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STABILITY AND HOPF BIFURCATION OF A MODIFIED DELAY PREDATOR-PREY MODEL WITH STAGE STRUCTURE

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Abstract In this paper, a modified delay predator-prey model with stage structure is established, which involves the economic factor and internal competition of all the prey and predator populations. By the methods of normal form and characteristic equation, we obtain the stability of the positive equilibrium point and the sufficient condition of the existence of Hopf bifurcation. We analyze the influence of the time delay on the equation and show the occurrence of Hopf bifurcation periodic solution. The simulation gives a visual understanding for the existence and direction of Hopf bifurcation of the model.

Keywords Delayed differential equation, hypernormal form, equilibrium point, stability, Hopf bifurcation.

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1. Introduction

In the ecosystem, many systems of interest such as the predator-prey models in population dynamics involve time delay [7, 17]. The study of dynamical behavior (Hopf bifurcation, periodic solution, chaos, etc.) for differential equations with delay has significant biological and mathematical meaning.

Hopf bifurcation is a very important phenomenon occurs in the differential equations with delay. It seems that the existence of Hopf bifurcations for delayed differential equations can be dated back to the work of Chafee [5] in 1971. However, the first proof of the Hopf bifurcation theorem for delay differential equations under analytically computable condition was presented by Chow and Mallet-Paret in 1977 [4]. Since then there have been various contributions on the stability and existence of Hopf bifurcation for time delay equations, especially for the delayed population dynamical equations. Shao and Dai [20] established an impulsive delay predator-prey model with stage structure and Beddington-type functional response concerning ratio-dependent. They obtained the existence and global attractivity of the predator-extinction periodic solution by using the discrete dynamical system determined by the stroboscopic map. Niu and Jiang [15] researched a predator-prey

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equation with two delays and stage structure for the prey according to the theory of bifurcation analysis of neutral delay differential equations. They obtained the stability and Hopf bifurcation of the positive equilibrium, and showed how the neutral terms affect the dynamical behavior of the prey and the predator in the equation. They found that the neutral delay which makes the predator-prey equation more complicated may induces double Hopf bifurcations. Meng and Huo [13] investigated a class of Lotka-Volterra mutualistic equation with time delays of benefit and feedback. The local stability of the positive equilibrium and the existence of Hopf bifurcation were obtained by analyzing the characteristic equation. They obtained explicit formulas to determine the properties of the Hopf bifurcation by using the normal form method and center manifold theorem. Guo et al. [8] investigated a class of three dimensional delayed Gause-type predator-prey models. They presented the boundedness of solutions and the permanence of system, and gave the global asymptotically stability of the positive equilibrium under some parameter conditions by the Lyapunov function.

In predator-prey models, as mentioned in [20], many species usually go through two or more life stages as they proceed from birth to death. Thus it is practical to introduce the stage structure into models. Bandyopadhyay and Banerjee [1] presented a class of predator-prey equation with time delay and stage structure as follows

$$\dot{x} = Lx + F(x), \tag{1.1}$$

where $x = (x_J, x_A, x_P)^{\mathrm{T}}$, and

$$L = \begin{pmatrix} -d_1-\alpha & r_1 & 0 \\ \alpha & -d_2 & 0 \\ 0 & 0 & -d_3 \end{pmatrix}, \quad F(x) = \begin{pmatrix} -s_1 x_J^2 - \beta x_J x_P \\ 0 \\ \beta x_J(t-\tau) x_P(t-\tau) - s_3 x_P^2 \end{pmatrix}.$$

Here x_J, x_A and x_P represent the population density of juvenile prey, adult prey and predator respectively. They obtained the sufficient condition for the local stability of the equation (1.1) near the positive equilibrium point based on the bifurcation theory of dynamical systems.

On the basis of the economic theoretical relationship of public fishery proposed in 1954 [9], Zhang et al. [22] added a term of harvest E about predator x_P into equation (1.1). Then it can be described by a differential-algebraic equation due to economic factor.

$$\begin{aligned} \dot{x} &= \tilde{L}x + F(x), \\ m &= E(px_P - c), \end{aligned} \tag{1.2}$$

where $\tilde{L} = \begin{pmatrix} -d_1-\alpha & r_1 & 0 \\ \alpha & -d_2 & 0 \\ 0 & 0 & -d_3-E \end{pmatrix}$. E represents the harvest effort about predator x_P. p and c represent the unit price of predator and the unit cost of harvest process respectively. m is the economic profit obtained from the harvest process. They found that the increase of delay destabilized the positive equilibrium point of the system and bifurcated into small amplitude periodic solution.

In [1] and [22], the stage structure and economic factor were considered, but the internal competition behavior of the adult prey population was not taken into account. From this point, we add the harvest terms in all the juvenile prey, adult prey and predator, and also consider the internal competitive term in the equation.

Then the differential-algebraic equation becomes

$$\begin{aligned} \dot{x} &= \hat{L}x + \tilde{F}(x), \\ m &= E_3(px_P - c), \end{aligned} \tag{1.3}$$

where

$$\hat{L} = \begin{pmatrix} -d_1-\alpha-E_1 & r_1 & 0 \\ \alpha & -d_2-E_2 & 0 \\ 0 & 0 & -d_3-E_3 \end{pmatrix}, \quad \tilde{F}(x) = \begin{pmatrix} -s_1x_J^2-\beta x_J x_P \\ -s_2x_A^2 \\ \beta x_J(t-\tau)x_P(t-\tau)-s_3x_P^2 \end{pmatrix}.$$

r_1 is the birth rate of juvenile prey, α is the conversion rate of juvenile prey transforms to the adult prey, β is the capture rate of juvenile prey by predator. The mortality of juvenile prey, adult prey and the predator is proportional to its own population density, the ratio coefficients are $d_i(i = 1, 2, 3)$. In general, the predator will not transfer the energy to the next generation immediately and we noted that the gestation time delay is τ. $E_i(i = 1, 2, 3)$ represent the harvest effort about juvenile prey, adult prey and predator respectively. $s_j(j = 1, 2, 3)$ represent the internal competitive coefficient of juvenile prey, adult prey and predator population. All of these parameters are positive.

The purpose of this paper is to study the stability of the positive equilibrium point of the modified equation (1.3), and analyze the parameter conditions of the existence of Hopf bifurcation as well as the direction of bifurcation. Due to the complexity of the equation, firstly, we should simplify equation (1.3) into a simpler one by normal form (hypernormal form) theory. The theories and calculation methods can refer to [11, 14, 16, 21] and references therein. And then we will focus on the stability and bifurcation analysis. According to the latest literatures, there are two common methods for studying the Hopf bifurcation of delayed differential equations. One is Lyapunov function (functional) and the other is the analysis approach based on the characteristic root of the characteristic equation for the delayed differential equation [6, 12, 18]. There is no general rule for constructing Lyapunov function (functional). Here we choose the latter method to analyze the distribution of the roots of characteristic equation. We obtain the varied conditions of the real part of characteristic root and it provides theoretical basis for the property analysis of Hopf bifurcation.

The rest of the present paper is organized as follows: In section 2, we obtain explicit expressions for the hypernormal form in terms of the original delay differential equation. This enables us to obtain not only existence but also stability and bifurcation direction. In section 3, we analyze the stability of the positive equilibrium point of equation that reduced by normal form method and obtain the existence condition of the Hopf bifurcation. In section 4, we research the properties of the Hopf bifurcation by combining the normal form theory and the center manifold method. In section 5, we present the numerical simulation of the Hopf bifurcation under given parameter conditions.

2. Hypernormal form of equation (1.3)

Based on the normal form theory, equation (1.3) is simplified in this section. Let H_n be the linear space spanned by all monomials of degree n, then equation (1.3) can be expressed as the following formal series

$$V^{(0)} = V_1^{(0)} + V_2^{(0)},$$

where $V_m^{(0)} \in H_m (m = 1, 2)$ yield

$$\begin{aligned} V_1^{(0)} =& (r_1 x_A - (d_1 + \alpha + E_1) x_J) \partial_{x_J} \\ &+ (-(d_2 + E_2) x_A + \alpha x_J) \partial_{x_A} - (d_3 x_P + E_3 x_P) \partial_{x_P}, \end{aligned}$$

and

$$\begin{aligned} V_2^{(0)} =& (-s_1 x_J^2 - \beta x_J x_P) x_J) \partial_{x_J} + (-s_2 x_A^2) \partial_{x_A} \\ &- (\beta x_J (t - \tau) x_P (t - \tau) - s_3 x_P^2) \partial_{x_P}. \end{aligned}$$

Denote the linear operator $L^{(1)}$,

$$L^{(1)} : \quad H_2 \to H_2, \quad Y_2 \mapsto [Y_2, V_1^{(0)}], \quad Y_2 \in H_2,$$

where operator $[\cdot, \cdot]$ defined by $[u, v] = Du \cdot v - Dv \cdot u$ $(u, v \in H_2)$.

Theorem 2.1. *The first order normal form (hypernormal form) of equation (1.3) with the parameter condition* $(d_1 + \alpha + E_1)(d_2 + E_2) - \alpha r_1 = 0$ *is*

$$\begin{aligned} \dot{y} &= \hat{L} y + F^*(y), \\ m &= E_3 (p y_P - c), \end{aligned} \tag{2.1}$$

where $y = (y_J, y_A, y_P)^T$, $F^*(y) = \begin{pmatrix} a_1 y_J^2 + a_2 y_J y_P \\ b_1 y_A^2 \\ c_1 y_J (t-\tau) y_P (t-\tau) \end{pmatrix}$, $a_1 = -s_1$, $a_2 = -\beta$, $b_1 = -s_2$, $c_1 = \beta - \alpha r_1 + (d_1 + \alpha + E_1)(d_2 + E_2)$.

