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PERIODIC ORBIT OF THE PENDULUM WITH A SMALL NONLINEAR DAMPING*

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Abstract We study the pendulum with a small nonlinear damping, which can be expressed by a Hamiltonian system with a small perturbation. We prove that a unique periodic orbit exists for any initial position between the equilibrium point and the heteroclinic orbit of the unperturbed system, depending on the choice of the bifurcation parameter in the damping. The main tools are bifurcation theory and Abelian integral technique, as well as the Zhang's uniqueness theorem on Liénard equations.

Keywords Periodic orbits, pendulum with a nonlinear damping, bifurcation, Abelian integrals.

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1. Introduction

We study the pendulum with a small nonlinear damping as follows

$$\ddot{x} + \mu(x^2 - \lambda)\dot{x} + a^2 \sin x = 0, \tag{1.1}$$

where $\mu > 0$ is a small parameter. As usual, we change equation (1.1) to a system

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= y, \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= -a^2 \sin x - \mu(x^2 - \lambda)y. \end{aligned} \tag{1.2}$$

If $\mu = 0$, then we have a Hamiltonian system

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= y, \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= -a^2 \sin x. \end{aligned} \tag{1.3}$$

It is well known that this is a mathematical model of the Simple Pendulum. The variable x is the angular displacement, the constant $a = \sqrt{\frac{g}{l}}$, where g is acceleration due to gravity, l is the length of the pendulum. The phase portraits of system (1.3) are shown in Fig. 1.

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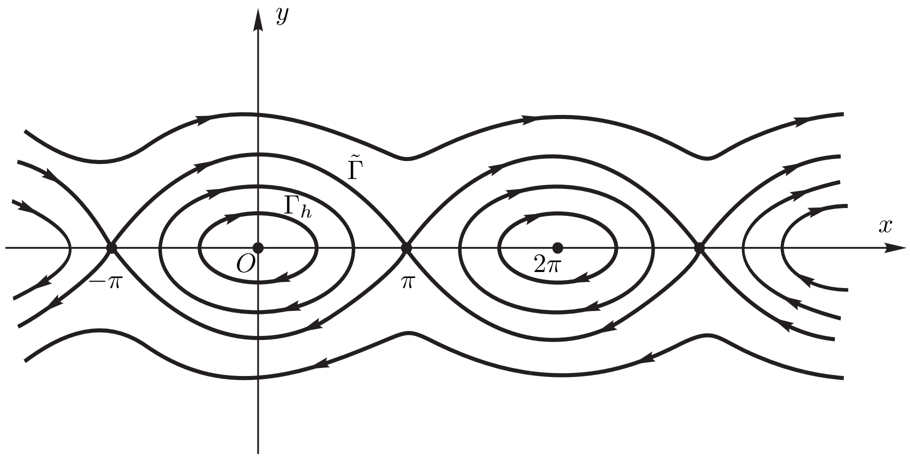

Figure 1. The phase portraits of system (1.3)

We let

$$H(x,y) = \frac{y^2}{2} - a^2 \cos x, \tag{1.4}$$

then a family of closed orbits of (1.3) can be expressed as

$$\Gamma_h = \{(x,y) : H(x,y) = h, -a^2 < h < a^2, -\pi < x < \pi\}, \tag{1.5}$$

which is continuous on h, Γ_h shrinks to the equilibrium point $O(0,0)$ as $h \to -a^2+0$, and Γ_h expands to the heteroclinic loop $\tilde{\Gamma}$ as $h \to a^2 - 0$.

Now we add the damping term and consider system (1.2), we have following two theorems.

Theorem 1.1. *If $\mu > 0$ and $\lambda \in (0, \pi^2 - 8)$, then system (1.2) has at most one closed orbit in the strip $-\pi < x < \pi$, and it is a stable and hyperbolic limit cycle if it exists.*

