# THE RIEMANN PROBLEM WITH DELTA INITIAL DATA FOR THE NONSYMMETRIC KEYFITZ-KRANZER SYSTEM WITH CHAPLYGIN PRESSURE*

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**Abstract** In this paper, we study the Riemann problem with the initial data containing the Dirac delta function for the nonsymmetric Keyfitz-Kranzer system with Chaplygin pressure. Under the generalized Rankine-Hugoniot conditions and entropy condition, we constructively obtain the global existence of generalized solutions including delta shock waves that explicitly exhibit four kinds of different structures. Moreover, we obtain the stability of generalized solutions by making use of the perturbation of the initial data.

**Keywords** Nonsymmetric Keyfitz-Kranzer system, Riemann problem, delta shock wave, generalized-Rankine-Hugoniot conditions, entropy condition.

**MSC(2010)** 35L65, 35L67.

## 1. Introduction

In this paper, we consider the following hyperbolic system of conservation laws

$$\begin{cases} \rho_t + \big(\rho(u - p)\big)_x = 0, \\ (\rho u)_t + \big(\rho u(u - p)\big)_x = 0, \end{cases} \tag{1.1}$$

where $p = p(\rho)$ and $\rho \geq 0$. Model (1.1) belongs to the nonsymmetric system of Keyfitz-Kranzer type [10, 12] as

$$\begin{cases} \rho_t + \big(\rho\phi(\rho, u_1, u_2, \ldots, u_n)\big)_x = 0, \\ (\rho u_i)_t + \big(\rho u_i\phi(\rho, u_1, u_2, \ldots, u_n)\big)_x = 0, \ \ i = 1, 2, \ldots, n, \end{cases} \tag{1.2}$$

which is of interest because it arises in such areas as elasticity theory, magnetohydrodynamics, and enhanced oil recovery. For delta shock waves, the nonsymmetric form is more convenient than the symmetric form. Model (1.1) is also a transformation of the traffic flow model introduced by Aw and Rascle [1], where $\rho$ and $u > 0$ are the density and velocity of cars on the roadway and the function $p(\rho)$ is smooth and strictly increasing and it satisfies

$$\rho p^{''}(\rho) + 2p^{'}(\rho) > 0, \ \ \text{for} \ \rho > 0. \tag{1.3}$$

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The equation of state is

$$p(\rho) = -\frac{1}{\rho}, \tag{1.4}$$

with $\rho > 0$. Equation (1.4) was introduced by Chaplygin [3] and Tsien [19] as a suitable mathematical approximation for calculating the lifting force on a wing of an airplane in aerodynamics. For a Chaplygin gas, Brenier [2] firstly studied the 1-D Riemann problem and obtained solutions with concentration when initial data belong to a certain domain in the phase plane. Furthermore, Guo, Sheng, and Zhang [7] abandoned this constrain and constructively obtained the global solutions to the 1-D Riemann problem, in which the $\delta$-shock developed. Moreover, they also systematically studied the 2-D Riemann problem for isentropic Chaplygin gas equations. For the 2-D case, we can also refer to [14] in which D. Serre studied the interaction of the pressure waves for the 2-D isentropic irrotational Chaplygin gas and constructively proved the existence of transonic solutions for two cases, saddle and vortex of 2-D Riemann problem. Recently, Wang and Zhang [20] studied the Riemann problem with delta initial data for the one-dimensional Chaplygin gas equations. However, it is noticed that few literatures contribute to system (1.1) for a Chaplygin gas so far. Recently, Cheng and Yang [6] proved the existence and uniqueness of delta shock solutions of Riemann problem for the relativistic Chaplygin Euler equations.

In particular, the delta shock waves appear in the Riemann solutions of (1.1) and (1.4). From the mathematical point of view, a delta shock wave is more compressive than an ordinary shock wave in the sense that more characteristics enter the discontinuity line of the delta shock wave. From the physical point of view, a delta shock represents the process of concentration of the mass. As for delta shock waves, we refer readers to [4–9, 11, 15–18, 20–25] and the references cited therein for more details.

In the present paper, we consider the Riemann problem (1.1) and (1.4) with initial data

$$\big(\rho, u\big)\big(t = 0, x\big) = \begin{cases} (\rho_-, u_-), & x < 0, \\ (m_0\delta, u_0), & x = 0, \\ (\rho_+, u_+), & x < 0, \end{cases} \tag{1.5}$$

where $\delta$ is the standard Dirac delta function (see [24]), and $m_0, u_0, \rho_\pm$ and $u_\pm$ are arbitrary constants. Because the delta shocks appear in Riemann solutions of (1.1) and (1.4), it is natural to consider system (1.1) and (1.4) with initial data (1.5) which contains Dirac delta functions. This kind of Riemann problem, which is also called the Randon measure initial data problem, was studied in [9, 11, 13, 15, 20–22] for the zero-pressure flow in gas dynamics and other related equations.

In our paper, we will solve the Riemann problem (1.1), (1.4) and (1.5). Under the generalized Rankine-Hugoniot conditions and suitable entropy condition, we constructively obtain the global existence and uniqueness of generalized solutions including delta shocks that explicitly exhibit four kinds of different structure. However, much more different from [11, 21, 22], the $\delta$-shock condition is not enough to guarantee the uniqueness of generalized solutions. As in [9, 20], we construct our solution on the basis of the stability theory of generalized solutions. Especially, when $m_0 = 0, u_0 = 0$, our results are consistent with those in [5].

The paper is organized as follows. In Section 2, we first present some preliminary knowledge about system (1.1) and (1.4); then display the Riemann solution of (1.1)

and (1.4) with constant initial data. In Section 3, we construct the Riemann solution of (1.1) and (1.4) with delta initial data case by case.

# 2. Riemann problem with constant initial data

In this section, we briefly review the Riemann solution of (1.1) and (1.4) with initial data

$$(\rho, u)(0, x) = (\rho_\pm, u_\pm), \qquad \pm x > 0, \tag{2.1}$$

where $\rho_\pm > 0$, the detailed study of which can be found in [5].

The eigenvalues of the system (1.1) and (1.4) are

$$\lambda_1 = u, \quad \lambda_2 = u + \frac{1}{\rho}. \tag{2.2}$$

Therefore, the system (1.1) and (1.4) is strictly hyperbolic for $\rho > 0$.

The corresponding right eigenvectors are

$$\overrightarrow{r}_1 = (1, 0)^T, \quad \overrightarrow{r}_2 = \left(1, \frac{1}{\rho^2}\right)^T. \tag{2.3}$$

By a direct calculation,

$$\nabla \lambda_i \cdot \overrightarrow{r}_i \equiv 0, \quad i = 1, 2.$$

Therefore, $\lambda_1$ and $\lambda_2$ are both linearly degenerate and the associated waves are both contact discontinuities.

