$\begin{aligned}
&u'(t)\\
=&A(t)u(t)+A_0(t)u(\gamma(t))+f\Big(t, x(t,(\tau,\xi,\eta))+u(t), y(t,(\tau,\xi,\eta))+v_\varphi(t,(\tau,\xi,\eta)),\\
&x(\gamma(t),(\tau,\xi,\eta))+u(\gamma(t)), y(\gamma(t),(\tau,\xi,\eta))+v_\varphi(\gamma(t),(\tau,\xi,\eta))\Big),
\end{aligned}\tag{4.4}$$

has a unique bounded solution $u_\varphi(t,(\tau,\xi,\eta))$ satisfying

$$|u_\varphi(t,(\tau,\xi,\eta))| \leqslant K\tilde{\rho}(A)\alpha^{-1}\mu.$$

Proof. Let $N_\tau = \{\psi | \psi : [\tau,+\infty) \to \mathbb{R}^{n_2}, \psi$ is a continuous function with $|\psi(t)| \leqslant \frac{1}{4}q_1(\xi) + \frac{1}{4}K\tilde{\rho}(A)\alpha^{-1}\mu + K\tilde{\rho}(B)\alpha^{-1}\mu\}$.

For any $\psi \in N_\tau$, define the following map

$$\begin{aligned}
T_1\psi(t) = -\int_t^{+\infty} G_2(t,s)\cdot g\Big(&s, x(s,(\tau,\xi,\eta))+\varphi(s), y(s,(\tau,\xi,\eta))+\psi(s),\\
&x(\gamma(s),(\tau,\xi,\eta))+\varphi(\gamma(s)), y(\gamma(s),(\tau,\xi,\eta))+\psi(\gamma(s))\Big)ds.
\end{aligned}$$

If $t \geqslant \tau$ and $\gamma(t) \geqslant \tau$, then $s \geqslant \tau$ and $\gamma(s) \geqslant \tau$. From **(A2)**, (3.1), Lemma 2.4 and Lemma 4.1, we have

$$\begin{aligned}
|T_1\psi(t)| \leqslant& \int_t^{+\infty} K\tilde{\rho}(B)e^{\alpha(t-s)}\{\ell|x(s,(\tau,\xi,\eta))+\varphi(s)|\\
&+\ell|x(\gamma(s),(\tau,\xi,\eta))+\varphi(\gamma(s))|+\mu\}ds\\
\leqslant& \int_t^{+\infty} K\tilde{\rho}(B)e^{\alpha(t-s)}(2\ell q_1(\xi)+2\ell K\tilde{\rho}(A)\alpha^{-1}\mu+\mu)ds
\end{aligned}$$

$$
\begin{aligned}
&= K\tilde{\rho}(B)\alpha^{-1}(2\ell K|\xi| + 2\ell K\tilde{\rho}(A)\alpha^{-1}\mu + \mu) \\
&\leqslant \frac{1}{4}q_1(\xi) + \frac{1}{4}K\tilde{\rho}(A)\alpha^{-1}\mu + K\tilde{\rho}(B)\alpha^{-1}\mu. \qquad (4.5)
\end{aligned}
$$

Therefore, T_1 is a map from N_τ to N_τ.

Moreover, owing to **(A3)** and Lemma 2.4,

$$
|T_1\psi_1(t) - T_1\psi_2(t)| \leqslant \int_t^{+\infty} K\tilde{\rho}(B)e^{\alpha(t-s)}\ell(|\psi_1(s)-\psi_2(s)|+|\psi_1(\gamma(s))-\psi_2(\gamma(s))|)ds.
$$

Define $\|\psi_1 - \psi_2\|_\tau = \sup\limits_{s\geqslant\tau}|\psi_1(s) - \psi_2(s)|$, then

$$
|T_1\psi_1(t) - T_1\psi_2(t)| \leqslant K\tilde{\rho}(B)\alpha^{-1}2\ell\|\psi_1 - \psi_2\|_\tau < \frac{1}{4}\|\psi_1 - \psi_2\|_\tau \quad (t \geqslant \tau, \gamma(t) \geqslant \tau).
$$

Thus T_1 is a contracting map and there exists $\psi_0(t) \in N_\tau$ such that

$$
\begin{aligned}
\psi_0(t) =& T_1\psi_0(t) \\
=& -\int_t^{+\infty} G_2(t,s)\cdot g\Big(s, x(s,(\tau,\xi,\eta)) + \varphi(s), y(s,(\tau,\xi,\eta)) + \psi_0(s), \\
& x(\gamma(s),(\tau,\xi,\eta)) + \varphi(\gamma(s)), y(\gamma(s),(\tau,\xi,\eta)) + \psi_0(\gamma(s))\Big)ds. \qquad (4.6)
\end{aligned}
$$

Calculating the derivative of $\psi_0(t)$, we obtain

$$
\begin{aligned}
\psi_0'(t) =& B(t)\psi_0(t) + B_0(t)\psi_0(\gamma(t)) + g\Big(t, x(t,(\tau,\xi,\eta)) + \varphi(t), y(t,(\tau,\xi,\eta)) + \psi_0(t), \\
& x(\gamma(t),(\tau,\xi,\eta)) + \varphi(\gamma(t)), y(\gamma(t),(\tau,\xi,\eta)) + \psi_0(\gamma(t))\Big), \qquad (4.7)
\end{aligned}
$$

which implies that $\psi_0(t)$ is a solution of system (4.3). It follows from (4.5) that $\psi_0(t)$ is bounded for $t \geqslant \tau$ and $\gamma(t) \geqslant \tau$.

Suppose that $\tau \in I_i$. If $\psi^*(t)$ is another solution of system (4.3) such that it is bounded for $t \geqslant \tau$ and $\gamma(t) \geqslant \tau$, by Lemma 2.3 we have

