

THE NEW EXACT SOLUTIONS OF VARIANT TYPES OF TIME FRACTIONAL COUPLED SCHRÖDINGER EQUATIONS IN PLASMA PHYSICS*

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Abstract In the present article, the new exact solutions of fractional coupled Schrödinger type equations have been studied by using a new reliable analytical method. We applied a relatively new method for finding some new exact solutions of time fractional coupled equations viz. time fractional coupled Schrödinger–KdV and coupled Schrödinger–Boussinesq equations. The fractional complex transform have been used here along with the property of local fractional calculus for reduction of fractional partial differential equations (FPDE) to ordinary differential equations (ODE). The obtained results have been plotted here for demonstrating the nature of the solutions.

Keywords Fractional complex transform, local fractional calculus, time fractional coupled Schrödinger–KdV equation, time-fractional coupled Schrödinger–Boussinesq equation.

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1. Introduction

Now a days study of nonlinear evolution equations play important role for describing the nonlinear wave phenomena [1–6] in mathematical physics. Especially Schrödinger types of equation are known to describe the quantum mechanical behaviour [7, 8]. The development of instability associated with the envelope modulation of an high frequency wave packet coupled to an low frequency wave field is presented by coupled nonlinear equations like coupled Schrödinger–KdV and coupled Schrödinger–Boussinesq equations in plasma physics [9].

Consider the time fractional coupled Schrödinger–KdV(SK) equation [10, 11]

$$\begin{aligned}iD_t^\alpha u - u_{xx} - uv &= 0, \\ D_t^\alpha v + 6vv_x + v_{xxx} + (|u|^2)_x &= 0,\end{aligned}\tag{1.1}$$

where the α symbolizes the order of fractional derivative, whose range is $0 < \alpha \leq 1$.

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Schrödinger-KdV equation describes various process such as dust-acoustic, Langmuir and electromagnetic waves in dusty plasma [12–14]. Various methods like unified algebraic method [15], hybrid of Fourier transform method [16], Variational iteration method [17] have been used for finding the solutions of Schrödinger-KdV equation. Consider time-fractional coupled Schrödinger–Boussinesq (SB) equation [18, 19]

$$\begin{aligned} i\varepsilon D_t^\alpha u + \frac{3}{2}u_{xx} - \frac{1}{2}uv &= 0, \\ D_t^{2\alpha}v - v_{xx} - v_{xxxx} - v_{xx}^2 - \frac{1}{4}(|u|^2)_{xx} &= 0, \end{aligned} \quad (1.2)$$

where u is the complex valued function which represents the short wave amplitude of media and v is the real-valued function which represents the long wave amplitude of media. The $\varepsilon > 0$ denotes the ratio between the electron number with respect to ion number and the α is the fractional order whose range is $0 < \alpha \leq 1$.

The coupled Schrödinger-Boussinesq equation is originated from nonlinear magnetosonic and upper-hybrid waves in magnetized plasma [20]. It also describes the diatomic lattice system [21], the dynamics of Langmuir soliton formation and the interaction in plasma [22–25]. Various methods like Multi-symplectic scheme [18], Fourier spectral method [19], conservative difference scheme [26], (G'/G) -expansion method [27], extended simplest equation method [28] have been used for finding solution for coupled Schrödinger-Boussinesq equation.

The fractional differential equations can be described best in discontinuous media and the fractional order is equivalent to its fractional dimensions. Fractal media which is complex, appears in different fields of engineering and physics. In this context, the local fractional calculus theory is very important for modelling problems for fractal mathematics and engineering on Cantorian space in fractal media.

Our main objective here to find new exact solutions of time fractional coupled Schrödinger-KdV (SK) and time-fractional coupled Schrödinger-Boussinesq (SB) equations by applying a reliable and relatively new analytical method.

The primary content of the article is arranged as following. Definitions of local fractional calculus with some properties are described in Section 2. The algorithm of new analytical method is presented in Section 3. The implementation of proposed method for establishing the exact solutions of time fractional coupled Schrödinger-KdV (SK) and time-fractional coupled Schrödinger-Boussinesq equations are presented in Section 4. The numerical simulation for newly proposed analytical method is presented in Section 5. A brief conclusion of the current study is presented in Section 6.

2. Preliminaries of local fractional calculus and proposed method

2.1. Local fractional continuity of a function

Definition 2.1. Suppose that $f(x)$ is defined throughout some interval containing x_0 and all point near x_0 , then $f(x)$ is said to be local fractional continuous at $x = x_0$, denote by $\lim_{x \rightarrow x_0} f(x) = f(x_0)$, if to each positive ε and some positive constant k

corresponds some positive δ such that [29–31]

$$|f(x) - f(x_0)| < k\varepsilon^\alpha, 0 < \alpha \leq 1, \quad (2.1)$$

whenever $|x - x_0| < \delta$, $\varepsilon, \delta > 0$ and $\varepsilon, \delta \in \mathbb{R}$. Consequently, the function $f(x)$ is called local fractional continuous on the interval (a, b) , denoted by

$$f(x) \in C_\alpha(a, b), \quad (2.2)$$

where α is fractal dimension with $0 < \alpha \leq 1$.

Definition 2.2. A function $f(x) : \mathbb{R} \rightarrow \mathbb{R}$, $X \mapsto f(X)$ is called a non-differentiable function of exponent α , $0 < \alpha \leq 1$, which satisfies Hölder function of exponent α , then for $x, y \in X$, we have [29–31]

$$|f(x) - f(y)| \leq C|x - y|^\alpha. \quad (2.3)$$

Definition 2.3. A function $f(x) : \mathbb{R} \rightarrow \mathbb{R}$, $X \mapsto f(X)$ is called to be local fractional continuous of order α , $0 < \alpha \leq 1$, or shortly α -local fractional continuous, when we have [29–31]

$$f(x) - f(x_0) = O((x - x_0)^\alpha). \quad (2.4)$$

Remark 2.1. A function $f(x)$ is said to be in the space $C_\alpha[a, b]$ if and only if it can be written as [29–31]

$$f(x) - f(x_0) = O((x - x_0)^\alpha)$$

with any $x_0 \in [a, b]$ and $0 < \alpha \leq 1$.

2.2. Local fractional derivative

Definition 2.4. Let $f(x) \in C_\alpha(a, b)$. Local fractional derivative of $f(x)$ of order α at $x = x_0$ is defined as [29–31]

$$f^{(\alpha)}(x_0) = \left. \frac{d^\alpha f(x)}{dx^\alpha} \right|_{x=x_0} = \lim_{x \rightarrow x_0} \frac{\Delta^\alpha(f(x) - f(x_0))}{(x - x_0)^\alpha}, \quad (2.5)$$

where $\Delta^\alpha(f(x) - f(x_0)) \cong \Gamma(1 + \alpha)\Delta(f(x) - f(x_0))$ and $0 < \alpha \leq 1$.