As usual, we seek the self-similar solution

$$(\rho, u)(t, x) = (\rho, u)(\xi), \quad \xi = \frac{x}{t}. \tag{2.4}$$

Then the Riemann problem (1.1), (1.4) and (2.1) is reduced to the following boundary value problem of the ordinary differential equations:

$$\begin{cases} -\xi \rho_\xi + (\rho u + 1)_\xi = 0, \\ -\xi (\rho u)_\xi + (\rho u^2 + u)_\xi = 0, \end{cases} \tag{2.5}$$

with

$$(\rho, u)(\pm \infty) = (\rho_\pm, u_\pm). \tag{2.6}$$

For any smooth solution, system (2.5) can be written as

$$\begin{pmatrix} u - \xi & \rho \\ -\xi u + u^2 & -\xi \rho + 2\rho u + 1 \end{pmatrix} \begin{pmatrix} \rho_\xi \\ u_\xi \end{pmatrix} = 0. \tag{2.7}$$

It provides either general solutions (constant states)

$$(\rho, u)(\xi) = \text{Constant} \quad (\rho > 0),$$

or singular solutions

$$\xi = u = u_-,$$

and

$$\xi = u + \frac{1}{\rho} = u_- + \frac{1}{\rho_-}. \tag{2.8}$$

For a bounded discontinuity at $\xi = \sigma$, the Rankine-Hugoniot conditions hold:

$$\begin{cases} -\sigma[\rho] + [\rho u + 1] = 0, \\ -\sigma[\rho u] + [\rho u^2 + u] = 0, \end{cases} \tag{2.9}$$

where $[\rho] = \rho - \rho_-$, etc. Solving (2.9), we have

$$\begin{aligned} &\sigma = u = u_-, \\ &\sigma = u + \frac{1}{\rho} = u_- + \frac{1}{\rho_-}. \end{aligned} \tag{2.10}$$

From (2.8) and (2.10), we find that the rarefaction waves and the shock waves are coincident in the phase plane, which correspond to contact discontinuities:

$$J_1 : \xi = u = u_-, \tag{2.11}$$

$$J_2 : \xi = u + \frac{1}{\rho} = u_- + \frac{1}{\rho_-}. \tag{2.12}$$

In the phase plane, through the point $(\rho_-, u_-)$, we draw a branch of curve (2.11) for $\rho > 0$, denoted by $J_1$. Through the point $(\rho_-, u_-)$, we also draw a branch of curve (2.12) for $\rho > 0$, which have two asymptotic lines $u = u_- + \frac{1}{\rho_-}$ and $\rho = 0$, denoted by $J_2$. Through the point $\left(\rho_-, u_- - \frac{1}{\rho_-}\right)$, we draw the curve $u + \frac{1}{\rho} = u_-$, denoted by $S$. It easy to know the phase plane can be divided into five regions, which is I, II, III, IV, and V.

For any given right state $(\rho_+, u_+)$, we can construct Riemann solutions of (1.1) and (2.1). when $(\rho_+, u_+) \in I \cup II \cup III \cup IV$, the Riemann solution contains a 1-contact discontinuity, a 2-contact discontinuity, a nonvacuum intermediate constant state $(\rho_*, u_*)$, where

$$u_* = u_-, \quad \frac{1}{\rho_*} = u_+ + \frac{1}{\rho_+} - u_-. \tag{2.13}$$

When $(\rho_+, u_+) \in V$, the characteristics originating from the origin will overlap in the domain $\Omega = \left\{(x, t) : u_+ + \frac{1}{\rho_+} < \frac{x}{t} < u_-\right\}$. So, singularity must happen in $\Omega$. It is easy to know that the singularity is impossible to be a jump with finite amplitude because the Rankine-Hugoniot condition is not satisfied on the bounded jump. In other words, there is no solutions which is piecewise smooth and bounded. Motivated by [18], we seek solutions with delta distribution at the jump. In fact, the appearance of delta shock wave is due to the overlap of linear degenerate characteristic lines.
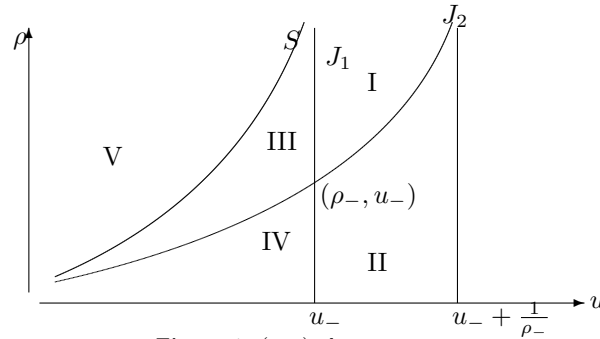
For system (1.1) and (1.4), the definition of solutions in the sense of distributions can be given as follows.

**Definition 2.1.** A pair $(\rho, u)$ constitutes a solution of (1.1) and (1.4) in the sense of distributions if it satisfies

$$\begin{cases} \int_0^{+\infty} \int_{-\infty}^{+\infty} (\rho \varphi_t + (\rho(u - p)) \varphi_x) dx dt = 0, \\ \int_0^{+\infty} \int_{-\infty}^{+\infty} ((\rho u) \varphi_t + (\rho u(u - p)) \varphi_x) dx dt = 0, \end{cases} \tag{2.14}$$

for all test functions $\varphi \in C_0^\infty(R^+ \times R^1)$.

Moreover, we define a two-dimensional weighted delta function in the following way.

**Figure 1.** $(\rho, u)$-plane.
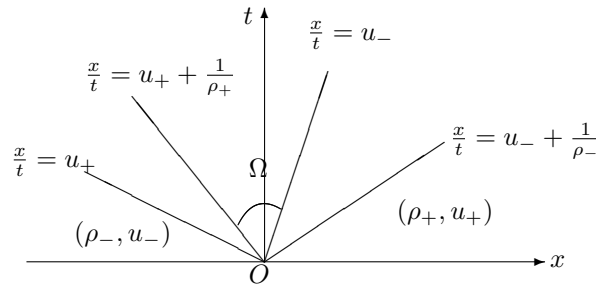


**Figure 2.** $(t, x)$-plane.

**Definition 2.2.** A two-dimensional weighted delta function $w(s)\delta_S$ supported on a smooth curve $S = \{(t(s), x(s)) : a < s < b\}$ is defined by

$$\langle w(s)\delta_S, \varphi \rangle = \int_a^b w(s)\varphi(t(s), x(s))ds,$$

for all test functions $\varphi \in C_0^\infty(R^2)$.

Let us consider a solution of (1.1) and (1.4) of the form

$$(\rho, u)(t, x) = \begin{cases} (\rho_-, u_-), \ x < \sigma t, \\ (w(t)\delta(x - \sigma t), \sigma), \ x = \sigma t, \\ (\rho_+, u_+), \ x > \sigma t, \end{cases} \tag{2.15}$$

where $\sigma$ is a constant, $w(t) \in C^1[0, +\infty)$, and $\delta(\cdot)$ is the standard Dirac measure. $x(t)$, $w(t)$ and $\sigma$ are the location, weight and velocity of the delta shock respectively. Then the following generalized Rankine-Hugoniot conditions hold:

$$\begin{cases} \dfrac{dx(t)}{dt} = \sigma, \\ \dfrac{dw(t)}{dt} = \sigma[\rho] - [\rho u + 1], \\ \dfrac{d(w(t)\sigma)}{dt} = \sigma[\rho u] - [\rho u^2 + u], \end{cases} \tag{2.16}$$

where $[\rho] = \rho_+ - \rho_-$, with initial data

$$(x, w)(0) = (0, 0).$$

By solving (2.16), we have, when $\rho_+ \neq \rho_-$,

$$w(t) = \sqrt{[\rho u]^2 - [\rho][\rho u^2 + u]}\, t,$$

$$x(t) = \frac{[\rho u] + \sqrt{[\rho u]^2 - [\rho][\rho u^2 + u]}}{[\rho]}\, t, \quad \sigma = \frac{[\rho u] + \sqrt{[\rho u]^2 - [\rho][\rho u^2 + u]}}{[\rho]}; \quad (2.17)$$

when $\rho_+ = \rho_-$,

$$w(t) = (\rho_- u_- - \rho_+ u_+)t, \quad x(t) = \frac{u_- + u_+ + \frac{1}{\rho_-}}{2}\, t,$$

$$\sigma = \frac{u_- + u_+ + \frac{1}{\rho_-}}{2}. \quad (2.18)$$

We also can justify the delta shock wave satisfies the entropy condition:

$$u_+ + \frac{1}{\rho_+} < \sigma < u_-, \quad (2.19)$$

which means that all the characteristics on both sides of the delta shock are coming. Thus, we obtain the global solutions to the 1-D Riemann problem for the nonsymmetric Keyfitz-Kranzer system with Chaplygin pressure.

