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DYNAMICAL BEHAVIOUR AND EXACT SOLUTIONS OF THIRTEENTH ORDER DERIVATIVE NONLINEAR SCHRÖDINGER EQUATION*

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Abstract In this paper, we considered the model of the thirteenth order derivatives of nonlinear Schrödinger equations. It is shown that a wave packet ansatz inserted into these equations leads to an integrable Hamiltonian dynamical sub-system. By using bifurcation theory of planar dynamical systems, in different parametric regions, we determined the phase portraits. In each of these parametric regions we obtain possible exact explicit parametric representation of the traveling wave solutions corresponding to homoclinic, hetroclinic and periodic orbits.

Keywords Coupled integrable system, exact solution, thirteenth order derivative nonlinear Schrödinger equation, homoclinic orbits, hetroclinic orbits, periodic orbits.

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1. Introduction

Many phenomena in physics, engineering and sciences are described by nonlinear partial differential equations (NPDEs). Exact traveling wave solutions of nonlinear evolution equations is one of the fundamental object of study in mathematical physics. When these exact solutions exist, they can help one to understand the mechanism of the complicated physical phenomena and dynamical process modeled by these nonlinear evolution equations.

Derivative nonlinear Schrödinger (DNLS) equations occur frequently in applications, for instance, in models incorporating self-steepening effects in optical pulses [8, 16]. Derivative nonlinear Schrödinger equations constitute a class of models which describe the evolution in physical media that has been drawn considerable attention both in a theoretical context and in many applied disciplines, notably in hydrodynamics, nonlinear optics and the study of Bose-Einstein condensates [2, 6, 19]. In the past decades a vast variety of the powerful and direct methods to find the explicit solutions of NPDE have been developed, such as inverse scattering transform

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method, Backlund and Darboux transforms method, Lie group method, homotopy perturbation method, planner dynamical method, algebraic method, Jacobi elliptic function expansion method [1,9,10,17,20,26] to list a few.

In this paper, we apply the planner dynamical method for solving coupled derivatives of nonlinear Schrodinger equations of thirteenth order:

$$iA_t + A_{xx} + f'(\Phi)A + j'(\Phi)iA_x + (g(\Phi))_x iA - \frac{s|A|_{xx}}{|A|}A = 0, \tag{1.1}$$

where $f = \frac{\mu}{2}\Phi^2 + \frac{\nu}{3}\Phi^3 + \frac{ac}{\lambda}\Phi^4 - \frac{a^2}{2\lambda}\Phi^7$, $j = 2a\Phi^4$ and $g = \frac{\alpha}{2}\Phi^2 + \frac{\beta}{3}\Phi^3$, are analytic functions which depend on the squared amplitude $\Phi = |A|^2 = \phi^2 + \psi^2$ and " $'$ " stands for the derivative with respect to Φ (see [11,12] and the reference there in). The constant $s > 1$ corresponds to the resonant nonlinear Schrödinger (NLS) equation so-called because it admits both fission and fusion resonant solitonic phenomena [11,21,23,25,27].

In a wide class of NLS equations with underlying Hamiltonian structure was shown to be reducible to the integrable resonant NLS equation [21,28]. This so-called resonant NLS equation has also been derived in plasma physics where it describes the transmission of long magneto-acoustic waves in a cold collisionless plasma subject to a transverse magnetic field. The resonant NLS equation and its (2+1)-dimensional integrable extension to a resonant Davey-Stewartson system were introduced in a capillarity model context in [25,27]. The nonlinear Schrödinger equation is an example of a universal nonlinear model that describes many physical nonlinear systems [5]. In the setting of optical fiber waveguides, terms involving λ, μ and α are associated with group velocity dispersion, Kerr (cubic) nonlinearity and self-steeping (or, more precisely, an effect which may be converted to self-steeping form following a gauge transformation).

Recent interests on coupled nonlinear Schrödinger systems, on a class of propagating wave patterns for families of derivative nonlinear Schrödinger equations of seventh (septic), ninth (nonic) and thirteenth order, which incorporates de Broglie-Bohm quantum potentials and which admit integrable Ermakov-Ray-Reid sub-systems have bought attention of researchers [4,29]. A procedure recently employed by [5,7,13,18] (the application of a pair of invariants of motion) is also applied here. To analyze the traveling wave solution with the form:

$$A(x,t) = [(\phi(\xi) + i\psi(\xi))]\exp(i(\nu x - \lambda t)), \quad \xi = x - \mu t, \tag{1.2}$$

where μ, ν and λ are related to the nonlinearity induced shifts in three quantities, namely, group delay, carrier frequency and propagation constant respectively. Substituting equation (1.2) into equation (1.1), separating the real and imaginary parts, one obtains the coupled nonlinear integrable system respectively,

$$\begin{aligned}
\frac{d\phi}{d\xi} = q_1, \quad \frac{dq_1}{d\xi} &= -(f' + \lambda - \nu^2)\phi + (j' + 2\nu - \mu)\dot{\psi} + g'(\phi^2 + \psi^2)^{\bullet}\psi \\
&\quad + s\left[\frac{(\dot{\phi}\phi + \dot{\psi}\psi)^{\bullet}}{\phi^2 + \psi^2} - \left(\frac{\dot{\phi}\phi + \dot{\psi}\psi}{\phi^2 + \psi^2}\right)^2\right]\phi, \\
\frac{d\psi}{d\xi} = q_2, \quad \frac{dq_2}{d\xi} &= -(f' + \lambda - \nu^2)\psi - (j' + 2\nu - \mu)\dot{\phi} - g'(\phi^2 + \psi^2)^{\bullet}\phi
\end{aligned} \tag{1.3}$$

$$+ s\left[\frac{(\dot{\phi}\phi+\dot{\psi}\psi)^{\bullet}}{\phi^2+\psi^2}-\left(\frac{\dot{\phi}\phi+\dot{\psi}\psi}{\phi^2+\psi^2}\right)^2\right]\psi.$$

where, dot indicates a derivative with respect to $x-\mu t$.

System (1.3) admits two independent integrals of motion and, accordingly, is integrable. Thus, we obtain the first integral of the dynamical invariant in the form of the Hamiltonian:

$$\begin{aligned}2\hat{H}(\phi,\psi,q_1,q_2) =& q_1^2+q_2^2-(\lambda-\nu^2)\Phi-\frac{\nu\alpha-\mu}{2\lambda}\Phi^2-\frac{(2\beta-1)\nu}{3\lambda}\Phi^3\\ &-\frac{a(\mu-2\nu)}{\lambda}\Phi^4+\frac{a^2}{\lambda^2}\Phi^7-\frac{s}{4}\frac{\dot{\Phi}^2}{\Phi}\equiv\hat{h}_1,\end{aligned}\tag{1.4}$$

where the constant $\hat{h}_1$ corresponds to the Hamiltonian invariant. The identification of constants of motion has been proven as key to the subsequent integration of other physically important nonlinear dynamical systems, such as the Ermakov - Ray - Reid systems with invariant of the type and the second integral [4,13,22,29]:

$$\hat{I}(\phi,\psi,q_1,q_2)=q_1\psi-q_2\phi-\frac{a}{\lambda^2}\Phi^4-\frac{\beta}{3\lambda}\Phi^3-\frac{\alpha}{4\lambda}\Phi^2+\frac{1}{2}(\mu-2\nu)\Phi=\hat{h}_2,\tag{1.5}$$

where, $\hat{h}_2$ is constant of motion. The pair of integrals of motion equation (1.4) and equation (1.5) allow explicit solution of the nonlinear coupled system (1.3) for ϕ and ψ in terms of quadrature.

Let $\hat{N}_{\hat{h}_1\hat{h}_2}$ be an invariant manifold family of system (1.3) given by

$$\hat{N}_{\hat{h}_1\hat{h}_2}=\{\hat{H}=\hat{h}_1,\ \hat{I}=\hat{h}_2,\ \hat{h}_1,\ \hat{h}_2\in\mathbf{R}\}\subset\mathbf{R}^{\mathbf{4}}.\tag{1.6}$$

Using the identity

$$(\phi^2+\psi^2)(\dot{\phi}^2+\dot{\psi}^2)-(\phi\dot{\phi}+\psi\dot{\psi})^2\equiv(\phi\dot{\psi}-\psi\dot{\phi})^2,\tag{1.7}$$

and combined with equation (1.4) and equation (1.5), for fixed $(\hat{h}_1,\hat{h}_2)$ gives rise

$$\begin{aligned}\left(\frac{d\Phi}{d\xi}\right)^2 =& \frac{4\Phi}{1-s}\left[2\hat{h}_1-(\lambda-\nu^2)\Phi-\frac{(\nu\alpha-\mu)}{2\lambda}\Phi^2-\frac{(2\beta-1)\nu}{3\lambda}\Phi^3-\frac{a(\mu-2\nu)}{\lambda}\Phi^4\right]\\ &+\frac{4a^2}{\lambda^2(1-s)}\Phi^8-\frac{4}{1-s}\left[\hat{h}_2-\frac{(\mu-2\nu)}{2}\Phi+\frac{\alpha}{4\lambda}\Phi^2+\frac{\beta}{3\lambda}\Phi^3+\frac{a}{\lambda}\Phi^4\right]^2\end{aligned}\tag{1.8}$$

From equation (1.5), [5] introduced the variable $\Theta=\arctan\Delta$. So that,

$$\cos\Theta=\frac{\psi}{\sqrt{\Phi}},\quad \sin\Theta=\frac{\phi}{\sqrt{\Phi}},$$

where $\Delta=\frac{\phi}{\psi}$. We see from equation (1.5) that,

$$\begin{aligned}\Theta(\xi)&=\int_0^{\xi}\frac{12\lambda\hat{h}_2-6\lambda(\mu-2\nu)\Phi+3\alpha\Phi^2+4\beta\Phi^3+12a\Phi^4}{12\lambda^2\Phi}d\xi\\ &=-\frac{1}{2}(\mu-2\nu)\xi+\hat{h}_2\int_0^{\xi}\frac{d\xi}{\Phi}+\frac{\alpha}{4\lambda}\int_0^{\xi}\Phi d\xi+\frac{\beta}{3\lambda}\int_0^{\xi}\Phi^2d\xi+\frac{a}{\lambda}\int_0^{\xi}\Phi^3d\xi,\end{aligned}\tag{1.9}$$

where ξ is a dummy variable of integration.

