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BIFURCATIONS IN A DELAYED DIFFERENTIAL-ALGEBRAIC PLANKTON ECONOMIC SYSTEM*

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Abstract This paper considers a phytoplankton-zooplankton bio-economic system with delay and harvesting, which is described by differential-algebraic equations. Local stability analysis of the system without delay reveals that a singularity-induced bifurcation phenomenon appears when a variation of the economic interest is taken into account, furthermore, a state feedback controller is designed to stabilize the system at the interior equilibrium. Then, we show that delay, which is considered in the toxic liberation, can induce stability switches, such that the positive equilibrium switches from stability to instability, to stability again and so on. Finally, some numerical simulations are performed to justify analytical findings.

Keywords Differential-algebraic equations, harvesting, delay, stability switches, singularity-induced bifurcation.

MSC(2010) 34C23, 34C25, 34G15, 34D20.

1. Introduction

The importance of plankton for the ocean ecosystems and ultimately for the planet itself is nowadays widely recognized [12]. Firstly, plankton is the basis of all aquatic food chain and eaten by terrestrial life. Secondly, plankton plays a key role in marine fisheries industries. Particularly, harmful algal blooms (HAB) are widely reported and have become a serious environmental problem world wide [1]. Therefore, a better understanding of mechanisms that determine the plankton dynamics is of considerable interest.

During the recent years, many authors [4,13,18,22] have studied the dynamics of phytoplankton-zooplankton models, and obtained complex dynamic behavior, such as stability of equilibrium, Hopf bifurcation, global Hopf-bifurcation, oscillations and so on. Quite a number of references [5, 9, 20, 23] have discussed persistence, chaos, periodic resonance and limit cycle of phytoplankton-zooplankton models. Most of these discussions on biological models are based on normal systems governed by differential equations or difference equations.

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In daily life, economic profit is a very important factor for governments, merchants and even every citizen, so it is necessary to research biological economic systems. Gordon [10] studies the effect of the harvest effort on the ecosystem from an economic perspective and proposed a algebraic equation which investigates the economic interest of the yield of the harvest effort: Net Economic Revenue(NER)=Total Revenue(TR)-Total Cost(TC). For the purpose of system modeling and analysis in biological economic systems, differential algebraic equations (DAEs) are considered as an essential tool. At present, differential-algebraic equations have been studied widely in power systems [2, 17], economic administration [15], mechanical engineering [3]. There are also several biological reports on differential algebraic equations [6, 16, 24, 25]. A lot of results have been obtained, such as local stability, optimal control, singularity induced bifurcation, feasibility regions, and so on. However, to our best knowledge, there are few reports on differential-algebraic equations in biological fields. This paper mainly studies a new biological economic system formulated by differential-algebraic equations. In what follows, we introduce the new biological economic system.

The basic model we consider is based on the following toxic phytoplankton and zooplankton system with harvesting

$$\begin{cases} \dot{P} = rP(1-\frac{P}{K}) - \frac{\beta PZ}{\alpha+P}, \\ \dot{Z} = \frac{\beta_1 PZ}{\alpha+P} - dZ - \frac{\rho PZ}{\alpha+P} - qZE. \end{cases} \tag{1.1}$$

where

1. (1) The variable P and Z are the density of phytoplankton population and zooplankton population at any instant of time t respectively;
2. (2) The parameter r is the intrinsic growth rate, d (>0) is the natural death rate of zooplankton and K is the environmental carrying capacity of population. The constat β (>0) is the maximum uptake rate for zooplankton species, β_1 (>0) denotes the ratio of biomass conversion (satisfying the obvious restriction $0<\beta_1<\beta$).
3. (3) The parameter ρ (>0) denotes the rate of toxic substances produced by per unit biomass of phytoplankton. It is assumed that $\beta_1<\rho$, that is, the ratio of biomass consumed by zooplankton is greater than the rate of toxic substance liberation by phytoplankton species. The constant q is the catchability coefficients of the zooplankton, E is the effort used to harvest the population.
4. (4) The term $\frac{\beta PZ}{\alpha+P}$ represents the functional response for the grazing of phytoplankton by zooplankton and α is the half saturation constant for a Holling type II functional response [26]. $\frac{\rho PZ}{\alpha+P}$ describes the distribution of toxic substance which ultimately contributes to the death of zooplankton populations.

The model (1.1) is no harvesting was considered by Chattopadhyay et al. [7] and Saha and Bandyopadhyay [18].

Gordon [10] studies the effect of the harvest effort on the ecosystem from an economic perspective and proposed the following equation which investigates the economic interest of the yield of the harvest effort:

$$\text{Net Economic Revenue(NER)} = \text{Total Revenue(TR)} - \text{Total Cost(TC)}.$$

Let $E(t)$ represents the harvest effort, $TR = pqZ(t)E(t)$ and $TC = cE(t)$, c is the constant fishing cost per unit effort, p is the constant price per unit biomass of harvested zooplankton population. Associated with the model system (1.1), an algebraic equation, which considers the net economic revenue m of the harvest effort on zooplankton population, is established as follows:

$$(pqZ - c)E - m = 0 \tag{1.2}$$

We assume that the liberation of toxic substances by phytoplankton species is not an instantaneous process, but is mediated by some time lag which is required for maturity of toxic-phytoplankton. If we assume τ is the discrete time delay, using (1.2), then the system (1.1) can be extended as a delay differential-algebraic equation as follows

$$\begin{cases} \dot{P} = rP(1 - \frac{P}{K}) - \frac{\beta PZ}{\alpha + P}, \\ \dot{Z} = \frac{\beta_1 PZ}{\alpha + P} - dZ - \frac{\rho P(t-\tau)Z}{\alpha + P(t-\tau)} - qZE, \\ (pqZ - c)E - m = 0. \end{cases} \tag{1.3}$$

The differential-algebraic model system (1.3) can be expressed in the following form

$$A(t)\dot{X}(t) = G(P(t), Z(t), E(t)). \tag{1.4}$$

where

$$X(t) = (P(t), Z(t), E(t))^T,\ A(t) = \begin{pmatrix} 1\ 0\ 0 \\ 0\ 1\ 0 \\ 0\ 0\ 0 \end{pmatrix},$$

$$G(P(t), Z(t), E(t)) = \begin{pmatrix} rP(1 - \frac{P}{K}) - \frac{\beta PZ}{\alpha + P} \\ \frac{\beta_1 PZ}{\alpha + P} - dZ - \frac{\rho P(t-\tau)Z}{\alpha + P(t-\tau)} - qZE \\ (pqZ - c)E - m \end{pmatrix}.$$

The algebraic equation in the differential-algebraic model system(1.4) contains no differentiated variables, hence the leading matrix $A(t)$ in model system (1.4) has a corresponding zero row.