Remark 2.2. The following rules are hold for local fractional derivative [31]

- (i) $\frac{d^\alpha x^{k\alpha}}{dx^\alpha} = \frac{\Gamma(1+k\alpha)}{\Gamma(1+(k-1)\alpha)}x^{(k-1)\alpha}$;
- (ii) $\frac{d^\alpha E_\alpha(kx^\alpha)}{dx^\alpha} = kE_\alpha(kx^\alpha)$, k is a constant.

Remark 2.3 ([29–32]). (i) If $y(x) = (f \circ u)(x)$ where $u(x) = g(x)$, then we have

$$\frac{d^\alpha y(x)}{dx^\alpha} = f^{(\alpha)}(g(x)) \left(g^{(1)}(x) \right)^\alpha, \quad (2.6)$$

when $f^{(\alpha)}(g(x))$ and $g^{(1)}(x)$ exist.

(ii) If $y(x) = (f \circ u)(x)$ where $u(x) = g(x)$, then we have

$$\frac{d^\alpha y(x)}{dx^\alpha} = f^{(1)}(g(x)) g^{(\alpha)}(x), \quad (2.7)$$

when $f^{(1)}(g(x))$ and $g^{(\alpha)}(x)$ exist.

3. Algorithm of the proposed new analytical method

In the present section, the algorithm of new analytical method has been presented. The major steps of the newly proposed method are described as follows:

Step 1: The nonlinear coupled time-fractional partial differential equation with two independent variables x and t is considered here as in the following form:

$$\begin{aligned} F(u, u_x, u_{xx}, u_{xxx}, \dots, D_t^\alpha u, v, v_x, v_{xx}, v_{xxx}, \dots, D_t^\alpha v, \dots) &= 0, 0 < \alpha \leq 1, \\ P(u, u_x, u_{xx}, u_{xxx}, \dots, D_t^\alpha u, v, v_x, v_{xx}, v_{xxx}, \dots, D_t^\alpha v, \dots) &= 0, 0 < \alpha \leq 1, \end{aligned} \quad (3.1)$$

where $u(x, t)$ and $v(x, t)$ are unknown function. Here the fractional derivative $D_t^\alpha u$, $D_t^\alpha v$ are considered in the modified Riemann-Liouville sense. F and P are the functions in $u(x, t)$ and $v(x, t)$ along with their highest order partial derivatives and nonlinear terms of $u(x, t)$ and $v(x, t)$ respectively.

Step 2: The exact solution of eqs. (3.1) is considered here with the help of fractional complex transform [33–37], which is given by

$$u(x, t) = \Phi(\zeta)e^{i\eta}, v(x, t) = \Psi(\zeta), \zeta = cx + \frac{\gamma t^\alpha}{\Gamma(\alpha + 1)}, \eta = kx + \frac{rt^\alpha}{\Gamma(\alpha + 1)}, \quad (3.2)$$

where c, γ, r, k are a constants, which are determined later.

By using the chain rule eq. (2.7) [34–37], we have

$$\begin{aligned} D_t^\alpha u &= \sigma_t \Phi_\zeta D_t^\alpha \zeta, \\ D_t^\alpha v &= \sigma_t \Psi_\zeta D_t^\alpha \zeta, \end{aligned}$$

where σ_t is the fractal indexes [36, 37], without loss of generality we can take $\sigma_t = \kappa$, where κ is a constant.

Using eqs. (3.2), the fractional partial differential equations (FPDEs) eqs. (3.1) is reduced to the following nonlinear ordinary differential equations (ODEs)

$$\begin{aligned} F(\Phi e^{i\eta}, ic\Phi' e^{i\eta}, -c^2\Phi'' e^{i\eta}, ic^3\Phi''' e^{i\eta}, \dots, \gamma i\Phi' e^{i\eta}, \Psi, c\Psi', c^2\Psi'', c^3\Psi''', \dots, \gamma\Psi') &= 0, \\ P(\Phi e^{i\eta}, ic\Phi' e^{i\eta}, -c^2\Phi'' e^{i\eta}, ic^3\Phi''' e^{i\eta}, \dots, \gamma i\Phi' e^{i\eta}, \Psi, c\Psi', c^2\Psi'', c^3\Psi''', \dots, \gamma\Psi') &= 0. \end{aligned} \quad (3.3)$$

Step 3: Here the exact solutions of eqs. (3.1) are assumed in the polynomial $\phi(\zeta)$ as follows:

$$\begin{aligned} \Phi(\zeta) &= a_0 + \sum_{i=1}^n a_i \phi^i(\zeta), \\ \Psi(\zeta) &= b_0 + \sum_{i=1}^m b_i \phi^i(\zeta), \end{aligned} \quad (3.4)$$

where $\phi(\zeta) = \frac{e^\zeta}{1+e^\zeta}$ and $\phi(\zeta)$ also satisfies following :

$$\phi_\zeta = \phi - \phi^2. \quad (3.5)$$

Step 4: According to the proposed method, we substitute $\Phi = \zeta^{-p}$ and $\Psi = \zeta^{-q}$ in all terms of eqs. (3.3) for determining the highest order singularity. Then the degree of all terms of eqs. (3.3) has been taken in to study and consequently the two or more terms of lower degree are chosen. The maximum value of p and q are

known as the pole and denoted as n and m respectively. For integer values of n and m , this proposed method only can be implemented. However, if n and m are non-integer, the above eqs. (3.3) can be transferred and then the above procedure can be repeated.

Step 5: The derivatives of the function $\Phi(\zeta)$ and $\Psi(\zeta)$ can be calculated by using eq. (3.5). Some derivatives of $\Phi(\zeta)$ are presented as follows:

$$\begin{aligned}\Phi'(\zeta) &= \sum_{i=1}^n a_i i (1-\phi) \phi^i; \quad \Psi'(\zeta) = \sum_{i=1}^m b_i i (1-\phi) \phi^i; \\ \Phi''(\zeta) &= \sum_{i=1}^n a_i i [\phi + i(1-\phi)] (1-\phi) \phi^i; \quad \Psi''(\zeta) = \sum_{i=1}^m b_i i [\phi + i(1-\phi)] (1-\phi) \phi^i; \\ \Phi'''(\zeta) &= \sum_{i=1}^n a_i i (i+1) (1-\phi)^2 \phi^{i+1} + \sum_{i=1}^n a_i i [\phi + i(1-\phi)] (1-\phi) \phi^{i+1}; \\ \Psi'''(\zeta) &= \sum_{i=1}^m b_i i (i+1) (1-\phi)^2 \phi^{i+1} + \sum_{i=1}^m b_i i [\phi + i(1-\phi)] (1-\phi) \phi^{i+1}.\end{aligned}\tag{3.6}$$

Step 6: Substituting eqs. (3.6) into eqs. (3.3) and equating the coefficient of ϕ^i ($i = 0, 1, 2, \dots$) into zero, we obtain the set of algebraic equations. By solving the obtained algebraic equations, we can get the unknowns a_i ($i = 0, 1, 2, \dots, n$), b_i ($i = 0, 1, 2, \dots, m$) and other constants. Then putting the all obtained unknowns in eq. (3.4), we get the required exact solutions for eqs. (3.1) instantly.