$$
\begin{aligned}
&\psi^*(t) \\
=& Z_2(t,\tau)\psi^*(\tau) + \int_\tau^{\xi_i} Z_2(t,\tau)\Phi_2(\tau,s)\cdot g\Big(s, x(s,(\tau,\xi,\eta)) + \varphi(s), y(s,(\tau,\xi,\eta)) \\
& + \psi^*(s), x(\gamma(s),(\tau,\xi,\eta)) + \varphi(\gamma(s)), y(\gamma(s),(\tau,\xi,\eta)) + \psi^*(\gamma(s))\Big)ds \\
& + \sum_{r=i+1}^{j}\int_{t_r}^{\zeta_r} Z_2(t,t_r)\Phi_2(t_r,s)\cdot g\Big(s, x(s,(\tau,\xi,\eta)) + \varphi(s), y(s,(\tau,\xi,\eta)) + \psi^*(s), \\
& x(\gamma(s),(\tau,\xi,\eta)) + \varphi(\gamma(s)), y(\gamma(s),(\tau,\xi,\eta)) + \psi^*(\gamma(s))\Big)ds \\
& + \sum_{r=i}^{j-1}\int_{\zeta_r}^{t_{r+1}} Z_2(t,t_{r+1})\Phi_2(t_{r+1},s)\cdot g\Big(s, x(s,(\tau,\xi,\eta)) + \varphi(s), y(s,(\tau,\xi,\eta)) \\
& + \psi^*(s), x(\gamma(s),(\tau,\xi,\eta)) + \varphi(\gamma(s)), y(\gamma(s),(\tau,\xi,\eta)) + \psi^*(\gamma(s))\Big)ds \\
& - \int_t^{\zeta_j} \Phi_2(t,s)\cdot g\Big(s, x(s,(\tau,\xi,\eta)) + \varphi(s), y(s,(\tau,\xi,\eta)) + \psi^*(s), \\
& x(\gamma(s),(\tau,\xi,\eta)) + \varphi(\gamma(s)), y(\gamma(s),(\tau,\xi,\eta)) + \psi^*(\gamma(s))\Big)ds.
\end{aligned}
$$

Similar to Lemma 4.2, we could prove that

$$\sum_{r=i+1}^{+\infty}\int_{t_r}^{\zeta_r} Z_2(\tau,t_r)\Phi_2(t_r,s)\cdot g\Big(s,x(s,(\tau,\xi,\eta))+\varphi(s),y(s,(\tau,\xi,\eta)) + \psi^*(s),x(\gamma(s),(\tau,\xi,\eta))+\varphi(\gamma(s)),y(\gamma(s),(\tau,\xi,\eta))+\psi^*(\gamma(s))\Big)ds,$$

and

$$\sum_{r=i}^{+\infty}\int_{\zeta_r}^{t_{r+1}} Z_2(\tau,t_{r+1})\Phi_2(t_{r+1},s)\cdot g\Big(s,x(s,(\tau,\xi,\eta))+\varphi(s),y(s,(\tau,\xi,\eta)) + \psi^*(s),x(\gamma(s),(\tau,\xi,\eta))+\varphi(\gamma(s)),y(\gamma(s),(\tau,\xi,\eta))+\psi^*(\gamma(s))\Big)ds$$

are convergent.

Recalling that $Z_2(t,t_r)=Z_2(t,\tau)Z_2(\tau,t_r)$, we have

$$\begin{aligned}
&\psi^*(t)\\
=&Z_2(t,\tau)\Big\{\psi^*(\tau)+\int_{\tau}^{\zeta_i}\Phi_2(\tau,s)\cdot g\Big(s,x(s,(\tau,\xi,\eta))+\varphi(s),y(s,(\tau,\xi,\eta))\\
&+\psi^*(s),x(\gamma(s),(\tau,\xi,\eta))+\varphi(\gamma(s)),y(\gamma(s),(\tau,\xi,\eta))+\psi^*(\gamma(s))\Big)ds\\
&\sum_{r=i+1}^{+\infty}\int_{t_r}^{\zeta_r} Z_2(\tau,t_r)\Phi_2(t_r,s)\cdot g\Big(s,x(s,(\tau,\xi,\eta))+\varphi(s),y(s,(\tau,\xi,\eta))\\
&+\psi^*(s),x(\gamma(s),(\tau,\xi,\eta))+\varphi(\gamma(s)),y(\gamma(s),(\tau,\xi,\eta))+\psi^*(\gamma(s))\Big)ds\\
&+\sum_{r=i}^{+\infty}\int_{\zeta_r}^{t_{r+1}} Z_2(\tau,t_{r+1})\Phi_2(t_{r+1},s)\cdot g\Big(s,x(s,(\tau,\xi,\eta))+\varphi(s),y(s,(\tau,\xi,\eta))\\
&+\psi^*(s),x(\gamma(s),(\tau,\xi,\eta))+\varphi(\gamma(s)),y(\gamma(s),(\tau,\xi,\eta))+\psi^*(\gamma(s))\Big)ds\Big\}\\
&-\sum_{r=j+1}^{+\infty}\int_{t_r}^{\zeta_r} Z_2(t,t_r)\Phi_2(t_r,s)\cdot g\Big(s,x(s,(\tau,\xi,\eta))+\varphi(s),y(s,(\tau,\xi,\eta))\\
&+\psi^*(s),x(\gamma(s),(\tau,\xi,\eta))+\varphi(\gamma(s)),y(\gamma(s),(\tau,\xi,\eta))+\psi^*(\gamma(s))\Big)ds\\
&-\sum_{r=j}^{+\infty}\int_{\zeta_r}^{t_{r+1}} Z_2(t,t_{r+1})\Phi_2(t_{r+1},s)\cdot g\Big(s,x(s,(\tau,\xi,\eta))+\varphi(s),y(s,(\tau,\xi,\eta))\\
&+\psi^*(s),x(\gamma(s),(\tau,\xi,\eta))+\varphi(\gamma(s)),y(\gamma(s),(\tau,\xi,\eta))+\psi^*(\gamma(s))\Big)ds\\
&-\int_{t}^{\zeta_j}\Phi_2(t,s)\cdot g\Big(s,x(s,(\tau,\xi,\eta))+\varphi(s),y(s,(\tau,\xi,\eta))+\psi^*(s),\\
&x(\gamma(s),(\tau,\xi,\eta))+\varphi(\gamma(s)),y(\gamma(s),(\tau,\xi,\eta))+\psi^*(\gamma(s))\Big)ds\\
\triangleq&Z_2(t,\tau)a+I.
\end{aligned}$$

If $t\geqslant\tau$ and $\gamma(t)\geqslant\tau$, similar to calculation in (4.5), we obtain

$$|I|\leqslant\frac{1}{4}q_1(\xi)+\frac{1}{4}K\tilde{\rho}(A)\alpha^{-1}\mu+K\tilde{\rho}(B)\alpha^{-1}\mu. \tag{4.8}$$

On the other hand, $|\psi^*(t)| < +\infty$ for $t \geqslant \tau$ and $\gamma(t) \geqslant \tau$. It follows from (4.8) that $Z_2(t,\tau)a$ is a bounded solution of subsystem (2.8). By Lemma 2.5 we get $a = 0$. Thus

$$\psi^*(t) = -\int_t^{+\infty} G_2(t,s) \cdot g\Big(s, x(s,(\tau,\xi,\eta)) + \varphi(s), y(s,(\tau,\xi,\eta)) + \psi^*(s), \\ x(\gamma(s),(\tau,\xi,\eta)) + \varphi(\gamma(s)), y(\gamma(s),(\tau,\xi,\eta)) + \psi^*(\gamma(s))\Big)ds.$$

From **(A3)**, (3.1) and Lemma 2.4, we have

$$\begin{aligned} |\psi_0(t) - \psi^*(t)| &\leqslant \int_t^{+\infty} K\tilde{\rho}(B)e^{\alpha(t-s)}\ell(|\psi_0(s) - \psi^*(s)| + |\psi_0(\gamma(s)) - \psi^*(\gamma(s))|)ds \\ &\leqslant \frac{1}{4}\sup_{s\geqslant\tau}|\psi_0(s) - \psi^*(s)| \quad (t \geqslant \tau, \gamma(t) \geqslant \tau). \end{aligned}$$

Thus $\psi^*(t) \equiv \psi_0(t)$ for $t \geqslant \tau$ and $\gamma(t) \geqslant \tau$.