Theorem 1.2. *For each $\lambda \in (0, \pi^2 - 8)$, there is a $\mu_\lambda > 0$, such that for $\mu \in (0, \mu_\lambda)$ system (1.2) has a unique periodic orbit $\gamma_{\mu,\lambda}$ in the strip $-\pi < x < \pi$, and $\gamma_{\mu,\lambda}$ is a stable and hyperbolic limit cycle. Moreover, when $\mu \to 0$, $\gamma_{\mu,\lambda} \to \Gamma_{P^{-1}(\lambda)}$ of system (1.3) (in Haursdorff distance), where $\lambda = P(h)$ is a strictly increasing function for $h \in (-a^2, a^2]$ and $\lim_{h \to -a^2+0} P(h) = 0, P(a^2) = \pi^2 - 8$. Hence for small $\mu > 0$, if $\lambda \sim 0^+$ then $\gamma_{\mu,\lambda}$ is near the equilibrium point $O(0,0)$; and if $\lambda \sim (\pi^2 - 8)^-$ then $\gamma_{\mu,\lambda}$ is close to the heteroclinic loop $\tilde{\Gamma}$ of system (1.3).*

The proofs of these two theorems are given in the following two sections.

2. Proof of theorem 1.1

For the generalized Liénard equations

$$\ddot{x} + f(x)\dot{x} + g(x) = 0, \tag{2.1}$$

the Zhang's theorem about the uniqueness of limit cycles can be stated below.

Theorem 2.1 (Theorem 4.7 of [12]). *Suppose that:*

(1) $g(x)$ is Lipshitz continuous; $xg(x) > 0$ for $x \in (\alpha, 0) \cup (0, \beta)$, where $\alpha < 0 < \beta$.

(2) $f(x)$ is continuous; $\frac{f(x)}{g(x)}$ is not decreasing when $x \in (\alpha, 0) \cup (0, \beta)$, and $\frac{f(x)}{g(x)}$ is not a constant when $|x|$ is small.

Then system (2.1) has at most one limit cycle in the strip $\alpha < x < \beta$, and it is stable and hyperbolic if exists.

Now we prove Theorem 1.1.

Proof. Comparing equation (1.1) with (2.1), we have that $f(x) = \mu(x^2-\lambda), g(x) = a^2 \sin x$, hence the most conditions in Theorem 2.1 are satisfied, where $\alpha = -\pi$ and $\beta = \pi$. We only need to check the monotonically increasing of $\frac{f(x)}{g(x)}$. Computation shows that

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{f(x)}{g(x)} = \frac{\mu\,\eta(x)}{a^2\sin^2 x}, \quad \eta(x) = 2x\sin x - (x^2-\lambda)\cos x.$$

Note that $\eta(x)$ is a even function, $\eta(0) = \lambda > 0$, $\eta(\pi) = \pi^2 - \lambda > 8$, and $\eta'(x) = (2 + x^2 - \lambda)\sin x > 0$ for $\lambda \in (0, \pi^2 - 8)$ and $x \in (0, \pi)$, hence $\frac{\mathrm{d}}{\mathrm{d}x}\frac{f(x)}{g(x)} > 0$ for $x \in (-\pi, 0) \cup (0, \pi)$. By Theorem 2.1, Theorem 1.1 is proved. □

3. Proof of theorem 1.2

The system (1.2) has the canonical form

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= \frac{\partial H}{\partial y} + \mu f(x,y), \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= -\frac{\partial H}{\partial x} + \mu g(x,y), \end{aligned} \tag{3.1}$$

where H is given in (1.4) and $f(x,y) = 0, g(x,y) = -(x^2-\lambda)y$. We first state two general lemmas, which are useful in our discussion.

Lemma 3.1 (A partial result of Theorem 2.4 of part (II), in the book [4]). *Suppose that H, f, g are analytic in x and y, let*

$$I(h) = \oint_{\Gamma_h^+} f(x,y)dy - g(x,y)dx,$$

where $\Gamma_h \subset H^{-1}(h)$ is a continuous family of closed orbits (for $h \in (c,d)$) of system (3.1) with $\mu = 0$, Γ_h^+ means the integral is taken counterclockwise. The following statements hold:

(1) If there is an $h^* \in (c,d)$, satisfying $I(h^*) = 0$ and $I'(h^*) \neq 0$, then there is a $\mu_{h^*} > 0$, such that system (3.1) has a unique periodic orbit γ_{μ,h^*} for $\mu \in (0, \mu_{h^*})$ with the property that γ_{μ,h^*} tends to Γ_{h^*} (in Hausdorff distance) as $\mu \to 0$. In this case we say that γ_{μ,h^*} bifurcates from Γ_{h^*}.