4. Implementation of new proposed method for the solutions of time-fractional coupled SK and coupled SB equations

In this part, the newly proposed method has been applied for obtaining the exact solutions for time-fractional coupled SK and coupled SB equation.

4.1. Exact solutions for time-fractional coupled SK equation

The newly proposed method has been applied here for finding the exact solutions for eqs. (1.1). By using the fractional complex transform (3.2) in eqs. (1.1), we have the following nonlinear ODE:

$$\Phi(\zeta)\Psi(\zeta) + (r - k^2)\Phi(\zeta) + c^2\Phi''(\zeta)^2 + i(-\gamma + 2kc)\Phi'(\zeta) = 0, \tag{4.1}$$

$$\gamma\Psi'(\zeta) + 6c\Psi(\zeta)\Psi'(\zeta) + c^3\Psi'''(\zeta) - 2c\Phi(\zeta)\Phi'(\zeta) = 0. \tag{4.2}$$

Again the eq. (4.1) can be written as

$$\Phi(\zeta)\Psi(\zeta) + (r - k^2)\Phi(\zeta) + c^2\Phi''(\zeta)^2 = 0, \tag{4.3}$$

where $\gamma = 2kc$.

By integrating eq. (4.2) once with respect to ζ and putting $\gamma = 2kc$, we have

$$2k\Psi(\zeta) + 3\Psi^2(\zeta) + c^2\Psi''(\zeta) - \Phi^2(\zeta) = 0. \tag{4.4}$$

Let

$$\Phi(\zeta) = a_0 + \sum_{i=1}^n a_i \phi^i \text{ and } \Psi(\zeta) = b_0 + \sum_{i=1}^m b_i \phi^i. \tag{4.5}$$

The dominant terms with highest order singularity of eq. (4.3) are $\Phi(\zeta)\Psi(\zeta)$ and $c^2\Phi''(\zeta)^2$. The maximum value of pole is 2 that means here $n = 2$. Similarly the dominant terms with highest order singularity of eq. (4.4) are $3\Psi^2(\zeta)$ and $c^2\Psi''(\zeta)$. The maximum value of pole is 2 that means here $m = 2$.

Therefore by eq. (4.5), we have the following ansatz:

$$\Phi(\zeta) = a_0 + a_1\phi + a_2\phi^2 \text{ and } \Psi(\zeta) = b_0 + b_1\phi + b_2\phi^2, \tag{4.6}$$

where ϕ satisfies eq. (3.5).

Substituting eq. (4.6) along with eqs. (3.6) into eqs. (4.3) and (4.4), then equating each coefficients of ϕ^i ($i = 0, 1, 2, \dots$) to zero, we can find a system of algebraic equations for $a_0, a_1, a_2, b_0, b_1, b_2, c, k$ and r as follows:

$$\begin{aligned} \phi^0 : a_0b_0 + a_0(-k^2 + r) &= 0; \\ &: -a_0^2 + 3b_0^2 + 2b_0k = 0; \\ \phi^1 : a_1b_0 + a_0b_1 + a_1c^2 + a_1(-k^2 + r) &= 0; \\ &: -2a_0a_1 + 6b_0b_1 + c^2b_1 + 2b_1k = 0; \\ \phi^2 : a_2b_0 + a_1b_1 + a_0b_2 - 3a_1c^2 + 4a_2c^2 + a_2(-k^2 + r) &= 0; \\ &: -a_1^2 - 2a_0a_2 + 3b_1^2 + 6b_0b_2 - 3c^2b_1 + 4c^2b_2 + 2b_2k = 0; \\ \phi^3 : a_2b_1 + a_1b_2 + 2a_1c^2 - 10a_2c^2 &= 0; \\ &: -2a_1a_2 + 6b_1b_2 + 2c^2b_1 - 10c^2b_2 = 0; \\ \phi^4 : a_2b_2 + 6a_2c^2 &= 0; \\ &: -a_2^2 + 3b_2^2 + 6c^2b_2 = 0. \end{aligned} \tag{4.7}$$

Solving the above algebraic eqs. (4.7), we have the following sets of coefficients for the solutions of eqs. (4.3) and (4.4) as given below:

Case 1:

$$\begin{aligned} c = c, k = -\frac{c^2}{2}, r = \frac{c^2(-4 + c^2)}{4}, a_0 = 0, a_1 = -6\sqrt{2}c^2, \\ a_2 = 6\sqrt{2}c^2, b_0 = 0, b_1 = 6c^2, b_2 = -6c^2. \end{aligned}$$

For **case 1**, we have the following solution

$$\begin{aligned} \Phi_{11} &= -\frac{6\sqrt{2}c^2(\cosh(\zeta) + \sinh(\zeta))}{1 + \cosh(\zeta) + \sinh(\zeta)} + \frac{6\sqrt{2}c^2(\cosh(2\zeta) + \sinh(2\zeta))}{(1 + \cosh(\zeta) + \sinh(\zeta))^2}, \\ \Psi_{11} &= \frac{6c^2(\cosh(\zeta) + \sinh(\zeta))}{1 + \cosh(\zeta) + \sinh(\zeta)} - \frac{6c^2(\cosh(2\zeta) + \sinh(2\zeta))}{(1 + \cosh(\zeta) + \sinh(\zeta))^2}, \end{aligned} \tag{4.8}$$

where $\zeta = cx + \frac{\gamma t^\alpha}{\Gamma(\alpha+1)}$.

Case 2:

$$\begin{aligned} c = c, k = -\frac{c^2}{2}, r = \frac{c^2(8 + 3c^2)}{12}, a_0 = \sqrt{2}c^2, \\ a_1 = -6\sqrt{2}c^2, a_2 = 6\sqrt{2}c^2, b_0 = -\frac{2c^2}{3}, b_1 = 6c^2, b_2 = -6c^2. \end{aligned}$$

For **case 2**, we have the following solution

$$\begin{aligned}\Phi_{21} &= \sqrt{2}c^2 - \frac{6\sqrt{2}c^2(\cosh(\zeta) + \sinh(\zeta))}{1 + \cosh(\zeta) + \sinh(\zeta)} + \frac{6\sqrt{2}c^2(\cosh(2\zeta) + \sinh(2\zeta))}{(1 + \cosh(\zeta) + \sinh(\zeta))^2}, \\ \Psi_{21} &= -\frac{2c^2}{3} + \frac{6c^2(\cosh(\zeta) + \sinh(\zeta))}{1 + \cosh(\zeta) + \sinh(\zeta)} - \frac{6c^2(\cosh(2\zeta) + \sinh(2\zeta))}{(1 + \cosh(\zeta) + \sinh(\zeta))^2},\end{aligned}\quad (4.9)$$

where $\zeta = cx + \frac{\gamma t^\alpha}{\Gamma(\alpha+1)}$.