Thus, if we know $\Phi(\xi)$ and $\Theta(\xi)$ from equation (1.8) and equation (1.9), then we may solve equation (1.1) and system (1.3) to obtain the following solutions:

$$A(x,t) = i\sqrt{\Phi}\exp\left[-i\Theta + i(\nu x - \lambda t)\right] \tag{1.10}$$

and

$$\phi(\xi) = \sqrt{\Phi}\sin\Theta \quad \text{and} \quad \psi(\xi) = \sqrt{\Phi}\cos\Theta. \tag{1.11}$$

The corresponding class of exact solutions of equation (1.1), is then given by equation (1.10). Let $\frac{d\Phi}{d\xi} = \frac{1}{2}(1-s)y$, then we have a system

$$\begin{aligned} \frac{d\Phi}{d\xi} &= \frac{1}{2}(1-s)y, \\ \frac{dy}{d\xi} &= \beta_0 - \beta_1\Phi - \beta_2\Phi^2 - \beta_3\Phi^3 - \beta_4\Phi^4 - \beta_5\Phi^5 - \beta_6\Phi^6, \end{aligned} \tag{1.12}$$

where,

$$\begin{aligned} \beta_0 &= 2\hat{h}_1 + \hat{h}_2(\mu - 2\nu); \\ \beta_1 &= \frac{1}{4\lambda}\left[8\lambda(\lambda - \nu^2) + \lambda(\mu - 2\nu)^2 + \alpha\hat{h}_2\right]; \\ \beta_2 &= \frac{1}{4\lambda}\left[(6\beta - 3(\mu - 2\nu))\alpha + 4\beta\hat{h}_2\right]; \\ \beta_3 &= \frac{4}{3\lambda}\left[2\nu\beta - \nu + \beta(\mu - 2\nu) - 2a\hat{h}_2\lambda\right] + \frac{1}{4\lambda^2}\alpha^2; \\ \beta_4 &= \frac{5\alpha\beta}{6\lambda^2}; \\ \beta_5 &= \frac{2}{3\lambda^2}\left(9\alpha^2 + 5\beta^2\right); \\ \beta_6 &= \frac{7\beta a}{3\lambda^2}. \end{aligned}$$

Clearly, system (1.12) is a seven parameter system depending on the eight parameter group $(\lambda, \mu, a, s, \nu, \delta, \hat{h}_1, \hat{h}_2)$. It is abound with dynamical system [11,23,24]. For a given parameter group $(\nu, \mu) \neq (0,0)$, we next take $(\hat{h}_1, \hat{h}_2)$ such that

$$\beta_0 \equiv 0, \quad \hat{h}_1 = \frac{(2\nu - \mu)}{4} \times \hat{h}_2. \tag{1.13}$$

Then, for $s = 2$ system (1.12) reduces to

$$\begin{aligned} \frac{d\Phi}{d\xi} &= -\frac{1}{2}y, \\ \frac{dy}{d\xi} &= -\beta_1\Phi - \beta_2\Phi^2 - \beta_3\Phi^3 - \beta_4\Phi^4 - \beta_5\Phi^5 - \beta_6\Phi^6, \end{aligned} \tag{1.14}$$

with the first integral

$$\widetilde{H}(\Phi, y) = \frac{1}{4}y^2 - \Phi^2\left(\frac{1}{2}\beta_1 + \frac{1}{3}\beta_2\Phi + \frac{1}{4}\beta_3\Phi^2 + \frac{1}{5}\beta_4\Phi^3 + \frac{1}{6}\beta_5\Phi^4 + \frac{1}{7}\beta_6\Phi^5\right) = \widetilde{h}. \tag{1.15}$$

We notice that, Rogers and Chow, [5] did not discuss the dynamical behavior for the cases when; $\beta_2 = \beta_4 = \beta_6 = 0$ when $\alpha = \beta = 0$, $\beta_5 \neq 0$. Moreover, they did not give all possible exact explicit parametric representations of the traveling wave solutions. Incorporating the above given conditions in equation (1.14), we have a new system:

$$\begin{aligned} \frac{d\Phi}{d\xi} &= -\frac{1}{2}y, \\ \frac{dy}{d\xi} &= -\Phi(\beta_1 + \beta_3\Phi^2 + \beta_5\Phi^4). \end{aligned} \tag{1.16}$$

In this paper, we use the method of dynamical systems [12,14,15] to investigate the dynamical behavior of system (1.16). In addition we give all possible exact explicit parametric representations of the traveling wave solutions of equation (1.1).

This paper is organized as follows. In section 2, we state the dynamical behavior of system (1.16). In section 3, by using the bifurcation of phase portraits we investigate the exact traveling wave solution of equation (1.1) in different parametric regions. In section 4, we give the main result of the study.

2. Bifurcations of phase portraits of system (1.16)

We consider system (1.16) which has the hamiltonian

$$H(\Phi, y) = \frac{1}{4}y^2 - \Phi^2(\frac{1}{2}\beta_1 + \frac{1}{4}\beta_3\Phi^2 + \frac{1}{6}\beta_5\Phi^4) = h. \tag{2.1}$$

Write $\Delta = \beta_3^2 - 4\beta_1\beta_5$. Clearly, system (1.16) has always the equilibrium point $E_0(0,0)$. If $\Delta < 0$ and $\beta_1 < 0 < \beta_5$, then system (1.16) has two equilibrium points $E_1(\Phi_1, 0)$ and $E_2(\Phi_2, 0)$, where $\Phi_1 = \left(\frac{-\beta_3 - \sqrt{\Delta}}{2\beta_5}\right)^{\frac{1}{2}}$ and $\Phi_2 = \left(\frac{-\beta_3 + \sqrt{\Delta}}{2\beta_5}\right)^{\frac{1}{2}}$. If $\Delta > 0$ and $\beta_5 < 0 < \beta_1$, system (1.16) has four equilibrium points $E_1(-\Phi_1, 0)$, $E_2(-\Phi_2, 0)$, $E_3(\Phi_2, 0)$ and $E_4(\Phi_1, 0)$. If $\Delta = 0$, and $\beta_1 = 0$, then system (1.16) has a double equilibrium point at $E_{12}(\Phi_1, 0) = E_1(\Phi_1, 0)$ and $E_{34}(\Phi_2, 0) = E_2(\Phi_2, 0)$.

Let $\hat{M}(\Phi_j, 0)$ be the coefficient matrix of the linearized system of system (1.16) at an equilibrium point E_j. We have

$$J(0,0) = \det\hat{M}(0,0) = \beta_1, \quad J(\Phi_j, y) = \det\hat{M}(\Phi_j, y) = \beta_1 + 3\beta_3\Phi^2 + 5\beta_5\Phi^4.$$

$J(0,0) = \beta_1$ implies that when $\beta_1 < 0$, the equilibrium point $E_0(0,0)$ is a center point, when $\beta_1 > 0$, equilibrium point $E_0(0,0)$ is a saddle point and while for $\beta_1 = 0$ and $\beta_5 < 0$, the equilibrium point $E_0(0,0)$ is a cusp point.

We write that for $H(\Phi, y) = h$ given by equation (2.1),

$$h_0 = H(0,0) = 0, \quad h_{1,2} = H(\pm\Phi_{1,2}, 0) = \frac{\Phi_{1,2}}{24\beta_5}\left(\beta_3\sqrt{\Delta} \pm (8\beta_1\beta_5 - \beta_3^2)\right). \tag{2.2}$$

For a fixed $\beta_1 < 0$ or $\beta_1 \geq 0$, we change the parameter space β_3 and β_5 in the parameter plane (β_3, β_5). There are two parameter curves $(L_1) : \beta_3 = \frac{4}{3}\sqrt{3\beta_1\beta_5}$ and $(L_2) : \beta_3 = 2\sqrt{\beta_1\beta_5}$, which partition this parameter plane into four regions: $(I), (II), (III), (IV)$ shown in Fig.1.

By using the above information to do qualitative analysis, we have the following bifurcations of phase portraits of system (1.16) in two cases as follows.

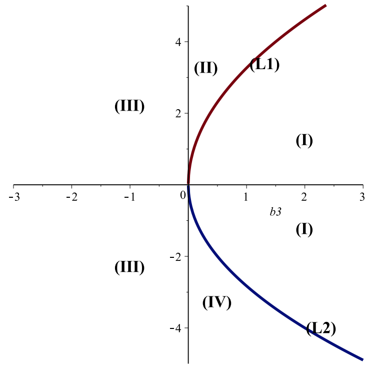
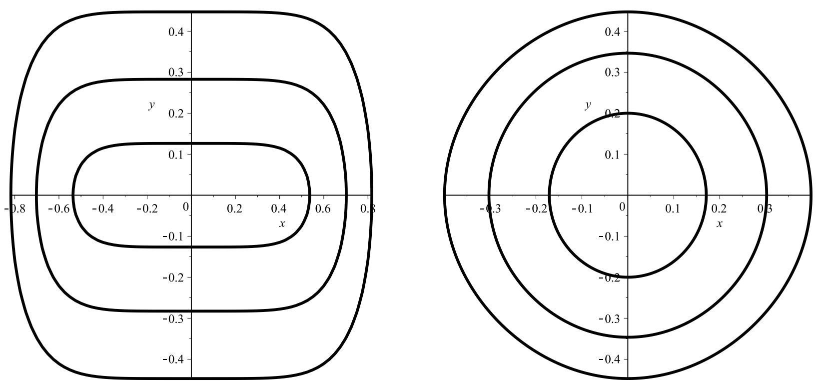


Figure 1. The parameter regions partitioned by bifurcation curves of (L_1) and (L_2).

Case 1. Assume that $\beta_1 \ll 0$. In this case, the equilibrium points $E_0(0,0)$ is a center. The bifurcations of phase portraits of system (1.16) are shown in Fig.2(a)–(b).



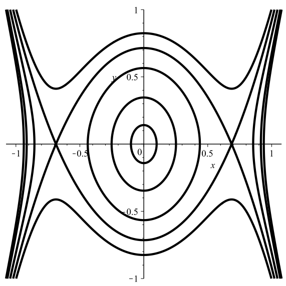
(a) $\beta_3 > 0,\ \beta_5 < 0,\ \Delta < 0.$

(b) $\beta_3 = 0,\ \beta_5 < 0.$

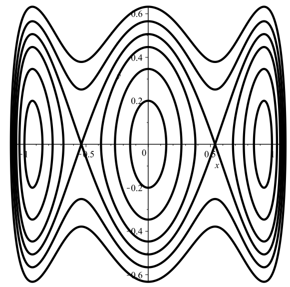
Figure 2. Bifurcations of phase portraits of system (1.16), $\beta_1 \ll 0$.