The organization of this paper is as follows: To begin with, we construct and briefly describe our model. In Section 2, we investigate the local stability analysis of the model, especially the phenomena of singularity induced bifurcation and Hopf bifurcation, and some numerical simulations are provided to support the analytical findings. Then, based on the normal-form theory and center manifold theorem introduced by Hassard et al. [11], we also derive the formula for determining the properties of the Hopf bifurcation of system (1.3). We end the paper with a brief conclusion and discussion in Section 3.

2. Local stability analysis

Considering the effects of harvest effort on zooplankton species and toxic liberation delay, the dynamical behaviors of the system (1.3) will be investigated by using the differential-algebraic system theory and bifurcation theory. Based on the local stability analysis of the system, the bifurcation phenomena which can reveal the instability mechanism of the system and their biological interpretations are investigated and discussed. In order to make zooplankton, phytoplankton and harvest effort on zooplankton all exist, particularly we care more about the positive interior equilibrium. The objective is to maintain the sustainable development of the zooplankton-phytoplankton ecosystem as well as keep the economic interest of harvesting at an ideal level.

2.1. Case I: The discrete time delay $\tau = 0$

In this subsection, the local stability of the differential-algebraic system (1.3) without time delay at the interior equilibrium is investigated. we get the occurrence of singularity-induced bifurcation by the variation of economic interest of harvest effort on zooplankton. Furthermore, a state feedback controller is designed to stabilize the system at the interior equilibrium.

2.1.1. Singularity induced bifurcation

Theorem 2.1. *The differential-algebraic system (1.3) without discrete time delay has a singularity induced bifurcation at the interior equilibrium when bifurcation value m increase through zero. Furthermore, the stability of the interior equilibrium point changes from being stable to being unstable.*

Proof. According to the economic theory of a common-property resource [10], there is a phenomenon of bio-economic equilibrium when the economic interest of harvesting is zero, i.e. $m = 0$. For the model system (1.3) without discrete time delay, an interior equilibrium $S^*(P^*, Z^*, E^*)$ can be obtained in the case of phenomenon of bio-economic equilibrium, where $Z^* = \frac{c}{pq}, E^* = \frac{(\beta_1-\rho)P^*}{q(\alpha+P^*)} - \frac{d}{q}$, and P^* is the solution of

$$(P^*)^2 + (\alpha - K)P^* - K\alpha + \frac{K\beta Z^*}{r} = 0. \tag{2.1}$$

If the condition $\beta c - \alpha rpq < 0$ is satisfied, then we get the unique positive solution of P^* from Eq.(2.1), where

$$P^* = \frac{1}{2}(K - \alpha + \sqrt{\Delta}), \qquad \Delta = (\alpha - K)^2 - 4(-K\alpha + \frac{K\beta Z^*}{r}). \tag{2.2}$$

According to the biological interpretation of the interior equilibrium, it follows that $P^* > 0, Z^* > 0, E^* > 0$. In order to guarantee the existence of interior equilibrium, some inequalities are satisfied,

$$\begin{cases} \beta c - \alpha rpq < 0, \\ P^* > \frac{d\alpha}{\beta_1-\rho-d}. \end{cases} \tag{2.3}$$

Let m be a bifurcation parameter, D be a differential operator, $H(t) = (P(t), Z(t))^T$,

$$h(H(t), E(t), m) = \begin{pmatrix} rP(1 - \frac{P}{K}) - \frac{\beta PZ}{\alpha+P} \\ \frac{\beta_1 PZ}{\alpha+P} - dZ - \frac{\rho P(t-\tau)Z}{\alpha+P(t-\tau)} - qZE \end{pmatrix}$$

and

$$g(H(t), E(t), m) = (pqZ - c)E - m.$$

It is evident that $D_E g = pqZ - c$ has a simple zero eigenvalue. Let us now define

$$\Delta(H(t), E(t), m) = D_E g = pqZ - c.$$

(i) From the existence of $S^*(P^*, Z^*, E^*)$, it follows that

$$Trace[D_E h[adj(D_E g)] D_H g]_{S^*} = -pq^2 Z^* E^* \neq 0.$$

(ii) Furthermore, it can be also calculated that

$$\begin{vmatrix} D_H h \ D_E h \\ D_H g \ D_E g \end{vmatrix}_{S^*} = \begin{vmatrix} r - \frac{2rP}{K} - \frac{\beta\alpha Z}{(\alpha+P)^2} & -\frac{\beta P}{\alpha+P} & 0 \\ \frac{(\beta_1-\rho)\alpha Z}{(\alpha+P)^2} & 0 & -qZ \\ 0 & pqE & 0 \end{vmatrix}_{S^*}$$
$$= pq^2 Z^* E^* (r - \frac{2rP^*}{K} - \frac{\beta\alpha Z^*}{(\alpha+P^*)^2}),$$

if $r - \frac{2rP^*}{K} - \frac{\beta\alpha Z^*}{(\alpha+P^*)^2} = 0$, then $K = \alpha + 2P^*$, because of $2P^* > K - \alpha$, Hence

$$\begin{vmatrix} D_H h \ D_E h \\ D_H g \ D_E g \end{vmatrix}_{S^*} \neq 0.$$

(iii) It can also be shown that

$$\begin{vmatrix} D_H h & D_E h & D_m h \\ D_H g & D_E g & D_m g \\ D_H \Delta & D_E \Delta & D_m \Delta \end{vmatrix}_{S^*} = \begin{vmatrix} r - \frac{2rP}{K} - \frac{\beta\alpha Z}{(\alpha+P)^2} & -\frac{\beta P}{\alpha+P} & 0 & 0 \\ \frac{(\beta_1-\rho)\alpha Z}{(\alpha+P)^2} & 0 & -qZ & 0 \\ 0 & pqE & 0 & -1 \\ 0 & pq & 0 & 0 \end{vmatrix}_{S^*}$$
$$= pq^2 Z^* E^* (r - \frac{2rP^*}{K} - \frac{\beta\alpha Z^*}{(\alpha+P^*)^2}) \neq 0.$$

The conditions for singularity induced bifurcation, which is introduced in Section III (A) of the reference [19], are all satisfied, hence the differential-algebraic model system (1.3) without discrete time delay has a singularity induced bifurcation when the bifurcation value $m = 0$.