Case 3:

$$\begin{aligned}c &= c, k = \frac{c^2}{2}, r = \frac{c^2(4 + c^2)}{4}, a_0 = \sqrt{2}c^2, a_1 = -6\sqrt{2}c^2, \\ a_2 &= 6\sqrt{2}c^2, b_0 = -c^2, b_1 = 6c^2, b_2 = -6c^2.\end{aligned}$$

For **case 3**, we have the following solution

$$\begin{aligned}\Phi_{31} &= \sqrt{2}c^2 - \frac{6\sqrt{2}c^2(\cosh(\zeta) + \sinh(\zeta))}{1 + \cosh(\zeta) + \sinh(\zeta)} + \frac{6\sqrt{2}c^2(\cosh(2\zeta) + \sinh(2\zeta))}{(1 + \cosh(\zeta) + \sinh(\zeta))^2}, \\ \Psi_{31} &= -c^2 + \frac{6c^2(\cosh(\zeta) + \sinh(\zeta))}{1 + \cosh(\zeta) + \sinh(\zeta)} - \frac{6c^2(\cosh(2\zeta) + \sinh(2\zeta))}{(1 + \cosh(\zeta) + \sinh(\zeta))^2},\end{aligned}\quad (4.10)$$

where $\zeta = cx + \frac{\gamma t^\alpha}{\Gamma(\alpha+1)}$.

Case 4:

$$\begin{aligned}c &= c, k = \frac{c^2}{2}, r = \frac{c^2(-8 + 3c^2)}{12}, a_0 = 0, a_1 = -6\sqrt{2}c^2, \\ a_2 &= 6\sqrt{2}c^2, b_0 = -\frac{c^2}{3}, b_1 = 6c^2, b_2 = -6c^2.\end{aligned}$$

For **case 4**, we have the following solution

$$\begin{aligned}\Phi_{41} &= -\frac{6\sqrt{2}c^2(\cosh(\zeta) + \sinh(\zeta))}{1 + \cosh(\zeta) + \sinh(\zeta)} + \frac{6\sqrt{2}c^2(\cosh(2\zeta) + \sinh(2\zeta))}{(1 + \cosh(\zeta) + \sinh(\zeta))^2}, \\ \Psi_{41} &= -\frac{c^2}{3} + \frac{6c^2(\cosh(\zeta) + \sinh(\zeta))}{1 + \cosh(\zeta) + \sinh(\zeta)} - \frac{6c^2(\cosh(2\zeta) + \sinh(2\zeta))}{(1 + \cosh(\zeta) + \sinh(\zeta))^2},\end{aligned}\quad (4.11)$$

where $\zeta = cx + \frac{\gamma t^\alpha}{\Gamma(\alpha+1)}$.

Case 5:

$$\begin{aligned}c &= c, k = -\frac{c^2}{2}, r = \frac{c^2(-4 + c^2)}{4}, a_0 = 0, a_1 = 6\sqrt{2}c^2, \\ a_2 &= -6\sqrt{2}c^2, b_0 = 0, b_1 = 6c^2, b_2 = -6c^2.\end{aligned}$$

For **case 5**, we have the following solution

$$\begin{aligned}\Phi_{51} &= \frac{6\sqrt{2}c^2(\cosh(\zeta) + \sinh(\zeta))}{1 + \cosh(\zeta) + \sinh(\zeta)} - \frac{6\sqrt{2}c^2(\cosh(2\zeta) + \sinh(2\zeta))}{(1 + \cosh(\zeta) + \sinh(\zeta))^2}, \\ \Psi_{51} &= \frac{6c^2(\cosh(\zeta) + \sinh(\zeta))}{1 + \cosh(\zeta) + \sinh(\zeta)} - \frac{6c^2(\cosh(2\zeta) + \sinh(2\zeta))}{(1 + \cosh(\zeta) + \sinh(\zeta))^2},\end{aligned}\quad (4.12)$$

where $\zeta = cx + \frac{\gamma t^\alpha}{\Gamma(\alpha+1)}$.

Case 6:

$$c = c, k = -\frac{c^2}{2}, r = \frac{c^2(8 + 3c^2)}{12}, a_0 = -\sqrt{2}c^2, a_1 = 6\sqrt{2}c^2, \\ a_2 = -6\sqrt{2}c^2, b_0 = -\frac{2c^2}{3}, b_1 = 6c^2, b_2 = -6c^2.$$

For **case 6**, we have the following solution

$$\Phi_{61} = -\sqrt{2}c^2 + \frac{6\sqrt{2}c^2(\cosh(\zeta) + \sinh(\zeta))}{1 + \cosh(\zeta) + \sinh(\zeta)} - \frac{6\sqrt{2}c^2(\cosh(2\zeta) + \sinh(2\zeta))}{(1 + \cosh(\zeta) + \sinh(\zeta))^2}, \\ \Psi_{61} = -\frac{2c^2}{3} + \frac{6c^2(\cosh(\zeta) + \sinh(\zeta))}{1 + \cosh(\zeta) + \sinh(\zeta)} - \frac{6c^2(\cosh(2\zeta) + \sinh(2\zeta))}{(1 + \cosh(\zeta) + \sinh(\zeta))^2}, \quad (4.13)$$

where $\zeta = cx + \frac{\gamma t^\alpha}{\Gamma(\alpha+1)}$.

Case 7:

$$c = c, k = \frac{c^2}{2}, r = \frac{c^2(4 + c^2)}{4}, a_0 = -\sqrt{2}c^2, a_1 = 6\sqrt{2}c^2, \\ a_2 = -6\sqrt{2}c^2, b_0 = -c^2, b_1 = 6c^2, b_2 = -6c^2.$$

For **case 7**, we have the following solution

$$\Phi_{71} = -\sqrt{2}c^2 + \frac{6\sqrt{2}c^2(\cosh(\zeta) + \sinh(\zeta))}{1 + \cosh(\zeta) + \sinh(\zeta)} - \frac{6\sqrt{2}c^2(\cosh(2\zeta) + \sinh(2\zeta))}{(1 + \cosh(\zeta) + \sinh(\zeta))^2}, \\ \Psi_{71} = -c^2 + \frac{6c^2(\cosh(\zeta) + \sinh(\zeta))}{1 + \cosh(\zeta) + \sinh(\zeta)} - \frac{6c^2(\cosh(2\zeta) + \sinh(2\zeta))}{(1 + \cosh(\zeta) + \sinh(\zeta))^2}, \quad (4.14)$$

where $\zeta = cx + \frac{\gamma t^\alpha}{\Gamma(\alpha+1)}$.