Case 2. Assume that $\beta_1 < 0$. In this case, system (1.16) has five equilibrium points $E_0(0,0)$ and $E_j(\Phi_j,0),\ j=1,2,3,4,5$. The origin $E_0(0,0)$, and $E_j(\Phi_j,0),\ j=1,4$ are centers, while $E_2(\Phi_2,0)$ and $E_4(\Phi_4,0)$ are saddle points. Specially, in Fig. 3(f), the equilibrium points $E_1(-\Phi_2,0)$ and $E_2(\Phi_2,0)$ are a cusp points. The bifurcations of phase portraits of system (1.16) are shown in Fig. 3(a)–3(f).

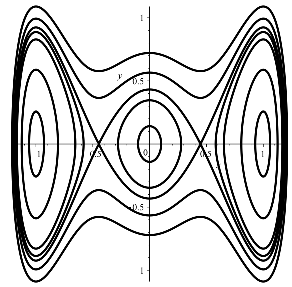
Case 3. Assume that $\beta_1 > 0$. When $\beta_5 \leq 0$, system (1.16) has three equilibrium points $E_0(0,0)$ and $E_j(\Phi_j,0),\ j=1,2$. The Equilibrium points $E_1(\Phi_1,0)$ and $E_2(\Phi_2,0)$ are center points while $E_0(0,0)$ is a saddle point. In addition, when $\beta_5 > 0$, system (1.16) has five equilibrium points $E_0(0,0)$ and $E_j(\Phi_j,0),\ j=1,2,3,4$. The Equilibrium points E_2 and E_3 are center points while E_0, E_1, and E_4 are saddle points. The bifurcations of phase portraits of system (1.16) are shown in Fig. 4(a)–4(f).



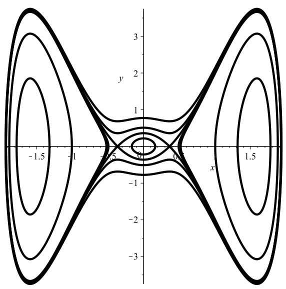
(a) $\beta_3 > 0,\ \beta_5 > 0,\ h_2 < 0.$



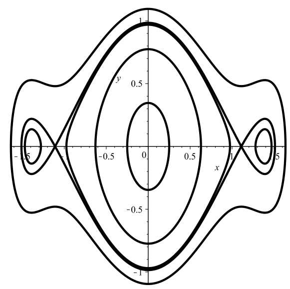
(b) $\beta_5 < \beta_3 < \frac{4}{3}\sqrt{3\beta_1\beta_5},\ 0 < h_1 < \infty.$



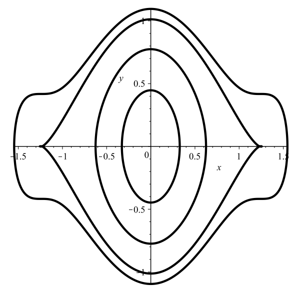
(c) $\beta_3 > 0,\ \beta_5 < 0,\ h_2 < h_1.$



(d) $\beta_3 = \frac{8}{3}\sqrt{3\beta_1\beta_5},\ \beta_5 < 0.$

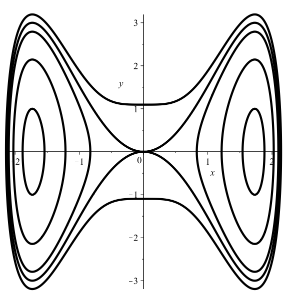


(e) $\beta_3 > 0,\ \beta_5 < 0.$

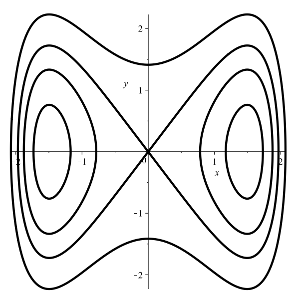


(f) $\beta_3 < 0,\ \beta_5 > 0.$

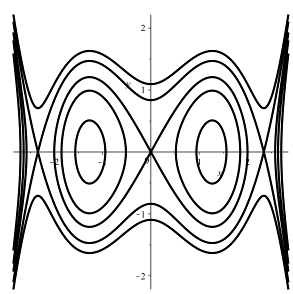
Figure 3. Bifurcations of phase portraits of system (1.16), $\beta_1 < 0$



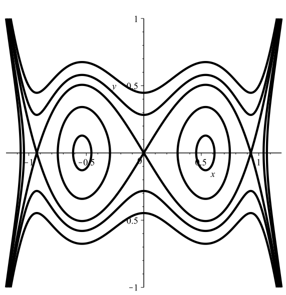
(a) $\beta_3 > 0,\ \beta_5 > 0,\ h_2 < 0.$



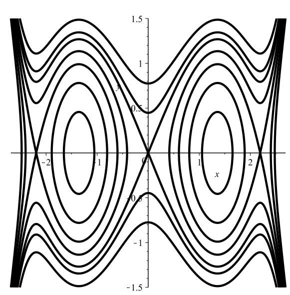
(b) $0 < \beta_3 < \frac{4}{3}\sqrt{3\beta_1\beta_5},\ \beta_5 < 0,\ 0 < h_1 < \infty.$



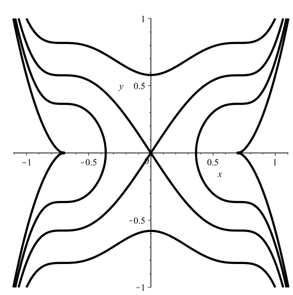
(c) $\beta_3 > 0,\ \beta_5 < 0,\ h_2 < h_1.$



(d) $0 < \beta_3 < 2\sqrt{\beta_1\beta_5},\ \beta_5 < 0.$



(e) $\beta_3 > 0,\ \beta_5 < 0.$



(f) $\beta_3 < 0,\ \beta_5 > 0.$

Figure 4. Bifurcations of phase portraits of system (1.16), $\beta_1 > 0$

3. Explicit parametric representations of the solutions of system (1.16)

We now consider the exact explicit parametric representations of the solutions of system (1.16) depending on $\sqrt{\Phi}$ and Θ. We see from equation (2.1) and the first equation of system (1.14) that

$$\sqrt{\frac{2\beta_5}{3}}\xi=\int_{\Phi_0}^{\Phi}\frac{d\Phi}{\sqrt{\frac{6h}{\beta_5}+\Phi^2(\frac{3\beta_1}{\beta_5}+\frac{3\beta_3}{2\beta_5}\Phi^2+\Phi^4)}}=\int_{\Psi_0}^{\Psi}\frac{d\Psi}{2\sqrt{\Psi[\frac{6h}{\beta_5}+\Psi(\frac{3\beta_1}{\beta_5}+\frac{3\beta_3}{2\beta_5}\Psi+\Psi^2)]}}. \tag{3.1}$$

To find the exact solutions, we consider all bounded orbits of system (1.16) with $\Phi_j=\sqrt{\Psi_j}>0,$ for $j=1,2,3,4,M$, where $\Psi_j>0$. In this section, we discuss the exact solutions of equation (1.1) in the different regions of parameter plane.

3.1. Explicit parametric representations of the solutions of system (1.16) when $\beta_1 \ll 0$

Consider case 1 in section 2 (see [Fig. 2(a), 2(b)]). In this subsection, we notice that $\beta_3\geq 0,\ \beta_5<0,\ \Delta=0$. For $h\in(h_1,h_2)$, the level curves defined by $H(\Phi,y)=h$, has a family of periodic orbits enclosing the origin.

Now, from equation (2.1) we have $y^2=\frac{2\beta_5}{3}\left(\frac{6h}{\beta_5}+\Phi^2(\frac{3\beta_1}{\beta_5}+\frac{3\beta_3}{2\beta_5}\Phi^2+\Phi^4)\right)=\frac{8\beta_5}{3}(\Psi-\alpha_0)\Psi(\Psi-\gamma)(\Psi-\bar{\gamma})=\frac{8\beta_5}{3}(\Psi-\alpha_0)\Psi\left[(\Psi-a_1)^2+b_1^2\right]$. Thus, we have the following parametric representation:

$$\Phi(\xi)=\sqrt{\Psi(\xi)}=\left(\frac{\alpha_0 B(1+\mathrm{cn}(\omega_1\xi,k))}{(B-A_1)+(A_1+B)\mathrm{cn}(\omega_1\xi,k)}\right)^{\frac{1}{2}}, \tag{3.2}$$

where, $A_1^2=(\alpha_0-b_1)^2+a_1^2,\ \omega_1=\sqrt{\frac{2|\beta_5|A_1B}{3}},\ B^2=a_1^2+b_1^2,\ k^2=\frac{(A_1+B)^2-\alpha_0^2}{4A_1B}$, $a_1^2=-\left(\frac{\gamma-\bar{\gamma}}{4}\right)^2,\ b_1^2=\frac{(\gamma+\bar{\gamma})^2}{2}$, $\mathrm{sn}(\cdot,k),\ \mathrm{cn}(\cdot,k),\ \mathrm{dn}(\cdot,k),\ \mathrm{sd}(\cdot,k)$ are Jacobin elliptic functions, $E(\cdot,k)$, is the elliptic integrals of the second kind and $\Pi(\cdot,\cdot,k)$, is the elliptic integrals of the third kind [3].