From Lemma 2.3, there exists a unique solution of system (4.3) satisfying the given initial value. Therefore,

$$\psi^*(t) \equiv \psi_0(t), \quad \forall t \in \mathbb{R}.$$

Denote $\psi_0(t) = v_\varphi(t,(\tau,\xi,\eta))$. By (4.6) we have

$$\begin{aligned} &v_\varphi(t,(\tau,\xi,\eta)) \\ &= -\int_t^{+\infty} G_2(t,s)g\Big(s, x(s,(\tau,\xi,\eta)) + \varphi(s), y(s,(\tau,\xi,\eta)) + v_\varphi(s,(\tau,\xi,\eta)), \\ &\quad x(\gamma(s),(\tau,\xi,\eta)) + \varphi(\gamma(s)), y(\gamma(s),(\tau,\xi,\eta)) + v_\varphi(\gamma(s),(\tau,\xi,\eta))\Big)ds. \end{aligned} \tag{4.9}$$

Now we prove that there exists a unique bounded solution $u_\varphi(t,(\tau,\xi,\eta))$ of system (4.4) for $v_\varphi(t,(\tau,\xi,\eta))$. Moreover, $|u_\varphi(t,(\tau,\xi,\eta))| \leqslant K\tilde{\rho}(A)\alpha^{-1}\mu$.

For any $\omega(t) \in \Omega$, we define map T_2 as follows:

$$\begin{aligned} &T_2\omega(t) \\ &= \sum_{-\infty}^{j} \int_{t_r}^{\zeta_r} Z_1(t,t_r)\Phi_1(t_r,s)f\Big(s, x(s,(\tau,\xi,\eta)) + \omega(s), y(s,(\tau,\xi,\eta)) + v_\varphi(s,(\tau,\xi,\eta)), \\ &\quad x(\gamma(s),(\tau,\xi,\eta)) + \omega(\gamma(s)), y(\gamma(s),(\tau,\xi,\eta)) + v_\varphi(\gamma(s),(\tau,\xi,\eta))\Big)ds \\ &\quad + \sum_{-\infty}^{j-1} \int_{\zeta_r}^{t_{r+1}} Z_1(t,t_{r+1})\Phi_1(t_{r+1},s)f\Big(s, x(s,(\tau,\xi,\eta)) + \omega(s), y(s,(\tau,\xi,\eta)) + v_\varphi \\ &\quad (s,(\tau,\xi,\eta)), x(\gamma(s),(\tau,\xi,\eta)) + \omega(\gamma(s)), y(\gamma(s),(\tau,\xi,\eta)) + v_\varphi(\gamma(s),(\tau,\xi,\eta))\Big)ds \\ &\quad + \int_{\zeta_j}^{t} \Phi_1(t,s)f\Big(s, x(s,(\tau,\xi,\eta)) + \omega(s), y(s,(\tau,\xi,\eta)) + v_\varphi(s,(\tau,\xi,\eta)), \\ &\quad x(\gamma(s),(\tau,\xi,\eta)) + \omega(\gamma(s)), y(\gamma(s),(\tau,\xi,\eta)) + v_\varphi(\gamma(s),(\tau,\xi,\eta))\Big)ds \\ &= \int_{-\infty}^{t} G_1(t,s)f\Big(s, x(s,(\tau,\xi,\eta)) + \omega(s), y(s,(\tau,\xi,\eta)) + v_\varphi(s,(\tau,\xi,\eta)), \\ &\quad x(\gamma(s),(\tau,\xi,\eta)) + \omega(\gamma(s)), y(\gamma(s),(\tau,\xi,\eta)) + v_\varphi(\gamma(s),(\tau,\xi,\eta))\Big)ds. \end{aligned}$$

In view of **(A2)** and Lemma 2.4, we obtain

$$|T_2\omega(t)| \leqslant \int_{-\infty}^{t} K\tilde{\rho}(A)e^{-\alpha(t-s)}\mu ds = K\tilde{\rho}(A)\alpha^{-1}\mu.$$

So T_2 is a map from Ω to Ω.

Due to **(A3)**, (3.1) and Lemma 2.4, we have

$$\begin{aligned}
&|T_2\omega_1(t) - T_2\omega_2(t)| \\
\leqslant &\int_{-\infty}^{t} K\tilde{\rho}(A)e^{-\alpha(t-s)}\ell(|\omega_1(s) - \omega_2(s)| + |\omega_1(\gamma(s)) - \omega_2(\gamma(s))|)ds \\
\leqslant &\frac{1}{4}\sup_{s\in R}|\omega_1(s) - \omega_2(s)|.
\end{aligned}$$

Therefore there exists a unique function $\omega_0(t) \in \Omega$ such that

$$\begin{aligned}
\omega_0(t) =&T_2\omega_0(t) \\
=&\int_{-\infty}^{t} G_1(t,s)f\Big(s, x(s,(\tau,\xi,\eta)) + \omega_0(s), y(s,(\tau,\xi,\eta)) + v_\varphi(s,(\tau,\xi,\eta)), \\
&x(\gamma(s),(\tau,\xi,\eta)) + \omega_0(\gamma(s), y(\gamma(s),(\tau,\xi,\eta)) + v_\varphi(\gamma(s),(\tau,\xi,\eta))\Big)ds
\end{aligned}$$

and $|\omega_0(t)| \leqslant K\tilde{\rho}(A)\alpha^{-1}\mu$.