On the other hand, we get

$$C_1 = -Trace[D_E h[adj(D_E g)] D_H g]_{S^*}] = pq^2 Z^* E^* \neq 0,$$
$$C_2 = \left[D_m \Delta - (D_H \Delta, D_E \Delta) \begin{pmatrix} D_H h \ D_E h \\ D_H g \ D_E g \end{pmatrix}^{-1} \begin{pmatrix} D_m h \\ D_m g \end{pmatrix} \right]_{S^*} = \frac{1}{E^*}.$$

Therefore, the existence of an interior equilibrium point implies that

$$\frac{C_1}{C_2} = pq^2 Z^* (E^*)^2 > 0.$$

The above inequality satisfies the Theorem 3 of reference [19]. According to the Theorem 3 of [19], when m increases through 0, one eigenvalue (denoted by λ_1) of the differential-algebraic model system (1.3) without discrete time delay moves from C^- to C^+ along the real axis by diverging through infinity, the movement behavior of this eigenvalue influences the stability of the differential-algebraic model system (1.3) without discrete time delay. □

Since the Jacobian of the differential-algebraic model system (1.3) without discrete time delay evaluated at S^* takes the form as follows:

$$J_{S^*} = \begin{pmatrix} r - \frac{2rP^*}{K} - \frac{\beta\alpha Z^*}{(\alpha+P^*)^2} & -\frac{\beta P^*}{\alpha+P^*} & 0 \\ \frac{(\beta_1-\rho)\alpha Z^*}{(\alpha+P^*)^2} & 0 & -qZ^* \\ 0 & pqE^* & 0 \end{pmatrix}$$

according to the leading matrix $A(t)$ in the model system (1.4) and J_{S^*}, the characteristic equation of the differential-algebraic model system (1.3) without discrete time delay at J_{S^*} is $\det(\lambda A - J_{S^*}) = 0$. By virtue of simple computation, the characteristic equation can be expressed as follows:

$$\lambda_2 = r - \frac{2rP^*}{K} - \frac{\beta\alpha Z^*}{(\alpha+P^*)^2} = \frac{rP^*(K-\alpha-2P^*)}{K(\alpha+P^*)} < 0.$$

Furthermore, the expressions (i)-(iii) satisfy the Theorem 1.1 of reference [21]. It follows from the Theorem 1.1 in the reference [21] that there is only one eigenvalue diverging to infinity as m increases through 0, and the other eigenvalue is continuous, nonzero and can not jump from on half open complex plane to another one as m increases through 0. It has been shown that λ_1 moves from C^- to C^+ along the real axis by diverging through infinity. However, λ_2 is continuous and bounded in the C^- half plane as m increases through 0 and its movement behavior has no influence on the stability of differential-algebraic model system (1.3) without discrete time delay at the interior equilibrium S^*.

Remark 2.1. (I) Local stability analysis of singular system often result in a new type of bifurcation phenomenon: singularity induced bifurcation (SIB). For the differential-algebraic equations with a one dimensional parameter m mentioned in [19], SIB occurs when an equilibrium point crosses the following singular surface: $S := \{(H,E,m)|g(H,E,m) = 0, det[D_E g(H,E,m)] = 0\}$. At the SIB, the equilibrium point undergoes stability exchanges and one eigenvalue of the system Jacobian matrix $(D_H h - D_E(D_E g)^{-1} D_H g)$ becomes diverging to infinity when the Jacobian $D_E g$ is singular. One important implication of the occurrence of SIB is that it causes impulsive phenomenon, which might yield catastrophic consequences. (II) In biological terms, the impulse phenomenon of the ecosystem is usually connected with the rapid growth of species population which will be beyond the carrying capacity of environment after a period of time. This is disastrous for the plankton ecosystem, such as the Harmful Algal Blooms [12].

2.1.2. State feedback control for singularity induced bifurcation

Owing to the above theorem, it is clear that the differential algebraic system (1.3) becomes unstable when the economic interest of the harvesting is considered to be positive. Fishery agencies are more interested in the positive economic interest of the harvesting from the economic perspective of the fishery. Thus, to stabilize system (1.3) without discrete time delay at the interior equilibrium S^* in the case of positive economic interest, By reference [8], a state feedback controller $\omega(t) = u(E(t)-E^*)$ (u stands for the net feedback gain, E^* is the component of the interior equilibrium S^*) can be applied to stabilize the differential-algebraic system (1.3) without discrete time delay at S^*, Let us introduce the state feedback controller to system (1.3) and rewrite the system as follows:

$$\begin{cases} \dot{P} = rP(1-\frac{P}{K}) - \frac{\beta PZ}{\alpha+P}, \\ \dot{Z} = \frac{\beta_1 PZ}{\alpha+P} - dZ - \frac{\rho PZ}{\alpha+P} - qZE, \\ (pqZ-c)E - m + u(E(t)-E^*) = 0. \end{cases} \tag{2.4}$$

Theorem 2.2. *If the feedback gain u satisfies the inequality $u > \max\{u_1^*, u_2^*\}$, then differential-algebraic system (2.4) is stable at S^*. Where*

$$u_1^* = \frac{pq^2E^*Z^*}{\frac{2rP^*}{K} + \frac{\beta\alpha Z^*}{(\alpha+P^*)^2} - r}, \quad u_2^* = \frac{pq^2E^*Z^*(\frac{2rP^*}{K} + \frac{\beta\alpha Z^*}{(\alpha+P^*)^2} - r)(\alpha+P^*)^3}{(\beta_1-\rho)\alpha\beta P^*Z^*}.$$

Proof. The Jacobian of the differential-algebraic model system (2.4) evaluated at the interior equilibrium S^* takes the form,

$$\widetilde{J}_{S^*} = \begin{pmatrix} r - \frac{2rP^*}{K} - \frac{\beta\alpha Z^*}{(\alpha+P^*)^2} & -\frac{\beta P^*}{\alpha+P^*} & 0 \\ \frac{(\beta_1-\rho)\alpha Z^*}{(\alpha+P^*)^2} & 0 & -qZ^* \\ 0 & pqE^* & u \end{pmatrix}$$

the characteristic equation of the differential- algebraic model system (2.4) at S^* is $\det(\lambda A - \widetilde{J}_{S^*}) = 0$, which can be expressed as follows:

$$\lambda^2 + \omega_1\lambda + \omega_2 = 0,$$

where

$$\omega_1 = -(r - \frac{2rP^*}{K} - \frac{\beta\alpha Z^*}{(\alpha+P^*)^2}) - \frac{pq^2E^*Z^*}{u},$$
$$\omega_1 = \frac{pq^2E^*Z^*}{u}(r - \frac{2rP^*}{K} - \frac{\beta\alpha Z^*}{(\alpha+P^*)^2}) + \frac{(\beta_1-\rho)\alpha\beta P^*Z^*}{(\alpha+P^*)^3}.$$

By using the Routh-Hurwitz criteria [14], the sufficient and necessary condition for the stability of the differential-algebraic model system (2.4) at S^* is that the feedback gain u satisfies: $u > max\{u_1^*, u_2^*\}$.