Write that,

$$\bar{\alpha}_0=\frac{A_1+B}{A_1-B},\quad \bar{\beta}_0=\frac{\alpha_0B}{A_1-B},\quad \bar{\gamma}_0=\frac{\alpha_0B}{A_1+B}.$$

Thus, we have from equation (1.9) that

$$\begin{aligned}\Theta(\xi)=&\frac{1}{2}(2\nu-\mu)\xi+\frac{\beta(\bar{\alpha}_0+1)}{3\lambda\bar{\gamma}_0\bar{\alpha}_0(1-\bar{\alpha}_0^2)}\times\left[\Pi\left(\arcsin(\mathrm{sn}(\omega_1\xi,k)),\frac{\bar{\alpha}_0^2}{\bar{\alpha}_0^2-1},k\right)-\bar{\alpha}_0f_1(\omega_1\xi,k)\right]\\&+\hat{h}_2\int_0^{\Psi}\left(-\frac{\bar{\alpha}_0}{\bar{\gamma}_0}+\frac{\left(\frac{1}{\bar{\beta}_0}+\frac{\bar{\alpha}_0}{\bar{\gamma}_0}\right)}{1-\mathrm{cn}(\omega_1\xi,k)}\right)^{\frac{1}{2}}d\xi+\frac{\alpha}{4\lambda}\int_0^{\Psi}\left(-\frac{\bar{\gamma}_0}{\bar{\alpha}_0}+\frac{\bar{\beta}_0+\frac{\bar{\gamma}_0}{\bar{\alpha}_0}}{1+\bar{\alpha}_0\mathrm{cn}(\omega_1\xi,k)}\right)^{\frac{1}{2}}d\xi\\&+\frac{a}{\lambda}\int_0^{\Psi}\left(-\frac{\bar{\gamma}_0}{\bar{\alpha}_0}+\frac{\bar{\beta}_0+\frac{\bar{\gamma}_0}{\bar{\alpha}_0}}{1+\bar{\alpha}_0\mathrm{cn}(\omega_1\xi,k)}\right)^{\frac{3}{2}}d\xi,\end{aligned} \tag{3.3}$$

where

$$f_1(t) = \sqrt{\frac{1-\alpha_0^2}{k^2+(k')^2\alpha_0^2}}\tan^{-1}\left(\sqrt{\frac{k^2+(k')^2\alpha_0^2}{1-\alpha_0^2}}\mathrm{sd}(t,k)\right), \quad \text{if } \frac{\alpha_0^2}{\alpha_0^2-1} < k^2,$$

$$= \mathrm{sd}(t,k), \quad \text{if } \frac{\alpha_0^2}{\alpha_0^2-1} = k^2,$$

$$= \frac{1}{2}\sqrt{\frac{\alpha_0^2-1}{k^2+(k')^2\alpha_0^2}}\ln\left[\frac{\sqrt{k^2+(k')^2\alpha_0^2}\mathrm{dn}(t,k)+\sqrt{\alpha_0^2-1}\mathrm{sn}(t,k)}{\sqrt{k^2+(k')^2\alpha_0^2}\mathrm{dn}(t,k)-\sqrt{\alpha_0^2-1}\mathrm{sn}(t,k)}\right], \text{if } \frac{\alpha_0^2}{\alpha_0^2-1} > k^2.$$

Hence, we have the following solution equation (1.1):

$$A(x,t) = i\left(\frac{\alpha_0 B(1+\mathrm{cn}(\omega_1\xi,k))}{(B-A_1)+(A_1+B)\mathrm{cn}(\omega_1\xi,k)}\right)^{\frac{1}{4}} \times \exp\left[-i\Theta + i(\nu x - \lambda t)\right], \quad (3.4)$$

where, $\Theta(\xi)$ is given by Eq. equation (3.3).

3.2. Explicit parametric representations of the solutions of system (1.16) when $\beta_1 < 0$

In this section, we consider case 2 of section 2 for $\beta_1 < 0$, (see [Fig. 3(a)–3(h)]).

(1) $\beta_3 > 0,\ \beta_5 > 0,\ h \in (0,\ h_1]$ (see [Fig. 3(a)])

(i) Corresponding to the family of periodic orbits defined by $H(\Phi, y) = h,\ h \in (0, h_1)$, we have from equation (2.1) that $y^2 = \frac{8\beta_5}{3}(r_1-\Psi)(r_2-\Psi)\Psi(\Psi-r_4)$, where $r_4 < 0 < r_2 < r_1$. Thus, we have the following parametric representation of the family of periodic orbits of system (1.16):

$$\Phi(\xi) = \pm\sqrt{\Psi(\xi)} = \pm\sqrt{r_4}\left(1 - \frac{1}{1-\alpha_1^2\mathrm{sn}^2(\omega_2\xi,k)}\right)^{\frac{1}{2}}, \quad (3.5)$$

where, $\alpha_1^2 = \frac{r_2}{r_2-r_4},\ k^2 = \alpha_1^2\left(\frac{r_1-r_4}{r_2-r_4}\right),\ \omega_2 = \sqrt{\frac{\beta_5 r_1(r_2-r_4)}{6}}$.

It follows from equation (1.9) that,

$$\begin{aligned}\Theta(\xi) = {} & \left(\frac{1}{2}(2\nu-\mu) - \frac{\hat{h}_2}{\sqrt{r_4}} + \frac{\beta r_4}{3\lambda}\right)\xi + \frac{\beta}{3\lambda}\Pi\left(\mathrm{arcsinh}(\mathrm{sn}(\omega_2\xi,k)), \frac{\alpha_1^2}{\alpha_1^2-1}, k\right) \\ & + \frac{\hat{h}_2}{\alpha_1\sqrt{r_4}}\ln\left(\frac{\mathrm{sn}(\omega_2\xi,k)}{\mathrm{cn}(\omega_2\xi,k)+\mathrm{dn}(\omega_2\xi,k)}\right) + \frac{\alpha\sqrt{r_4}}{4\lambda}\int_0^{\Psi}\left(1-\frac{1}{1-\alpha_1^2\mathrm{sn}^2(\omega_2\xi,k)}\right)^{\frac{1}{2}}d\xi \\ & + \frac{ar_4^{\frac{3}{2}}}{\lambda}\int_0^{\Psi}\left(1-\frac{1}{1-\alpha_1^2\mathrm{sn}^2(\omega_2\xi,k)}\right)^{\frac{3}{2}}d\xi. \end{aligned} \quad (3.6)$$

Hence, we have the following solution equation (1.1):

$$A(x,t) = i\left(r_4\left(1-\frac{1}{1-\alpha_1^2\mathrm{sn}^2(\omega_2\xi,k)}\right)\right)^{\frac{1}{4}} \times \exp\left[-i\Theta + i(\nu x - \lambda t)\right], \quad (3.7)$$

where, $\Theta(\xi)$ is given by equation (3.6).

(ii) The level curve defined by $H(\Phi, y) = h_1$, have two hetroclinic orbits. In this case $\Phi_2 = -\Phi_1$ and the origin is a center. Now, from equation (2.1) we have

$$y^2 = \frac{8\beta_5}{3}\left(\frac{6h}{\beta_5}\Psi + \frac{3\beta_1}{\beta_5}\Psi^2 + \frac{3\beta_3}{2\beta_5}\Psi^3 + \Psi^4\right) = \frac{8\beta_5}{3}(\Psi_2 - \Psi)^2\Psi(\Psi - r_1),$$

where $0 < r_1 < \Psi_2 = \Phi_2^2$.

Then, the hetroclinic orbit has a parametric representations:

$$\Phi(\xi) = \pm\sqrt{\Psi(\xi)} = \pm\left(\Psi_2 - \frac{2\Psi_2(\Psi_2 - r_1)}{r_1\cosh(\sqrt{(\Psi_2 - r_1)\Psi_2}\xi) + (2\Psi_2 - r_1)}\right)^{\frac{1}{2}}. \tag{3.8}$$

It follows from equation (1.9) that,

$$\begin{aligned}\Theta(\xi) = &\left(\frac{1}{2}(2\nu-\mu)+\frac{\beta}{3\lambda}\Psi_2\right)\xi - \frac{2\beta}{3\lambda}\sqrt{(\Psi_2-r_1)\Psi_2}\arctan\sqrt{\frac{\Psi_2-r_1}{\Psi_2}}\tan\frac{\sqrt{(\Psi_2-r_1)\Psi_2}}{2}\xi\\ &+ \frac{\hat{h}_2}{\sqrt{\Psi_2}}\int_0^\Psi\sqrt{\frac{r_1\cosh(\sqrt{(\Psi_2 - r_1)\Psi_2}\xi) + r_1 - 2\Psi_2}{r_1\cosh(\sqrt{(r_1 - \Psi_2)\Psi_2}\xi) + \Psi_2}}d\xi\\ &+ \frac{\alpha\sqrt{\Psi_2}}{4\lambda}\int_0^\Psi\left(\frac{r_1\cosh(\sqrt{(\Psi_2 - r_1)\Psi_2}\xi + \Psi_2}{r_1\cosh(\sqrt{(\Psi_2 - r_1)\Psi_2}\xi) + (2\Psi_2 - r_1)}\right)^{\frac{1}{2}}d\xi\\ &+ \frac{a\sqrt{\Psi_2}}{\lambda}\int_0^\Psi\left(\frac{r_1\cosh(\sqrt{(\Psi_2 - r_1)\Psi_2}\xi + \Psi_2}{r_1\cosh(\sqrt{(\Psi_2 - r_1)\Psi_2}\xi) + (2\Psi_2 - r_1)}\right)^{\frac{3}{2}}d\xi.\end{aligned} \tag{3.9}$$

Hence, we have the following solution equation (1.1):

$$A(x,t) = i\left(\Psi_2 - \frac{2\Psi_2(\Psi_2 - r_1)}{r_1\cosh(\sqrt{(\Psi_2 - r_1)\Psi_2}\xi) + (2\Psi_2 - r_1)}\right)^{\frac{1}{4}} \times\exp\left[-i\Theta + i(\nu x - \lambda t)\right], \tag{3.10}$$

where, $\Theta(\xi)$ is given by equation (3.9).

(2) $\beta_5 < \beta_3 < \frac{4}{3}\sqrt{3\beta_1\beta_5},\ 0 < h_1 < \infty$ (see [Fig. 3(b)]).

(i) Corresponding to the level curves defined by $H(\Phi, y) = h,\ h \in (0, h_1)$, there exist three families of periodic orbits of system (1.16), enclosing the equilibrium points $E_1(-\Phi_1, 0)$ and $E_4(\Phi_1, 0)$, $E_0(0, 0)$ respectively.

Now, equation (3.1) has the forms

$$\sqrt{\frac{8\beta_5}{3}}\xi = \int_{r_2}^\Psi \frac{d\Psi}{\sqrt{(\Psi - r_3)(\Psi - r_2)\Psi(r_1 - \Psi)}}$$

and

$$\sqrt{\frac{8\beta_5}{3}}\xi = \int_0^\Psi \frac{d\Psi}{\sqrt{\Psi(r_3 - \Psi)(r_2 - \Psi)(r_1 - \Psi)}},$$

where $0 < r_3 < r_2 < \Phi_1 < r_1$.

Therefore, the periodic orbits enclosing the equilibrium points $E_1(-\Phi_1, 0)$ and $E_4(\Phi_1, 0)$ has a parametric representations:

$$\Phi(\xi) = \pm\sqrt{\Psi(\xi)} = \pm\left(r_3 + \frac{r_2 - r_3}{1 - \alpha_1^2\mathrm{sn}^2(\omega_3\xi, k)}\right)^{\frac{1}{2}}, \tag{3.11}$$

where $k^2 = \frac{(r_1-r_2)r_3}{(r_1-r_3)r_2}$, $\alpha_2^2 = \frac{r_1-r_2}{r_1-r_3}$, $\omega_3 = \sqrt{\frac{\beta_5 r_2(r_1-r_3)}{6}}$.