Proof. Let the basic vector Y_2 in linear space H_2 is

$$\begin{aligned} Y_2 =& (x_J (x_J + x_A + x_P) + x_A (x_A + x_P) + x_P^2) \partial_{x_J} \\ &+ (x_J (x_J + x_A + x_P) + x_A (x_A + x_P) + x_P^2) \partial_{x_A} \\ &+ (x_J (x_J + x_A + x_P) + x_A (x_A + x_P) + x_P^2) \partial_{x_P}, \end{aligned}$$

we have

$$\begin{aligned} [Y_2, V_1^{(0)}] =& (m_2 + m_1 (d_3 + E_3)) \partial_{x_J} \\ &+ (m_2 - \alpha m_1 + m_1 (d_2 + E_2)) \partial_{x_A} \\ &+ (m_2 + m_1 (d_1 + \alpha + E_1) - r_1 m_1) \partial_{x_P}, \end{aligned}$$

where

$$m_1 = x_J (x_J + x_A + x_P) + x_A (x_A + x_P) + x_P^2,$$

and

$$\begin{aligned} m_2 =& (-(x_A + 2x_J + x_P)(d_1 + \alpha + E_1) + \alpha (2x_A + x_J + x_P)) x_J \\ &+ (r_1 (x_A + 2x_J + x_P) - (2x_A + x_J + x_P)(d_2 + E_2)) x_A \\ &- (x_A + x_J + 2x_P)(d_3 + E_3) x_P. \end{aligned}$$

With the Maple software, under following parameter condition

$$(d_1 + \alpha + E_1)(d_2 + E_2) - \alpha r_1 = 0,$$

we get the complementary space $C_2^{(1)}$ to $\mathrm{Im}L^{(1)}$ in H_2,

$$C_2^{(1)} = (x_J^2 + x_J x_P)\partial_{x_J} + x_A^2 \partial_{x_A} + x_J(t-\tau)x_P(t-\tau)\partial_{x_P}.$$

According to the proof of theorem 3.1 in paper [16], the first order normal form (2.1) is the hypernormal form of equation (1.3). □

Theorefore, we obtain the complementary space $C_2^{(1)}$ to $\mathrm{Im}L^{(1)}$ in H_2 with the parameter condition above based on normal form theory in this section, and get the first order normal form (hypernormal form) of equation (1.3). The normal form can give the critical information about not only the stability of a positive equilibrium point but also the existence and direction of bifurcation.

3. The stability of a positive equilibrium point and the existence of Hopf bifurcation

In this section, we detect the positive equilibrium point of system (2.1), and consider time delay τ as the bifurcation parameter. We obtain bifurcation parameter value τ_0 according to the distribution of roots of the characteristic equation of the linearized system near the positive equilibrium point, and give the sufficient condition of the existence of Hopf bifurcation near the positive equilibrium point.

3.1. The positive equilibrium point of equation (2.1)

Let the right side of equation (2.1) be zero and solving these nonlinear equations. We have that system (2.1) has a boundary equilibrium point $P_0(0,0,0)$. If $Q_1 < 0$, system (2.1) has only one positive equilibrium point $P_+(y_J^*, y_A^*, y_P^*)$, where $y_J^* = \frac{d_3+E_3}{c_1}$, $y_A^* = \frac{Q_2}{2b_1}$, $y_P^* = \frac{(d_1+\alpha+E_1)y_J^* - a_1(y_J^*)^2 - r_1 y_A^*}{a_2 y_J^*}$, $Q_1 = 2b_1c_1(d_1+\alpha+E_1)(d_3+E_3) - 2a_1b_1(d_3+E_3)^2 - r_1c_1^2Q_2$, $Q_2 = (d_2+E_2) - \sqrt{(d_2+E_2)^2 - 4b_1\frac{d_3+E_3}{c_1}}$.

3.2. The characteristic equation of (2.1) at P_+ and the type of the equilibrium point

Let $\tilde{y}_J = y_J - y_J^*$, $\tilde{y}_A = y_A - y_A^*$, $\tilde{y}_P = y_P - y_P^*$, system (2.1) becomes

$$\begin{aligned} \dot{\tilde{y}} &= \bar{L}_1\tilde{y} + \bar{L}_2\tilde{y}_{(t-\tau)} + \bar{F}(\tilde{y}), \\ m &= E_3(p\tilde{y}_P - c), \end{aligned} \tag{3.1}$$

where

$$\bar{L}_1 = \begin{pmatrix} n_1 & r_1 & n_2 \\ \alpha & n_3 & 0 \\ 0 & 0 & n_6 \end{pmatrix}, \bar{L}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ n_4 & 0 & n_5 \end{pmatrix}, \bar{F}(\tilde{y}) = \begin{pmatrix} a_1\tilde{y}_J^2 + a_2\tilde{y}_J\tilde{y}_P \\ b_1\tilde{y}_A^2 \\ c_1\tilde{y}_J(t-\tau)\tilde{y}_P(t-\tau) \end{pmatrix}.$$

The linearization equation of equation (3.1) at P_+ is

$$\dot{\tilde{y}} = \bar{L}_1\tilde{y} + \bar{L}_2\tilde{y}_{(t-\tau)}, \tag{3.2}$$

whose characteristic equation yields

$$\lambda^3 - q_1\lambda^2 - q_2\lambda + q_3 - (n_5\lambda^2 - q_4\lambda + q_5)e^{-\lambda\tau} = 0. \tag{3.3}$$

And

$$\lambda_1 = \gamma_1, \quad \lambda_{2,3} = \gamma_2 \pm i\gamma_3, \tag{3.4}$$

are the roots of the equation (3.3). $P_n(n_1, n_2, n_3, n_4, n_5, n_6)$, $P_q(q_1, q_2, q_3, q_4, q_5)$ and $P_\gamma(\gamma_1, \gamma_2, \gamma_3)$ denote the parameter condition in equation (3.1), equation (3.3) and equation (3.4), respectively. The relations between them and the coefficients in equation (2.1) are given in the appendix.

According to qualitative theory of differential equation, the type of the equilibrium point P_+ of equation (2.1) are showed in Table 1.

Table 1. Characteristic equation of equation (2.1) at P_+ and the type of P_+.

Characteristic equation	$\lambda^3 - q_1\lambda^2 - q_2\lambda + q_3 - (n_5\lambda^2 - q_4\lambda + q_5)e^{-\lambda\tau} = 0$		
Parameter condition	$\gamma_1\gamma_2 < 0$	$\gamma_1 > 0$, $\gamma_2 > 0$	$\gamma_1 < 0$, $\gamma_2 < 0$
Type of p_+	Saddle focus	Unstable focus	Stable focus

Remark 3.1. Hartman Grobman theorem ensures that when the singularities of the linearized system is hyperbolic type, the trajectory of nonlinear system and corresponding linearization system keep the topology equivalence in the neighborhood of singular point. We analyze the type and the stability of singularities in nonlinear system requires the center manifold approach when the singularities are not hyperbolic type.

3.3. Computation of bifurcation parameter value τ_0 of equation (2.1)

In order to study the Hopf bifurcation of equation (2.1), we denote $\lambda = \pm i\omega (\omega > 0)$ as the solution of characteristic equation (3.3). Hence,

$$-i\omega^3 + q_1\omega^2 - iq_2\omega + q_3 - (n_2\omega^2 - iq_4\omega + q_5)e^{-i\omega\tau} = 0$$

is obtained. By separating the real part and the imaginary part leads to

$$\begin{cases} -\omega^3 - q_2\omega + q_4\omega\cos(\omega\tau) - (n_2\omega^2 - q_5)\sin(\omega\tau) = 0, \\ q_1\omega^2 + q_3 + (n_2\omega^2 - q_5)\cos(\omega\tau) + q_4\omega\sin(\omega\tau) = 0. \end{cases}$$

Clearly, the solution of equations mentioned above can be written as

$$\cos(\omega\tau) = -\frac{(n_2q_1 - q_4)\omega^4 + (n_2q_3 - q_1q_5 - q_2q_4)\omega^2 - q_3q_5}{(n_2\omega^2 - q_5)^2 + \omega^2 q_4^2},$$
$$\sin(\omega\tau) = -\frac{\omega(n_2\omega^4 + (n_2q_2 + q_1q_4 - q_5)\omega^2 - q_2q_5 + q_3q_4)}{(n_2\omega^2 - q_5)^2 + \omega^2 q_4^2},$$

and ω is the root of the equation

$$\omega^6 + p_1\omega^4 + p_2\omega^2 + p_3 = 0. \tag{3.5}$$

We use $P_p(p_1, p_2, p_3)$ denote the parameter condition in equation (3.5). The relationships between P_p and coefficients in equation (2.1) are showed in the appendix.

Let $z=\omega^2$ and equation (3.5) becomes

$$z^3+p_1z^2+p_2z+p_3=0. \tag{3.6}$$

Assume that equation (3.6) has three positive roots $z_k(k=1,2,3)$. Then, equation (3.5) has positive root $\omega_k(=\sqrt{z_k})$ with

$$\tau_k^{(j)}=\frac{1}{\omega_k}(\arccos\delta_1+2j\pi), \tag{3.7}$$

and

$$\tau_k^{(j)}=\frac{1}{\omega_k}(\arcsin\delta_2+2j\pi), \tag{3.8}$$

where

$$\delta_1=-\frac{(n_2q_1-q_4)\omega^4+(n_2q_3-q_1q_5-q_2q_4)\omega^2-q_3q_5}{(n_2\omega^2-q_5)^2+\omega^2q_4^2},$$

and

$$\delta_2=-\frac{\omega(n_2\omega^4+(n_2q_2+q_1q_4-q_5)\omega^2-q_2q_5+q_3q_4)}{(n_2\omega^2-q_5)^2+\omega^2q_4^2},$$

$k=1,\,2,\,3,\quad j=0,\,1,\,2,\cdots$.