Hence, it is possible to design a suitable controller function such that the singularity-induced bifurcation can be eliminated. Thus, the impulsive phenomenon of a sustainable ecosystem can also be removed. Again, the economic interest of fishery managers may be achieved using a suitably designed state feedback controller. □

For the parameter values $r = 8, k = 4, \alpha = 1, \beta = 1, \beta_1 = 0.4, \rho = 0.1, c = 2, p = 2.5, d = 0.1, m = 0$, the system (1.3) without delay has a positive equilibrium point

$S^*(3, 8, 1.25)$. In the neighbor of $m = 0$, two values of economic profit are chosen: when $m = -0.181$, the eigenvalues of the system (1.3) at S^* are $\lambda_1 = -1.4207, \lambda_2 = -4.6671$, when $m = 0.01$, the eigenvalues are $\lambda_1 = 32.3577, \lambda_2 = -4.791$. Therefore, it is clearly observed from the Table 1 that when m increases through zero, one eigenvalue of the characteristic polynomial of the system (1.3) without delay remains the same, but one eigenvalue of model moves from C^- to C^+ along the real axis by diverging through ∞. Hence, the stability of the system (1.3) without delay at the interior equilibrium point S^* changes from being stable to being unstable.

Table 1. Interior equilibrium and eigenvalues of system (1.3) without delay for different net revenue.

Net revenue	Interior equilibrium	eigenvalues
$m = -0.181$	$P^* = 3.100, Z^* = 7.380, E^* = 1.168$	$-1.4207, -4.6671$
$m = 0.01$	$P^* = 2.9947, Z^* = 8.0319, E^* = 1.249$	$32.3577, -4.791$

To stabilize the system (1.3) without delay in the case of positive economic interest, let us consider a state feedback controller of the form $\omega(t) = u(E(t) - E^*)$.

It is possible to evaluate the numerical value of the net feedback gain using Theorem 2.2. We compute $u > max\{0.056, 10\}$ from Theorem 2.2 for system (2.4). Let us take the numerical value of the net feedback gain as $u = 11$, using the system (2.4), and get the values shown in Table 2.

Table 2. Interior equilibrium and eigenvalues of system (2.4) for different net revenue.

Net revenue and state feedback gain	Interior equilibrium in the presence of a s-tate feedback controller	eigenvalues
$m = -0.181, u = 11$	$P^* = 2.7059, Z^* = 9.5915, E^* = 1.1905$	$-0.0129, -3.4840$
$m = 0.01, u = 11$	$P^* = 3.054, Z^* = 7.6698, E^* = 1.2600$	$-0.0004, -4.6602$

It is clearly observed that when m increases through zero, all the eigenvalues of the characteristic polynomial of system (2.4) remain negative, that is, the stability of system (2.4) may be resumed at the interior equilibrium point in the case of positive economic profit. Hence, any singularity-induced bifurcation can be eliminated from the differential algebraic model system (2.4) at the interior equilibrium point when the net economic profit increases through zero and is taken to be positive. The result is clearly shown in Fig. 1.

2.2. Case II: The discrete time delay $\tau > 0$

In this subsection, we shall now investigate the dynamics of delay system (1.3). a stability switch in the model system (1.3) due to variation of the delay $\tau > 0$ will be investigated in the case of positive economic interest of harvesting. Furthermore, a phenomenon of Hopf bifurcation is also studied.

For the system (1.3), let $\widetilde{S}^*(\widetilde{P}^*, \widetilde{Z}^*, \widetilde{E}^*)$ denote interior equilibrium of the model system (1.3) in the case of positive economic interest of harvesting, where $\widetilde{P}^* = \frac{(d+q\widetilde{E}^*)\alpha}{\beta_1-\rho-d-q\widetilde{E}^*}$, $\widetilde{Z}^* = \frac{r(K-\widetilde{P}^*)(\alpha+\widetilde{P}^*)}{\beta K}$, $\widetilde{E}^* = \frac{m}{pq\widetilde{Z}^*-c}$. it is easy to get the interior positive solution of $\widetilde{P}^*$, $\widetilde{Z}^*$, and $\widetilde{E}^*$, provided that

$$0 < \widetilde{P}^* < K, \quad \widetilde{Z}^* > \frac{c}{pq}, \quad 0 < \widetilde{E}^* < \frac{\beta_1 - \rho - d}{q}. \tag{H_1}$$

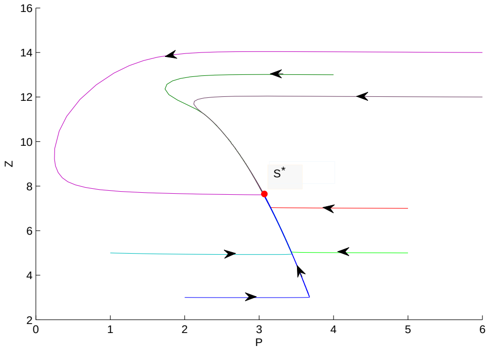


Figure 1. Phase plane trajectories of phytoplankton and zooplankton biomass beginning with different initial levels when $m = 0.01$ and $u = 11$. The trajectories clearly indicate that the interior equilibrium is asymptotically stable.