## 3.   Riemann problem with delta initial data

In this section, we construct Riemann solutions of the system (1.1) and (1.4) with initial data (1.5). According to the relations among $u_-$, $u_0$ and $u_+ + \frac{1}{\rho_+}$, we discuss the Riemann problem case by case.

Case 3.1.   $u_- \leq u_0 \leq u_+ + \frac{1}{\rho_+}$.

According to the value of $u_0$, we divide our discussion into the following three subcases.

Subcase 3.1.1.   $u_- < u_0 < u_+ + \frac{1}{\rho_+}$.

To construct the solution of (1.1), (1.4) and (1.5), here we first consider the initial value problem (1.1) and (1.4) with the following initial data:

$$(\rho, u)(0, x) = \begin{cases} (\rho_-, u_-), & x < -\varepsilon, \\ (\frac{m_0}{2\varepsilon}, u_0), & -\varepsilon < x < \varepsilon, \\ (\rho_+, u_+), & x > \varepsilon, \end{cases} \quad (3.1)$$

where $\varepsilon > 0$ is sufficiently small. On the basis of the stability theory of weak solutions, if we obtain a solution of (1.1), (1.4) and (3.1), then by letting $\varepsilon \to 0$, we can get a solution of (1.1), (1.4) and (1.5).

Because $u_- < u_0 < u_0 + \frac{m_0}{2\epsilon}$, $u_0 < u_+ + \frac{1}{\rho_+}$, when $t$ is small, the solution of the initial value problem (1.1), (1.4) and (3.1) can be expressed briefly as follows (see Fig. 3):

$$(\rho_-, u_-) + \widehat{J_1^-} + (\widehat{\rho}_1, \widehat{u}_1) + \widehat{J_2^-} + (\frac{m_0}{2\varepsilon}, u_0) + \widehat{J_1^+} + (\widehat{\rho}_2, \widehat{u}_2) + \widehat{J_2^+} + (\rho_+, u_+), \quad (3.2)$$

where "+" means "followed by", $\widehat{J}_1^{\pm}$ and $\widehat{J}_2^{\pm}$ denote a 1-contact discontinuity and a 2-contact discontinuity, respectively.
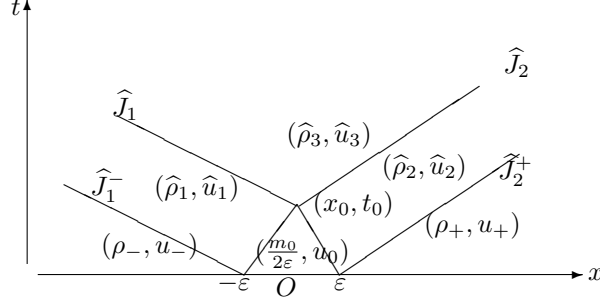


**Figure 3.**

Furthermore, $(\widehat{\rho}_1, \widehat{u}_1)$ and $(\widehat{\rho}_2, \widehat{u}_2)$ are given by

$$\begin{cases} \widehat{u}_1 = u_-, \\ \widehat{\rho}_1 = \dfrac{m_0}{(u_0 - u_-)m_0 + 2\varepsilon}, \end{cases} \tag{3.3}$$

and

$$\begin{cases} \widehat{u}_2 = u_0, \\ \widehat{\rho}_2 = \dfrac{\rho_+}{(u_+ - u_0)\rho_+ + 1}, \end{cases} \tag{3.4}$$

respectively. The propagation speed of $\widehat{J}_2^-$ is $u_0 + \frac{2\varepsilon}{m_0}$, and that of $\widehat{J}_1^+$ is $u_0$. Since $u_0 + \frac{2\varepsilon}{m_0} > u_0$, the contact discontinuity $\widehat{J}_2^-$ will overtake the contact discontinuity $\widehat{J}_1^+$ in a finite time. The intersection point $(x_0, t_0)$ is determined by

$$\begin{cases} x_0 + \varepsilon = \left(u_0 + \dfrac{2\varepsilon}{m_0}\right)t_0, \\ x_0 - \varepsilon = u_0 t_0. \end{cases} \tag{3.5}$$

A simple calculation leads to

$$(x_0, t_0) = (u_0 m_0 + \varepsilon, m_0). \tag{3.6}$$

It is clear that a new Riemann problem is formed when two elementary waves intersect at a finite time. At the time $t = t_0$, we again have a Riemann problem with initial data:

$$(\rho, u)(t_0, x) = \begin{cases} (\widehat{\rho}_1, \widehat{u}_1), & x < x_0, \\ (\widehat{\rho}_2, \widehat{u}_2), & x > x_0. \end{cases} \tag{3.7}$$

Since $\widehat{u}_1 = u_- < u_+ + \frac{1}{\rho_+} = \widehat{u}_2 + \frac{1}{\widehat{\rho}_2}$, the Riemann solution contains a 1-contact discontinuity $\widehat{J}_1$, a 2-contact discontinuity $\widehat{J}_2$ and an intermediate state $(\widehat{\rho}_3, \widehat{u}_3)$, where $(\widehat{\rho}_3, \widehat{u}_3)$ is given by

$$\begin{cases} \widehat{u}_3 = u_-, \\ \widehat{\rho}_3 = \dfrac{\rho_+}{(u_+ - u_-)\rho_+ + 1}. \end{cases} \tag{3.8}$$

Therefore, when $t > t_0$, the solution of (1.1), (1.4) and (3.1) can be expressed as

$$(\rho_-, u_-) + \widehat{J}_1^- + (\widehat{\rho}_1, \widehat{u}_1) + \widehat{J}_1 + (\widehat{\rho}_3, \widehat{u}_3) + \widehat{J}_2 + (\widehat{\rho}_2, \widehat{u}_2) + \widehat{J}_2^+ + (\rho_+, u_+).$$

So far, we have completely constructed a solution of (1.1), (1.4) and (3.1). Letting $\varepsilon \to 0$, we obtain a solution of (1.1), (1.4) and (1.5), which is shown in Fig. 4.
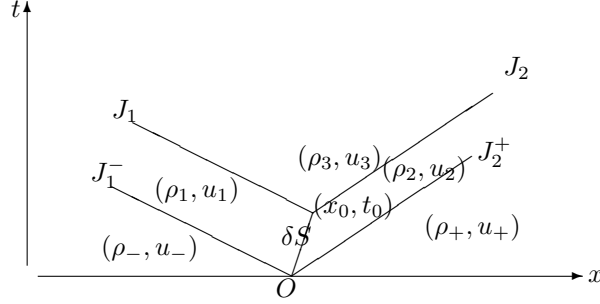


**Figure 4.**

In Fig. 4, we have

$$
\begin{cases}
u_1 = u_-, \\
\dfrac{1}{\rho_1} = u_0 - u_-,
\end{cases}
\quad
\begin{cases}
u_2 = u_0, \\
\dfrac{1}{\rho_2} = u_+ + \dfrac{1}{\rho_+} - u_0,
\end{cases}
$$
$$
\begin{cases}
u_3 = u_-, \\
\dfrac{1}{\rho_3} = u_+ + \dfrac{1}{\rho_+} - u_-,
\end{cases}
\quad (x_0, t_0) = (u_0 m_0, m_0),
$$
(3.9)

and a $\delta$-shock wave $\delta S$ with

$$x(t) = u_0 t, \quad w(t) = m_0 - t, \quad u_\delta(t) = u_0, \ \text{for} \ \ 0 \le t \le m_0,$$

where $x(t)$, $w(t)$ and $u_\delta(t)$ respectively denote the location, weight and propagation speed of the $\delta$-shock.