Solving equation (3.7) and (3.8), we obtain ω_0 and bifurcation parameter τ_0

$$\omega_0=\omega_0(\tau_0),\quad \tau_0=\min\{\tau_k^0\}\ (1\le k\le 3),$$

respectively.

3.4. The sufficient conditions of the Hopf bifurcation of equation (2.1) near P_+

Let $z_0=\frac{-p_2+\sqrt{\Delta}}{3}$, $\Delta=p_1^2-3p_2$ and $h(z)=z^3+p_1z^2+p_2z+p_3$ in equation (3.6) and we have the following lemma.

Lemma 3.1. *Assuming that parameter conditions $q_1+n_5<0$, $q_3-q_5>0$ and $(q_1+n_5)(q_2-q_4)-q_3+q_5>0$ hold, then*

(i) If $p_3\ge 0$ and $\Delta<0$, then all the roots of the characteristic equation (3.3) have strict negative real part;

(ii) If $p_3<0$ or $(p_3\ge 0,\Delta\ge 0,z_0>0,h(z_0)\le 0)$, then all the roots of the characteristic equation (3.3) have strict negative real part with $\tau\in[0,\tau_0)$.

Proof. (i) According to $\tau=0$, equation (3.3) can be written as

$$\lambda^3-(q_1+n_5)\lambda^2-(q_2-q_4)\lambda+q_3-q_5=0.$$

The necessary and sufficient conditions for all the roots of the equation (3.6) have strict negative real part, according to Routh-Hurwitz criterion, is

$$q_1+n_5<0,\quad q_3-q_5>0\quad (q_1+n_5)(q_2-q_4)-q_3+q_5>0. \tag{3.9}$$

If $p_3 \geq 0$ and $\Delta < 0$, equation (3.6) has no positive real root and equation (3.3) has no pure imaginary roots. From the proof of theorem 3.1 in paper [19], we know that all roots of equation (3.3) have strict negative real parts for $\tau \in [0, \infty)$.

(ii) If $p_3 < 0 (p_3 \geq 0, \Delta \geq 0, z_1 > 0, h(z_1) \leq 0)$, equation (3.6) has positive real root and equation (3.3) has pure imaginary roots only when $\tau \in [0, \tau_0)$. □

Let $\lambda(\tau) = \alpha(\tau) + i\omega(\tau)$ be the root of equation (3.3) with condition $\alpha(\tau_0) = 0$ and $\omega(\tau_0) = \omega_0$. Denote that

$$G = -\frac{\omega_0^2(\omega_0^4 f_1 + \omega_0^2 f_2 + f_3)}{f_4},$$

and $P_{f_{1-4}}(f_1, f_2, f_3, f_4)$ is the parameter condition above. The relationships between $P_{f_{1-4}}$ and coefficients in equation (2.1) are showed in the appendix. Then, we have the theorem as follows.

Theorem 3.1 (The sufficient condition of the existence of Hopf bifurcation)**.** *If the conditions in Lemma 3.1(ii) hold and $G > 0$, we have*

(i) The positive equilibrium point P_+ of equation (2.1) is asymptotically stable when $0 \leq \tau \leq \tau_0$;

(ii) The positive equilibrium point P_+ of equation (2.1) is unstable when $\tau > \tau_0$;

(iii) Equation (3.3) has a pair of pure imaginary roots when $\tau = \tau_0$, and τ_0 is the Hopf bifurcation point of equation (2.1).

Proof. (i) From Lemma 3.1(ii), if $p_3 < 0$ $(p_3 \geq 0, \Delta \geq 0, z_0 > 0, h(z_0) \leq 0)$, all roots of the characteristic equation (3.3) have strict negative real part with $\tau \in [0, \tau_0)$. According to qualitative theory of differential equation, the positive equilibrium P_+ of equation (2.1) is asymptotically stable.

(ii) For that $\lambda(\tau) = \alpha(\tau) + i\omega(\tau)$ is the root of equation (3.3) with condition $\alpha(\tau_0) = 0$ and $\omega(\tau_0) = \omega_0$, we calculate

$$\frac{\mathrm{d}\lambda(\tau)}{\mathrm{d}\tau}|_{\lambda = i\omega_0, \tau = \tau_0} = \frac{f_5 + i f_6}{f_7 + i f_8}.$$

Denote that the parameter condition above is $P_{f_{5-8}}(f_5, f_6, f_7, f_8)$, Moreover, the relationships between $P_{f_{5-8}}$ and coefficients of equation (2.1) are showed in the appendix. So, we have

$$\begin{aligned}
\frac{\mathrm{dRe}\lambda(\tau)}{\mathrm{d}\tau}|_{\lambda = i\omega_0, \tau = \tau_0} &= \frac{f_5 f_7 + f_6 f_8}{f_7^2 + f_8^2} \\
&= -\frac{\omega_0^2(\omega_0^4 f_1 + \omega_0^2 f_2 + f_3)}{(q_4^2\omega_0^2 + n_5^2\omega_0^4 - 2n_5\omega_0^2 q_5 + q_5^2)(f_7^2 + f_8^2)} \\
&= -\frac{\omega_0^2(\omega_0^4 f_1 + \omega_0^2 f_2 + f_3)}{f_4(f_7^2 + f_8^2)} \\
&= \frac{G}{f_7^2 + f_8^2}.
\end{aligned}$$

If $G > 0$, we have that $\frac{\mathrm{dRe}\lambda(\tau)}{\mathrm{d}\tau}|_{\lambda = i\omega_0, \tau = \tau_0} > 0$. Then, the Equation (3.3) has at least one characteristic root with strict positive real part when $\tau > \tau_0$, and the positive equilibrium point P_+ of equation (2.1) is unstable according to qualitative theory of differential equation.

(iii) Equation (3.3) has a pair of pure imaginary roots $\lambda = \pm i\omega_0$ when $\tau = \tau_0$. Combine (i) and (ii), we get the conclusion that $\tau = \tau_0$ is the Hopf bifurcation point of equation (2.1). □

We get the positive equilibrium point P_+ of equation (2.1) and discuss its type. According to the distribution of roots of the characteristic equation for the linearized system at P_+, we obtain the bifurcation value τ_0 and $\frac{\mathrm{dRe}\lambda(\tau)}{\mathrm{d}\tau}|_{\lambda=i\omega_0,\tau=\tau_0}$ across $\tau = t_0$. Based on the differential equation qualitative theory, we get the sufficient condition of the existence of Hopf bifurcation near P_+.

4. The properties of the Hopf bifurcation

In this section, we transfer equation (2.1) into abstract ordinary differential equations by Riesz representation theorem and the infinitesimal generator approach of functional differential equation. We apply normal form theory and center manifold approach [2,3,10] to obtain the differential equations restricted to the flow of center manifold of that abstract equations. We analyze the direction, stability and periodicity of the periodic solution of Hopf bifurcation by means of the normal form obtained above.

4.1. Calculation of eigenvectors $q(\theta)$ and $q^*(\theta)$

Let $\tau = \tau_0 + \mu$ and equation (2.1) exists Hopf bifurcation at $\mu = 0$. We denote $u(t) = (\tilde{y}_J(t), \tilde{y}_A(t), \tilde{y}_P(t))^{\mathrm{T}} = (u_J(t), u_A(t), u_P(t))^{\mathrm{T}}$ and $u(t) = u(t+\theta)$, $\theta \in [-\tau, 0]$. The equation (2.1) can be rewritten as a functional differential equation with the following form

$$\dot{u} = L_\mu u + F(u), \tag{4.1}$$

where

$$L_\mu u = B_1 u(t) + B_2 u(t-\tau),$$

$$B_1 = \begin{pmatrix} n_1 & r_1 & n_2 \\ \alpha & n_3 & 0 \\ 0 & 0 & n_6 \end{pmatrix}, B_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ n_4 & 0 & n_5 \end{pmatrix}, F(u) = \begin{pmatrix} -s_1 u_J^2 - \beta u_J u_P \\ -s_2 u_A^2 \\ \beta u_J(t-\tau) u_P(t-\tau) \end{pmatrix},$$

for any initial condition $\varphi(\theta) = (\varphi_1(\theta), \varphi_2(\theta), \varphi_3(\theta))^{\mathrm{T}} \in C^3$ with $\varphi(0) = u(t)$ and $\varphi(-\tau) = u(t-\tau)$, we have

$$L_\mu u = B_1 \varphi(0) + B_2 \varphi(-\tau). \tag{4.2}$$

According to spectral decomposition theory, infinitesimal generator $A(0)$ associated with linearized equation of equation (4.2)

$$A(0)\varphi(\theta) = \begin{cases} \dfrac{\mathrm{d}\varphi(\theta)}{\mathrm{d}\theta}, & \theta \in [-\tau, 0), \\ \displaystyle\int_{-\tau}^{0} \varphi(s)\mathrm{d}\eta(s), & \theta = 0, \end{cases} \tag{4.3}$$

where $\varphi \in C^3$. And $\eta(\theta)$ is three order matrix function with

$$\eta(\mu, \theta) = B_1 \delta(\theta) + B_2 \delta(\theta + \tau),$$

where $\delta(\theta)$ is Dirac-delta function. We also define operator $R(0)$ as

$$R(0)\varphi(\theta) = \begin{cases} 0, & \theta \in [-\tau, 0), \\ F(u), & \theta = 0. \end{cases}$$

Then, equation (4.1) is equivalent to the abstract differential equation

$$\dot{u} = A(0)u + R(0)u. \tag{4.4}$$

Let $\psi \in C^3$, we represent the formal adjoint operator of $A(0)$ as A^* and we have

$$A^*(0)\psi(s) = \begin{cases} -\dfrac{\mathrm{d}\psi(s)}{\mathrm{d}s}, & s \in (0, \tau], \\ \displaystyle\int_{-\tau}^{0} \psi(-t)\mathrm{d}\eta^{\mathrm{T}}(t), & s = 0. \end{cases} \tag{4.5}$$

For ψ and φ, we define the bilinear form of the inner product as follows

$$\langle \psi, \varphi \rangle = \bar{\psi}(0)\varphi(0) - \int_{-\tau}^{0}\int_{0}^{\theta} \bar{\psi}(\xi - \theta)\varphi(\xi)\mathrm{d}\eta(\theta)\mathrm{d}\xi. \tag{4.6}$$

According to the bilinear inner product (4.6), we have $\langle \psi, A\varphi \rangle = \langle A^*\psi, \varphi \rangle$.