Similar to (4.7), we could check that $\omega_0(t)$ is a solution of system (4.4). Moreover, we could verify that the bounded solution of system (4.4) is unique. Denote

$$\begin{aligned}
&u_\varphi(t,(\tau,\xi,\eta)) = \omega_0(t) \\
=&\int_{-\infty}^{t} G_1(t,s)f\Big(s, x(s,(\tau,\xi,\eta)) + u_\varphi(s,(\tau,\xi,\eta)), y(s,(\tau,\xi,\eta)) + v_\varphi(s,(\tau,\xi,\eta)), \\
&x(\gamma(s),(\tau,\xi,\eta)) + u_\varphi(\gamma(s),(\tau,\xi,\eta)), y(\gamma(s),(\tau,\xi,\eta)) + v_\varphi(\gamma(s),(\tau,\xi,\eta))\Big)ds.
\end{aligned} \tag{4.10}$$

The proof is complete. □

For $\varphi(t) \in \Omega$, there exists a unique $u_\varphi(t,(\tau,\xi,\eta)) \in \Omega$. Thus, for any fixed $\tau \in \mathbb{R}$, $\xi \in \mathbb{R}^{n_1}$ and $\eta \in \mathbb{R}^{n_2}$, we could define operator $T : \Omega \to \Omega$ as follows:

$$T\varphi(t) = u_\varphi(t,(\tau,\xi,\eta)). \tag{4.11}$$

Lemma 4.4. *For any fixed $\tau \in \mathbb{R}$, $\xi \in \mathbb{R}^{n_1}$ and $\eta \in \mathbb{R}^{n_2}$, the operator T has a unique fixed point in Ω.*

Proof. For any $\varphi_1, \varphi_2 \in \Omega$, by **(A3)**, (4.10) and Lemma 2.4, we obtain

$$\begin{aligned}
&|T\varphi_1(t) - T\varphi_2(t)| \\
\leqslant &\int_{-\infty}^{t} K\tilde{\rho}(A)e^{-\alpha(t-s)}\ell\big\{|u_{\varphi_1}(s,(\tau,\xi,\eta)) - u_{\varphi_2}(s,(\tau,\xi,\eta))| + |v_{\varphi_1}(s,(\tau,\xi,\eta)) \\
&- v_{\varphi_2}(s,(\tau,\xi,\eta))| + |u_{\varphi_1}(\gamma(s),(\tau,\xi,\eta)) - u_{\varphi_2}(\gamma(s),(\tau,\xi,\eta))| \\
&+ |v_{\varphi_1}(\gamma(s),(\tau,\xi,\eta)) - v_{\varphi_2}(\gamma(s),(\tau,\xi,\eta))|\big\}ds.
\end{aligned} \tag{4.12}$$

Since both v_{φ_1} and v_{φ_2} are bounded for $t \geqslant \tau$ and $\gamma(t) \geqslant \tau$, we could define

$$\| v_{\varphi_1} - v_{\varphi_2} \|_{[t,+\infty)} = \sup_{s\in[t,+\infty)} |v_{\varphi_1}(s,(\tau,\xi,\eta)) - v_{\varphi_2}(s,(\tau,\xi,\eta))|,$$
$$\| u_{\varphi_1} - u_{\varphi_2} \| = \sup_{s\in\mathbb{R}} |u_{\varphi_1}(s,(\tau,\xi,\eta)) - u_{\varphi_2}(s,(\tau,\xi,\eta))|.$$

From **(A3)**, (4.9), (3.1) and Lemma 2.4, for any mixed t_0, if $t \geqslant t_0$ and $\gamma(t) \geqslant t_0$, then

$$\begin{aligned}
&|v_{\varphi_1}(t,(\tau,\xi,\eta)) - v_{\varphi_2}(t,(\tau,\xi,\eta))| \\
\leqslant &\int_t^{+\infty} K\tilde{\rho}(B)e^{\alpha(t-s)}\ell\big\{|\varphi_1(s)-\varphi_2(s)| + |\varphi_1(\gamma(s))-\varphi_2(\gamma(s))| + |v_{\varphi_1}(s,(\tau,\xi,\eta)) \\
&- v_{\varphi_2}(s,(\tau,\xi,\eta))| + |v_{\varphi_1}(\gamma(s),(\tau,\xi,\eta)) - v_{\varphi_2}(\gamma(s),(\tau,\xi,\eta))|\big\}ds \\
\leqslant &K\tilde{\rho}(B)\alpha^{-1}\ell\big(2 \| \varphi_1 - \varphi_2 \| + 2 \| v_{\varphi_1} - v_{\varphi_2} \|_{[\min(t,\gamma(t)),+\infty)}\big) \\
\leqslant &\frac{1}{4} \| \varphi_1 - \varphi_2 \| + \frac{1}{4} \| v_{\varphi_1} - v_{\varphi_2} \|_{[t_0,+\infty)},
\end{aligned}$$

which gives rise to

$$\| v_{\varphi_1} - v_{\varphi_2} \|_{[t_0,+\infty)} \leqslant \frac{1}{4} \| \varphi_1 - \varphi_2 \| + \frac{1}{4} \| v_{\varphi_1} - v_{\varphi_2} \|_{[t_0,+\infty)} .$$

Thus,

$$\| v_{\varphi_1} - v_{\varphi_2} \|_{[t_0,+\infty)} \leqslant \frac{1}{3} \| \varphi_1 - \varphi_2 \| .$$

Since the right-hand side of the above inequality is independent of t_0, the above inequality is valid for all $t_0 \in \mathbb{R}$. Therefore, $\sup_{s\in\mathbb{R}} |v_{\varphi_1}(s,(\tau,\xi,\eta)) - v_{\varphi_2}(s,(\tau,\xi,\eta))|$ exists and we denote

$$\| v_{\varphi_1} - v_{\varphi_2} \| = \sup_{s\in\mathbb{R}} |v_{\varphi_1}(s,(\tau,\xi,\eta)) - v_{\varphi_2}(s,(\tau,\xi,\eta))|.$$

Thus,

$$\| v_{\varphi_1} - v_{\varphi_2} \| \leqslant \frac{1}{3} \| \varphi_1 - \varphi_2 \| . \tag{4.13}$$

From **(A3)**, (3.1), (4.11), (4.12) and (4.13), we obtain

$$\begin{aligned}
|T\varphi_1(t) - T\varphi_2(t)| &\leqslant \int_{-\infty}^t K\tilde{\rho}(A)e^{-\alpha(t-s)}2\ell(\| T\varphi_1 - T\varphi_2 \| + \| v_{\varphi_1} - v_{\varphi_2} \|)ds \\
&\leqslant \frac{1}{4}(\| T\varphi_1 - T\varphi_2 \| + \frac{1}{3} \| \varphi_1 - \varphi_2 \|),
\end{aligned}$$

which gives rise to

$$\| T\varphi_1 - T\varphi_2 \| \leqslant \frac{4}{9} \| \varphi_1 - \varphi_2 \| .$$