The $\delta S$ satisfies the following generalized Rankine-Hugoniot conditions

$$
\begin{cases}
\dfrac{dx(t)}{dt} = u_\delta(t), \\
\dfrac{dw(t)}{dt} = u_\delta(t)[\rho] - [\rho u + 1], \\
\dfrac{d(w(t)u_\delta(t))}{dt} = u_\delta(t)[\rho u] - [\rho u^2 + u],
\end{cases}
$$
(3.10)

where $[\rho] = \rho_2 - \rho_1$, with initial data

$$(x, w, u_\delta)(0) = (0, m_0, u_0).$$

From (3.9) and (3.10), we can calculate that

$$\frac{dw(t)}{dt} = u_\delta(t)[\rho] - [\rho u + 1] = u_0(\rho_2 - \rho_1) - (\rho_2 u_2 - \rho_1 u_1)$$

$$= \rho_2(u_0 - u_2) + \rho_1(u_1 - u_0) = \frac{1}{u_0 - u_1}(u_1 - u_0) = -1. \tag{3.11}$$

Solving (3.11) with $w(0) = m_0$, we obtain

$$w(t) = w(0) - t = m_0 - t.$$

Subcase 3.1.2.    $u_- = u_0 < u_+ + \frac{1}{\rho_+}$.

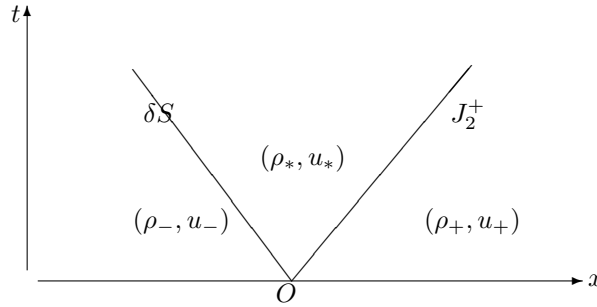Similar to Subcase 3.1.1, we have the Riemann solution of (1.1), (1.4) and (1.5) as shown in Fig. 5.



**Figure 5.**

In Fig. 5, $(\rho_*, u_*)$ is given by

$$\begin{cases} u_* = u_0, \\ \dfrac{1}{\rho_*} = u_+ + \dfrac{1}{\rho_+} - u_0, \end{cases} \tag{3.12}$$

and the $\delta$-shock wave $\delta S$ has the following expression:

$$x(t) = u_0 t, \quad w(t) = m_0, \quad u_\delta(t) = u_0, \quad \text{for} \quad t \geq 0. \tag{3.13}$$

From (3.12) and (3.13), we can calculate that

$$\frac{dw(t)}{dt} = u_\delta(t)[\rho] - [\rho u + 1] = u_0(\rho_* - \rho_-) - (\rho_* u_* - \rho_- u_-)$$

$$= \rho_*(u_0 - u_*) + \rho_-(u_- - u_0) = 0, \tag{3.14}$$

which implies

$$w(t) = w(0) = m_0.$$

Subcase 3.1.3.    $u_- < u_0 = u_+ + \frac{1}{\rho_+}$.

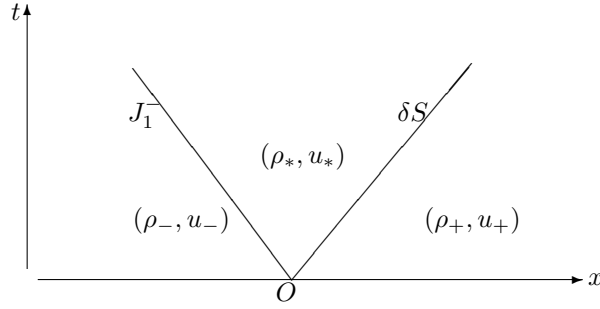Similar to Subcase 3.1.2, we have the Riemann solution of (1.1), (1.4) and (1.5) as shown in Fig. 6.

**Figure 6.**

In Fig. 6, $(\rho_*, u_*)$ is given by

$$\begin{cases} u_* = u_-, \\ \dfrac{1}{\rho_*} = u_0 - u_-, \end{cases}$$

and the $\delta$-shock wave $\delta S$ has the following expression:

$$x(t) = u_0 t, \quad w(t) = m_0, \quad u_\delta(t) = u_0, \quad \text{for} \quad t \geq 0.$$

Case 3.2.    $u_0 < u_- < u_+ + \frac{1}{\rho_+}$. (If $u_- < u_+ + \frac{1}{\rho_+} < u_0$, then the structure of the solution is similar.)

It is seen that the particles $x_0 < 0$ collide with the particles $x_0 = 0$ at the start, while the particles $x_0 \leq 0$ never collide with the particles $x_0 > 0$. Thus the solution can be expressed as (see Fig. 7)

$$(\rho, u)(t, x) = \begin{cases} (\rho_-, u_-), & x < x(t), \\ (w(t)\delta(x - x(t)), u_\delta(t)), & x = x(t), \\ (\overline{\rho}, \overline{u})(t, x), & x(t) < x < \left(u_+ + \dfrac{1}{\rho_+}\right)t, \\ (\rho_+, u_+), & x > \left(u_+ + \dfrac{1}{\rho_+}\right)t, \end{cases} \tag{3.15}$$

where $(\overline{\rho}, \overline{u})(t, x) = (\rho_*, u_*)(\overline{t})$ along the straight line

$$\left(u_+ + \frac{1}{\rho_+}\right)t - x = \left(u_+ + \frac{1}{\rho_+}\right)\overline{t} - x(\overline{t}), \text{ for } \overline{t} \geq 0.$$

Here, $(\rho_*, u_*)(t)$ is the right state of the $\delta$-shock wave $\delta S$ defined by

$$\begin{cases} u_* = u_\delta, \\ u_* + \dfrac{1}{\rho_*} = u_+ + \dfrac{1}{\rho_+}. \end{cases} \tag{3.16}$$

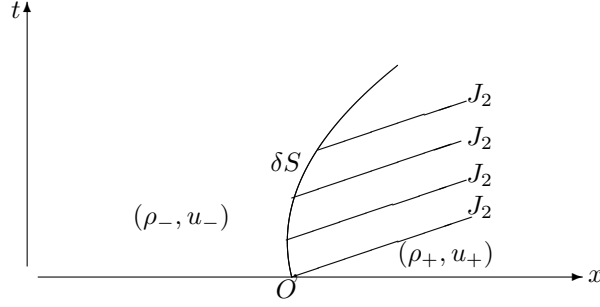The $\delta$-shock wave $\delta S$ satisfies the following generalized Rankine-Hugoniot condi-

**Figure 7.**

tions:

$$\begin{cases} \dfrac{dx(t)}{dt} = u_\delta(t), \\[2mm] \dfrac{dw(t)}{dt} = u_\delta(t)[\rho] - [\rho u + 1], \\[2mm] \dfrac{d(w(t)u_\delta(t))}{dt} = u_\delta(t)[\rho u] - [\rho u^2 + u], \end{cases} \tag{3.17}$$

where $[\rho] = \rho_*(t) - \rho_-$, with initial data

$$(x, w, u_\delta)(0) = (0, m_0, u_0). \tag{3.18}$$

Next, we only need to solve the initial value problem (3.17) and (3.18). From (3.16) and $(3.17)_2$, we have

$$\frac{dw(t)}{dt} = \rho_*(u_\delta - u_*) - \rho_- u_\delta + \rho_- u_- = \rho_-(u_- - u_\delta). \tag{3.19}$$