Theorem 4.1. *Let $q(\theta)$ and $q^*(\theta)$ be eigenvectors of $A(0)$ and $A^*(0)$ associated with $i\omega_0$ and $-i\omega_0$ respectively, we have*

$$q(\theta) = Ve^{i\omega_0\theta}, \quad q^*(\theta) = DV^*e^{i\omega_0\theta},$$

and

$$\langle q^*(\theta), q(\theta) \rangle = 1, \quad \langle q^*(\theta), \bar{q}(\theta) \rangle = 0,$$

where

$$\begin{aligned} V &= (-1, -1, \rho_1)^T, \quad \rho_1 = \frac{n_1 - i\omega_0 + r_1}{n_2}, \\ V^* &= (\rho_2, -1, -1)^T, \quad \rho_2 = \frac{n_2 + r_1}{n_1 + i\omega_0}, \\ \bar{D} &= \frac{1}{\bar{V^*}^T V + \tau e^{-i\omega_0\tau}\bar{V^*}^T B_2 V}. \end{aligned}$$

Proof. According to equation (4.3) and $A(0)q(\theta) = i\omega_0 q(\theta)$, we have

$$A(0)q(\theta) = \begin{cases} \dfrac{\mathrm{d}q(\theta)}{\mathrm{d}\theta} = i\omega_0 q(\theta), & \theta \in [-\tau, 0), \\ \displaystyle\int_{-\tau}^{0} q(s)\mathrm{d}\eta(s) = i\omega_0 q(\theta), & \theta = 0, \end{cases}$$

and therefore

$$q(\theta) = Ve^{i\omega_0\theta}, \quad \theta \in [-\tau, 0],$$

where $V = (v_1, v_2, v_3)^{\mathrm{T}} \in C^3$. From equation (4.5) and $A^*(0)q^*(\theta) = -i\omega_0 q^*(\theta)$, we have

$$A^*(0)q^*(s) = \begin{cases} -\dfrac{\mathrm{d}q^*(s)}{\mathrm{d}s}, & s \in (0, \tau], \\ \displaystyle\int_{-\tau}^{0} q^*(-s)\mathrm{d}\eta^{\mathrm{T}}(s), & s = 0, \end{cases}$$

and therefore

$$q^*(\theta) = DV^* e^{i\omega_0\theta}, \quad \theta \in [-\tau, 0],$$

where $D \in R$. From equation (4.2), it follows that

$$B_1 V + B_2 V e^{-i\omega_0\theta} = i\omega_0 I,$$

where I is three order unit matrix. We solve the equation above and obtain the solution as follows

$$V = \Big(-1, -1, \frac{n_1 - i\omega_0 + r_1}{n_2} \Big)^{\mathrm{T}}.$$

Similarly, we have

$$V^* = \Big(\frac{n_2 + r_1}{n_1 + i\omega_0}, -1, -1 \Big)^{\mathrm{T}}.$$

From equation (4.8), the inner product of $q^*(\theta)$ with $q(\theta)$ is

$$\begin{aligned} \langle q^*(\theta), q(\theta) \rangle &= \bar{q^*}^{\mathrm{T}}(0)q(0) - \int_{-\tau}^{0} \int_{0}^{\theta} \bar{q^*}^{\mathrm{T}}(\xi - \theta)q(\xi)\mathrm{d}\eta(\theta)\mathrm{d}\xi \\ &= \bar{D}\bar{V^*}^{\mathrm{T}} V - \int_{-\tau}^{0} \int_{0}^{\theta} \bar{D}\bar{V^*}^{\mathrm{T}} e^{-i\omega_0(\xi-\theta)} V e^{i\omega_0\xi} \mathrm{d}\eta(\theta)\mathrm{d}\xi \\ &= \bar{D}\bar{V^*}^{\mathrm{T}} V - \int_{-\tau}^{0} \bar{D}\bar{V^*}^{\mathrm{T}} e^{i\omega_0\theta} V\theta \mathrm{d}\eta(\theta) \\ &= \bar{D}\bar{V^*}^{\mathrm{T}} V - \bar{D}\bar{V^*}^{\mathrm{T}} (-\tau B_2 e^{-i\omega_0\tau}) V \\ &= \bar{D}(\bar{V^*}^{\mathrm{T}} V + \tau e^{-i\omega_0\tau} \bar{V^*}^{\mathrm{T}} B_2 V). \end{aligned}$$

Let

$$\bar{D} = \frac{1}{\bar{V^*}^{\mathrm{T}} V + \tau e^{-i\omega_0\tau} \bar{V^*}^{\mathrm{T}} B_2 V},$$

we have

$$\langle q^*(\theta), q(\theta) \rangle = 1.$$

Similarly, we have

$$\langle q^*(\theta), \bar{q}(\theta) \rangle = 0.$$

Therefore, the theorem is proved. $\square$

We define the local coordinates of the equation (4.4) in the direction of $q^*(\theta)$ and $\bar{q}^*(\theta)$ as

$$z(t) = \langle q^*, u\rangle, \tag{4.7}$$

where u is the solution of equation (4.4) at $\mu = 0$. We denote the real space which differ from eigenvalue $\pm i\omega_0$ of equation (4.4) is $Q_{\pm i\omega_0}$. So, we obtain the following equation on the center manifold $C_{\mu=0} = W(z(t), \bar{z}(t), \theta)$ of equation (4.4),

$$\begin{aligned} W(z(t), \bar{z}(t), \theta) &= W(t, \theta) \\ &= u(t) - z(t)q(\theta) - \bar{z}(t)\bar{q}(\theta) \\ &= u(t) - 2\mathrm{Re}(z(t)q(\theta)), \end{aligned} \tag{4.8}$$

where $W(t, \theta) \in Q_{\pm i\omega_0}$ and $W(z(t), \bar{z}(t), \theta)$ could be expressed as power series expansion about $z(t)$ and $\bar{z}(t)$,

$$W(z(t), \bar{z}(t), \theta) = W_{20}(\theta)\frac{z(t)^2}{2} + W_{11}(\theta)z(t)\bar{z}(t) + W_{02}(\theta)\frac{\bar{z}(t)^2}{2} + \cdots .$$

According to equation (4.7) and (4.8), the flow of equation (14) on center manifold $C_{\mu=0}$ could be determined by

$$\begin{aligned} \dot{z}(t) &= \langle q^*, \dot{u}\rangle \\ &= \langle q^*, A(0)u + R(0)u\rangle \\ &= \langle q^*, A(0)u\rangle + \langle q^*, R(0)u\rangle \\ &= \langle A^*(0)q^*, u\rangle + \bar{q^*}^{\mathrm{T}}(0)F_0 \\ &= i\omega_0 z(t) + \bar{q^*}^{\mathrm{T}}(0)F_0 \end{aligned} \tag{4.9}$$

where $F_0 = F(W(z(t), \bar{z}(t), \theta) + z(t)q(\theta) + \bar{z}(t)\bar{q}(\theta), 0)$.

Hence, we need to calculate the undetermined coefficients $W_{20}(\theta)$, $W_{11}(\theta)$ and $W_{02}(\theta)$, and bring these coefficients into equation (4.9). And then, we obtain the equation restricted to the center manifold $C_{\mu=0}$. We denote that

$$\begin{aligned} g(z(t), \bar{z}(t)) =& \bar{q^*}^{\mathrm{T}}(0)F_0 \\ =& \bar{q^*}^{\mathrm{T}}(0)F(W(z(t), \bar{z}(t), \theta) + z(t)q(\theta) + \bar{z}(t)\bar{q}(\theta), 0) \\ =& g_{20}(\theta)\frac{z(t)^2}{2} + g_{11}(\theta)z(t)\bar{z}(t) + g_{02}(\theta)\frac{\bar{z}(t)^2}{2} \\ &+ g_{21}\frac{z(t)^2\bar{z}(t)}{2} + \cdots . \end{aligned} \tag{4.10}$$

From equation (4.10), we know that, as long as the coefficients g_{20}, g_{11}, g_{02} and g_{21} were calculated, the equation restricted to the center manifold (4.9) will be obtained.