Case 8:

$$c = c, k = \frac{c^2}{2}, r = \frac{c^2(-8 + 3c^2)}{12}, a_0 = 0, a_1 = 6\sqrt{2}c^2, \\ a_2 = -6\sqrt{2}c^2, b_0 = -\frac{c^3}{3}, b_1 = 6c^2, b_2 = -6c^2.$$

For **case 8**, we have the following solution

$$\Phi_{81} = \frac{6\sqrt{2}c^2(\cosh(\zeta) + \sinh(\zeta))}{1 + \cosh(\zeta) + \sinh(\zeta)} - \frac{6\sqrt{2}c^2(\cosh(2\zeta) + \sinh(2\zeta))}{(1 + \cosh(\zeta) + \sinh(\zeta))^2}, \\ \Psi_{81} = -\frac{c^2}{3} + \frac{6c^2(\cosh(\zeta) + \sinh(\zeta))}{1 + \cosh(\zeta) + \sinh(\zeta)} - \frac{6c^2(\cosh(2\zeta) + \sinh(2\zeta))}{(1 + \cosh(\zeta) + \sinh(\zeta))^2}, \quad (4.15)$$

where $\zeta = cx + \frac{\gamma t^\alpha}{\Gamma(\alpha+1)}$.

4.2. Exact solutions for time-fractional coupled SB equation

The newly proposed method has been used here for getting the exact solutions for eqs. (1.1). By using the fractional complex transform (3.2) in eqs. (1.1), we have

the following nonlinear ODE:

$$-\left(r\varepsilon + \frac{3}{2}k^2\right)\Phi(\zeta) + \frac{3}{2}c^2\Phi''(\zeta) - \frac{1}{2}\Phi(\zeta)\Psi(\zeta) + i(r\varepsilon + 3kc)\Phi'(\zeta) = 0, \quad (4.16)$$

$$(\gamma^2 - c^2)\Psi''(\zeta) - c^4\Psi^{IV}(\zeta) - c^2(\Psi^2(\zeta))'' - \frac{1}{4}c^2(\Phi^2(\zeta))'' = 0. \quad (4.17)$$

Again the eq. (4.16) can be written as

$$-(2r\varepsilon + 3k^2)\Phi(\zeta) + 3c^2\Phi''(\zeta) - \Phi(\zeta)\Psi(\zeta) = 0, \quad (4.18)$$

where $\gamma = -\frac{3kc}{\varepsilon}$.

By integrating eq. (4.17) twice with respect to ζ and putting $\gamma = -\frac{3kc}{\varepsilon}$, we have

$$\left(\left(-\frac{3kc}{\varepsilon}\right)^2 - c^2\right)\Psi(\zeta) - c^4\Psi''(\zeta) - c^2\Psi^2(\zeta) - \frac{1}{4}c^2\Phi^2(\zeta) = 0. \quad (4.19)$$

Let

$$\Phi(\zeta) = a_0 + \sum_{i=1}^n a_i\phi^i \text{ and } \Psi(\zeta) = b_0 + \sum_{i=1}^m b_i\phi^i. \quad (4.20)$$

The dominant terms with highest order singularity of eq. (4.18) are $\Phi(\zeta)\Psi(\zeta)$ and $3c^2\Phi''(\zeta)$. The maximum value of pole is 2 that means here $n = 2$. Similarly the dominant terms with highest order singularity in eq. (4.19) are $c^4\Psi''(\zeta)$ and $c^2\Psi^2(\zeta)$. The maximum value of pole is 2 that means here $m = 2$.

Therefore by eq. (4.20), we have the following ansatz:

$$\Phi(\zeta) = a_0 + a_1\phi + a_2\phi^2 \text{ and } \Psi(\zeta) = b_0 + b_1\phi + b_2\phi^2, \quad (4.21)$$

where ϕ satisfies eq. (3.5).

Substituting eq. (4.21) along with eqs. (3.6) into eqs. (4.18) and (4.19), then equating each coefficients of ϕ^i ($i = 0, 1, 2, \dots$) to zero, we can find a system of algebraic equations for $a_0, a_1, a_2, b_0, b_1, b_2, c, k$ and r as follows:

$$\begin{aligned} \phi^0 : & -a_0b_0 + a_0(3k^2 + 2\varepsilon r) = 0; \\ & : -\frac{1}{4}a_0^2c^2 - b_0^2c^2 + b_0\left(-c^2 + \frac{9c^2k^2}{\varepsilon^2}\right) = 0; \\ \phi^1 : & -a_1b_0 - a_0b_1 - 3a_1c^2 + a_1(3k^2 + 2\varepsilon r) = 0; \\ & : -\frac{1}{2}a_0a_1c^2 - 2b_0b_1c^2 + c^4b_1 + b_1\left(-c^2 + \frac{9c^2k^2}{\varepsilon^2}\right) = 0; \\ \phi^2 : & -a_2b_0 - a_1b_1 - a_0b_2 - 9a_1c^2 + 12a_2c^2 + a_2(3k^2 + 2\varepsilon r) = 0; \\ & : -\frac{1}{4}a_1^2c^2 - \frac{1}{2}a_0a_2c^2 - b_1^2c^2 - 2b_0b_2c^2 + 3c^4b_1 - 4c^4b_2 + b_2\left(-c^2 + \frac{9c^2k^2}{\varepsilon^2}\right) = 0; \\ \phi^3 : & -a_2b_1 - a_1b_2 + 6a_1c^2 - 30a_2c^2 = 0; \\ & : -\frac{1}{2}a_1a_2c^2 - 2b_1b_2c^2 - 2c^4b_1 + 10c^4b_2 = 0; \\ \phi^4 : & -a_2b_2 + 18a_2c^2 = 0; \\ & : -\frac{1}{4}a_2^2c^2 - b_2^2c^2 - 6c^4b_2 = 0. \end{aligned} \quad (4.22)$$

Solving the above algebraic eqs. (4.22), we have the following sets of coefficients for the solutions of eqs. (4.18) and (4.19) as given below:

Case 1:

$$c = c, k = -\frac{1}{3}\sqrt{1+c^2}\varepsilon, r = \frac{-9c^2 - \varepsilon^2 - c^2\varepsilon^2}{6\varepsilon},$$

$$a_0 = 0, a_1 = -24i\sqrt{3}c^2, a_2 = 24i\sqrt{3}c^2, b_0 = 0, b_1 = -18c^2, b_2 = 18c^2.$$

For **case 1**, we have the following solution

$$\Phi_{11} = -\frac{24i\sqrt{3}c^2(\cosh(\zeta) + \sinh(\zeta))}{1 + \cosh(\zeta) + \sinh(\zeta)} + \frac{24i\sqrt{3}c^2(\cosh(2\zeta) + \sinh(2\zeta))}{(1 + \cosh(\zeta) + \sinh(\zeta))^2},$$

$$\Psi_{11} = -\frac{18c^2(\cosh(\zeta) + \sinh(\zeta))}{1 + \cosh(\zeta) + \sinh(\zeta)} + \frac{18c^2(\cosh(2\zeta) + \sinh(2\zeta))}{(1 + \cosh(\zeta) + \sinh(\zeta))^2},$$
(4.23)

where $\zeta = cx + \frac{\gamma t^\alpha}{\Gamma(\alpha+1)}$.