Thus, we have from equation (1.9) that

$$\Theta(\xi) = \left(\frac{1}{2}(2\nu-\mu) + \frac{\beta r_4}{3\lambda}\right)\xi + \frac{\beta(r_2-r_3)}{3\lambda} \times \Pi\left(\operatorname{arcsinh}\left(\operatorname{sn}\left(\omega_3\xi,k\right)\right), \alpha_2^2, k\right)$$
$$+ \hat{h}_2 \int_{r_2}^{\Psi} \left(\frac{1-\alpha_2^2\operatorname{sn}^2(\omega_3\xi,k)}{r_2-r_3\alpha_2^2\operatorname{sn}^2(\omega_3\xi,k)}\right)^{\frac{1}{2}} d\xi + \frac{a}{\lambda}\int_{r_2}^{\Psi}\left(r_3 + \frac{r_2-r_3}{1-\alpha_2^2\operatorname{sn}^2(\omega_3\xi,k)}\right)^{\frac{3}{2}} d\xi. \tag{3.12}$$

Hence, we have the following solution equation (1.1):

$$A(x,t) = i\left(r_3 + \frac{r_2-r_3}{1-\alpha_2^2\operatorname{sn}^2(\omega_3\xi,k)}\right)^{\frac{1}{4}} \times \exp\left[-i\Theta + i(\nu x - \lambda t)\right], \tag{3.13}$$

where, $\Theta(\xi)$ is given by equation (3.12).

The periodic orbits enclosing the equilibrium points $E_0(0,0)$ has the parametric representation:

$$\Phi(\xi) = \pm\sqrt{\Psi(\xi)} = \pm\left(r_1 - \frac{r_1}{1-\alpha_2^2\operatorname{sn}^2(\omega_3\xi,k)}\right)^{\frac{1}{2}}, \tag{3.14}$$

where $k^2 = \frac{(r_1-r_2)r_3}{(r_1-r_3)r_2}$, $\alpha_3^2 = \frac{r_3}{|r_3-r_1|}$.

Thus, we have from equation (1.9) that

$$\Theta(\xi) = \left(\frac{1}{2}(2\nu-\mu) + \frac{\beta r_1}{3\lambda}\right)\xi - \frac{\beta r_1}{3\lambda}\Pi\left(\operatorname{arcsinh}(\operatorname{sn}(\omega_3\xi,k)), \alpha_3^2, k\right)$$
$$+ \frac{\hat{h}_2}{\sqrt{r_1}}\int_0^{\Psi}\sqrt{\frac{1}{1-\frac{1}{1-\alpha_3^2\operatorname{sn}^2(\omega_3\xi,k)}}}d\xi + \frac{\alpha\sqrt{r_1}}{4\lambda}\int_0^{\Psi}\sqrt{1-\frac{1}{1-\alpha_3^2\operatorname{sn}^2(\omega_3\xi,k)}}d\xi$$
$$+ \frac{a}{\lambda}\int_0^{\Psi}\left(r_1 - \frac{r_1}{1-\alpha_3^2\operatorname{sn}^2(\omega_3\xi,k)}\right)^{\frac{3}{2}} d\xi. \tag{3.15}$$

Hence, we have the following solution equation (1.1):

$$A(x,t) = i\left(r_1 - \frac{r_1}{1-\alpha_3^2\operatorname{sn}^2(\omega_3\xi,k)}\right)^{\frac{1}{4}} \times \exp\left[-i\Theta + i(\nu x - \lambda t)\right], \tag{3.16}$$

where, $\Theta(\xi)$ is given by equation (3.15).

(ii) The level curves defined by $H(\Phi,y) = h_1$ there exist two homoclinic orbits of system (1.16), enclosing the equilibrium points $E_1(-\Phi_1,0)$ and $E_4(\Phi_1,0)$; two hetroclinic orbits connecting $E_2(-\Phi_2,0)$ and $E_3(\Phi_2,0)$, enclosing the equilibrium point $E_0(0,0)$.

Corresponding to the two homoclinic orbits, equation (3.1) becomes

$$\sqrt{\frac{8\beta_5}{3}}\xi = \pm\int_{\Psi}^{\Psi_M}\frac{d\Psi}{(\Psi-\Psi_3)\sqrt{(\Psi_M-\Psi)\Psi}},$$

where, $0 < \Phi_3 < \Phi_4 < \Phi_M$, and $\omega_4 = \sqrt{\frac{2\beta_5\Psi_3|\Psi_3-\Psi_M|}{3}}$.

Thus, we obtain the parametric representations of the homoclinic orbit:

$$\Phi(\xi) = \pm\sqrt{\Psi(\xi)} = \pm\left(\Psi_3 + \frac{2\Psi_3(\Psi_M - \Psi_3)}{\Psi_M \cosh(\omega_4\xi) - (\Psi_M - 2\Psi_3)}\right)^{\frac{1}{2}}. \tag{3.17}$$

Thus, we have from equation (1.9) that

$$\begin{aligned}\Theta(\xi) = &\left(\frac{1}{2}(2\nu-\mu)+\frac{\beta}{3\lambda}\Psi_3\right)\xi+\frac{\beta}{3\lambda}\sqrt{\Psi_3(\Psi_M-\Psi_3)}\arctan\left(\sqrt{\frac{|\Psi_3-\Psi_M|}{\Psi_3}}\right)\tanh\left(\frac{1}{2}\omega_4\xi\right)\\ &+\hat{h}_2\int_{\Psi}^{\Psi_M}\left(\frac{\Psi_M\cosh(\omega_4\xi)+(2\Psi_3-\Psi_M)}{\Psi_3\Psi_M\cosh(\omega_4\xi)+\Psi_M\Psi_3)}\right)^{\frac{1}{2}}d\xi\\ &+\frac{\alpha}{4\lambda}\int_{\Psi}^{\Psi_M}\left(\frac{\Psi_3\Psi_M\cosh(\omega_4\xi)+\Psi_M\Psi_3)}{\Psi_M\cosh(\omega_4\xi)+(2\Psi_3-\Psi_M)}\right)^{\frac{1}{2}}d\xi\\ &+\frac{a}{\lambda}\int_{\Psi}^{\Psi_M}\left(\frac{\Psi_3\Psi_M\cosh(\omega_4\xi)+\Psi_M\Psi_3)}{\Psi_M\cosh(\omega_4\xi)+(2\Psi_3-\Psi_M)}\right)^{\frac{3}{2}}d\xi,\end{aligned} \tag{3.18}$$

Hence, we have the following solution equation (1.1):

$$A(x,t) = i\left(\Psi_3 + \frac{2\Psi_3(\Psi_M - \Psi_3)}{\Psi_M\cosh(\omega_4\xi) - (\Psi_M - 2\Psi_3)}\right)^{\frac{1}{4}} \times \exp\left[-i\Theta + i(\nu x - \lambda t)\right], \tag{3.19}$$

where, $\Theta(\xi)$ is given by equation (3.18).

Next, for the two hetroclinic orbits, equation (3.1) becomes

$$\sqrt{\frac{8\beta_5}{3}}\xi = \pm\int_0^{\Psi}\frac{d\Psi}{(\Psi_3-\Psi)\sqrt{(\Psi_M-\Psi)\Psi}}.$$

Thus, we obtain the parametric representation of a hetroclinic orbits as follows:

$$\Phi(\xi) = \pm\sqrt{\Psi(\xi)} = \pm\left(\Psi_3 - \frac{2\Psi_3(\Psi_M-\Psi_3)}{\Psi_M\cosh(\omega_4\xi)+(\Psi_M-2\Psi_3)}\right)^{\frac{1}{2}}, \quad \xi\in(0,\infty). \tag{3.20}$$

Thus, we have from equation (1.9) that

$$\begin{aligned}\Theta(\xi) = &\left(\frac{1}{2}(2\nu-\mu)+\frac{\beta\Psi_3}{3\lambda}\right)\xi+\frac{\beta}{3\lambda}\sqrt{\Psi_3(\Psi_M-\Psi_3)}\arctan\left(\sqrt{\frac{\Psi_3}{\Psi_M-\Psi_3}}\tanh(\frac{1}{2}\omega_5\xi)\right)\\ &+\frac{\alpha}{4\lambda}\int_0^{\Psi}\left(\Psi_3-\frac{2(\Psi_M-\Psi_3)\Psi_3}{\Psi_M\cosh(\omega_5\xi)+(\Psi_M-2\psi_3)}\right)^{\frac{1}{2}}\\ &+\frac{a}{\lambda}\int_0^{\Psi}\left(\Psi_3-\frac{2(\Psi_M-\Psi_3)\Psi_3}{\Psi_M\cosh(\omega_5\xi)+(\Psi_M-2\psi_3)}\right)^{\frac{3}{2}}d\xi\\ &-\Sigma(\xi)\hat{h}_2,\end{aligned} \tag{3.21}$$

where $\omega_5 = \sqrt{\frac{2\beta_5\Psi_3|\Psi_M-\Psi_3|}{3}}$.

Hence, we have the following solution equation (1.1):

$$A(x,t) = i\left(\Psi_3 - \frac{2\Psi_3(\Psi_M - \Psi_3)}{\Psi_M\cosh(\omega_4\xi) + (\Psi_M - 2\Psi_3)}\right)^{\frac{1}{4}} \times \exp\left[-i\Theta + i(\nu x - \lambda t)\right], \tag{3.22}$$

where, $\Theta(\xi)$ is given by equation (3.21).

(iii) Corresponding to the level curves defined by $H(\Phi, y) = h, h \in (h_1, \infty)$, there exists a family of periodic orbits of system (1.16), enclosing five equilibrium points. It has the same parametric representation as equation (3.4).

(3) $\beta_3 = \frac{8}{3}\sqrt{3\beta_1\beta_5},\ \beta_5 < 0$ (see [Fig. 3 (d)])

In this case, we have that $h_2 < 0 < h_1$.

(i) Corresponding to the level curves defined by $H(\Phi, y), h \in (h_2, 0]$, there exists a family of two periodic orbits of system (1.16), enclosing the equilibrium point $E_1(-\Phi_1, 0)$ and $E_4(\Phi_1, 0)$. It has the same parametric representation as equation (3.13).

(ii) Corresponding to the level curves defined by $H(\Phi, y) = h, h \in (0, h_1)$, there exist three families of periodic orbits of system (1.16), enclosing the equilibrium points $E_1(-\Phi_1, 0)$, $E_4(\Phi_1, 0)$ and $E_0(0, 0)$ respectively. They have the same parametric representation as equation (3.13) and equation (3.16).