4.2. Computation of coefficients g_{20}, g_{11} and g_{02} on center manifold $C_{\mu=0}$

From equation (4.8), we have

$$\dot{W}(t, \theta) = \dot{u}(t) - \dot{z}(t)q(\theta) - \dot{\bar{z}}(t)\bar{q}(\theta). \tag{4.11}$$

According to equation (4.4) and (4.9), it follows that

$$\dot{W}(t,\theta)=\begin{cases}AW(t,\theta)-2\mathrm{Re}(\bar{q^*}^{\mathrm{T}}(0)F_0q(\theta)), & \theta\in[-\tau,0),\\ AW(t,\theta)-2\mathrm{Re}(\bar{q^*}^{\mathrm{T}}(0)F_0q(\theta))+F_0, & \theta=0.\end{cases}\tag{4.12}$$

Further more, the equation (4.12) could be expresses as

$$\dot{W}(t,\theta)=AW(t,\theta)+H(z(t),\bar{z}(t),\theta),$$

where

$$H(z(t),\bar{z}(t),\theta)=H_{20}(\theta)\frac{z(t)^2}{2}+H_{11}(\theta)z(t)\bar{z}(t)+H_{02}(\theta)\frac{\bar{z}(t)^2}{2}+\cdots.\tag{4.13}$$

On the other hand, from equation (4.13), we have

$$\begin{aligned}\dot{W}(z,\bar{z},\theta)=&W_{z(t)}\dot{z}(t)+W_{\bar{z}(t)}\dot{\bar{z}}(t)\\ =&(W_{20}(\theta)z+W_{11}(\theta)\bar{z})\dot{z}+(W_{20}(\theta)z+W_{11}(\theta)\bar{z})\dot{\bar{z}}\\ =&(W_{20}(\theta)z+W_{11}(\theta)\bar{z})(i\omega_0z+\bar{q^*}^{\mathrm{T}}(0)F_0)\\ &+(W_{20}(\theta)z+W_{11}(\theta)\bar{z})(-i\omega_0z+\bar{q^*}^{\mathrm{T}}(0)F_0)\\ =&i\omega_0W_{20}(\theta)z^2-i\omega_0W_{02}(\theta)\bar{z}^2+\cdots.\end{aligned}\tag{4.14}$$

Substituting equation (4.14) into equation (4.12) and comparing the coefficients about the term $\frac{z^2}{2}$ and $z\bar{z}$, we have

$$\begin{cases}(A-2i\omega_0I)W_{20}(\theta)=-H_{20}(\theta), & \theta\in[-\tau,0),\\ (A-2i\omega_0I)W_{20}(\theta)=g_{20}q(0)+\bar{g}_{02}\bar{q}(0)-F_{z^2}, & \theta=0,\end{cases}\tag{4.15}$$

and

$$\begin{cases}AW_{11}(\theta)=-H_{11}(\theta), & \theta\in[-\tau,0),\\ AW_{11}(\theta)=g_{11}q(0)+\bar{g}_{11}\bar{q}(0)-F_{z\bar{z}}, & \theta=0.\end{cases}\tag{4.16}$$

From (4.8), we have

$$u(t)=u(t+\theta)=W(t,\theta)+z(t)q(\theta)+\bar{z}(t)\bar{q}(\theta),$$

and

$$\begin{aligned}u_J(t+\theta)&=W^{(1)}(t,\theta)+z(t)(-1)+\bar{z}(t)(-1)\\ &=W_{20}^{(1)}(\theta)\frac{z^2}{2}+W_{11}^{(1)}(\theta)z\bar{z}+W_{02}^{(1)}(\theta)\frac{\bar{z}^2}{2}-z(t)e^{i\omega_0\theta}-\bar{z}(t)e^{-i\omega_0\theta},\\ u_A(t+\theta)&=W^{(2)}(t,\theta)+z(t)(-1)+\bar{z}(t)(-1)\\ &=W_{20}^{(2)}(\theta)\frac{z^2}{2}+W_{11}^{(2)}(\theta)z\bar{z}+W_{02}^{(2)}(\theta)\frac{\bar{z}^2}{2}-z(t)e^{i\omega_0\theta}-\bar{z}(t)e^{-i\omega_0\theta},\\ u_P(t+\theta)&=W^{(3)}(t,\theta)+z(t)\rho_1+\bar{z}(t)\bar{\rho}_1\\ &=W_{20}^{(3)}(\theta)\frac{z^2}{2}+W_{11}^{(3)}(\theta)z\bar{z}+W_{02}^{(3)}(\theta)\frac{\bar{z}^2}{2}-z(t)\rho_1e^{i\omega_0\theta}-\bar{z}(t)\bar{\rho}_1e^{-i\omega_0\theta}.\end{aligned}$$

For equation (4.1), we denote that

$$F(\varphi)=\begin{pmatrix}-s_1u_J^2-\beta u_Ju_P\\ -s_2u_A^2\\ \beta u_J(t-\tau)u_P(t-\tau)\end{pmatrix}=\begin{pmatrix}K_1 & K_2 & K_3 & K_4\\ K_5 & K_6 & K_7 & K_8\\ K_9 & K_{10} & K_{11} & K_{12}\end{pmatrix}\begin{pmatrix}z^2\\ z\bar{z}\\ \bar{z}^2\\ z^2\bar{z}\end{pmatrix},$$

where $\varphi(J)$, $\varphi(A)$, $\varphi(P)$ and $K_i(i=1,2,\cdots,12)$ are showed in the appendix.

According to equation (4.10), it follows that

$$\begin{aligned}g(z(t),\bar{z}(t))=&\bar{q^*}^{\mathrm{T}}(0)F_0\\ =&\bar{D}\bar{V^*}^{\mathrm{T}}\begin{pmatrix}K_1 & K_2 & K_3 & K_4\\ K_5 & K_6 & K_7 & K_8\\ K_9 & K_{10} & K_{11} & K_{12}\end{pmatrix}\begin{pmatrix}z^2\\ z\bar{z}\\ \bar{z}^2\\ z^2\bar{z}\end{pmatrix}\\ =&\bar{D}(\bar{\rho}_2K_1-K_5-K_9)z^2+\bar{D}(\bar{\rho}_2K_2-K_6-K_{10})z\bar{z}\\ &+\bar{D}(\bar{\rho}_2K_3-K_7-K_{11})\bar{z}^2+\bar{D}(\bar{\rho}_2K_4-K_8-K_{12})z^2\bar{z}.\end{aligned}$$

Comparing the coefficients of each term with equation (4.10) yields

$$\begin{aligned}g_{20}&=2\bar{D}(\bar{\rho}_2K_1-K_5-K_9),\\ g_{11}&=\bar{D}(\bar{\rho}_2K_2-K_6-K_{10}),\\ g_{02}&=2\bar{D}(\bar{\rho}_2K_3-K_7-K_{11}).\end{aligned}\tag{4.17}$$

Furthermore, coefficients g_{20}, g_{11} and g_{02} are obtained through these equations above. However, coefficient $g_{21}=2\bar{D}(\bar{\rho}_2K_4-K_8-K_{12})$ depends on terms $W_{11}(\theta)$ and $W_{20}(\theta)$. We should calculate $W_{11}(\theta)$ and $W_{20}(\theta)$ in equation.

4.3. Computation of coefficients g_{21} on center manifold $C_{\mu=0}$

From equation (4.12) at $\theta\in[-\tau,0)$, it follows that

$$\begin{aligned}H(z(t),\bar{z}(t),\theta)=&-2\mathrm{Re}\big(\bar{q^*}^{\mathrm{T}}(0)F_0q(\theta)\big)\\ =&-\Big(g_{20}\frac{z^2}{2}+g_{11}z\bar{z}+g_{02}\frac{\bar{z}^2}{2}+g_{21}\frac{z^2\bar{z}}{2}+\cdots\Big)q(\theta)\\ &-\Big(\bar{g}_{20}\frac{\bar{z}^2}{2}+\bar{g}_{11}z\bar{z}+\bar{g}_{02}\frac{z^2}{2}+\bar{g}_{21}\frac{\bar{z}^2z}{2}+\cdots\Big)\bar{q}(\theta).\end{aligned}\tag{4.18}$$

Comparing the coefficients about the term $\frac{z^2}{2}$ and $z\bar{z}$ between equation (4.13) and (4.18), we have

$$H_{20}(\theta)=-g_{20}q(\theta)-\bar{g}_{02}\bar{q}(\theta),$$

and

$$H_{11}(\theta)=-g_{11}q(\theta)-\bar{g}_{11}\bar{q}(\theta).$$

From equation (4.15) at $\theta \in [-\tau, 0)$, it follows that

$$\begin{aligned}\dot{W}_{20}(\theta) &= 2i\omega_0 W_{20}(\theta) - H_{20}(\theta)\\ &= 2i\omega_0 W_{20}(\theta) + g_{20}q(\theta) + \bar{g}_{02}\bar{q}(\theta)\\ &= 2i\omega_0 W_{20}(\theta) + g_{20}Ve^{i\omega_0\theta} + \bar{g}_{02}V^* e^{-i\omega_0\theta}.\end{aligned}$$

We solve equation and obtain the solution

$$W_{20}(\theta) = \frac{ig_{20}}{\omega_0}Ve^{i\omega_0\theta} + \frac{i\bar{g}_{02}}{3\omega_0}\bar{V}e^{-i\omega_0\theta} + M_1 e^{2i\omega_0\theta}. \tag{4.19}$$

From equation (4.16) at $\theta \in [-\tau, 0)$, it follows that

$$\begin{aligned}\dot{W}_{11}(\theta) &= -H_{11}(\theta)\\ &= g_{11}q(\theta) + \bar{g}_{11}\bar{q}(\theta)\\ &= g_{11}Ve^{i\omega_0\theta} + \bar{g}_{11}V^* e^{-i\omega_0\theta}.\end{aligned}$$

We solve equation and obtain the solution

$$W_{11}(\theta) = -\frac{ig_{11}}{\omega_0}Ve^{i\omega_0\theta} + \frac{i\bar{g}_{11}}{\omega_0}\bar{V}e^{-i\omega_0\theta} + M_2. \tag{4.20}$$