Case 2:

$$c = c, k = -\frac{1}{3}\sqrt{1+c^2}\varepsilon, r = \frac{-12c^2 - \varepsilon^2 + c^2\varepsilon^2}{6\varepsilon},$$

$$a_0 = 0, a_1 = -24i\sqrt{3}c^2, a_2 = 24i\sqrt{3}c^2, b_0 = -c^2, b_1 = -18c^2, b_2 = 18c^2.$$

For **case 2**, we have the following solution

$$\Phi_{21} = -\frac{24i\sqrt{3}c^2(\cosh(\zeta) + \sinh(\zeta))}{1 + \cosh(\zeta) + \sinh(\zeta)} + \frac{24i\sqrt{3}c^2(\cosh(2\zeta) + \sinh(2\zeta))}{(1 + \cosh(\zeta) + \sinh(\zeta))^2},$$

$$\Psi_{21} = -c^2 - \frac{18c^2(\cosh(\zeta) + \sinh(\zeta))}{1 + \cosh(\zeta) + \sinh(\zeta)} + \frac{18c^2(\cosh(2\zeta) + \sinh(2\zeta))}{(1 + \cosh(\zeta) + \sinh(\zeta))^2},$$
(4.24)

where $\zeta = cx + \frac{\gamma t^\alpha}{\Gamma(\alpha+1)}$.

Case 3:

$$c = c, k = -\frac{1}{3}\sqrt{1+c^2}\varepsilon, r = \frac{-9c^2 - \varepsilon^2 - c^2\varepsilon^2}{6\varepsilon},$$

$$a_0 = 0, a_1 = 24i\sqrt{3}c^2, a_2 = -24i\sqrt{3}c^2, b_0 = 0, b_1 = -18c^2, b_2 = 18c^2.$$

For **case 3**, we have the following solution

$$\Phi_{31} = \frac{24i\sqrt{3}c^2(\cosh(\zeta) + \sinh(\zeta))}{1 + \cosh(\zeta) + \sinh(\zeta)} - \frac{24i\sqrt{3}c^2(\cosh(2\zeta) + \sinh(2\zeta))}{(1 + \cosh(\zeta) + \sinh(\zeta))^2},$$

$$\Psi_{31} = -\frac{18c^2(\cosh(\zeta) + \sinh(\zeta))}{1 + \cosh(\zeta) + \sinh(\zeta)} + \frac{18c^2(\cosh(2\zeta) + \sinh(2\zeta))}{(1 + \cosh(\zeta) + \sinh(\zeta))^2},$$
(4.25)

where $\zeta = cx + \frac{\gamma t^\alpha}{\Gamma(\alpha+1)}$.

Case 4:

$$c = c, k = -\frac{1}{3}\sqrt{1-c^2}\varepsilon, r = \frac{-12c^2 - \varepsilon^2 + c^2\varepsilon^2}{6\varepsilon},$$

$$a_0 = 0, a_1 = 24i\sqrt{3}c^2, a_2 = -24i\sqrt{3}c^2, b_0 = -c^2, b_1 = -18c^2, b_2 = 18c^2.$$

For **case 4**, we have the following solution

$$\begin{aligned}\Phi_{41} &= \frac{24i\sqrt{3}c^2(\cosh(\zeta) + \sinh(\zeta))}{1 + \cosh(\zeta) + \sinh(\zeta)} - \frac{24i\sqrt{3}c^2(\cosh(2\zeta) + \sinh(2\zeta))}{(1 + \cosh(\zeta) + \sinh(\zeta))^2}, \\ \Psi_{41} &= -c^2 - \frac{18c^2(\cosh(\zeta) + \sinh(\zeta))}{1 + \cosh(\zeta) + \sinh(\zeta)} + \frac{18c^2(\cosh(2\zeta) + \sinh(2\zeta))}{(1 + \cosh(\zeta) + \sinh(\zeta))^2},\end{aligned}\quad (4.26)$$

where $\zeta = cx + \frac{\gamma t^\alpha}{\Gamma(\alpha+1)}$.

Case 5:

$$\begin{aligned}c &= c, k = -\frac{1}{3}\sqrt{1 + c^2}\varepsilon, r = \frac{12c^2 - \varepsilon^2 - c^2\varepsilon^2}{6\varepsilon}, \\ a_0 &= 4i\sqrt{3}c^2, a_1 = -24i\sqrt{3}c^2, a_2 = 24i\sqrt{3}c^2, b_0 = 4c^2, b_1 = -18c^2, b_2 = 18c^2.\end{aligned}$$

For **case 5**, we have the following solution

$$\begin{aligned}\Phi_{51} &= 4i\sqrt{3}c^2 - \frac{24i\sqrt{3}c^2(\cosh(\zeta) + \sinh(\zeta))}{1 + \cosh(\zeta) + \sinh(\zeta)} + \frac{24i\sqrt{3}c^2(\cosh(2\zeta) + \sinh(2\zeta))}{(1 + \cosh(\zeta) + \sinh(\zeta))^2}, \\ \Psi_{51} &= 4c^2 - \frac{18c^2(\cosh(\zeta) + \sinh(\zeta))}{1 + \cosh(\zeta) + \sinh(\zeta)} + \frac{18c^2(\cosh(2\zeta) + \sinh(2\zeta))}{(1 + \cosh(\zeta) + \sinh(\zeta))^2},\end{aligned}$$

where $\zeta = cx + \frac{\gamma t^\alpha}{\Gamma(\alpha+1)}$.

Case 6:

$$\begin{aligned}c &= c, k = -\frac{1}{3}\sqrt{1 - c^2}\varepsilon, r = \frac{9c^2 - \varepsilon^2 - c^2\varepsilon^2}{6\varepsilon}, \\ a_0 &= 4i\sqrt{3}c^2, a_1 = -24i\sqrt{3}c^2, a_2 = 24i\sqrt{3}c^2, b_0 = 3c^2, b_1 = -18c^2, b_2 = 18c^2.\end{aligned}$$

For **case 6**, we have the following solution

$$\begin{aligned}\Phi_{61} &= 4i\sqrt{3}c^2 - \frac{24i\sqrt{3}c^2(\cosh(\zeta) + \sinh(\zeta))}{1 + \cosh(\zeta) + \sinh(\zeta)} + \frac{24i\sqrt{3}c^2(\cosh(2\zeta) + \sinh(2\zeta))}{(1 + \cosh(\zeta) + \sinh(\zeta))^2}, \\ \Psi_{61} &= 3c^2 - \frac{18c^2(\cosh(\zeta) + \sinh(\zeta))}{1 + \cosh(\zeta) + \sinh(\zeta)} + \frac{18c^2(\cosh(2\zeta) + \sinh(2\zeta))}{(1 + \cosh(\zeta) + \sinh(\zeta))^2},\end{aligned}\quad (4.27)$$

where $\zeta = cx + \frac{\gamma t^\alpha}{\Gamma(\alpha+1)}$.