(iii) Corresponding to the level curves defined by $H(\Phi, y) = h_1$, there exist two homoclinic orbits of system (1.16), enclosing the equilibrium points $E_1(-\Phi_1, 0)$ and $E_4(\Phi_1, 0)$, and two hetroclinic orbits, enclosing the equilibrium point $E_0(0, 0)$. They have the same parametric representation as equation (3.19) and equation (3.22).

(iv) Corresponding to the level curves defined by $H(\Phi, y) = h, h \in (h_1, \infty)$, there exists a family of periodic orbits of system (1.16), enclosing five equilibrium points. It has the same parametric representation as equation (3.4).

(4) $\beta_3 > 0,\ \beta_5 < 0$. (see [Fig. 3(e)])

In this case, we have that $\Delta = 0,\ h_2 < 0 < h_1$.

(i) For $h \in (h_2, 0)$, the level curves defined by $H(\Phi, y) = h$, there exists a family of periodic orbits of system (1.16), enclosing the equilibrium point $E_0(0, 0)$. It has the same parametric representation as equation (3.4).

(ii) For $h \in (0, h_1)$, the level curves defined by $H(\Phi, y) = h$, there exist three families of periodic orbits of system (1.16), enclosing the equilibrium points $E_1(-\Phi_1, 0)$ and $E_4(\Phi_1, 0)$, and $E_0(0, 0)$, respectively. They have the same parametric representation as equation (3.19) and equation (3.22).

(iii) Corresponding to the level curves defined by $H(\Phi, y) = h_1$, there exist two homoclinic orbits of system (1.16), enclosing the equilibrium points $E_1(-\Phi_1, 0)$ and $E_4(\Phi_1, 0)$ and two hetroclinic orbits, enclosing the equilibrium point $E_0(0, 0)$. They have the same parametric representation as equation (3.13) and equation (3.16).

(iv) For $h \in (h_1, \infty)$, to the level curves defined by $H(\Phi, y) = h$, there exists a family of periodic orbits of system (1.16), enclosing five equilibrium points. It has the same parametric representation as equation (3.4).

(5) $\beta_3 < 0,\ \beta_5 > 0$. (see [Fig. 3(f)])

In this case, we have that $\Phi_1 = \Phi_2,\ h_2 = h_1$.

(i) For $h \in (0, h_1)\bigcup(h_1, \infty)$ the level curves defined by $H(\Phi, y) = h$, there exists a family of periodic orbits of system (1.16), enclosing the equilibrium point $E_0(0, 0)$. It has the same solution as equation (3.4).

(ii) Corresponding to the level curves defined by $H(\Phi, y) = h_1$, there exist two hetroclinic orbits connecting the equilibrium point (cusp) $E_2(-\Phi_2, 0)$ and $E_3(\Phi_2, 0)$.

For two hetroclinic orbits, equation (3.1) becomes

$$\sqrt{\frac{8\beta_5}{3}}\xi = \pm\int_0^{\Psi} \frac{d\Psi}{(\Psi_2-\Psi)\sqrt{(\Psi_2-\Psi)\Psi}},$$

where, $\Phi_2 = \pm\sqrt{\Psi_2}$. Then, we have the following parametric representation of $\Phi(\xi)$:

$$\Phi(\xi) = \pm\sqrt{\Psi(\xi)} = \pm\sqrt{\Psi_2}\left(1-\frac{4}{4-(\Psi_2\xi)^2}\right)^{\frac{1}{2}},\ \xi \in (0,\infty). \tag{3.23}$$

Thus, we have from equation (1.9) that

$$\Theta(\xi) = \left(\frac{1}{2}(2\nu-\mu)+\frac{\beta}{2\lambda}\right)\xi + \frac{\hat{h}_2}{|\Psi_2|}\left(\sqrt{4-(\Psi_2\xi)^2} - 2\ln\left(\frac{2-\sqrt{4-(\Psi_2\xi)^2}}{\Psi_2\xi}\right)\right)$$
$$+\frac{\beta|\Psi_2|}{3\lambda}\sqrt{4-(\Psi_2\xi)^2}+\frac{\alpha|\Psi_2|}{4\lambda}\sqrt{4-(\Psi_2\xi)^2}+\frac{a}{\lambda}\int_0^{\Psi}\sqrt{\left(1-\frac{4}{4-(\Psi_2\xi)^2}\right)^3}d\xi. \tag{3.24}$$

Hence, we have the following solution equation (1.1):

$$A(x,t) = i\left(1-\frac{4}{4-(\Psi_2\xi)^2}\right)^{\frac{1}{4}} \times \exp\left[-i\Theta + i(\nu x-\lambda t)\right], \tag{3.25}$$

where, $\Theta(\xi)$ is given by equation (3.24).

3.3. Explicit parametric representations of the solutions of system (1.16) when $\beta_1 > 0$

In this section we consider case 4 of section 2, for $\beta_1 > 0$. (see [Fig. 4(a)–4(g)])

(1) $\beta_3 > 0,\ \beta_5 > 0,\ h_2 < h_1 = 0$. (see [Fig. 4(a)])

(i) Corresponding to the level curves defined by $H(\Phi, y) = h, h \in (h_2, h_1)$, there exist two families of periodic orbits of system (1.16). Equation (3.1) has the form

$$\sqrt{\frac{8\beta_5}{3}}\xi = \int_{r_2}^{\Psi} \frac{d\Psi}{\sqrt{(\Psi-r_4)\Psi(\Psi-r_2)(r_1-\Psi)}},$$

where $r_4 < 0 < r_2 < r_1$. It gives rise to the parametric representations of two families of the periodic orbits of system (1.16) as follows:

$$\Phi(\xi) = \pm\sqrt{\Psi(\xi)} = \pm\left(\frac{r_2}{1-\alpha_3^2\mathrm{sn}^2(\omega_3\xi,k)}\right)^{\frac{1}{2}}, \tag{3.26}$$

where $k^2 = \frac{(r_1-r_2)|r_4|}{r_1(r_2-r_4)},\ \alpha_4^2 = \frac{r_1-r_2}{r_1} < 1,\ \omega_6 = \frac{1}{2}\sqrt{\frac{\beta_5}{6}r_1(r_2-r_4)}$.

Thus, we have from equation (1.9) that

$$\Theta(\xi) = \frac{1}{2}(2\nu-\mu)\xi + \frac{\beta r_2}{3\lambda}\Pi\big(\arcsin\left(\mathrm{sn}\left(\omega_6\xi,k\right)\right),\alpha_4^2,k\big) + \left(\frac{\hat{h}_2}{\sqrt{r_2}}\right)\int_{r_2}^{\Psi}\sqrt{1-\alpha_4^2\mathrm{sn}^2(\omega_6\xi,k)}d\xi$$

$$+\left(\frac{\alpha}{4\lambda}\sqrt{r_2}\right)\int_{r_2}^{\Psi}\frac{d\xi}{\sqrt{1-\alpha_4^2\mathrm{sn}^2(\omega_6\xi,k)}}+\left(\frac{a}{\lambda}r_2^{\frac{3}{2}}\right)\int_{r_2}^{\Psi}\frac{d\xi}{\sqrt[3]{1-\alpha_4^2\mathrm{sn}^2(\omega_6\xi,k)}}. \tag{3.27}$$

Hence, we have the following solution equation (1.1):

$$A(x,t)=i\left(\frac{r_2}{1-\alpha_4^2\mathrm{sn}^2(\omega_6\xi,k)}\right)^{\frac{1}{4}}\times\exp\left[-i\Theta+i(\nu x-\lambda t)\right], \tag{3.28}$$

where, $\Theta(\xi)$ is given by equation (3.27).

(ii) Corresponding to the level curves defined by $H(\Phi,y)=h_1$, there exist two homoclinic orbits of system (1.16) to the multiple equilibrium point at $E_0(0,0)$. Equation (3.1) has the form

$$\sqrt{\frac{8\beta_5}{3}}\xi=\pm\int_{\Psi}^{\Psi_M}\frac{d\Psi}{\Psi\sqrt{\Psi\left(\frac{4}{3}|\beta_3|-\Psi\right)}}.$$

Hence, we obtain the following parametric representations of system (1.16):

$$\Phi(\xi)=\pm\sqrt{\Psi(\xi)}=\pm\left(\frac{12|\beta_3|}{9+16\beta_3^2\xi^2}\right)^{\frac{1}{2}}. \tag{3.29}$$

Thus, we have from equation (1.9) that

$$\Theta(\xi)=-\frac{1}{2}(\mu-2\nu)+\frac{\hat{h}_2}{2\sqrt{3\beta_3}}\left[\sqrt{9+16\beta_3^2\xi^2}+3\ln\left(\xi+\frac{\sqrt{9+16\beta_3^2\xi^2}}{4\beta_3^2}\right)\right]+\frac{\beta}{3\lambda}\arctan(4\beta_3\xi)$$
$$+\frac{\alpha\sqrt{3}}{8\lambda\sqrt{\beta_3}}\ln\left(\xi+\frac{\sqrt{9+16\beta_3^2\xi^2}}{4\beta_3}\right)+\frac{a\sqrt[3]{12\beta_3}}{\lambda}\int_{\Psi}^{\Psi_M}\frac{d\xi}{\sqrt[3]{9+16\beta_3^2\xi^2}}. \tag{3.30}$$

Hence, we have the following solution equation (1.1):

$$A(x,t)=i\left(\frac{12|\beta_3|}{9+16\beta_3^2\xi^2}\right)^{\frac{1}{4}}\times\exp\left[-i\Theta+i(\nu x-\lambda t)\right], \tag{3.31}$$

where, $\Theta(\xi)$ is given by equation (3.30).

(iii) For $h\in(0,\infty)$, the level curves defined by $H(\Phi,y)=h$, there exists a family of periodic orbits of system (1.16), enclosing three equilibrium points. It has the same parametric representation of solution as equation (3.4).

(2) $0<\beta_3<\frac{4}{3}\sqrt{3\beta_1\beta_5}$, $\beta_5<0$. (see [Fig. 4(b)])

(i) Corresponding to the level curves defined by $H(\Phi,y)=h, h\in(h_2,0)$, there exist two families of periodic orbits of system (1.16). The parametric representations of solution of equation (1.1) is the same as equation (3.28).