The following Jacobian matrix A_1 of the differential algebraic system (1.3) at the interior equilibrium point $\widetilde{S}^*(\widetilde{P}^*, \widetilde{Z}^*, \widetilde{E}^*)$, which is introduced in Appendix B of the reference [27], is found from the following variational matrix:

$$A_1 = D_H h - D_E(D_E g)^{-1} D_H g = \begin{pmatrix} a_{10} & a_{01} \\ b_{10} + c_{10}e^{-\lambda\tau} & b_{01} \end{pmatrix}, \tag{2.5}$$

where

$$a_{10} = r - \frac{2r\widetilde{P}^*}{K} - \frac{\beta\alpha\widetilde{Z}^*}{(\alpha + \widetilde{P}^*)^2}, \quad a_{01} = -\frac{\beta\widetilde{P}^*}{\alpha + \widetilde{P}^*},$$
$$b_{10} = \frac{\beta_1\alpha\widetilde{Z}^*}{(\alpha + \widetilde{P}^*)^2}, \quad c_{10} = -\frac{\rho\alpha\widetilde{Z}^*}{(\alpha + \widetilde{P}^*)^2}, \quad b_{01} = \frac{pq^2\widetilde{E}^*\widetilde{Z}^*}{pq\widetilde{Z}^* - c}.$$

The characteristic equation of delay system (1.3) is a transcendental equation of the following form:

$$G(\lambda, \tau) = \lambda^2 + A\lambda + B + Ce^{-\lambda\tau} = 0, \tag{2.6}$$

where

$$A = -(a_{10} + b_{01}), \quad B = a_{10}b_{01} - a_{01}b_{10}, \quad C = -a_{01}c_{10}.$$

It is easy to see that $\widetilde{S}^*$ is locally asymptotically stable for the system (1.3) when $\tau = 0$ if

$$A > 0, \quad B + C > 0. \tag{H_2}$$

At $\tau = 0$, $\widetilde{S}^*$ is locally asymptotically stable whenever the condition (H_2) is satisfied. Now $\widetilde{S}^*$ will be locally asymptotically stable for $\tau \geq 0$ if the real parts of all the roots of $G(\lambda, 0) = 0$ are negative and $G(i\omega, \tau) \neq 0$ for every real ω and $\tau \geq 0$. Assume that a purely imaginary solution of the form $\lambda = i\omega$ exists in the Equation (2.6), where ω is a positive real number. Substituting it into (2.6) and equating its real and imaginary parts give

$$\begin{cases} \omega^2 - B - C\cos\omega\tau = 0, \\ A\omega - C\sin\omega\tau = 0, \end{cases} \tag{2.7}$$

which leads to

$$\omega^4 + (A^2 - 2B)\omega^2 + B^2 - C^2 = 0. \tag{2.8}$$

Let us denote

$$\Delta = (A^2 - 2B)^2 - 4(B^2 - C^2).$$

Then the roots of biquadratic equation (2.8) are given by

$$\omega_{\pm}^2 = \frac{1}{2}[-(A^2 - 2B) \pm \sqrt{\Delta}].$$

Now as we need $\omega(> 0)$ as a real quantity, we have to consider the following cases

(i) $\Delta < 0$ imply no purely imaginary roots of the form $i\omega$,

(ii) $\Delta > 0, B^2 > C^2, A^2 - 2B > 0$, imply no purely imaginary roots of the form $i\omega$,

(iii) $\Delta > 0, B^2 < C^2, A^2 - 2B > 0$, imply one purely imaginary root $i\omega_+$,

(iv) $\Delta > 0, B^2 < C^2, A^2 - 2B < 0$, imply one purely imaginary root $i\omega_+$,

(v) $\Delta > 0, B^2 > C^2, A^2 - 2B < 0$, imply two purely imaginary roots $i\omega_{\pm}$.

From the condition (H_2), we have $B+C > 0$, since $B-C > B+C > 0$, i.e.,$B^2 > C^2$, the cases (iii) and (iv) can be excluded. For the cases (i) and (ii), it does not have any purely imaginary roots of the characteristic Eq.(2.6). This shows that the positive interior equilibrium point $\widetilde{S}^*$ is locally asymptotically stable for all $\tau \geq 0$ under the parametric restrictions (H_1) with the conditions (i) and (ii).

We now consider the case (v), in this case there are two purely imaginary roots given by $\omega_{\pm} = \sqrt{\frac{1}{2}[-(A^2 - 2B) \pm \sqrt{\Delta}]}$ where $0 < \omega_- < \omega_+$. From (2.7), we have

$$\begin{cases} M = \cos(\omega_{\pm}\tau) = \frac{-A^2 \pm \sqrt{\Delta}}{2C}, \\ N = \sin(\omega_{\pm}\tau) = \frac{A\omega_{\pm}}{C} < 0, \end{cases} \tag{2.9}$$

Thus

$$\tau_k^{\pm} = \begin{cases} \frac{1}{\omega_{\pm}}[2\pi + \arcsin\frac{A\omega_{\pm}}{C} + 2k\pi] \ M \geq 0, \\ \frac{1}{\omega_{\pm}}[\pi - \arcsin\frac{A\omega_{\pm}}{C} + 2k\pi] \ \ M < 0, \end{cases} \quad k = 0, 1, 2, \ldots. \tag{2.10}$$

It is easy to prove $\tau_0^+ < \tau_0^-$.

Lemma 2.1. *We have the following transversality conditions:*

$$sign\left[Re\left(\frac{d\lambda}{d\tau}\right)\right]_{\tau=\tau_k^+,\omega=\omega^+} > 0 \text{ and } sign\left[Re\left(\frac{d\lambda}{d\tau}\right)\right]_{\tau=\tau_k^-,\omega=\omega^-} < 0,$$

where $k = 0, 1, 2, \ldots$.

Proof. From (2.6) we obtain

$$(\frac{d\lambda}{d\tau})^{-1} = \frac{2\lambda + A}{-\lambda(\lambda^2 + A\lambda + B)} - \frac{\tau}{\lambda}.$$

Thus,

$$\begin{aligned} sign\left[Re\left(\frac{d\lambda}{d\tau}\right)\right]_{\lambda=i\omega} &= sign\left[Re\left(\frac{d\lambda}{d\tau}\right)^{-1}\right]_{\lambda=i\omega} \\ &= sign\left[\frac{\omega^2(A^2+2\omega^2-2B)}{\omega^4A^2+(\omega^3+\omega B)^2}\right]. \end{aligned}$$

And it can be easily verified that the following transversality conditions

$$sign\left[Re\left(\frac{d\lambda}{d\tau}\right)\right]_{\tau=\tau_k^+,\omega=\omega^+} > 0 \text{ and } sign\left[Re\left(\frac{d\lambda}{d\tau}\right)\right]_{\tau=\tau_k^-,\omega=\omega^-} < 0 \text{ hold.}$$

□

Now, summarizing the above results, and using the center manifold and normal form theories presented by Hassard et al. [11], whose details are given in Appendix A, we have the following theorem:

Theorem 2.3. *For the delayed system (1.3),*

(i) if $A > 0, B + C > 0$, either $\Delta < 0$ or $\Delta > 0$ and $A^2 - 2B > 0$, then $\widetilde{S}^*$ is locally asymptotically stable for all $\tau \geq 0$.