Therefore, the operator T has a unique fixed point φ in Ω satisfying that

$$\varphi(t) = T\varphi(t) = u_\varphi(t,(\tau,\xi,\eta)). \tag{4.14}$$

The proof is completed. □

Denote

$$u_\varphi(t,(\tau,\xi,\eta)) = u(t,(\tau,\xi,\eta)), \quad v_\varphi(t,(\tau,\xi,\eta)) = v(t,(\tau,\xi,\eta)).$$

In view of (4.9), (4.10) and (4.14), we get

$$\begin{aligned}
&u(t,(\tau,\xi,\eta))\\
=&\int_{-\infty}^{t} G_1(t,s) f\Big(s, x(s,(\tau,\xi,\eta)) + u(s,(\tau,\xi,\eta)), y(s,(\tau,\xi,\eta)) + v(s,(\tau,\xi,\eta)),\\
&x(\gamma(s),(\tau,\xi,\eta)) + u(\gamma(s),(\tau,\xi,\eta)), y(\gamma(s),(\tau,\xi,\eta)) + v(\gamma(s),(\tau,\xi,\eta))\Big)ds
\end{aligned}$$

and

$$\begin{aligned}
&v(t,(\tau,\xi,\eta))\\
=&-\int_{t}^{\infty} G_2(t,s) g\Big(s, x(s,(\tau,\xi,\eta)) + u(s,(\tau,\xi,\eta)), y(s,(\tau,\xi,\eta)) + v(s,(\tau,\xi,\eta)),\\
&x(\gamma(s),(\tau,\xi,\eta)) + u(\gamma(s),(\tau,\xi,\eta)), y(\gamma(s),(\tau,\xi,\eta)) + v(\gamma(s),(\tau,\xi,\eta))\Big)ds.
\end{aligned}$$

Lemma 4.5. *For any $\xi \in \mathbb{R}^{n_1}$, $\eta \in \mathbb{R}^{n2}$, $\tau \in \mathbb{R}$, $t \in \mathbb{R}$ and $\omega \in \mathbb{R}$,*

$$\begin{aligned}
u(\omega,(t,x(t,(\tau,\xi,\eta)),y(t,(\tau,\xi,\eta)))) &= u(\omega,(\tau,\xi,\eta)),\\
v(\omega,(t,x(t,(\tau,\xi,\eta)),y(t,(\tau,\xi,\eta)))) &= v(\omega,(\tau,\xi,\eta)).
\end{aligned}$$

In particular, if $\omega = t$, then

$$u(t,(t,x(t,(\tau,\xi,\eta)),y(t,(\tau,\xi,\eta)))) \equiv u(t,(\tau,\xi,\eta))$$

and

$$v(t,(t,x(t,(\tau,\xi,\eta)),y(t,(\tau,\xi,\eta)))) \equiv v(t,(\tau,\xi,\eta)).$$

Proof. Denote

$$\begin{aligned}
&u_1(\omega) = u(\omega,(t,x(t,(\tau,\xi,\eta)),y(t,(\tau,\xi,\eta)))), \quad u_2(\omega) = u(\omega,(\tau,\xi,\eta)),\\
&v_1(\omega) = v(\omega,(t,x(t,(\tau,\xi,\eta)),y(t,(\tau,\xi,\eta)))), \quad v_2(\omega) = v(\omega,(\tau,\xi,\eta)),\\
&x_1(s) = x(s,(t,x(t,(\tau,\xi,\eta)),y(t,(t,(\tau,\xi,\eta)))))
\end{aligned}$$

and

$$y_1(s) = y(s,(t,x(t,(\tau,\xi,\eta)),y(t,(t,(\tau,\xi,\eta))))).$$

Observing that $x_1(s) = x(s,(\tau,\xi,\eta))$ and $y_1(s) = y(s,(\tau,\xi,\eta))$, we have

$$\begin{aligned}
&f\Big(s, x_1(s) + u_1(s), y_1(s) + v_1(s), x_1(\gamma(s)) + u_1(\gamma(s)), y_1(\gamma(s)) + v_1(\gamma(s))\Big)\\
=&f\Big(s, x(s,(\tau,\xi,\eta)) + u_1(s), y(s,(\tau,\xi,\eta)) + v_1(s), x(\gamma(s),(\tau,\xi,\eta)) + u_1(\gamma(s)),\\
&y(\gamma(s),(\tau,\xi,\eta)) + v_1(\gamma(s))\Big).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&g\Big(s, x_1(s) + u_1(s), y_1(s) + v_1(s), x_1(\gamma(s)) + u_1(\gamma(s)), y_1(\gamma(s)) + v_1(\gamma(s))\Big)\\
=&g\Big(s, x(s,(\tau,\xi,\eta)) + u_1(s), y(s,(\tau,\xi,\eta)) + v_1(s), x(\gamma(s),(\tau,\xi,\eta)) + u_1(\gamma(s)),\\
&y(\gamma(s),(\tau,\xi,\eta)) + v_1(\gamma(s))\Big).
\end{aligned}$$

From **(A3)**, (3.1) and Lemma 2.4, for any mixed t_0, if $\omega \geqslant t_0$ and $\gamma(\omega) \geqslant t_0$, then

$$\begin{aligned}|v_1(\omega) - v_2(\omega)| \leqslant & \int_\omega^{+\infty} K\tilde{\rho}(B)e^{\alpha(\omega-s)}\ell\big\{|u_1(s) - u_2(s)| + |v_1(s) - v_2(s)| \\ & + |u_1(\gamma(s)) - u_2(\gamma(s))| + |v_1(\gamma(s)) - v_2(\gamma(s))|\big\}ds \\ \leqslant & \frac{1}{4}\sup_{s\in\mathbb{R}}|u_1(s) - u_2(s)| + \frac{1}{4}\sup_{s\in[\min(\omega,\gamma(\omega)),+\infty)}|v_1(s) - v_2(s)| \\ \leqslant & \frac{1}{4}\sup_{s\in\mathbb{R}}|u_1(s) - u_2(s)| + \frac{1}{4}\sup_{s\in[t_0,+\infty)}|v_1(s) - v_2(s)|.\end{aligned}$$

Therefore,

$$\sup_{s\in[t_0,+\infty)}|v_1(s) - v_2(s)| \leqslant \frac{1}{4}\sup_{s\in\mathbb{R}}|u_1(s) - u_2(s)| + \frac{1}{4}\sup_{s\in[t_0,+\infty)}|v_1(s) - v_2(s)|.$$

Consequently,

$$\sup_{s\in[t_0,+\infty)}|v_1(s) - v_2(s)| \leqslant \frac{1}{3}\sup_{s\in\mathbb{R}}|u_1(s) - u_2(s)|.$$