From (3.16) and $(3.17)_3$, we have

$$\begin{aligned} \frac{d\big(w(t)u_\delta(t)\big)}{dt} &= \rho_* u_*(u_\delta - u_*) - \rho_- u_- u_\delta + \rho_- u_-^2 + u_- - u_* \\ &= \rho_-\big(u_- + \frac{1}{\rho_-}\big)(u_- - u_\delta). \end{aligned} \tag{3.20}$$

Combining (3.19) and (3.20), we have

$$\frac{d\big(w(t)u_\delta(t)\big)}{dt} = \big(u_- + \frac{1}{\rho_-}\big)\frac{dw(t)}{dt}. \tag{3.21}$$

Integrating (3.21) from 0 to $t$, we have

$$w\big(u_- + \frac{1}{\rho_-} - u_\delta\big) = m_0\big(u_- + \frac{1}{\rho_-} - u_0\big) > 0. \tag{3.22}$$

Combining (3.22) and (3.19), we obtain

$$\frac{dw}{dt} = \frac{A - w}{w}, \tag{3.23}$$

where

$$A = \rho_- m_0 \left( u_- + \frac{1}{\rho_-} - u_0 \right). \tag{3.24}$$

In addition, the delta shock wave should satisfy the entropy condition:

$$u_0 < u_\delta < u_-.$$

This, together with (3.19) and (3.22) implies

$$\frac{dw}{dt} > 0, \tag{3.25}$$

and

$$w > 0. \tag{3.26}$$

Combining (3.23), we have

$$0 < w < A. \tag{3.27}$$

Solving (3.23) with initial data $w(0) = m_0$, we have

$$m_0 - w + A\ln(A - m_0) - A\ln(A - w) = t. \tag{3.28}$$

Setting $f(w) = m_0 - w + A\ln(A - m_0) - A\ln(A - w)$, then, from (3.27) we have

$$f'(w) = \frac{w}{A - w} > 0. \tag{3.29}$$

Thus, from (3.11), there exists a unique inverse function $f^{-1}(t)$, such that $w = w(t) = f^{-1}(t)$. Then from (3.22), we obtain

$$u_\delta(t) = u_- + \frac{1}{\rho_-} - \frac{A}{\rho_- w(t)}. \tag{3.30}$$

Furthermore, we have

$$x(t) = \int_0^t u_\delta(\tau)d\tau. \tag{3.31}$$

**Remark 3.1.** From (3.18), we have $\lim\limits_{t \to +\infty} w(t) = A$. Then from (3.20), $\lim\limits_{t \to +\infty} u_\delta(t) = u_- < u_+ + \frac{1}{\rho_+}$. This implies that the delta shock wave $\delta S$ will never overtake those 2-contact discontinuities.

Case 3.3.    $u_+ + \frac{1}{\rho_+} < u_0 < u_-$.

This is a typical case, a delta shock wave emits from the origin. We seek the solution in the following form (see Fig. 8)

$$(\rho, u)(t, x) = \begin{cases} (\rho_-, u_-), & x < x(t), \\ (w(t)\delta(x - x(t)), u_\delta(t)), & x = x(t), \\ (\rho_+, u_+), & x > x(t), \end{cases} \tag{3.32}$$

which satisfies (3.17), where $[\rho] = \rho_+ - \rho_-$, with initial data

$$(x, w, u_\delta)(0) = (0, m_0, u_0). \tag{3.33}$$

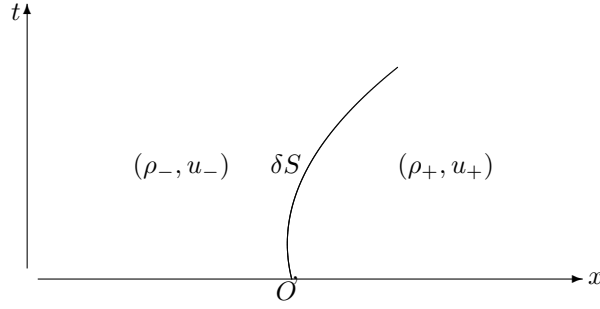Now, we are going to solve the initial value problem (3.32) and (3.33).

**Figure 8.**

Integrating (3.17) from 0 to t with initial data (3.33), we have

$$\begin{cases} w - m_0 = [\rho]x - [\rho u + 1]t, \\ wu_\delta - m_0 u_0 = [\rho u]x - [\rho u^2 + u]t. \end{cases} \tag{3.34}$$

Cancelling w(t) in (3.34), we have

$$m_0 u_0 - m_0 u_\delta = [\rho]xu_\delta - [\rho u]tu_\delta - [\rho u]x + [\rho u^2 + u]t, \tag{3.35}$$

or

$$\frac{d}{dt}\left\{\frac{1}{2}[\rho]x^2 + (m_0 - [\rho u]t)x + \frac{1}{2}[\rho u^2 + u]t^2 - m_0 u_0 t\right\} = 0. \tag{3.36}$$

Integrating (3.35) from 0 to $t$, we obtain

$$\frac{1}{2}[\rho]x^2 + (m_0 - [\rho u]t)x + \frac{1}{2}[\rho u^2 + u]t^2 - m_0 u_0 t = 0. \tag{3.37}$$

From $u_+ + \frac{1}{\rho_+} < u_0 < u_-$, we know that

$$u_+ < u_+ + \frac{1}{\rho_+} < u_0 < u_- < u_- + \frac{1}{\rho_-}, \tag{3.38}$$

if $[\rho] \neq 0$, we have

$$\begin{aligned} \Delta =&(m_0 - [\rho u]t)^2 - 2[\rho]\left(\frac{1}{2}[\rho u^2 + u]t^2 - m_0 u_0 t\right) \\ =&\left([\rho u]^2 - [\rho][\rho u^2 + u]\right)t^2 + 2m_0 t\left([\rho]u_0 - [\rho u]\right) + m_0^2 \\ =&\rho_+\rho_-(u_- - u_+)\left(u_- + \frac{1}{\rho_-} - \left(u_+ + \frac{1}{\rho_+}\right)\right)t^2 \\ &+ 2m_0 t\left(\rho_+(u_0 - u_+) + \rho_-(u_- - u_0)\right) + m_0^2 > 0. \end{aligned} \tag{3.39}$$

We solve Eq. (3.37) to obtain

$$x(t) = \frac{-m_0 + [\rho u]t \pm \sqrt{\Delta}}{[\rho]}. \tag{3.40}$$

From (3.40), we have

$$u_\delta(t) = \frac{[\rho u] \pm \Delta^{-\frac{1}{2}}\left(m_0([\rho]u_0 - [\rho u]) + t([\rho u]^2 - [\rho][\rho u^2 + u])\right)}{[\rho]}.$$

Thus,

$$\lim_{t\to\infty} u_\delta(t) = \frac{[\rho u] \pm \sqrt{[\rho u]^2 - [\rho][\rho u^2 + u]}}{[\rho]}.$$

Substituting (3.40) into $(3.34)_1$, we have

$$w(t) = \pm\sqrt{\Delta}.$$

In addition, to guarantee uniqueness, the delta shock wave should satisfy the entropy condition:

$$u_+ + \frac{1}{\rho_+} < u_\delta(t) < u_-.$$

So, we have

$$w(t) = \sqrt{\Delta}.$$

Then, we obtain a unique solution

$$\begin{cases} x(t) = \dfrac{-m_0 + [\rho u]t + \sqrt{\Delta}}{[\rho]}, \\[2mm] u_\delta(t) = \dfrac{[\rho u] + \Delta^{-\frac{1}{2}}\left(m_0([\rho]u_0 - [\rho u]) + ([\rho u]^2 - [\rho][\rho u^2 + u])t\right)}{[\rho]}, \\[2mm] w(t) = \sqrt{\Delta}. \end{cases} \qquad (3.41)$$