(i) if $A > 0, B+C > 0$, and $\Delta > 0, A^2-2B < 0$ then there exists a positive integer n, such that the equilibrium $\widetilde{S}^*$ switches n times from stability to instability to stability and so on, such that $\widetilde{S}^*$ is locally asymptotically stable whenever $\tau \in [0, \tau_0^+)\cup(\tau_0^-, \tau_1^+)\cup\cdots\cup(\tau_{n-1}^-, \tau_n^+)$ and is unstable whenever $\tau \in (\tau_0^+, \tau_0^-)\cup(\tau_1^+, \tau_1^-)\cup\cdots\cup(\tau_{n-1}^+, \tau_{n-1}^-)\cup(\tau_n^+, +\infty)$. The system (1.3) undergoes a Hopf-bifurcation around $\widetilde{S}^*$ for every $\tau = \tau_k^{\pm}$.

(1) If $\beta_2 > 0(\beta_2 < 0)$ when $\tau = \tau_k^+$ the bifurcating periodic solutions are backward (forward), and they are unstable (stable) on the center manifold.

(2) If $\beta_2 > 0(\beta_2 < 0)$ when $\tau = \tau_k^-$, the bifurcating periodic solutions are forward (backward), and they are unstable (stable) on the center manifold.

Let $r = 9, k = 1.6, \alpha = 1, \beta = 1, \beta_1 = 0.7, \rho = 0.1, q = 0.1, m = 1.2, p = 2.5, c = 0.1, d = 0.1$, the system (1.3) has a positive equilibrium point $P^* = 0.3415, Z^* = 9.4965, E^* = 0.5275$, As the conditions $\Delta > 0, B^2 > C^2, A^2 - 2B < 0$ are satisfied, we find two purely imaginary roots $i\omega_{\pm}$ with

$$\omega_+ = 1.0253, \quad \omega_- = 0.9103.$$

After some usual algebraic calculations one can find the minimum value of the delay parameter τ for which the stability behaviour changes and the first critical values are given by

$$\tau_k^+ = 3.5594 + 6.1281k,\ k = 0, 1, 2\cdots,\ sign\left[Re\left(\frac{d\lambda}{d\tau}\right)\right]_{\tau=\tau_k^+,\omega=\omega^+} > 0,$$

$$C_1(0)_{\omega^+} = -0.3696 - 3.3812i,$$

$$\tau_k^- = 6.4812 + 6.9712k,\ k = 0, 1, 2, \cdots,\ sign\left[Re\left(\frac{d\lambda}{d\tau}\right)\right]_{\tau=\tau_k^-,\omega=\omega^-} < 0.$$

$$C_1(0)_{\omega^-} = -0.2525 - 3.1947i.$$

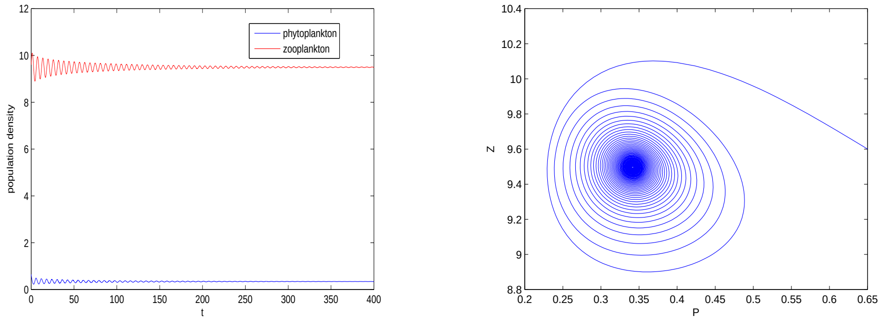
Especially, $\tau_0^+ = 3.5594 < \tau_0^- = 6.4812 < \tau_1^+ = 9.6875 < \tau_1^- = 13.4524 < \tau_2^+ = 15.8156 < \tau_2^- = 20.4236 < \tau_3^+ = 21.9437 < \tau_3^- = 27.3948 < \tau_4^+ = 28.0714 < \tau_5^+ = 34.1999 < \tau_4^- = 34.3660$. Therefore, by Theorems 2.3, we get the following theorem.

Theorem 2.4. *For system (1.3) with* $r = 9, k = 1.6, \alpha = 1, \beta = 1, \beta_1 = 0.7, \rho = 0.1, q = 0.1, m = 1.2, p = 2.5, c = 0.1, d = 0.1$,

(1) *Equilibrium* $\widetilde{S}^*$ *is stable when* $\tau \in [0, \tau_0^+) \cup (\tau_0^-, \tau_1^+) \cup (\tau_1^-, \tau_2^+) \cup (\tau_2^-, \tau_3^+) \cup (\tau_3^-, \tau_4^+)$.

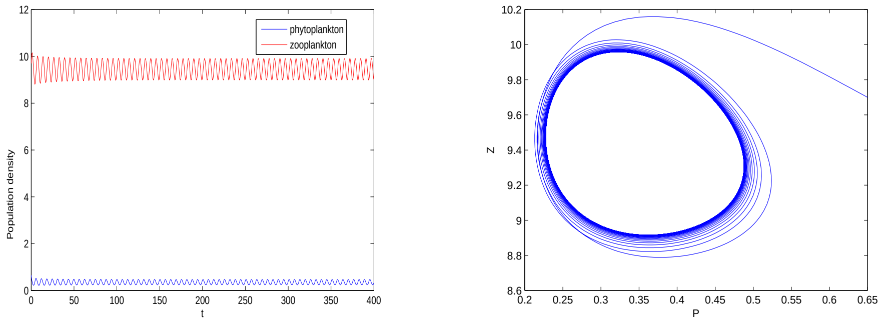
(2) *Equilibrium* $\widetilde{S}^*$ *is unstable when* $\tau \in (\tau_0^+, \tau_0^-) \cup (\tau_1^+, \tau_1^-) \cup (\tau_2^+, \tau_2^-) \cup (\tau_3^+, \tau_3^-) \cup (\tau_4^+, +\infty)$.