Since the right-hand side of the above inequality is independent of t_0,

$$\sup_{s\in\mathbb{R}}|v_1(s) - v_2(s)| \leqslant \frac{1}{3}\sup_{s\in\mathbb{R}}|u_1(s) - u_2(s)|. \tag{4.15}$$

Similarly, by **(A3)**, (3.1), (4.15) and Lemma 2.4, we get

$$\begin{aligned}|u_1(\omega) - u_2(\omega)| \leqslant & \int_{-\infty}^{\omega} K\tilde{\rho}(A)e^{-\alpha(\omega-s)}\ell\big\{|u_1(s) - u_2(s)| + |v_1(s) - v_2(s)| \\ & + |u_1(\gamma(s)) - u_2(\gamma(s))| + |v_1(\gamma(s)) - v_2(\gamma(s))|\big\}ds \\ \leqslant & \frac{1}{4}\sup_{s\in\mathbb{R}}|u_1(s) - u_2(s)| + \frac{1}{4}\sup_{s\in\mathbb{R}}|v_1(s) - v_2(s)| \\ \leqslant & \frac{1}{3}\sup_{s\in\mathbb{R}}|u_1(s) - u_2(s)|.\end{aligned}$$

Therefore, $u_1(\omega) \equiv u_2(\omega)$. That is

$$u(\omega, (t, x(t, (\tau, \xi, \eta)), y(t, (\tau, \xi, \eta)))) \equiv u(\omega, (\tau, \xi, \eta)).$$

Moreover, by (4.15),

$$v(\omega, (t, x(t, (\tau, \xi, \eta)), y(t, (\tau, \xi, \eta)))) \equiv v(\omega, (\tau, \xi, \eta)).$$

□

Now, we are in a position to define topological equivalent functions H and L. In the next section, we will prove that H and L are homeomorphisms and $L = H^{-1}$.

Let $(X(t, (\tau, x, y)), Y(t, (\tau, x, y)))^T$ be the solution of the nonlinear system (2.1) with $X(\tau) = x$ and $Y(\tau) = y$.

Define

$$
\begin{aligned}
L_1(t,\xi,\eta) =& \xi + u(t,(t,\xi,\eta)), \quad L_2(t,\xi,\eta) = \eta + v(t,(t,\xi,\eta)),\\
H_1(t,x,y) =& x - \int_{-\infty}^{t} G_1(t,s) f\Big(s, X(s,(t,x,y)), Y(s,(t,x,y)),\\
& X(\gamma(s),(t,x,y)), Y(\gamma(s),(t,x,y))\Big) ds,\\
H_2(t,x,y) =& y + \int_{t}^{+\infty} G_2(t,s) g\Big(s, X(s,(t,x,y)), Y(s,(t,x,y)),\\
& X(\gamma(s),(t,x,y)), Y(\gamma(s),(t,x,y))\Big) ds,\\
H(t,x,y) =& \begin{pmatrix} H_1(t,x,y)\\ H_2(t,x,y)\end{pmatrix} \quad \text{and} \quad L(t,x,y) = \begin{pmatrix} L_1(t,\xi,\eta)\\ L_2(t,\xi,\eta)\end{pmatrix}.
\end{aligned}
$$

5. Proof of Theorem 3.1

This section is devoted to proving Theorem 3.1. We divide the proof into several lemmas.

Lemma 5.1. *H sends solutions of the nonlinear system* (2.1) *onto those of its linear system* (2.2)*, and L sends solutions of system* (2.2) *onto those of system* (2.1).

Proof. Denote

$$X(s,(t,X(t,(\tau,x,y)),Y(t,(\tau,x,y)))) \triangleq X_1(s),$$

and

$$Y(s,(t,X(t,(\tau,x,y)),Y(t,(\tau,x,y)))) \triangleq Y_1(s).$$

Observe that $X_1(s) = X(s,(\tau,x,y))$ and $Y_1(s) = Y(s,(\tau,x,y))$, we get

$$
\begin{aligned}
& H_1(t,X(t,(\tau,x,y)),Y(t,(\tau,x,y)))\\
=& X(t,(\tau,x,y)) - \int_{-\infty}^{t} G_1(t,s) f\Big(s, X_1(s), Y_1(s), X_1(\gamma(s)), Y_1(\gamma(s))\Big) ds\\
=& X(t,(\tau,x,y)) - \int_{-\infty}^{t} G_1(t,s)\cdot f\Big(s, X(s,(\tau,x,y)), Y(s,(\tau,x,y)),\\
& X(\gamma(s),(\tau,x,y)), Y(\gamma(s),(\tau,x,y))\Big) ds.
\end{aligned}
$$

Denote $H_1(t,X(t,(\tau,x,y)),Y(t,(\tau,x,y)))$ by $H_1(t)$. By simple calculation, the derivative of $H_1(t)$ is

$$H_1'(t) = A(t)H_1(t) + A_0(t)H_1(\gamma(t)). \tag{5.1}$$

Similarly,

$$
\begin{aligned}
&H_2(t,X(t,(\tau,x,y)),Y(t,(\tau,x,y)))\\
=&Y(t,(\tau,x,y))+\int_t^{+\infty}G_2(t,s)g\Big(s,X_1(s),Y_1(s),X_1(\gamma(s)),Y_1(\gamma(s))\Big)ds\\
=&Y(t,(\tau,x,y))+\int_t^{+\infty}G_2(t,s)\cdot g\Big(s,X(s,(\tau,x,y)),Y(s,(\tau,x,y)),\\
&X(\gamma(s),(\tau,x,y)),Y(\gamma(s),(\tau,x,y))\Big)ds.
\end{aligned}
$$

Denote $H_2(t,X(t,(\tau,x,y)),Y(t,(\tau,x,y)))$ by $H_2(t)$, we have

$$H_2'(t)=B(t)H_2(t)+B_0(t)H_2(\gamma(t)),$$

together with (5.1), we get that $H(t)$ is a solution of system (2.2). Therefore, $H(t)$ sends the solutions of system (2.1) onto those of system (2.2).