If $[\rho] = 0$, solving (3.37), we have

$$x(t) = \frac{m_0 u_0 t - \frac{1}{2}[\rho u^2 + u]t^2}{m_0 - [\rho u]t}.$$

Then, we have

$$u_\delta(t) = \frac{m_0^2 u_0 + \frac{1}{2}[\rho u^2 + u][\rho u]t^2 - [\rho u^2 + u]m_0 t}{(m_0 - [\rho u]t)^2},$$

and

$$w(t) = m_0 - [\rho u]t.$$

**Remark 3.2.** It is seen that

$$\lim_{t\to\infty} u_\delta(t) = \begin{cases} \dfrac{[\rho u^2 + u]}{2[\rho u]}, & [\rho] = 0, \\[3mm] \dfrac{[\rho u] + \sqrt{[\rho u]^2 - [\rho][\rho u^2 + u]}}{[\rho]}, & [\rho] \neq 0. \end{cases}$$

So, the delta-shock satisfies the entropy condition

$$u_+ + \frac{1}{\rho_+} < \lim_{t\to\infty} u_\delta(t) < u_-,$$

which means that all the characteristics on both sides of the delta shock are incoming.

**Remark 3.3.** If $m_0 = 0$, $u_0 = 0$, then

$$(x, w, u_\delta)(t)$$
$$= \begin{cases} \left( \frac{[\rho u^2 + u]}{2[\rho u]}t, -[\rho u]t, \frac{[\rho u^2 + u]}{2[\rho u]} \right), & [\rho] = 0, \\ \left( \frac{[\rho u] + \sqrt{[\rho u]^2 - [\rho][\rho u^2 + u]}}{[\rho]}t, \sqrt{[\rho u]^2 - [\rho][\rho u^2 + u]}t, \frac{[\rho u] + \sqrt{[\rho u]^2 - [\rho][\rho u^2 + u]}}{[\rho]} \right), & [\rho] \neq 0. \end{cases}$$

This is consistent with the results in [5]. It implies that the solution constructed here is stable under some perturbations.

Case 3.4. $u_0 < u_+ + \frac{1}{\rho_+} < u_-$. (If $u_+ + \frac{1}{\rho_+} < u_- < u_0$, then the structure of the solution is similar.)

Similar to the analysis in Case 3.2, we know that, in this case, when $t$ is small enough, the solution is the same as that in Case 3.2. From (3.27) and (3.30), we have

$$u_\delta'(t) = \frac{m_0 \left( u_- + \frac{1}{\rho_-} - u_0 \right)}{w^2} \frac{dw}{dt} > 0, \quad \text{for} \ \ t > 0, \tag{3.42}$$

which shows that $u_\delta(t)$ is a strictly monotonic increasing function of $t$ for $t \in [0, +\infty)$. On the other hand, $u_\delta(0) = u_0$, $\lim\limits_{t \to +\infty} u_\delta(t) = u_-$ and $u_0 < u_+ + \frac{1}{\rho_+} < u_-$. Thus we can apply the intermediate value theorem in mathematical analysis, and conclude that there exists a unique $t^*$ in $[0, +\infty)$ such that $u_\delta(t^*) = u_+ + \frac{1}{\rho_+}$.

When $0 \leq t \leq t^*$, the solution is the same as that in Case 3.2, which can be expressed as (see Fig. 9)

$$(\rho, u)(t, x) = \begin{cases} (\rho_-, u_-), & x < x(t), \\ (w(t)\delta(x - x(t)), u_\delta(t)), & x = x(t), \\ (\overline{\rho}, \overline{u})(t, x), & x(t) < x < \left( u_+ + \frac{1}{\rho_+} \right)t, \\ (\rho_+, u_+), & x > \left( u_+ + \frac{1}{\rho_+} \right)t, \end{cases} \tag{3.43}$$

where $x(t)$, $w(t)$ and $u_\delta(t)$ are the same as those in Case 3.2. When $t > t^*$, the delta shock wave will overtake all the 2-contact discontinuities and penetrate them in finite time. Suppose that the penetration ends at time $t = t^\#$.

When $t^* \leq t < t^\#$, the solution can be written in the following form (see Fig. 9)

$$(\rho, u)(t, x) = \begin{cases} (\rho_-, u_-), & x < x^1(t), \\ (w^1(t)\delta(x - x^1(t)), u_\delta^1(t)), & x = x^1(t), \\ (\overline{\rho}, \overline{u})(t, x), & x^1(t) < x < \left( u_+ + \frac{1}{\rho_+} \right)t, \\ (\rho_+, u_+), & x > \left( u_+ + \frac{1}{\rho_+} \right)t. \end{cases} \tag{3.44}$$

Here, for any point $(x^1(t), t)$ on the delta shock wave $\delta S_1$, there exists a unique point $(x(t_1), t_1)$ $(0 \leq t_1 \leq t^*)$ on the delta shock wave $\delta S$, such that

$$\left( u_+ + \frac{1}{\rho_+} \right)t - x^1(t) = \left( u_+ + \frac{1}{\rho_+} \right)t_1 - x(t_1). \tag{3.45}$$
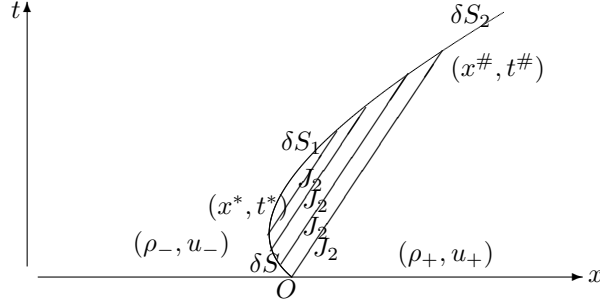
**Figure 9.**

Let $(x, w, u_\delta)(t^*) = (x^*, w^*, u_\delta^*)$, then $u_\delta^* = u_+ + \frac{1}{\rho_+}$, and the $\delta$-shock wave $\delta S_1$ satisfies the following generalized Rankine-Hugoniot conditions:

$$\begin{cases} \dfrac{dx^1(t)}{dt} = u_\delta^1(t), \\[2mm] \dfrac{dw^1(t)}{dt} = u_\delta^1(t)[\rho] - [\rho u + 1], \\[2mm] \dfrac{d(w^1(t)u_\delta^1(t))}{dt} = u_\delta^1(t)[\rho u] - [\rho u^2 + u], \end{cases} \tag{3.46}$$

where $[\rho] = \rho_*(t_1) - \rho_-$, with initial data

$$(x^1, w^1, u_\delta^1)(t^*) = (x^*, w^*, u_\delta^*). \tag{3.47}$$

Here, $(\rho_*, u_*)(t_1)$ is the right state of the $\delta$-shock wave $\delta S_1$ defined by

$$\begin{cases} u_*(t_1) = u_\delta^1(t_1), \\[2mm] u_*(t_1) + \dfrac{1}{\rho_*(t_1)} = u_+ + \dfrac{1}{\rho_+}. \end{cases} \tag{3.48}$$

When $t^\# \le t < +\infty$, the solution can be expressed as (see Fig. 9)

$$(\rho, u)(t, x) = \begin{cases} (\rho_-, u_-), & x < x^2(t), \\ (w^2(t)\delta(x - x^2(t)), u_\delta^2(t)), & x = x^2(t), \\ (\rho_+, u_+), & x > x^2(t). \end{cases} \tag{3.49}$$