(3) *Hopf bifurcation occurs at equilibrium* $\widetilde{S}^*$ *when* $\tau = \tau_k^\pm$, *when* $\tau = \tau_0^+, \tau_1^+, \tau_2^+, \tau_3^+, \tau_4^+$ $(\tau = \tau_0^-, \tau_1^-, \tau_2^-, \tau_3^-)$ *the bifurcating periodic solutions are forward (backward), and they are stable.*



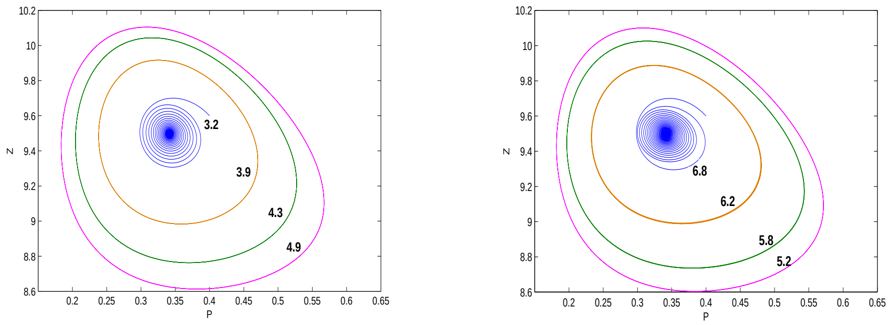
(a) (b)

Figure 2. **(a)** The trajectories of phytoplankton and zooplankton densities versus time when $\tau = 3.4 < \tau_0^+ = 3.5594$, **(b)** The equilibrium $\widetilde{S}^*$ is asymptotically stable.



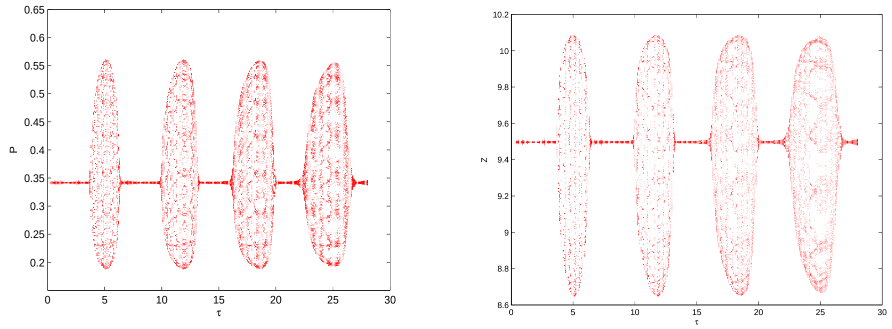
(a) (b)

Figure 3. **(a)** The trajectories of phytoplankton and zooplankton densities versus time when $\tau = 4 > \tau_0^+ = 3.5594$, **(b)** Hopf-bifurcating periodic orbit is stable.



(a) (b)

Figure 4. **(a)** when $\tau = 3.2 < \tau_0^+$, we have a stable $\widetilde{S}^*$, which looses stability via a Hopf bifurcation and the stable periodic solutions globally exist when $\tau = 3.9; 4.3; 4.9$, respectively. **(b)** when $\tau = 6.8 > \tau_0^-$, we have a stable $\widetilde{S}^*$, the stable periodic solutions globally exist when $\tau = 6.2; 5.8; 5.2$, respectively.



(a) (b)

Figure 5. The phytoplankton density P and zooplankton density Z with respect to time delay τ.

By we give numerical simulations to the (1.3) with different delays τ(see Figs. 2, 3.). It clear that the numerical simulations agree with the analytical predictions. The normal form method presented in this paper is for local dynamical behaviors, however, we concern that whether two families of stable periodic solutions can occur in the large region of delay. Therefore, we give the phytoplankton density P and zooplankton density Z with respect to time delay τ, respectively (see Figs. 4, 5). Surprisingly, we find that even for parameter values not chosen in the neighborhood of the Hopf critical points, two families of stable periodic solutions occur in a large region time delay τ. Namely, stable global bifurcation periodic exists.

3. Conclusion and Discussion

Considering the economic theory of fishery resource proposed by [10], a differential-algebraic system is established to investigate the effects of harvesting and toxic liberation delay of phytoplankton on the dynamic behaviors of the phytoplankton-zooplankton system in this paper.

The biological resources in the phytoplankton-zooplankton system are mostly harvested with the aim of achieving economic interest, the theoretical analysis show that the harvest effort on zooplankton influences the stability of the system, and an singular induced bifurcation occurs when economic interest of harvesting increases through 0. A direct damage done by the singular induced bifurcation to the system is impulse phenomenon, which may lead to the collapse of the phytoplankton-zooplankton ecosystem. With the purpose of maintaining ecological balance and sustainable development of the system, a feedback controller is designed which can stabilize the ecosystem to the interior equilibrium in the case of positive economic interest.

Another attempt is tried to understand the effect of toxic liberation delay of phytoplankton on the stability of the ecosystem. In section 2, we show the system enters into Hopf bifurcations and has stable periodic orbits around the coexisting equilibrium point as the time delay crosses its different threshold values $\tau_k^{\pm}$. This result indicates that there is some threshold limits of toxin liberation by the phytoplankton species below which the system does not have any excitable nature and above which the system shows excitability.

Generally speaking, toxic liberation delay of phytoplankton can be regarded as an inherence of biological population. Thus, it is easier to adjust economic profit than toxic liberation delay in order to eliminate bifurcation phenomenon and keep biological population stay at steady state, such as adjusting revenue, drawing out favorable policy to encourage or improve fishery and so on.

Although the study we have been has some theoretical and practical meanings in a certain extent, on the other hand, there still exist lots of work to do by the mathematical methods. For example, we have analyzed the existence and stability of the periodic solutions in small neighbourhoods of the critical values, and verify that the periodic solutions are global existence by numerical simulations, however, for the global continuation of local bifurcating periodic solutions, We leave it as a future work.