According to equation (4.16), we obtain $H(z, \bar{z}, 0)$ at $\theta = 0$ as follows

$$\begin{aligned}H(z,\bar{z},0) = &- 2\mathrm{Re}\big(\bar{q^*}^{\mathrm{T}}(0)F_0 q(\theta)\big) + F_0\\ = &- \Big(g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + g_{21}\frac{z^2\bar{z}}{2} + \cdots\Big)q(0)\\ &- \Big(\bar{g}_{20}\frac{\bar{z}^2}{2} + \bar{g}_{11}z\bar{z} + \bar{g}_{02}\frac{z^2}{2} + \bar{g}_{21}\frac{\bar{z}^2 z}{2} + \cdots\Big)\bar{q}(0) + F_0,\\ = &- \Big(g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + g_{21}\frac{z^2\bar{z}}{2} + \cdots\Big)V\\ &- \Big(\bar{g}_{20}\frac{\bar{z}^2}{2} + \bar{g}_{11}z\bar{z} + \bar{g}_{02}\frac{z^2}{2} + \bar{g}_{21}\frac{\bar{z}^2 z}{2} + \cdots\Big)\bar{V}\\ &+ (U_1, U_2, U_3)^{\mathrm{T}},\end{aligned} \tag{4.21}$$

where

$$\begin{aligned}U_1 &= K_1 z^2 + K_2 z\bar{z} + K_3\bar{z}^2 + K_4 z^2\bar{z},\\ U_2 &= K_5 z^2 + K_6 z\bar{z} + K_7\bar{z}^2 + K_8 z^2\bar{z},\\ U_3 &= K_9 z^2 + K_{10} z\bar{z} + K_{11}\bar{z}^2 + K_{12} z^2\bar{z}.\end{aligned}$$

Comparing the coefficients about the term $\frac{z^2}{2}$ and $z\bar{z}$ between equation (4.21) and (4.13), we have

$$H_{20}(0) = -g_{20}q(0) - \bar{g}_{02}\bar{q}(0) + (K_1, K_5, K_9)^{\mathrm{T}},$$

and

$$H_{11}(0) = -g_{11}q(0) - \bar{g}_{11}\bar{q}(0) + (K_2, K_6, K_{10})^{\mathrm{T}}.$$

From equation (4.4) and (4.15), we calculate the integrals as following at $\theta = 0$,

$$\begin{aligned}\int_{-\tau}^{0} W_{20}(\theta)\mathrm{d}\eta(\theta) =& 2i\omega_0 W_{20}(0) - H_{20}(0)\\ =& 2i\omega_0 W_{20}(0) + g_{20}q(0) + \bar{g}_{02}\bar{q}(0) - (K_1, K_5, K_9)^{\mathrm{T}}\\ =& 2i\omega_0(\frac{ig_{20}}{\omega_0}V + \frac{i\bar{g}_{02}}{3\omega_0}\bar{V} + M_1) + g_{20}V + \bar{g}_{02}\bar{V}\\ & - (K_1, K_5, K_9)^{\mathrm{T}},\end{aligned} \tag{4.22}$$

and

$$\begin{aligned}\int_{-\tau}^{0} W_{11}(\theta)\mathrm{d}\eta(\theta) &= -H_{11}(0)\\ &= g_{11}q(0) + \bar{g}_{11}\bar{q}(0) - (K_2, K_6, K_{10})^{\mathrm{T}}\\ &= g_{11}V + \bar{g}_{11}\bar{V} - (K_2, K_6, K_{10})^{\mathrm{T}}.\end{aligned} \tag{4.23}$$

Because that $q(\theta)$ is characteristic vector of $A(0)$ associated with $i\omega_0$, when $\theta = 0$, from equation (4.3) we have

$$A(0)q(0) = \int_{-\tau}^{0} q(s)\mathrm{d}\eta(s) = i\omega_0 q(0). \tag{4.24}$$

From equation (4.24), it follows that

$$\Big(i\omega_0 I - \int_{-\tau}^{0} e^{i\omega_0 s}\mathrm{d}\eta(s)\Big)q(0) = 0.$$

So, for $\bar{\eta} = \eta$, we have

$$\Big(-i\omega_0 I - \int_{-\tau}^{0} e^{-i\omega_0 s}\mathrm{d}\eta(s)\Big)q(0) = 0.$$

Hence, from equations (4.19) and (4.22), we obtain

$$M_1 = \Big(2i\omega_0 I - \int_{-\tau}^{0} e^{2i\omega_0 s}\mathrm{d}\eta(s)\Big)^{-1}(K_1, K_5, K_9)^{\mathrm{T}}. \tag{4.25}$$

Similarly, from equations (4.20) and (4.23), we also obtain

$$M_2 = \Big(-\int_{-\tau}^{0} \mathrm{d}\eta(s)\Big)^{-1}(K_2, K_6, K_{10})^{\mathrm{T}}. \tag{4.26}$$

Moreover, $M_1 = \big(M_1^{(1)}, M_1^{(2)}, M_1^{(3)}\big)^{\mathrm{T}}$ and $M_2 = \big(M_2^{(1)}, M_2^{(2)}, M_2^{(3)}\big)^{\mathrm{T}}$, $M_i^{(j)}(i = 1, 2,\ j = 1, 2, 3)$ are showed in the appendix.

Because of equations (4.19) and (4.20), we calculate $W_{20}^{(i)}(\theta)$ and $W_{11}^{(i)}(\theta)(i = 1, 2, 3)$ and obtain the coefficient

$$g_{21} = 2\bar{D}(\bar{\rho}_2 K_4 - K_8 - K_{12}), \tag{4.27}$$

where $W_{20}^{(i)}(\theta)$, $W_{11}^{(i)}(\theta)(i = 1, 2, 3)$ are showed in the appendix.

We calculate all these coefficients and bring g_{20}, g_{11}, g_{02} and g_{21} into equation (4.9). Therefore, the equation can be written as

$$\begin{aligned}\dot{z}(t) &= i\omega_0 z + g(z, \bar{z})\\ &= i\omega_0 z + g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + g_{21}\frac{z^2\bar{z}}{2} + \cdots.\end{aligned}$$

4.4. The properties of Hopf bifurcation of equation (2.1) and corresponding algorithm

Based on the Hopf bifurcation theory of dynamical systems, equation (4.9) can be transformed into

$$\dot{w} = i\omega_0 w + c_1(0)w^2\bar{w} + O(|w|^4), \tag{4.28}$$

where $c_1(0) = \frac{i}{2\omega_0}\big(g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\big) + \frac{g_{21}}{2}$. We denote the first Lyapunov coefficient of equation (4.28) as

$$l_1(0) = \frac{\mathrm{Re}c_1(0)}{\omega_0} = \frac{1}{2\omega_0^2}\mathrm{Re}(ig_{20}g_{11} + \omega_0 g_{21}),$$

and

$$\mu_2 = -\frac{\omega_0 l_1(0)}{\mathrm{Re}\lambda'(0)},$$

where μ_2 and $l_1(0)$ determines the direction and stability of Hopf bifurcation respectively, we have the following theorem.

Theorem 4.2 (The properties of the Hopf bifurcation). *If $\mu_2 > 0\,(\mu_2 < 0)$, the Hopf bifurcation of equation (2.1) is supercritical (subcritical), and there exits bifurcation periodic solution when $\tau > \tau_0\,(\tau < \tau_0)$; If $l_1(0) < 0\,(l_1(0) > 0)$, the bifurcation periodic solution of equation (2.1) is stable (unstable).*

Combined with analysis the properties of Hopf bifurcation periodic solution of equation (2.1), we give the algorithm process as follows: Rewrite the equation (3.1) in abstract form $\dot{u} = A(0)u + R(0)u$ and obtain the local coordinate on the direction of $q^*(\theta)$ and $\bar{q}^*(\theta)$ of equation (4.4) according to the theory of spectral decomposition of equation; Obtain the equation (4.9) restricted on the center manifold $C_{\mu=0}$ and calculate the coefficients; Calculate μ_2 and $l_1(0)$, research the bifurcation direction and stability of Hopf bifurcation periodic solution of equation (2.1). The corresponding algorithm process is shown in Figure 1.

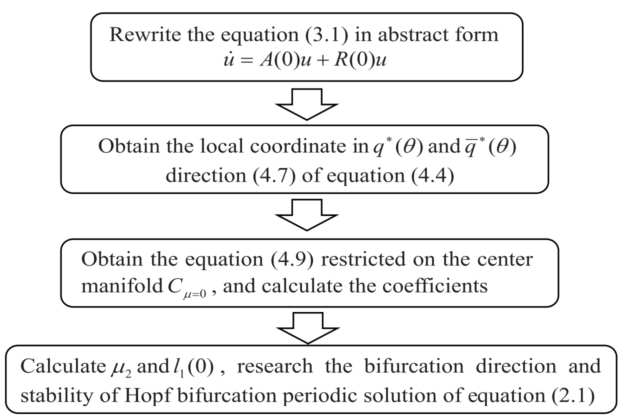


Figure 1. Algorithm process.

The spectral decomposition theory of equation is basic approach for studying bifurcation problems by applying center manifold method. We give the equation which restricted on the flow of center manifold of equation (2.1) by using of spectral decomposition theory and center manifold method, and obtain the algorithm shows the properties such as stability and direction of Hopf bifurcation periodic solution of equation (2.1).

5. Numerical simulations

In this section, we carry out the numerical simulation of the Hopf bifurcation periodic solution of the equation (2.1). We obtain τ_0 and $\frac{\mathrm{dRe}\lambda(\tau)}{\mathrm{d}\tau}|_{\lambda=i\omega_0,\tau=\tau_0}$ according to lemma 3.1 and theorem 3.1, and calculate μ_2 and $l_1(0)$ to determines the direction and stability of Hopf bifurcation respectively.

We choose the parameters in equation (1.3) as $P = (r_1, \alpha, \beta, d, s, E)$, where $r_1 = 80$, $\alpha = 0.3$, $\beta = 1.2$, $d = (d_1, d_2, d_3) = (1.5, 1, 0.6)$, $s = (s_1, s_2, s_3) = (1.2, 0.3, 0.8)$, $E = (E_1, E_2, E_3) = (1.5, 1.8, 2)$. Under this parameter condition, equation (2.1) has a unique positive equilibrium point $P_+ = (2.167, 0.227, 2.057)$. For the ecological (biological) equation, the positive equilibrium point means that all the speices in the equation are exist. We also get that $\omega_0 = 0.361$, $\tau_0 = 7.650$, $p_3 = -2.96 \times 10^4 < 0$, $G = 568.4 > 0$, $\frac{\mathrm{dRe}\lambda(\tau)}{\mathrm{d}\tau}|_{\lambda=i\omega_0,\tau=\tau_0} > 0$. All the characteristic roots of the characteristic equation (3.3) have strictly negative real part with $\tau \in [0, \tau_0)$. Hence, the positive equilibrium point P_+ of equation (2.1) is asymptotically stable as showed in Figure 2.