It is easy to know that $\rho_*$ is a function of $t_1$. Next, our aim is to express $\rho_*$ as a function of $t$. Integrating (3.19) from 0 to $t_1$, we have

$$\frac{1}{\rho_-}(w(t_1) - m_0) = u_- t_1 - x(t_1). \tag{3.50}$$

From (3.28), we have

$$w(t_1) - m_0 + A \ln(A - w(t_1)) - A \ln(A - m_0) = -t_1. \tag{3.51}$$

Letting $a = \left(u_- - \left(u_+ + \frac{1}{\rho_+}\right)\right) > 0$, then calculating $(3.51) \times a + (3.50)$, we have

$$\left(a + \frac{1}{\rho_-}\right)w(t_1) - \left(a + \frac{1}{\rho_-}\right)m_0 + aA\ln(A - w(t_1)) - aA\ln(A - m_0)$$
$$= \left(u_+ + \frac{1}{\rho_+}\right)t_1 - x(t_1). \tag{3.52}$$

Substituting $(3.45)$ into $(3.52)$, we have

$$\left(a + \frac{1}{\rho_-}\right)w(t_1) - \left(a + \frac{1}{\rho_-}\right)m_0 + aA\ln(A - w(t_1)) - aA\ln(A - m_0)$$
$$= \left(u_+ + \frac{1}{\rho_+}\right)t - x^1(t). \tag{3.53}$$

From $(3.30)$ and $(3.48)$, we have

$$w(t_1) = \frac{A}{\rho_-\left(u_- + \frac{1}{\rho_-} - u_\delta\right)} = \frac{A}{\rho_-\left(u_- + \frac{1}{\rho_-} - \left(u_+ + \frac{1}{\rho_+}\right) + \frac{1}{\rho_*}\right)}. \tag{3.54}$$

Substituting $(3.54)$ into $(3.53)$, we have

$$F\left(\frac{1}{\rho_*}\right) = \left(u_+ + \frac{1}{\rho_+}\right)t - x^1(t), \tag{3.55}$$

where

$$F(s) = \left(a + \frac{1}{\rho_-}\right)\frac{A}{\rho_-\left(u_- + \frac{1}{\rho_-} - \left(u_+ + \frac{1}{\rho_+}\right) + s\right)} - \left(a + \frac{1}{\rho_-}\right)m_0 - aA\ln(A - m_0)$$
$$+ aA\ln\left(A - \frac{A}{\rho_-\left(u_- + \frac{1}{\rho_-} - \left(u_+ + \frac{1}{\rho_+}\right) + s\right)}\right).$$

For $s > 0$,

$$F'(s) = -\frac{As}{\rho_-^2\left(u_- - \left(u_+ + \frac{1}{\rho_+}\right) + s\right)\left(u_- + \frac{1}{\rho_-} - \left(u_+ + \frac{1}{\rho_+}\right) + s\right)^2} < 0, \tag{3.56}$$

which shows that $F(s)$ is a strictly monotonic decreasing function of $s$ for $s \in [0, +\infty)$.

And from $(3.55)$ together with $(3.56)$, we have

$$\frac{1}{\rho_*} = G\left(\left(u_+ + \frac{1}{\rho_+}\right)t - x^1(t)\right), \tag{3.57}$$

where $G = F^{-1}$ and $\frac{1}{G}$ is integrable. From $(3.46)$ and $(3.48)$, we have

$$\frac{dw^1(t)}{dt} = u_\delta^1(\rho_* - \rho_-) - (\rho_* u_* - \rho_- u_-)$$
$$= \rho_*(u_\delta^1 - u_*) - \rho_- u_\delta^1 + \rho_- u_-$$
$$= \rho_*\left(u_\delta^1 - \left(u_+ + \frac{1}{\rho_+}\right)\right) - \rho_- u_\delta^1 + \rho_-\left(u_- + \frac{1}{\rho_-}\right), \tag{3.58}$$

and

$$
\begin{aligned}
\frac{dw^1(t)u_\delta^1(t)}{dt} =& u_\delta^1(\rho_* u_* - \rho_- u_-) - (\rho_* u_*^2 + u_* - \rho_- u_-^2 - u_-) \\
=& u_\delta^1 \big(\rho_*\big(u_+ + \frac{1}{\rho_+}\big) - 1 - \rho_- u_-\big) - \big(u_*(\rho_* u_* + 1) - \rho_- u_-\big(u_- + \frac{1}{\rho_-}\big)\big) \\
=& u_\delta^1 \rho_*\big(u_+ + \frac{1}{\rho_+}\big) - \rho_-\big(u_- + \frac{1}{\rho_-}\big)u_\delta^1 + \rho_- u_-\big(u_- + \frac{1}{\rho_-}\big) \\
&- \big(\rho_*\big(u_+ + \frac{1}{\rho_+}\big) - 1\big)\big(u_+ + \frac{1}{\rho_+}\big) \\
=& \rho_*\big(u_+ + \frac{1}{\rho_+}\big)\big(u_\delta^1 - \big(u_+ + \frac{1}{\rho_+}\big)\big) + \big(u_+ + \frac{1}{\rho_+}\big) \\
&- \rho_-\big(u_- + \frac{1}{\rho_-}\big)u_\delta^1 + \rho_- u_-\big(u_- + \frac{1}{\rho_-}\big).
\end{aligned} \tag{3.59}
$$

Substituting (3.57) into (3.58) and (3.59) respectively, and integrating from $t^*$ to $t$, we have

$$
w^1 - w^* = \int_{t^*}^t \frac{u_\delta^1 - \big(u_+ + \frac{1}{\rho_+}\big)}{G\big(\big(u_+ + \frac{1}{\rho_+}\big)\tau - x^1(\tau)\big)} d\tau - \rho_-(x^1 - x^*) + \rho_-\big(u_- + \frac{1}{\rho_-}\big)(t - t^*),
\tag{3.60}
$$

and

$$
\begin{aligned}
w^1 u_\delta^1 - w^* u_\delta^* =& \big(u_+ + \frac{1}{\rho_+}\big)\int_{t^*}^t \frac{u_\delta^1 - \big(u_+ + \frac{1}{\rho_+}\big)}{G\big(\big(u_+ + \frac{1}{\rho_+}\big)\tau - x^1(\tau)\big)} d\tau - \rho_-\big(u_- + \frac{1}{\rho_-}\big)(x^1 - x^*) \\
&+ \big(u_+ + \frac{1}{\rho_+}\big)(t - t^*) + u_- \rho_-\big(u_- + \frac{1}{\rho_-}\big)(t - t^*).
\end{aligned} \tag{3.61}
$$

Calculating $(3.61) - (3.60) \times u_\delta^1$, and noting the fact that $u_\delta^* = u_+ + \frac{1}{\rho_+}$, we have

$$
\begin{aligned}
&\big(u_+ + \frac{1}{\rho_+} - u_\delta^1\big)\int_{t^*}^t \frac{u_\delta^1 - \big(u_+ + \frac{1}{\rho_+}\big)}{G\big(\big(u_+ + \frac{1}{\rho_+}\big)\tau - x^1(\tau)\big)} d\tau + \big(u_+ + \frac{1}{\rho_+}\big)(t - t^*) \\
&+ \rho_- u_\delta^1(x^1 - x^*) - \rho_-\big(u_- + \frac{1}{\rho_-}\big)(x^1 - x^*) - \rho_-\big(u_- + \frac{1}{\rho_-}\big)(t - t^*) \\
&+ u_- \rho_-\big(u_- + \frac{1}{\rho_-}\big)(t - t^*) - w^*\big(u_\delta^1 - \big(u_+ + \frac{1}{\rho_+}\big)\big) = 0.
\end{aligned} \tag{3.62}
$$