(ii) For $h=0$, to the level curves defined by $H(\Phi,y)=h$, there exist two homoclinic orbits of system (1.16). Equation (3.1) has the form

$$\sqrt{\frac{8\beta_5}{3}}\xi=\int_{\Psi}^{\Psi_M}\frac{d\Psi}{\Psi\sqrt{(\Psi_M-\Psi)(\Psi-r_4)}},$$

where $r_4 < 0 < \Psi_M, r_4 = -\frac{3\beta_3+\sqrt{\Delta}}{4\beta_5},\ \Psi_M = \frac{-3\beta_3+\sqrt{\Delta}}{4\beta_5},\ \Delta = 9\beta_3^2 + 12\beta_1|\beta_5|$. It gives rise to the parametric representations of homoclinic orbit of system (1.3) as follows:

$$\Phi(\xi) = \pm\sqrt{\Psi(\xi)} = \pm\left(\frac{3\beta_1}{\sqrt{\Delta}\cosh\left(\sqrt{\frac{3\beta_1}{4\beta_5}}\xi\right) + 3\beta_3}\right)^{\frac{1}{2}}. \tag{3.32}$$

Thus, we have from equation (1.9) that

$$\begin{aligned}\Theta(\xi) =& \frac{1}{2}(2\nu-\mu)\xi + \frac{\beta}{18\lambda\beta_1\sqrt{|\beta_5|}}\operatorname{arctan}\left(\frac{2\sqrt{3\beta_1|\beta_5|}}{3\beta_3+\sqrt{\Delta}}\right)\tanh\left(\frac{1}{4}\sqrt{\frac{3\beta_1}{|\beta_5|}}\xi\right)\\ &+ \frac{\alpha}{4\lambda}\int_{\Psi}^{\Psi_M}\left(\frac{3\beta_1}{\sqrt{\Delta}\cosh\left(\sqrt{\frac{3\beta_1}{4\beta_5}}\xi\right)+3\beta_3}\right)^{\frac{1}{2}}d\xi\\ &+ \frac{\hat{h}_2}{\sqrt{6\beta_1}}\int_{\Psi}^{\Psi_M}\sqrt{3\beta_3+\sqrt{\Delta}\cosh\left(\sqrt{\frac{3\beta_1}{4\beta_5}}\xi\right)}d\xi\\ &+ \frac{a}{\lambda}\int_{\Psi}^{\Psi_M}\left(\frac{3\beta_1}{\sqrt{\Delta}\cosh\left(\sqrt{\frac{3\beta_1}{4\beta_5}}\xi\right)+3\beta_3}\right)^{\frac{3}{2}}d\xi.\end{aligned} \tag{3.33}$$

Hence, we have the following solution of equation (1.1):

$$A(x,t) = i\left(\frac{3\beta_1}{\sqrt{\Delta}\cosh\left(\sqrt{\frac{3\beta_1}{4\beta_5}}\xi\right)+3\beta_3}\right)^{\frac{1}{4}} \times \exp\left[-i\Theta + i(\nu x - \lambda t)\right], \tag{3.34}$$

where, $\Theta(\xi)$ is given by equation (3.33).

(iii) Corresponding to the level curves defined by $H(\Phi, y) = h, h \in (0,\infty)$, there exists a family of periodic orbits of system (1.16), enclosing three equilibrium points. It has the same parametric representation as equation (3.4).

(3) $\beta_3 > 0,\ \beta_5 < 0$. (see [Fig. 4(c)])

In this case $h_2 < 0 < h_1$, $\Phi_1 = -\Phi_4$ and $\Phi_2 = -\Phi_3$.

(i) Corresponding to the level curves defined by $H(\Phi, y) = h, h \in (h_2, 0)$, there exist two families of periodic orbits of system (1.16). The parametric representations of solution of equation (1.1) is the same as equation (3.28).

(ii) Corresponding to the level curves defined by $H(\Phi, y) = 0$, there exist two homoclinic orbits of system (1.16) with the following parametric representation:

$$\Phi(\xi) = \pm\sqrt{\Psi(\xi)} = \pm\left(\frac{2\Psi_M\Psi_L}{(\Psi_L-\Psi_M)\cosh(\omega_7\xi)+(\Psi_M+\Psi_L)}\right)^{\frac{1}{2}}, \tag{3.35}$$

where $0 < \Psi_M < \Psi_L$.

Thus, we have from equation (1.9) that

$$\Theta(\xi) = \frac{1}{2}(2\nu-\mu)\xi + \frac{\beta}{3\lambda\sqrt{\Psi_M\Psi_L}}\operatorname{arccoth}\left(\sqrt{\frac{\Psi_M}{\Psi_L}}\right)\tanh\left(\frac{1}{2}\sqrt{\Psi_M\Psi_L}\xi\right)$$

$$+\frac{\hat{h}_2}{\sqrt{2\Psi_M\Psi_L}}\int_{\Psi}^{\Psi_M}\sqrt{(\Psi_M+\Psi_L)+(\Psi_L-\Psi_M)\cosh(\omega_7\xi)}d\xi$$
$$+\frac{\alpha}{4\lambda}\int_{\Psi}^{\Psi_M}\left(\frac{2\Psi_M\Psi_L}{(\Psi_L-\Psi_M)\cosh(\omega_7\xi)+(\Psi_M+\Psi_L)}\right)^{\frac{1}{2}}$$
$$+\frac{a}{\lambda}\int_{\Psi}^{\Psi_M}\left(\frac{2\Psi_M\Psi_L}{(\Psi_L-\Psi_M)\cosh(\omega_7\xi)+(\Psi_M+\Psi_L)}\right)^{\frac{3}{2}}d\xi. \tag{3.36}$$

Hence, we have the following solution equation (1.1):

$$A(x,t)=i\left(\frac{2\Psi_M\Psi_L}{(\Psi_L-\Psi_M)\cosh(\omega_7\xi)+(\Psi_M+\Psi_L)}\right)^{\frac{1}{4}}\times\exp\left[-i\Theta+i(\nu x-\lambda t)\right], \tag{3.37}$$

where, $\omega_7=\sqrt{\frac{\beta_5\Psi_M\Psi_L}{6}}$ and $\Theta(\xi)$ is given by equation (3.36).

(iii) For $h\in(0,h_1)$, the level curves defined by $H(\Phi,y)=h$, there exists a family of periodic orbits of system (1.16), enclosing three equilibrium points $E_1(-\Phi_1,0)$, $E_2(\Phi_2,0)$, and $E_0(0,0)$.

Now, equation (3.1) has the forms

$$\sqrt{\frac{8\beta_5}{3}}\xi=\int_0^{\Psi}\frac{d\Psi}{\sqrt{(\Psi-\Psi_l)\Psi(r_1-\Psi)(\Psi_L-\Psi)}},$$

where $\Psi_l<0<r_1<\Psi_L$. Therefore, the periodic orbits has a parametric representations:

$$\Phi(\xi)=\pm\sqrt{\Psi(\xi)}=\pm\left(\Psi_l-\frac{\Psi_l}{1-\alpha_2^2\mathrm{sn}^2(\omega_4\xi,k)}\right)^{\frac{1}{2}}, \tag{3.38}$$

where $\alpha_5^2=\frac{r_1}{r_1-\Psi_l}$, $\omega_8=\sqrt{\frac{\beta_5\Psi_L(r_1-\Psi_l)}{6}}$, $k^2=\alpha_5^2\left(1-\frac{\Psi_l}{\Psi_L}\right)$.

Thus, we have from equation (1.9) that

$$\Theta(\xi)=\left(\frac{1}{2}(2\nu-\mu)+\frac{\beta\Psi_l}{3\lambda}\right)\xi-\frac{\beta\Psi_l}{3\lambda}\Pi\left(\mathrm{arcsinh}(\mathrm{sn}(\omega_8\xi,k)),\alpha_5^2,k\right)$$
$$+\frac{\hat{h}_2}{\sqrt{\Psi_l}}\int_0^{\Psi}\sqrt{\frac{1}{1-\frac{1}{1-\alpha_5^2\mathrm{sn}^2(\omega_8\xi,k)}}}d\xi+\frac{\alpha\sqrt{\Psi_l}}{4\lambda}\int_0^{\Psi}\sqrt{1-\frac{1}{1-\alpha_5^2\mathrm{sn}^2(\omega_8\xi,k)}}d\xi$$
$$+\frac{a}{\lambda}\int_0^{\Psi}\left(\Psi_l-\frac{\Psi_l}{1-\alpha_5^2\mathrm{sn}^2(\omega_8\xi,k)}\right)^{\frac{3}{2}}d\xi. \tag{3.39}$$

Hence, we have the following solution equation (1.1):

$$A(x,t)=i\left(\Psi_l-\frac{\Psi_l}{1-\alpha_5^2\mathrm{sn}^2(\omega_8\xi,k)}\right)^{\frac{1}{4}}\times\exp\left[-i\Theta+i(\nu x-\lambda t)\right], \tag{3.40}$$

where, $\Theta(\xi)$ is given by equation (3.39).

(iv) Corresponding to the level curves defined by $H(\Phi,y)=h_1$, there exist two hetroclinic orbits, enclosing the equilibrium points $E_2(-\Phi_2,0)$, $E_0(0,0)$ and $E_3(\Phi_2,0)$ connecting the equilibrium points $E_1(-\Phi_1,0)$ and $E_4(\Phi_1,0)$.

Then, we have the following parametric representation as system (1.16),

$$\Phi(\xi) = \pm\sqrt{\Psi(\xi)} = \pm\left(\Psi_1 - \frac{2\Psi_1(r_M - \Psi_1)}{r_M\cosh(\omega_9\xi) - (2\Psi_1 - r_M)}\right)^{\frac{1}{2}}, \tag{3.41}$$

where $\omega_9 = \sqrt{\frac{\beta_5\Psi_1|r_M-\Psi_1|}{6}}$, $0 < r_M < \Psi_1$.

Thus, from equation (1.9) we have

$$\begin{aligned}\Theta(\xi) = &\left(\frac{1}{2}(2\nu-\mu) + \frac{\beta\Psi_1}{3\lambda}\right)\xi - \frac{\beta}{3\lambda}\arctan\left(\sqrt{\frac{\Psi_1}{|r_M-\Psi_1|}}\right)\tanh\left(\frac{1}{2}\omega_9\xi\right)\\ &+ \hat{h}_2\int_0^{\Psi}\left(\frac{r_M\cosh(\omega_9\xi) - (2\Psi_1 - r_M)}{\Psi_1 r_M\cosh(\omega_9\xi) - \Psi_1 r_M}\right)^{\frac{1}{2}} d\xi\\ &+ \frac{\alpha}{4\lambda}\int_0^{\Psi}\left(\frac{\Psi_1 r_M\cosh(\omega_9\xi) - \Psi_1 r_M}{r_M\cosh(\omega_9\xi) - (2\Psi_1 - r_M)}\right)^{\frac{1}{2}} d\xi\\ &+ \frac{a}{\lambda}\int_0^{\Psi}\left(\frac{\Psi_1 r_M\cosh(\omega_9\xi) - \Psi_1 r_M}{r_M\cosh(\omega_9\xi) - (2\Psi_1 - r_M)}\right)^{\frac{3}{2}} d\xi.\end{aligned} \tag{3.42}$$

Hence, we have the following solution equation (1.1):

$$A(x,t) = i\left(\Psi_1 - \frac{2\Psi_1(r_M - \Psi_1)}{r_M\cosh(\omega_9\xi) - (2\Psi_1 - r_M)}\right)^{\frac{1}{4}} \times \exp\left[-i\Theta + i(\nu x - \lambda t)\right], \tag{3.43}$$

where, $\Theta(\xi)$ is given by equation (3.42).