The second assertion follows by similar arguments. □

Lemma 5.2. *For any $x\in\mathbb{R}^{n_1}$, $y\in\mathbb{R}^{n_2}$, $\xi\in\mathbb{R}^{n_1}$, $\eta\in\mathbb{R}^{n_2}$, and $t\in\mathbb{R}$, we have*

$$
\begin{aligned}
&|H_1(t,x,y)-x|\leqslant K\tilde\rho(A)\alpha^{-1}\mu,\\
&|H_2(t,x,y)-y|\leqslant \frac14 q_2(x)+K\tilde\rho(B)\alpha^{-1}\mu,\\
&|L_1(t,\xi,\eta)-\xi|\leqslant K\tilde\rho(A)\alpha^{-1}\mu,\\
&|L_2(t,\xi,\eta)-\eta|\leqslant \frac14 q_1(\xi)+\frac14K\tilde\rho(A)\alpha^{-1}\mu+K\tilde\rho(B)\alpha^{-1}\mu.
\end{aligned}
$$

Proof. From **(A2)**, (3.1) and Lemma 2.4, we have

$$
\begin{aligned}
|H_1(t,x,y)-x|=&\Big|\int_{-\infty}^t G_1(t,s)f\Big(s,X(s,(t,x,y)),Y(s,(t,x,y)),X(\gamma(s),(t,x,y)),\\
&Y(\gamma(s),(t,x,y))\Big)ds\Big|\\
\leqslant&\int_{-\infty}^t K\tilde\rho(A)e^{-\alpha(t-s)}\mu ds\\
=&K\tilde\rho(A)\alpha^{-1}\mu,
\end{aligned}
$$

and

$$
\begin{aligned}
&|H_2(t,x,y)-y|\\
=&\Big|\int_t^{+\infty}G_2(t,s)g\Big(s,X(s,(t,x,y)),Y(s,(s,x,y)),X(\gamma(s),(t,x,y)),\\
&Y(\gamma(s),(s,x,y))\Big)ds\Big|\\
\leqslant&\int_t^{+\infty}K\tilde\rho(B)e^{\alpha(t-s)}\{\ell(|X(s,(t,x,y))|+|X(\gamma(s),(t,x,y))|)+\mu\}ds.
\end{aligned}\tag{5.2}
$$

Since $s\geqslant t$ holds in (5.2), it follows from Lemma 4.1 that

$$
\begin{aligned}
|H_2(t,x,y)-y|\leqslant&\int_t^{+\infty}K\tilde\rho(B)e^{\alpha(t-s)}\{\ell(q_2(x)+q_2(x))+\mu\}\\
\leqslant&\frac14 q_2(x)+K\tilde\rho(B)\alpha^{-1}\mu.
\end{aligned}
$$

The other inequalities can be proved in a similar way. □

Lemma 5.3. *For any $t \in \mathbb{R}$, $\xi \in \mathbb{R}^{n_1}$ and $\eta \in \mathbb{R}^{n_2}$, we have*

$$H(t, L(t, \xi, \eta)) = (\xi, \eta)^T.$$

Proof. Suppose that $(\xi(t), \eta(t))^T$ is a solution of the linear system (2.2). It follows from Lemma 5.1 that $L(t, \xi(t), \eta(t))$ is a solution of system (2.1) and $H(t, L(t, \xi(t), \eta(t)))$ is a solution of system (2.2).

Therefore, $(J_1(t), J_2(t))^T = H(t, L(t, \xi(t), \eta(t))) - (\xi(t), \eta(t))^T$ is also a solution of system (2.2).

Moreover,

$$\begin{aligned}|J_1(t)| =& |H_1(t, L_1(t, \xi(t), \eta(t)), L_2(t, \xi(t), \eta(t))) - \xi(t)|\\ \leqslant& |H_1(t, L_1(t, \xi(t), \eta(t)), L_2(t, \xi(t), \eta(t))) - L_1(t, \xi(t), \eta(t))|\\ &+ |L_1(t, \xi(t), \eta(t)) - \xi(t)|.\end{aligned}$$

From Lemma 5.2, we obtain

$$|J_1(t)| \leqslant 2K\tilde{\rho}(A)\alpha^{-1}\mu.$$

Therefore, $J_1(t)$ is a bounded solution of subsystem (2.7). Due to Lemma 2.5, we have $J_1(t) \equiv 0$. That is,

$$H_1(t, L_1(t, \xi(t), \eta(t)), L_2(t, \xi(t), \eta(t))) \equiv \xi(t).$$

Similarly, by Lemma 5.2, we get

$$\begin{aligned}|J_2(t)| =& |H_2(t, L_1(t, \xi(t), \eta(t)), L_2(t, \xi(t), \eta(t))) - \eta(t)|\\ \leqslant& |H_2(t, L_1(t, \xi(t), \eta(t)), L_2(t, \xi(t), \eta(t))) - L_2(t, (\xi(t), \eta(t)))|\\ &+ |L_2(t, (\xi(t), \eta(t))) - \eta(t)|\\ \leqslant& \frac{1}{4}q_2(L_1(t, \xi(t), \eta(t))) + \frac{1}{4}q_1(\xi(t)) + \frac{1}{4}K\tilde{\rho}(A)\alpha^{-1}\mu + 2K\tilde{\rho}(B)\alpha^{-1}\mu.\end{aligned}$$

It follows from Lemma 5.1 that $L_1(t, \xi(t), \eta(t))$ is a solution of subsystem (2.5). Moreover, in view of Lemma 4.1, for $t \geqslant 0$ we have

$$|\xi(t)| \leqslant q_1(\xi(0)),$$

and

$$|L_1(t, \xi(t), \eta(t))| \leqslant q_2(L_1(0, \xi(0), \eta(0))).$$

Thus, for $t \geqslant 0$ we have

$$|J_2(t)| \leqslant \frac{1}{4}q_2(q_2(L_1(0, \xi(0), \eta(0)))) + \frac{1}{4}q_1(q_1(\xi(0))) + \frac{1}{4}K\tilde{\rho}(A)\alpha^{-1}\mu + 2K\tilde{\rho}(B)\alpha^{-1}\mu,$$

which implies that $|J_2(t)|$ is bounded for $t \geqslant 0$. Noting that $J_2(t)$ is a solution of subsystem (2.8), from Lemma 2.5 we obtain $J_2(t) \equiv 0$. Therefore,

$$H_2(t, L_1(t, \xi(t), \eta(t)), L_2(t, \xi(t), \eta(t))) \equiv \eta(t).$$

□

Lemma 5.4. *Suppose that* $(X(t),Y(t))^T$ *is a solution of the nonlinear system* (2.1) *and* $(\varrho_0(t),\omega_0(t))^T$ *is a solution of the system*

$$\begin{cases} \varrho'(t) = A(t)\varrho(t) + A_0(t)\varrho(\gamma(t)) + f\Big(t, X(t)+\varrho(t), Y(t)+\omega(t), X(\gamma(t)) \\ \qquad\qquad + \varrho(\gamma(t)), Y(\gamma(t))+\omega(\gamma(t))\Big) - f\big(t, X(t), Y(t), X(\gamma(t)), Y(\gamma(t))\big), \\ \omega'(t) = B(t)\omega(t) + B_0(t)\omega(\gamma(t)) + g\Big(t, X(t)+\varrho(t), Y(t)+\omega(t), X(\gamma(t)) \\ \qquad\qquad + \varrho(\gamma(t)), Y(\gamma(t))+\omega(\gamma(t))\Big) - g\big(t, X(t), X(\gamma(t)), Y(t), Y(\gamma(t))\big), \end{cases} \tag{5.3}$$

satisfying

$$|\varrho_0(t)| < +\infty, \quad t \in \mathbb{R}, \tag{5.4}$$

$$|\omega_0(t)| < +\infty, \quad t \geqslant 0. \tag{5.5}$$

Then $(\varrho_0(t),\omega_0(t))^T \equiv 0$.