Case 7:

$$\begin{aligned}c &= c, k = -\frac{1}{3}\sqrt{1 + c^2}\varepsilon, r = \frac{12c^2 - \varepsilon^2 - c^2\varepsilon^2}{6\varepsilon}, \\ a_0 &= -4i\sqrt{3}c^2, a_1 = 24i\sqrt{3}c^2, a_2 = -24i\sqrt{3}c^2, b_0 = 4c^2, b_1 = -18c^2, b_2 = 18c^2.\end{aligned}$$

For **case 7**, we have the following solution

$$\begin{aligned}\Phi_{71} &= -4i\sqrt{3}c^2 + \frac{24i\sqrt{3}c^2(\cosh(\zeta) + \sinh(\zeta))}{1 + \cosh(\zeta) + \sinh(\zeta)} - \frac{24i\sqrt{3}c^2(\cosh(2\zeta) + \sinh(2\zeta))}{(1 + \cosh(\zeta) + \sinh(\zeta))^2}, \\ \Psi_{71} &= 4c^2 - \frac{18c^2(\cosh(\zeta) + \sinh(\zeta))}{1 + \cosh(\zeta) + \sinh(\zeta)} + \frac{18c^2(\cosh(2\zeta) + \sinh(2\zeta))}{(1 + \cosh(\zeta) + \sinh(\zeta))^2},\end{aligned}$$

where $\zeta = cx + \frac{\gamma t^\alpha}{\Gamma(\alpha+1)}$.

Case 8:

$$c = c, k = -\frac{1}{3}\sqrt{1 - c^2}\varepsilon, r = \frac{9c^2 - \varepsilon^2 + c^2\varepsilon^2}{6\varepsilon},$$

$$a_0 = -4i\sqrt{3}c^2, a_1 = 24i\sqrt{3}c^2, a_2 = -24i\sqrt{3}c^2, b_0 = 3c^2, b_1 = -18c^2, b_2 = 18c^2.$$

For **case 8**, we have the following solution

$$\Phi_{81} = -4i\sqrt{3}c^2 + \frac{24i\sqrt{3}c^2(\cosh(\zeta) + \sinh(\zeta))}{1 + \cosh(\zeta) + \sinh(\zeta)} - \frac{24i\sqrt{3}c^2(\cosh(2\zeta) + \sinh(2\zeta))}{(1 + \cosh(\zeta) + \sinh(\zeta))^2},$$

$$\Psi_{81} = 3c^2 - \frac{18c^2(\cosh(\zeta) + \sinh(\zeta))}{1 + \cosh(\zeta) + \sinh(\zeta)} + \frac{18c^2(\cosh(2\zeta) + \sinh(2\zeta))}{(1 + \cosh(\zeta) + \sinh(\zeta))^2},$$
(4.28)

where $\zeta = cx + \frac{\gamma t^\alpha}{\Gamma(\alpha+1)}$.

5. Numerical simulations for time-fractional SK and SB equations

In this present section, we have presented the numerical simulations of time-fractional coupled SK and SB equations by newly proposed analytical method. Here the solutions presented in eqs. (4.8) and (4.23) have been used to draw the 3-D and the corresponding 2-D solution graphs for fractional coupled SK and coupled SB equations respectively.

5.1. Numerical simulations for time-fractional coupled SK equation

We have used here eqs. (4.8) for presenting the solution graphs for time-fractional SK equation in case of both classical and fractional orders.

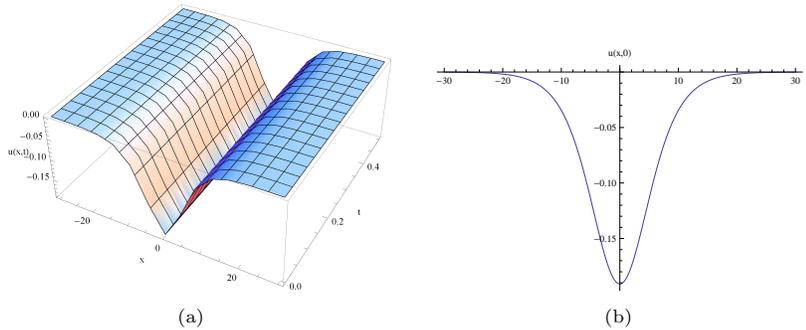


Figure 1. (a) The 3-D solitary wave graph for $u(x, t)$ appears in eq. (4.8) as Φ_{11} in **Case 1**, when $c = 0.3$ and $\alpha = 1$ (Classical order), (b) the corresponding 2-D graph for $u(x, t)$ when $t = 0$.

5.2. Numerical simulations for time-fractional coupled SB equation

We have used here eqs. (4.23) for presenting the solution graphs for time-fractional coupled SB equation in case of both classical and fractional orders. As the solution

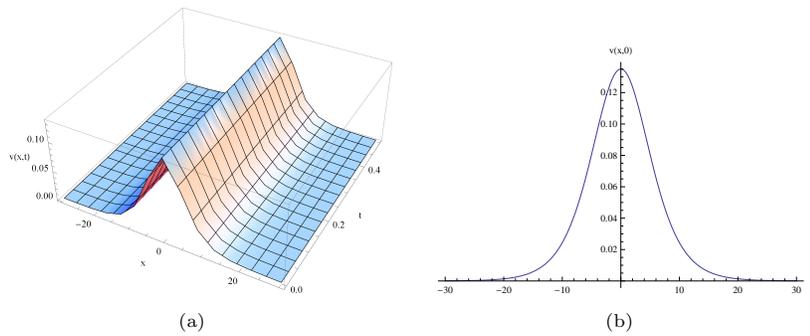


Figure 2. (a) The 3-D solitary wave graph for $v(x,t)$ appears in eq. (4.8) as Ψ_{11} in **Case 1**, when $c = 0.3$ and $\alpha = 1$ (Classical order), (b) the corresponding 2-D graph for $v(x,t)$ when $t = 0$.

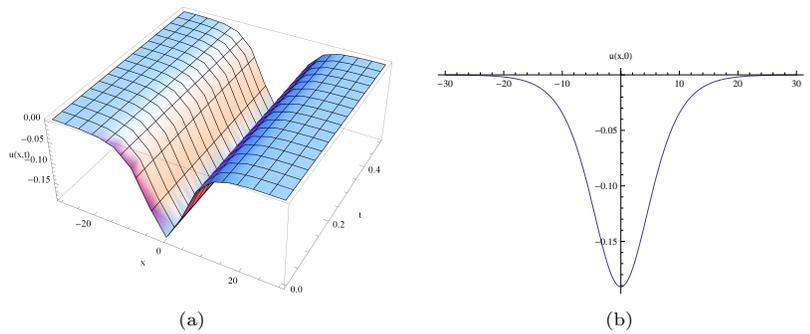


Figure 3. (a) The 3-D solitary wave graph for $u(x,t)$ appears in eq. (4.8) as Φ_{11} in **Case 1**, when $c = 0.3$ and $\alpha = 0.5$ (Fractional order), (b) the corresponding 2-D graph for $u(x,t)$ when $t = 0$.