(4) $0 < \beta_3 < 2\sqrt{\beta_1\beta_5}$, $\beta_5 < 0$. (see [Fig. 4 (d)])

In this case $0 < h_1 < \infty$, $\Phi_4 = -\Phi_1$ and $\Phi_3 = -\Phi_2$.

(i) Corresponding to the level curves defined by $H(\Phi, y) = h, h \in (0, h_1)$, there exist two families of periodic orbits of system (1.16). The parametric representations of solution of equation (1.1) is the same as equation (3.28).

(ii) Corresponding to the level curves defined by $H(\Phi, y) = h_1$, there exist pairs of hetroclinic orbits connecting $E_0(0,0)$ and $E_1(\pm\Phi_1, 0)$, enclosing the equilibrium points $E_2(\pm\Phi_2, 0)$. Then, we have the following parametric representation as system (1.16).

$$\Phi(\xi) = \pm\sqrt{\Psi(\xi)} = \pm\left(\frac{\Psi_4}{1 + \exp(\Psi_4\xi)}\right)^{\frac{1}{2}}. \tag{3.44}$$

Thus, from equation (1.9) we have

$$\begin{aligned}\Theta(\xi) = &\left(\frac{1}{2}(2\nu-\mu) + \frac{\beta\Psi_4}{3\lambda} - \frac{\alpha\sqrt{\Psi_4}}{2\lambda}\right)\xi - \frac{\beta}{3\lambda}\ln(1+\exp(\Psi_4\xi))\\ &- \frac{\alpha}{2\lambda\sqrt{\Psi_4}}\ln\left(1 + \sqrt{1+\exp(\Psi_4\xi)}\right) + \frac{\hat{h}_2}{\sqrt{\Psi_4}}\int_{r_0}^{\Psi}\sqrt{1+\exp(\Psi_4\xi)}d\xi\\ &+ \frac{a}{\lambda}\int_{r_0}^{\Psi}\left(\frac{\Psi_4}{1+\exp(\Psi_4\xi)}\right)^{\frac{3}{2}} d\xi.\end{aligned} \tag{3.45}$$

Hence, we have the following solution equation (1.1)

$$A(x,t) = i\left(\frac{\Psi_4}{1+\exp(\Psi_4\xi)}\right)^{\frac{1}{4}} \times \exp\left[-i\Theta + i(\nu x - \lambda t)\right], \tag{3.46}$$

where, $r_0 \in (0, \Psi_4)$ and $\Theta(\xi)$ is given by equation (3.45).

(5) $\beta_3 > 0,\ \beta_5 < 0$. (see [Fig. 4(e)])

In this case $h_2 < 0 < h_1$, $\Phi_4 = -\Phi_1$ and $\Phi_3 = -\Phi_2$.

(i) For $h \in (h_2, h_1)$, the level curves defined by $H(\Phi, y) = h$, there exist a pair of periodic orbits, enclosing the equilibrium points $E_2(\pm\Phi_2, 0)$. Then, we have the following parametric representation as system (1.16).

$$\Phi(\xi) = \pm\sqrt{\Psi(\xi)} = \pm\left(r_2 - \frac{r_2 - r_4}{1-\alpha_6^2\mathrm{sn}(\Omega_1\xi,k)}\right)^{\frac{1}{2}}, \tag{3.47}$$

where $r_4 < \Psi_2 < r_3 < 0 < r_2,\ \ \alpha_6^2 = \frac{|r_4-r_3|}{r_2-r_3}\ \ k^2 = \frac{r_2}{|r_4|}\alpha_6^2$ and $\Omega_1 = \sqrt{\frac{\beta_5(r_2-r_3)|r_4|}{6}}$.

Thus, from equation (1.9) we have

$$\begin{aligned}\Theta(\xi) = &\left(\frac{1}{2}(2\nu-\mu) + \frac{\beta r_2}{3\lambda}\right)\xi + \frac{\beta}{3\lambda\Omega_1}(r_4 - r_2)\times\Pi\left(\mathrm{arcsinh}\left(\mathrm{sn}\left(\Omega_1\xi,k\right)\right), \alpha_6^2, k\right)\\ &+ \hat{h}_2\int_{r_4}^{\Psi}\left(\frac{1-\alpha_6^2\mathrm{sn}(\Omega_1\xi,k)}{r_4 - r_2\alpha_6^2\mathrm{sn}(\Omega_1\xi,k)}\right)^{\frac{1}{2}}d\xi + \frac{\alpha}{4\lambda}\int_{r_4}^{\Psi}\left(\frac{r_4 - r_2\alpha_6^2\mathrm{sn}(\Omega_1\xi,k)}{1-\alpha_6^2\mathrm{sn}(\Omega_1\xi,k)}\right)^{\frac{1}{2}}d\xi\\ &+ \frac{a}{\lambda}\int_{r_4}^{\Psi}\left(\frac{r_4 - r_2\alpha_6^2\mathrm{sn}(\Omega_1\xi,k)}{1-\alpha_6^2\mathrm{sn}(\Omega_1\xi,k)}\right)^{\frac{3}{2}}d\xi.\end{aligned} \tag{3.48}$$

Hence, we have the following solution equation (1.1):

$$A(x,t) = i\left(r_2 - \frac{r_2 - r_4}{1-\alpha_6^2\mathrm{sn}(\Omega_1\xi,k)}\right)^{\frac{1}{4}} \times \exp\left[-i\Theta + i(\nu x - \lambda t)\right], \tag{3.49}$$

where, $\Theta(\xi)$ is given by equation (3.48).

(ii) Corresponding to the level curves defined by $H(\Phi, y) = h_1$, there exist two homoclinic orbits, enclosing the equilibrium points $E_2(-\Phi_2, 0)$ and $E_3(\Phi_2, 0)$ at $E_1(-\Phi_1, 0)$ and $E_4(\Phi_1, 0)$, respectively.

Then, we have the following parametric representation of system (1.16).

$$\Phi(\xi) = \pm\sqrt{\Psi(\xi)} = \pm\left(\Psi_1 + \frac{2\Psi_1(\Psi_M - \Psi_1)}{\Psi_M\cosh\left(\Omega_2\xi\right) - (2\Psi_1 - \Psi_M)}\right)^{\frac{1}{2}}, \tag{3.50}$$

where $\Psi_1 < \Psi_M < 0$ and $\Omega_2 = \sqrt{\frac{\beta_5\Psi_1(\Psi_M - \Psi_1)}{6}}$.

Thus, from equation (1.9) we have

$$\begin{aligned}\Theta(\xi) = &\left(\frac{1}{2}(2\nu-\mu) + \frac{\beta\Psi_1}{3\lambda}\right)\xi + \frac{\beta}{3\lambda}\arctan\left(\sqrt{\frac{\Psi_1}{\Psi_M - \Psi_1}}\right)\tanh\left(\frac{1}{2}\Omega_2\xi\right)\\ &+ \hat{h}_2\int_0^{\Psi}\left(\frac{\Psi_1\Psi_M\cosh\left(\Omega_2\xi\right) - 4\Psi_1^2 + 3\Psi_1\Psi_M}{\Psi_M\cosh\left(\Omega_2\xi\right) - (2\Psi_1 - \Psi_M)}\right)^{\frac{1}{2}}d\xi\end{aligned}$$

$$+\frac{\alpha}{4\lambda}\int_0^\Psi\left(\frac{\Psi_M\cosh(\Omega_2\xi)-(2\Psi_1-\Psi_M)}{\Psi_1\Psi_M\cosh(\Omega_2\xi)-4\Psi_1^2+3\Psi_1\Psi_M}\right)^{\frac{1}{2}}$$

$$+\frac{a}{\lambda}\int_0^\Psi\left(\frac{\Psi_M\cosh(\Omega_2\xi)-(2\Psi_1-\Psi_M)}{\Psi_1\Psi_M\cosh(\Omega_2\xi)-4\Psi_1^2+3\Psi_1\Psi_M}\right)^{\frac{3}{2}}d\xi. \tag{3.51}$$

Hence, we have the following solution equation (1.1):

$$A(x,t)=i\left(\Psi_1+\frac{2\Psi_1(\Psi_M-\Psi_1)}{\Psi_M\cosh(\Omega_2\xi)-(2\Psi_1-\Psi_M)}\right)^{\frac{1}{4}}\times\exp\left[-i\Theta+i(\nu x-\lambda t)\right], \tag{3.52}$$

where, $\Theta(\xi)$ is given by equation (3.51).

4. Conclusion

To sum up, we have proved the following Theorems:

Theorem 4.1. *Suppose that the parametric conditions* $\beta_5\neq 0,\ \beta_2=\beta_4=\beta_6=0$ *of system* (1.14) *holds. Depending on the changes of system parameters, the bifurcations of phase portraits of system* (1.16) *are shown in Fig.1–Fig.4.*

Theorem 4.2. *(i) By using the method of dynamical systems we have found fourteen solutions depending on change of parameters regions of* (β_3,β_5) *for* $\beta_1<0$ *and* $\beta_1>0$*, corresponding to the periodic, homoclinic and hetroclinic orbits of system* (1.16)*. The thirteenth order derivative nonlinear Schrödinger equation* (1.1) *has sixteen exact solutions given by equations* (3.4)*,* (3.7)*,* (3.10)*,* (3.13)*,* (3.16)*,* (3.19)*,* (3.22)*,* (3.25)*,* (3.28)*,* (3.31)*,* (3.34)*,* (3.37)*,* (3.40)*,* (3.43)*,* (3.46)*,* (3.49) *and* (3.52)*.*

(ii) System (1.3) *has sixteen exact explicit solutions* $\phi(\xi)=\sqrt{\Phi}\sin\Theta$*, and* $\psi(\xi)=\sqrt{\Phi}\cos\Theta$*, where* $\phi(\xi)$ *and* $\psi(\xi)$ *are given in section 1.*

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