A. Properties of Hopf bifurcation

In this section, based on numerical evaluation of Hopf bifurcation (see Hassard et al. [[11], Chap. 3]), and using a similar computation process as in [11], we obtain the coefficients determining the important quantities:

$$\begin{aligned} g_{20} &= 2\bar{D}(\Delta_{11}+\bar{\rho}^*\Delta_{21}), \quad g_{11} = \bar{D}(\Delta_{12}+\bar{\rho}^*\Delta_{22}), \\ g_{02} &= 2\bar{D}(\Delta_{13}+\bar{\rho}^*\Delta_{23}), \quad g_{21} = 2\bar{D}(\Delta_{14}+\bar{\rho}^*\Delta_{24}). \end{aligned}$$

where

$$\begin{aligned} \Delta_{11} =& a_{20}+\rho a_{11}, \quad \Delta_{12} = 2a_{20}+(\rho+\bar{\rho})a_{11}, \quad \Delta_{13} = a_{20}+\bar{\rho}a_{11}, \\ \Delta_{14} =& a_{20}(W_{20}^1(0)+2W_{11}^1(0))+a_{11}(W_{11}^2(0)+\frac{1}{2}W_{20}^2(0)+\rho W_{11}^1(0)+\frac{1}{2}\bar{\rho}W_{20}^1(0)), \\ \Delta_{21} =& b_{20}+\rho b_{11}+\rho^2 b_{02}+c_{20}e^{-2i\omega^*\tau^*}+c_{11}\rho e^{-i\omega^*\tau^*}, \\ \Delta_{22} =& 2b_{20}+(\rho+\bar{\rho})b_{11}+2|\rho|^2 b_{02}+2c_{20}+c_{11}(\rho e^{i\omega^*\tau^*}+\bar{\rho}e^{-i\omega^*\tau^*}), \\ \Delta_{23} =& b_{20}+\bar{\rho}b_{11}+\bar{\rho}^2 b_{02}+c_{20}e^{2i\omega^*\tau^*}+c_{11}\bar{\rho}e^{i\omega^*\tau^*}, \end{aligned}$$

$$\begin{aligned}\Delta_{24} =& b_{20}(W^1_{20}(0) + 2W^1_{11}(0)) + b_{11}(W^2_{11}(0) + \frac{1}{2}W^2_{20}(0) + \rho W^1_{11}(0) + \frac{1}{2}\bar{\rho}W^1_{20}(0))\\ &+ b_{02}(\bar{\rho}W^2_{20}(0) + 2\rho W^2_{11}(0)) + c_{20}(2e^{-i\omega^*\tau^*}W^1_{11}(-\tau^*) + e^{i\omega^*\tau^*}W^1_{20}(-\tau^*))\\ &+ c_{11}(e^{-i\omega^*\tau^*}W^2_{11}(0) + \frac{1}{2}e^{i\omega^*\tau^*}W^2_{20}(0) + \frac{1}{2}\bar{\rho}W^1_{20}(-\tau^*) + \rho W^1_{11}(-\tau^*))\end{aligned}$$

and

$$\begin{aligned}&a_{10} = r - \frac{2r\widetilde{P}^*}{K} - \frac{\beta\alpha\widetilde{Z}^*}{(\alpha+\widetilde{P}^*)^2}, \quad a_{01} = -\frac{\beta\widetilde{P}^*}{(\alpha+\widetilde{P}^*)},\\ &b_{10} = \frac{\beta_1\alpha\widetilde{Z}^*}{(\alpha+\widetilde{P}^*)^2}, \quad b_{01} = \frac{pq^2\widetilde{E}^*\widetilde{Z}^*}{pq\widetilde{Z}^* - c}, \quad c_{10} = -\frac{\rho\alpha\widetilde{Z}^*}{(\alpha+\widetilde{P}^*)^2},\\ &a_{20} = -\frac{r}{K} + \frac{\beta\alpha\widetilde{Z}^*}{(\alpha+\widetilde{P}^*)^3}, \quad a_{11} = -\frac{\beta\alpha}{(\alpha+\widetilde{P}^*)^2}, \quad b_{20} = -\frac{\beta_1\alpha\widetilde{Z}^*}{(\alpha+\widetilde{P}^*)^3},\\ &b_{11} = \frac{\beta_1\alpha}{(\alpha+\widetilde{P}^*)^2}, \quad b_{02} = -\frac{pq^2\widetilde{E}^*c}{2(pq\widetilde{Z}^*-c)^2}, \quad c_{20} = \frac{\rho\alpha\widetilde{Z}^*}{(\alpha+\widetilde{P}^*)^3}, \qquad c_{11} = -\frac{\rho\alpha}{(\alpha+\widetilde{P}^*)^2},\\ &W_{20}(\theta) = \frac{ig_{20}q(0)}{\omega^*}e^{i\omega^*\theta} + \frac{i\bar{g}_{02}\bar{q}(0)}{3\omega^*}e^{-i\omega^*\theta} + E_1e^{2i\omega^*\theta},\\ &W_{11}(\theta) = \frac{-ig_{11}q(0)}{\omega^*}e^{i\omega^*\theta} + \frac{i\bar{g}_{11}\bar{q}(0)}{\omega^*}e^{-i\omega^*\theta} + E_2,\\ &\rho = \frac{i\omega^* - a_{10}}{a_{01}}, \quad \rho^* = -\frac{a_{01}}{b_{01}+i\omega^*}, \quad D = \frac{1}{1+\rho^*\overline{\rho}+\tau^*\rho^*c_{10}e^{i\omega^*\tau^*}}.\end{aligned}$$

with

$$\begin{aligned}E_1 &= (E^1_1, E^2_1),\ E_2 = (E^1_2, E^2_2),\\ q(\theta) &= (1,\rho)^Te^{i\omega^*\theta}(\theta\in[-\tau,0]),\ q^*(s) = D(1,\rho^*)e^{i\omega^*s}(s\in(0,\tau]),\\ E^1_1 &= \frac{2}{M}\{(2i\omega^* - b_{01})\Delta_{11} + a_{01}\Delta_{21}\},\\ E^2_1 &= \frac{2}{M}\{(2i\omega^* - a_{10})\Delta_{21} + \Delta_{11}(b_{10} + c_{10}e^{-2i\omega^*\tau^*})\},\\ E^1_2 &= \frac{1}{N}(-b_{01}\Delta_{12} + a_{01}\Delta_{22}),\\ E^2_2 &= \frac{1}{N}\{-a_{10}\Delta_{22} + (b_{10}+c_{10})\Delta_{12}\},\\ M &= (2i\omega^* - a_{10})(2i\omega^* - b_{01}) - a_{01}(b_{10} + c_{10}e^{-2i\omega^*\tau^*}),\\ N &= a_{10}b_{01} - a_{01}(b_{10}+c_{10}).\end{aligned}$$

Therefore, we can compute the following values

$$\begin{aligned}C_1(0) &= \tfrac{i}{2\omega_0}[g_{20}g_{11} - 2|g_{11}|^2 - \tfrac{1}{3}|g_{02}|^2] + \tfrac{g_{21}}{2},\\ \mu_2 &= -\tfrac{Re(C_1(0))}{Re(\lambda'(\tau^*))},\\ \beta_2 &= 2Re(C_1(0)).\end{aligned}$$

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