Figure 2 shows that the juvenile prey, adult prey and predator can reach a stable state after a period of time in the harvest condition. The characteristic equation (3.3) has, at least, one characteristic root with strictly positive real part at $\tau > \tau_0$. Hence, the positive equilibrium point P_+ of equation (2.1) is unstable.

The coefficients of the center manifold $C_{\mu=0}$ are obtained as follows

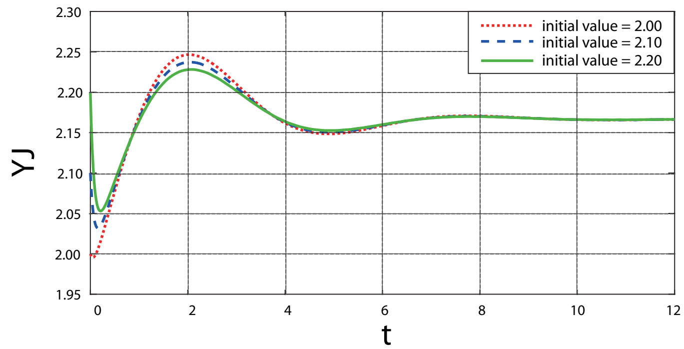
$$\begin{aligned} g_{20} &= -0.0705 + 0.0217i, \\ g_{11} &= -0.0707 + 0.0283i, \\ g_{02} &= -0.0658 + 0.0327i, \\ g_{21} &= 12.5734 - 4.5615i. \end{aligned}$$

From equation (4.28), we obtained $c_1(0) = 6.2920 - 2.2932i$, $\mu_2 < 0$ and $l_1(0) > 0$. According to theorem 4.2, we know that this Hopf bifurcation is subcritical and there exists the bifurcation periodic solution at $\tau < \tau_0$ in equation (2.1) as showed in Figure 3.

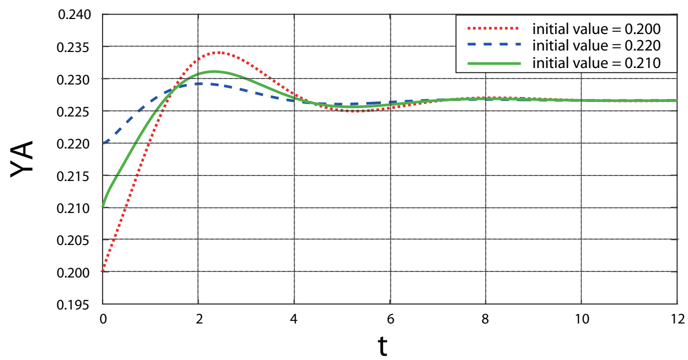
Based on the algorithm in section 4, we obtain that these coefficients of equation which restricted on the flow of center manifold $C_{\mu=0}$, and give μ_2 and $l_1(0)$ to determines the direction and stability of Hopf bifurcation respectively in this section. Numerical simulation shows that there exists subcritical Hopf bifurcation near P_+ in equation (2.1). Hopf bifurcation periodic solution exists when $\tau < \tau_0$ and disappears when $\tau \geq \tau_0$.

6. Conclusion

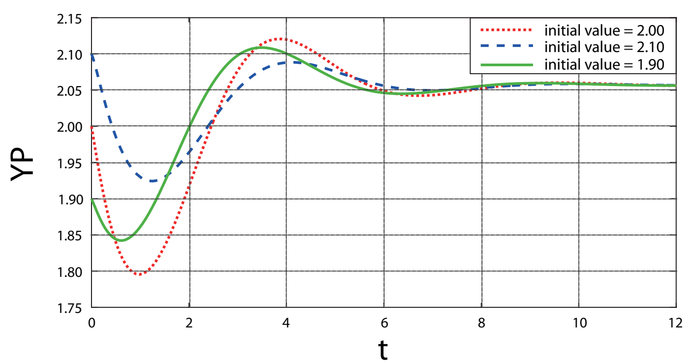
The response of a biological and financial equations to a particular input is often not immediate but is delayed. Even though a small time delay may leads to complex dynamic behavior.



(1)



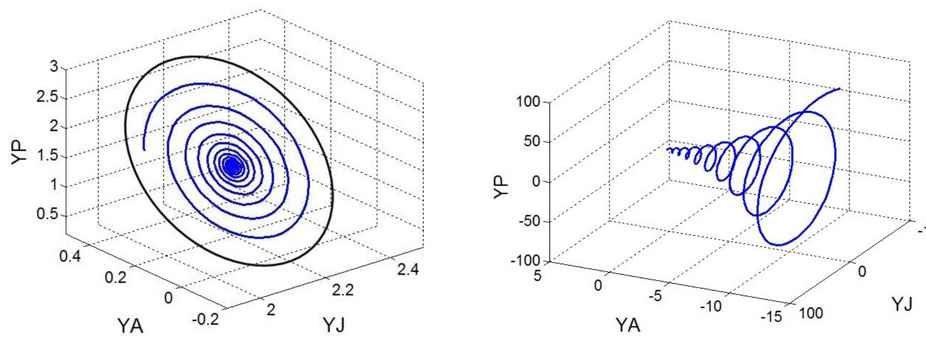
(2)



(3)

Figure 2. The positive equilibrium point P_+ of the equation (2.1) is asymptotically stable. (1),(2),(3) shows the population of juvenile prey, adult prey and predator variety process about time t respectively.

In this paper, we have detected the harvest term about juvenile prey, adult prey and predator as well as the internal competition, based on the result obtained in papers [1, 22]. We analyze the stability of the positive equilibrium point of equa-



(1) (2)

Figure 3. Subcritical Hopf bifurcation of equation (2.1) near P_+. (1) $\tau < \tau_0$. (2) $\tau \geq \tau_0$.

tion (2.1), and obtain the existence condition of Hopf bifurcation. When τ passes through the critical value τ_0, the equation loses its stability and generates Hopf bifurcation. We calculate the bifurcation value τ_0 and $\frac{\mathrm{dRe}\lambda(\tau)}{\mathrm{d}\tau}|_{\lambda=i\omega_0,\tau=\tau_0}$ across $\tau = \tau_0$. Based on differential equation qualitative theory, we present the sufficient condition of the existence of Hopf bifurcation near P_+. By using of spectral decomposition theory and center manifold method, we obtain the equation which restricted on the flow of center manifold of equation (2.1), and also present the algorithm which shows the properties such as stability and direction of Hopf bifurcation periodic solution of equation (2.1). Numerical simulation indicates that equation (2.1) exists subcritical Hopf bifurcation near P_+ under certain parameter conditions. Hopf bifurcation periodic solution exists when $\tau < \tau_0$ and disappears when $\tau \geq \tau_0$.

We have obtained the critical value of time delay which could affect the stable coexistence of both prey and predator species at the equilibrium point. However, the conditions to preserve stability or generate Hopf bifurcation are dependent upon the system parameters (e.g. harvest efforts, internal competition). It would be more interesting to give a thorough discussion about these parameters and their implications on the global behavior of the solutions. Furthermore, it is necessary to impose some control to prevent the possible abnormal oscillation in population density which can be carried out in a separate paper.

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Appendix

Section 3.2

(1) Relationships between parameter condition $P_n(n_1, n_2, n_3, n_4, n_5, n_6)$ and coefficients in equation (2.1) are shown as follows.

$$\begin{aligned}
n_1 &= -(d_1 + \alpha + E_1) + 2a_1 y_J^* + a_2 y_P^*,\\
n_2 &= a_2 y_J^*,\\
n_3 &= -(d_2 + E_2) - 2s_2 y_A^*,\\
n_4 &= c_1 y_P^*,\\
n_5 &= c_1 y_J^*,\\
n_6 &= -(d_3 + E_3).
\end{aligned}$$

(2) Relationships between parameter condition $P_q(q_1, q_2, q_3, q_4, q_5)$ and coefficients in equation (2.1) are shown as follows.

$$\begin{aligned}
q_1 &= n_1 + n_3 + n_6,\\
q_2 &= \alpha r_1 - n_1 n_3 - n_1 n_6 - n_3 n_6,\\
q_3 &= \alpha n_6 r_1 - n_1 n_3 n_6,\\
q_4 &= n_1 n_5 + n_3 n_5 - n_2 n_4,\\
q_5 &= n_1 n_3 n_5 - n_2 n_3 n_4 - n_5 \alpha r_1.
\end{aligned}$$

(3) Relationships between parameter condition $P_\gamma(\gamma_1, \gamma_2, \gamma_3)$ and coefficients in equation (2.1) are shown as follows.