Integrating (3.62) from $t^*$ to $t$, we have

$$
\begin{aligned}
&\int_{t^*}^t \big(u_+ + \frac{1}{\rho_+} - u_\delta^1\big)\int_{t^*}^s \frac{u_\delta^1 - \big(u_+ + \frac{1}{\rho_+}\big)}{G\big(\big(u_+ + \frac{1}{\rho_+}\big)\tau - x^1(\tau)\big)} d\tau ds + \frac{1}{2}\big(u_+ + \frac{1}{\rho_+}\big)(t - t^*)^2 \\
&+ \frac{1}{2}\rho_-(x^1 - x^*)^2 - \rho_-\big(u_- + \frac{1}{\rho_-}\big)(x^1 - x^*)(t - t^*) + \frac{1}{2}\rho_- u_-\big(u_- + \frac{1}{\rho_-}\big)(t - t^*)^2 \\
&- w^*\big((x^1 - x^*) - \big(u_+ + \frac{1}{\rho_+}\big)(t - t^*)\big) = 0.
\end{aligned} \tag{3.63}
$$

Letting $Y = \big(u_+ + \frac{1}{\rho_+}\big)\tau - x^1(\tau)$, $Z = \big(u_+ + \frac{1}{\rho_+}\big)s - x^1(s)$, then the first term

on the left-hand side of (3.63) equals to

$$-\int_{(u_++\frac{1}{\rho_+})t^*-x^*}^{(u_++\frac{1}{\rho_+})t-x^1}\int_{(u_++\frac{1}{\rho_+})t^*-x^*}^{Z}\frac{1}{G(Y)}dY\,dZ. \tag{3.64}$$

So, (3.63) can be written as

$$H(x^1,t)=0, \tag{3.65}$$

where

$$\begin{aligned}
H(x^1,t)=&-\int_{(u_++\frac{1}{\rho_+})t^*-x^*}^{(u_++\frac{1}{\rho_+})t-x^1}\int_{(u_++\frac{1}{\rho_+})t^*-x^*}^{Z}\frac{1}{G(Y)}dY\,dZ+\frac{1}{2}(u_++\frac{1}{\rho_+})(t-t^*)^2\\
&-\rho_-(u_-+\frac{1}{\rho_-})(x^1-x^*)(t-t^*)\\
&+\frac{1}{2}\rho_-(x^1-x^*)^2+\frac{1}{2}u_-\rho_-(u_-+\frac{1}{\rho_-})(t-t^*)^2\\
&-w^*(x^1-x^*-(u_++\frac{1}{\rho_+})(t-t^*)).
\end{aligned} \tag{3.66}$$

For $t^*<t<t^\#$, we have

$$\begin{aligned}
&H\big|_{x^1=x^*+(u_++\frac{1}{\rho_+})(t-t^*)}\\
=&\frac{1}{2}(u_++\frac{1}{\rho_+})(t-t^*)^2-\rho_-(u_-+\frac{1}{\rho_-})(u_++\frac{1}{\rho_+})(t-t^*)^2+\frac{1}{2}\rho_-(u_++\frac{1}{\rho_+})^2(t-t^*)^2\\
&+\frac{1}{2}\rho_-u_-(u_-+\frac{1}{\rho_-})(t-t^*)^2\\
=&\frac{1}{2}\rho_-(t-t^*)^2\Big((u_++\frac{1}{\rho_+})(\frac{1}{\rho_-}-(u_-+\frac{1}{\rho_-}))+(u_++\frac{1}{\rho_+})(u_++\frac{1}{\rho_+}-(u_-+\frac{1}{\rho_-}))\\
&+u_-(u_-+\frac{1}{\rho_-})\Big)\\
=&\frac{1}{2}\rho_-(t-t^*)^2(u_--(u_++\frac{1}{\rho_+}))(u_-+\frac{1}{\rho_-}-(u_++\frac{1}{\rho_+}))>0, \tag{3.67}
\end{aligned}$$

and

$$\begin{aligned}
&H\big|_{x^1=x^*+(u_-+\frac{1}{\rho_-})(t-t^*)}\\
\leq&\frac{1}{2}(u_++\frac{1}{\rho_+})(t-t^*)^2-\rho_-(u_-+\frac{1}{\rho_-})^2(t-t^*)^2+\frac{1}{2}\rho_-(u_-+\frac{1}{\rho_-})^2(t-t^*)^2\\
&+\frac{1}{2}\rho_-u_-(u_-+\frac{1}{\rho_-})(t-t^*)^2-w^*(u_-+\frac{1}{\rho_-}-(u_++\frac{1}{\rho_+}))(t-t^*)\\
=&\frac{1}{2}(t-t^*)^2(\rho_-(u_-+\frac{1}{\rho_-})(u_--u_--\frac{1}{\rho_-})+(u_++\frac{1}{\rho_+}))\\
&-w^*(u_-+\frac{1}{\rho_-}-(u_++\frac{1}{\rho_+})(t-t^*))\\
=&\frac{1}{2}(t-t^*)^2(u_++\frac{1}{\rho_+}-u_--\frac{1}{\rho_-})-w^*(u_-+\frac{1}{\rho_-}-(u_++\frac{1}{\rho_+})(t-t^*))<0. \\
&\tag{3.68}
\end{aligned}$$

Moreover, for $x^* + \left(\left(u_+ + \frac{1}{\rho_+}\right)(t - t^*) < x^1 < x^* + \left(\left(u_- + \frac{1}{\rho_-}\right)(t - t^*)\right)$, we have

$$\frac{\partial H}{\partial x^1} = \int_{\left(\left(u_+ + \frac{1}{\rho_+}\right)t^* - x^*\right)}^{\left(\left(u_+ + \frac{1}{\rho_+}\right)t - x^1\right)} \frac{1}{G(Y)} dY + \rho_-(x^1 - x^* - \left(\left(u_- + \frac{1}{\rho_-}\right)(t - t^*)\right)) - w^* < 0. \quad (3.69)$$

On account of (3.67), (3.68) and (3.69), there exists a unique function $x^1 = x^1(t) \in (x^* + \left(u_+ + \frac{1}{\rho_+}\right)(t - t^*), x^* + \left(u_- + \frac{1}{\rho_-}\right)(t - t^*))$, such that $H(x^1, t) = 0$ for $t^* < t < t^\#$. Futhermore, we have $u_\delta^1(t) = \frac{dx^1(t)}{dt}$. From (3.60), we have

$$w^1(t) = \int_{t^*}^{t} \frac{u_\delta^1(\tau) - \left(u_+ + \frac{1}{\rho_+}\right)}{G(\left(u_+ + \frac{1}{\rho_+}\right)\tau - x^1(\tau))} d\tau - \rho_-(x^1(t) - x^*) + \rho_-\left(u_- + \frac{1}{\rho_-}\right)(t - t^*) + w^*.$$
$$(3.70)$$

For $t^\# \leq t < +\infty$, where $t^\#$ is determined by $x^1(t^\#) = \left(u_+ + \frac{1}{\rho_+}\right)t^\#$, the solution (3.49) is similar to that in case 3.3, where $x^2(t)$, $w^2(t)$ and $u_\delta^2(t)$ are determined by the Rankine-Hugoniot condition (3.34) with initial data

$$(x^2, w^2, u_\delta^2)(t^\#) = (x^1(t^\#), w^1(t^\#), u_\delta^1(t^\#)).$$

The details are omitted.

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