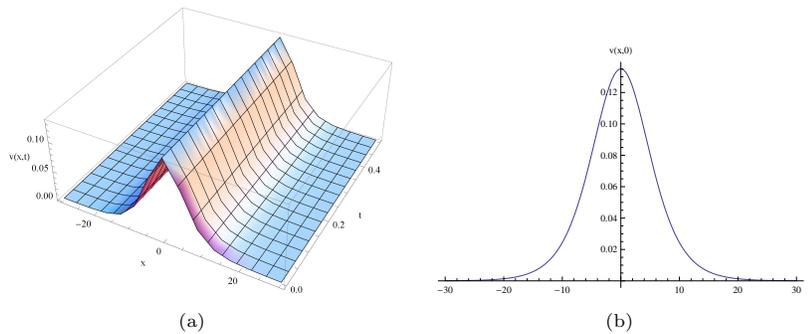


Figure 4. (a) The 3-D solitary wave graph for $v(x,t)$ appears in eq. (4.8) as Ψ_{11} in **Case 1**, when $c = 0.3$ and $\alpha = 0.5$ (Fractional order), (b) the corresponding 2-D graph for $v(x,t)$ when $t = 0$.

obtaining in eqs. (4.23) for $u(x,t)$ is complex in nature, we have taken here the absolute value of $u(x,t)$ for obtaining the 3-D and the corresponding 2-D graphs.

The present section contains the numerical simulations for both time-fractional coupled SK and SB equations. We have presented the solution graphs for time-fractional coupled SK and for time-fractional SB equations for both classical and fractional order in Sections 5.1 and 5.2 respectively. Figs. 1, 2, 3 and 4 show the evolution of the solitary wave solutions for eqs. (1.1) and Figs. 5, 6, 7 and 8 show

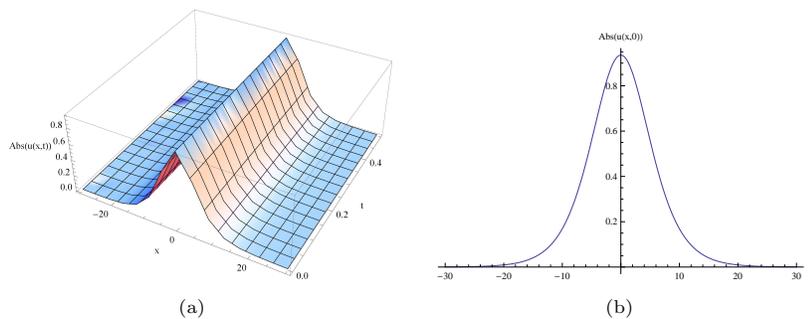


Figure 5. (a) The 3-D solitary wave graph for $Abs(u(x,t))$ appears in eq. (4.23) as Φ_{11} in **Case 1**, when $\varepsilon = 0.5$, $c = 0.3$ and $\alpha = 1$ (Classical order), (b) the corresponding 2-D graph for $Abs(u(x,t))$ when $t = 0$.

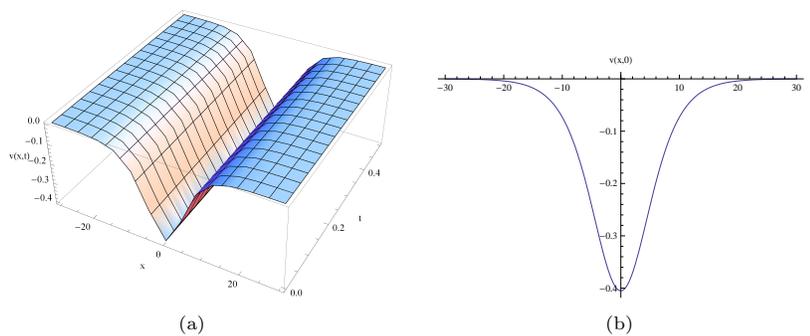


Figure 6. (a) The 3-D solitary wave graph for $v(x,t)$ appears in eq. (4.23) as Ψ_{11} in **Case 1**, when $c = 0.3$ and $\alpha = 1$ (Classical order), (b) the corresponding 2-D graph for $v(x,t)$ when $t = 0$.

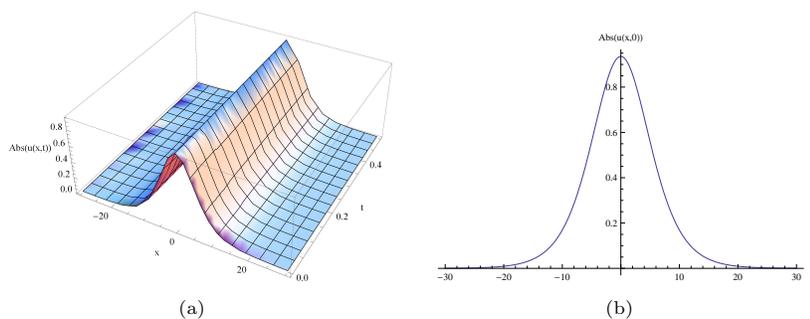


Figure 7. (a) The 3-D solitary wave graph for $Abs(u(x,t))$ appears in eq. (4.23) as Φ_{11} in **Case 1**, when $\varepsilon = 0.5$, $c = 0.3$ and $\alpha = 0.5$ (Fractional order), (b) the corresponding 2-D solution graph for $Abs(u(x,t))$ when $t = 0$.

the evolution of the solitary wave solutions for eqs. (1.2).

6. Conclusion

In the present article, we have implemented a new analytical method for getting exact solutions of time fractional time-fractional coupled SK and SB equations. We

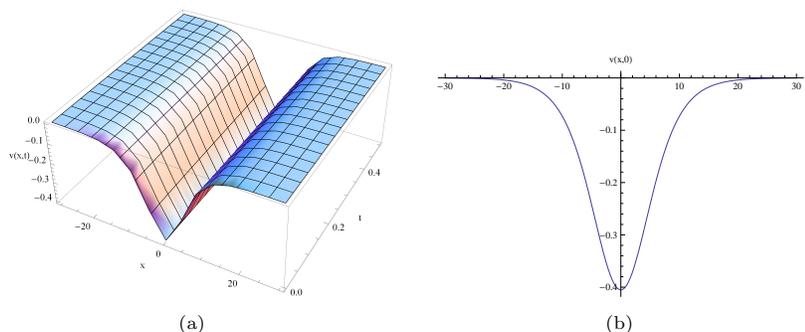


Figure 8. (a) The 3-D solitary wave solution graph for $v(x, t)$ appears in eq. (4.23) as Ψ_{11} in **Case 1**, when $c = 0.3$ and $\alpha = 0.5$ (Fractional order), (b) the corresponding 2-D graph for $v(x, t)$ when $t = 0$.

have used here fractional complex transform for transformation of the nonlinear fractional differential equations to nonlinear ordinary differential equations. The most essential interest of the newly proposed method is that it takes less computation for obtaining the exact solutions. The exact solutions obtained from newly proposed method have also been used here for presenting the numerical simulations. From the numerical simulations, we have analyzed the nature of solution in physical form as solitary waves. The newly proposed method is an effective and powerful technique for handling the fractional nonlinear differential equation to obtain the exact solutions.

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