$$\begin{aligned}
\gamma_1 &= Q_3 + \frac{1}{3}q_1,\\
\gamma_2 &= -\frac{1}{2}Q_3 + \frac{1}{3}q_1,\\
\gamma_3 &= \frac{1}{2}\sqrt{3}\Big(Q_5 + \frac{6Q_6}{Q_5}\Big),
\end{aligned}$$

where

$$\begin{aligned}
Q_3 &= \frac{1}{6}Q_5 - \frac{6Q_6}{Q_5},\\
Q_4 &= -12q_1^3 q_3 - 3q_1^2 1_2^2 - 54q_1 q_2 q_3 - 12q_2^3 + 81q_3^2 \geq 0,\\
Q_5 &= \sqrt[3]{36q_1 q_2 - 108q_3 + 8q_1^3 + 12\sqrt{Q_4}},\\
Q_6 &= -\frac{1}{3}q_2 - \frac{1}{9}q_1^2,
\end{aligned}$$

with

$$\begin{aligned}
&n_5\big(Q_3 + \frac{1}{3}q_1\big)^2 - q_4\big(Q_3 + \frac{1}{3}q_1\big) + q_5 = 0,\\
&n_5\Big(\big(-\frac{1}{2}Q_3 + \frac{1}{3}q_1\big)^2 - \big(3\big(Q_5 + \frac{6Q_5}{Q_4}\big)\big)^2\Big) - q_4\big(-\frac{1}{2}Q_3 + \frac{1}{3}q_1\big) + q_5 = 0,\\
&2n_5\big(-\frac{1}{2}Q_3 + \frac{1}{3}q_1\big)\big(-\frac{1}{2}\sqrt{3}\big(Q_5 + \frac{6Q_6}{Q_5}\big)\big) - q_4\big(\frac{1}{2}\sqrt{3}\big(Q_5 + \frac{6Q_6}{Q_5}\big)\big) = 0.
\end{aligned}$$

Section 3.3

Relationships between parameter condition $P_p(p_1, p_2, p_3)$ and coefficients in equation (2.1) are shown as follows.

$$\begin{aligned} p_1 &= n_5^2 - 2q_2 - q_1^2, \\ p_2 &= q_4^2 - 2n_5 q_5 - q_2^2 - 2q_1 q_3, \\ p_3 &= q_5^2 - q_3^2. \end{aligned}$$

Section 3.4

Relationships between parameter condition $P_{f_{1-8}}(f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8)$ and coefficients in equation (2.1) are shown as follows.

$$\begin{aligned} f_1 =& n_5^2(q_2^2 + 2q_1q_3) + n_5(2q_5q_1^2 + 4q_5q_2) - q_1^2q_4^2 - 3q_5^2 - 2q_4^2q_2, \\ f_2 =& 2n_5^2q_3^2 - 4q_5^2q_2 - 2q_1^2q_5^2, \\ f_3 =& -2n_5q_5q_3^2 - 2q_1q_3q_5^2 + q_3^2q_4^2 - q_5^2q_2^2, \\ f_4 =& q_4^2\omega_0^2 + n_5^2\omega_0^4 - 2n_5\omega_0^2q_5 + q_5^2, \\ f_5 =& -q_4\omega_0^2\cos(\omega_0\tau_0) + (n_5\omega_0^3 - q_5\omega_0)\sin(\omega_0\tau_0), \\ f_6 =& (n_5\omega_0^3 - q_5\omega_0)\cos(\omega_0\tau_0) + q_4\omega_0^2\sin(\omega_0\tau_0), \\ f_7 =& -3\omega_0^2 - q_2 + (-\tau_0n_5\omega_0^2 + \tau_0q_5)\cos(\omega_0\tau_0) - \omega_0(2n_5 + \tau_0q_4)\sin(\omega_0\tau_0), \\ f_8 =& -2q_1\omega_0 - -\omega_0(2n_5 + \tau_0q_4)\cos(\omega_0\tau_0) - (-\tau_0n_5\omega_0^2 + \tau_0q_5)\sin(\omega_0\tau_0). \end{aligned}$$

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$$\begin{aligned} \varphi_J(0) &= -z - \bar{z} + W_{20}^{(1)}(0)\frac{z^2}{2} + W_{11}^{(1)}(0)z\bar{z} + W_{02}^{(1)}(0)\frac{\bar{z}^2}{2} + \cdots, \\ \varphi_P(0) &= -z\rho_1 - \bar{z}\bar{\rho}_1 + W_{20}^{(3)}(0)\frac{z^2}{2} + W_{11}^{(3)}(0)z\bar{z} + W_{02}^{(3)}(0)\frac{\bar{z}^2}{2} + \cdots, \\ \varphi_A(0) &= -z - \bar{z} + W_{20}^{(2)}(0)\frac{z^2}{2} + W_{11}^{(2)}(0)z\bar{z} + W_{02}^{(2)}(0)\frac{\bar{z}^2}{2} + \cdots, \\ \varphi_J(-\tau) &= -ze^{-i\omega_0\tau} - \bar{z}e^{i\omega_0\tau} + W_{20}^{(1)}(-\tau)\frac{z^2}{2} + W_{11}^{(1)}(-\tau)z\bar{z} + W_{02}^{(1)}(-\tau)\frac{\bar{z}^2}{2} + \cdots, \\ \varphi_P(-\tau) &= -z\rho_1e^{-i\omega_0\tau} - \bar{z}\bar{\rho}_1e^{i\omega_0\tau} + W_{20}^{(3)}(-\tau)\frac{z^2}{2} + W_{11}^{(3)}(-\tau)z\bar{z} + W_{02}^{(3)}(-\tau)\frac{\bar{z}^2}{2} + \cdots, \\ \varphi_A(-\tau) &= -ze^{-i\omega_0\tau} - \bar{z}e^{i\omega_0\tau} + W_{20}^{(2)}(-\tau)\frac{z^2}{2} + W_{11}^{(2)}(-\tau)z\bar{z} + W_{02}^{(2)}(-\tau)\frac{\bar{z}^2}{2} + \cdots, \end{aligned}$$

and

$$\begin{aligned} K_1 =& -s_1 - \beta\rho_1, \\ K_2 =& -2s_1 - \beta(\rho_1 + r\bar{h}o_1), \\ K_3 =& -s_1 - \beta\bar{\rho}_1, \\ K_4 =& -s_1\big(-W_{20}^{(1)}(0) - W_{11}^{(1)}(0)\big) \\ & -\beta\Big(-W_{11}^{(3)}(0) - \frac{W_{20}^{(3)}(0)}{2} - \rho_1W_{11}^{(1)}(0) - \bar{\rho}_1\frac{W_{20}^{(1)}(0)}{2}\Big), \end{aligned}$$

$$
\begin{aligned}
K_5 =& -s_2,\\
K_6 =& -2s_2,\\
K_7 =& -s_2,\\
K_8 =& -s_2\big(W_{20}^{(2)}(0)-W_{11}^{(2)}(0)\big),\\
K_9 =& \beta\rho_1 e^{-2i\omega_0\tau},\\
K_{10} =& \beta(\rho_1+\bar{\rho}_1),\\
K_{11} =& \beta\bar{\rho}_1 e^{2i\omega_0\tau},\\
K_{12} =& \beta\Big(\big(-W_{11}^{(3)}(-\tau)-W_{11}^{(1)}(-\tau)\rho_1\big)e^{-i\omega_0\tau}\\
&+\Big(-\frac{W_{20}^{(3)}(-\tau)}{2}-\frac{W_{20}^{(1)}(-\tau)}{2}\bar{\rho}_1\Big)e^{i\omega_0\tau}\Big).
\end{aligned}
$$

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$$
\begin{aligned}
M_1^{(1)} &= \frac{K_1m_4m_5-r_1K_5m_5+n_2K_9m_4}{m_3m_4m_5+r_1\alpha m_5-n_2n_4m_4e^{-2i\omega_0\tau}},\\
M_1^{(2)} &= -\frac{K_5+\alpha M_1^{(1)}}{m_4}, M_1^{(3)} = \frac{n_4e^{-2i\omega_0\tau}M_1^{(1)}+K_9}{m_5},
\end{aligned}
$$

and

$$
\begin{aligned}
M_2^{(1)} &= \frac{K_2n_3(n_2-n_6)-K_6r_1(n_2-n_6)+K_{10}n_2n_3}{\alpha r_1(n_2-n_6)-n_2n_3n_4-n_2n_3(n_2-n_6)},\\
M_2^{(2)} &= -\frac{K_6+\alpha M_2^{(1)}}{n_3}, M_2^{(3)} = \frac{n_4M_2^{(1)}+K_{10}}{(n_2-n_6)},
\end{aligned}
$$

where $m_3 = 2i\omega_0-n_1$, $m_4 = 2i\omega_0-n_3$, $m_5 = 2i\omega_0-n_6+n_2e^{-2i\omega_0\tau}$. From equation (4.21), we obtain $W_{20}^{(i)}(\theta)$ and $W_{11}^{(i)}(\theta)(i=1,2,3)$,

$$
\begin{aligned}
W_{20}^{(1)}(\theta) &= -\frac{ig_{20}}{\omega_0}e^{i\omega_0\theta}-\frac{i\bar{g}_{02}}{3\omega_0}e^{-i\omega_0\theta}+M_1^{(1)}e^{2i\omega_0\theta},\\
W_{20}^{(2)}(\theta) &= -\frac{ig_{20}}{\omega_0}e^{i\omega_0\theta}-\frac{i\bar{g}_{02}}{3\omega_0}e^{-i\omega_0\theta}+M_1^{(2)}e^{2i\omega_0\theta},\\
W_{20}^{(3)}(\theta) &= -\frac{ig_{20}}{\omega_0}e^{i\omega_0\theta}-\frac{i\bar{g}_{02}}{3\omega_0}e^{-i\omega_0\theta}+M_1^{(3)}e^{2i\omega_0\theta}.
\end{aligned}
$$

From equation(4.22), we have

$$
\begin{aligned}
W_{11}^{(1)}(\theta) &= \frac{ig_{11}}{\omega_0}e^{i\omega_0\theta}-\frac{i\bar{g}_{11}}{\omega_0}e^{-i\omega_0\theta}+M_2^{(1)},\\
W_{11}^{(2)}(\theta) &= \frac{ig_{11}}{\omega_0}e^{i\omega_0\theta}-\frac{i\bar{g}_{11}}{\omega_0}e^{-i\omega_0\theta}+M_2^{(2)},\\
W_{11}^{(3)}(\theta) &= \frac{ig_{11}}{\omega_0}e^{i\omega_0\theta}-\frac{i\bar{g}_{11}}{\omega_0}e^{-i\omega_0\theta}+M_2^{(3)}.
\end{aligned}
$$