Proof. It is clear that $(\varrho_0(t),\omega_0(t))^T \equiv 0$ is a solution of system (5.3) satisfying (5.4) and (5.5). Arguments similar to those in Lemmas 4.3 and 4.4 show that $(\varrho_0(t),\omega_0(t))^T \equiv 0$ is the unique solution satisfying (5.4) and (5.5). □

Lemma 5.5. *For any* $t \in \mathbb{R}$, $x \in \mathbb{R}^{n_1}$ *and* $y \in \mathbb{R}^{n_2}$, *we have*

$$L(t, H(t,x,y)) = (x,y)^T.$$

Proof. Suppose that $(X(t),Y(t))^T$ is a solution of the nonlinear system (2.1). It follows from Lemma 5.1 that $H(t,X(t),Y(t))$ is a solution of the linear system (2.2) and $L(t,H(t,X(t),Y(t)))$ is also a solution of system (2.1). Denote

$$J_1(t) = L_1(t, H_1(t,X(t),Y(t)), H_2(t,X(t),Y(t))) - X(t) \triangleq L_1(t) - X(t),$$
$$J_2(t) = L_2(t, H_1(t,X(t),Y(t)), H_2(t,X(t),Y(t))) - Y(t) \triangleq L_2(t) - Y(t).$$

Moreover, we could verify that $(J_1(t),J_2(t))^T$ is a solution of system (5.3).

By Lemma 5.2, we have

$$\begin{aligned} |J_1(t)| =& |L_1(t, H_1(t,X(t),Y(t)), H_2(t,X(t),Y(t))) - H_1(t,X(t),Y(t))| \\ &+ |H_1(t,X(t),Y(t)) - X(t)| \leqslant 2K\tilde{\rho}(A)\alpha^{-1}\mu \end{aligned}$$

and

$$\begin{aligned} |J_2(t)| =& |L_2(t, H_1(t,X(t),Y(t)), H_2(t,X(t),Y(t))) - H_2(t,X(t),Y(t))| \\ &+ |H_2(t,X(t),Y(t)) - X(t)| \\ \leqslant& \frac{1}{4}q_1(H_1(t,X(t),Y(t))) + \frac{1}{4}q_2(X(t)) + \frac{1}{4}K\tilde{\rho}(A)\alpha^{-1}\mu + 2K\tilde{\rho}(B)\alpha^{-1}\mu. \end{aligned}$$

Since $H_1(t,X(t),Y(t))$ is a solution of subsystem (2.7) and $X(t)$ is a solution of subsystem (2.5), for $t \geqslant 0$, it follows from Lemma 4.1 that

$$|H_1(t,X(t),Y(t))| \leqslant q_1(H_1(0,X(0),Y(0))) \quad \text{and} \quad |X(t)| \leqslant q_2(X(0)).$$

Thus, for $t \geqslant 0$ we get

$$|J_2(t)| \leqslant \frac{1}{4} q_1(q_1(H_1(0, X(0), Y(0)))) + \frac{1}{4} q_2(q_2(X(0))) + \frac{1}{4} K\tilde{\rho}(A)\alpha^{-1}\mu + 2K\tilde{\rho}(B)\alpha^{-1}\mu.$$

Owing to Lemma 5.4, we have $J_1(t) \equiv 0$ and $J_2(t) \equiv 0$. That is

$$L(t, H(t, X(t), Y(t))) = (X(t), Y(t))^T.$$

Therefore, for any $t \in \mathbb{R}$, $x \in \mathbb{R}^{n_1}$ and $y \in \mathbb{R}^{n_2}$, we have

$$L(t, H(t, x, y)) = (x, y)^T.$$

□

Lemma 5.6 (Lemma 5.1, [25]). *Let $t \mapsto z(t, \tau, \xi)$ and $t \mapsto z(t, \tau, \xi')$ be the solutions of* (2.1) *passing respectively through ξ and ξ' at $t = \tau$. If* (3.2) *is valid, then it follows that*

$$|z(t, \tau, \xi') - z(t, \tau, \xi)| \leqslant |\xi - \xi'| e^{p(\ell)|t-\tau|},$$

where $p(\ell)$ is defined by

$$p(\ell) = \eta_1 + \frac{\eta_2 e^{\eta_1 \theta}}{1 - \upsilon} \quad \text{with} \quad \eta_1 = M + \ell, \quad \eta_2 = M_0 + \ell,$$

and $\upsilon \in [0, 1)$ is defined by (3.2)*.*

Remark 5.1. If $h(t, z(t), z(\gamma(t))) = 0$, take $\ell = 0$, Lemma 5.6 reduces to Lemma 5.2 in [25]. Moreover, since $p(\ell) > p(0)$ and $F(\ell, \theta) \geqslant F(0, \theta)$ in (3.2), Lemma 5.6 is also valid for system (2.2).

Proof of Theorem 3.1. For any $t \in \mathbb{R}$, it follows from Lemmas 5.3 and 5.5 that $H(t, \cdot)$ is a bijection of $\mathbb{R}^n$ and $H^{-1}(t, \cdot) = L(t, \cdot)$.

According to Lemma 5.6 and Remark 3, solutions of systems (2.1) and (2.2) are continuous with respect to initial values. Moreover, similar to Lemma 5.6, we obtain solutions of systems (4.3) and (4.4) are continuous with respect to initial values.

By definitions of $H(t, \cdot)$ and $L(t, \cdot)$, both $H(t, \cdot)$ and $L(t, \cdot)$ are continuous. Thus $H(t, \cdot)$ and $L(t, \cdot)$ are homeomorephisms of $\mathbb{R}^n$.

Moreover, Lemma 5.1 implies that $H(t, \cdot)$ sends the solutions of system (2.1) onto those of system (2.2) and $L(t, \cdot)$ sends the solutions of system (2.2) onto those of system (2.1). It follows from Lemma 5.2 that H and L are topological equivalent functions. Therefore, system (2.1) and system (2.2) are topologically conjugated.

□

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