(2) If system (3.1) has a limit cycle bifurcating from Γ_{h^*}, then $I(h^*) = 0$.

Remark 3.1. In Theorem 2.4 of part (II) of [4] it is supposed that H, f, g are polynomials, but most results there, especially the results above, can be generalized to the analytic case without any difficulty, because Theorem 2.4 is based on Theorem 2.1 of [4] (Poincaré-Pontryagin Theorem), which is valid in analytic case. This result has been successfully applied to many problems, and is generalized to 3-dimensional case in [9].

Lemma 3.2 (A simplified result of [10, Theorem 1])**.** *Consider a Hamiltonian function*

$$H(x,y) = \frac{y^2}{2} + \Phi(x), \tag{3.2}$$

where $\Phi(x)$ *is analytic, satisfying* $\Phi'(x)x > 0\,(or\ < 0)$ *for* $x \in (\alpha, 0) \cup (0, \beta)$. *Let* $\{\Gamma_h \subset H^{-1}(h)\}$ *be a continuous family of closed ovals, surrounding the origin* $(0,0)$*, for* $h \in (c,d)$*, and* u_h *and* v_h *be the intersection points of* Γ_h *with the* $x-$*axis. Hence for each* Γ_h *a unique function* $\tilde{x} = \tilde{x}(x)$ *can be defined by* $\Phi(\tilde{x}) = \Phi(x)$ *for* $u_h < x < 0 < \tilde{x} < v_h$*. Along with a ratio of two Abelian integrals*

$$P(h) = \frac{\oint_{\Gamma_h} f_2(x) y\, dx}{\oint_{\Gamma_h} f_1(x) y\, dx}, \tag{3.3}$$

we define a function

$$\xi(x) = \left. \frac{f_2(x)\Phi'(\tilde{x}) - f_2(\tilde{x})\Phi'(x)}{f_1(x)\Phi'(\tilde{x}) - f_1(\tilde{x})\Phi'(x)} \right|_{\tilde{x}=\tilde{x}(x)}, \tag{3.4}$$

where $x \in (\alpha, 0)$ *and* $f_1(x) f_1(\tilde{x}(x)) > 0$*. Then* $\xi'(x) < 0\ (> 0)$ *for* $x \in (\alpha, 0)$ *implies* $P'(h) > 0\ (< 0)$ *for* $h \in (c,d)$*.*

Remark 3.2. Lemma 3.2 is used to study the number of zeros of Abelian integral with 2 generating terms like $I(h) = c_1 I_1(h) + c_2 I_2(h)$, it has been generalized to the case that the Abelian integral has n generating terms in [5] and [11], see also the application of [5] in [2].

We prove Theorem 1.2 below.

Proof. System (1.2) is a perturbation of the Hamiltonian system (1.3) for small μ, we can use the Poincaré-Pontryagin theory to study the number of closed orbits, bifurcating from the family of $\{\Gamma_h : h \in (-a^2, a^2)\}$, see, for example, Section 2.1 of part (II) of [4]. The corresponding Abelian integral is given by

$$I(h) = \oint_{\Gamma_h} (x^2 - \lambda) y\, dx = I_2(h) - \lambda\, I_0(h) = I_0(h)(P(h) - \lambda), \tag{3.5}$$

where $I_k(h) = \oint_{\Gamma_h} x^k y\, dx$, the orientation of the integral is clockwise by the first equation of (1.2), and $P(h) = \frac{I_2(h)}{I_0(h)}$. Since $I_0(h)$ is the area of the region surrounded by Γ_h, $I_0(h) > 0$ for $h \in (-a^2, a^2]$, hence the zero of $I(h)$ is given by $\lambda = P(h)$. Let us prove that

$$P(-a^2) = \lim_{h \to -a^2+0} P(h) = 0, P(a^2) = \pi^2 - 8;\ P'(h) > 0, h \in (-a^2, a^2). \tag{3.6}$$

Changing the line integral along Γ_h to the definite integral with respect to x from u_h to v_h, and by the mean-value theorem of integrals we have

$$P(h) = \frac{\int_{u_h}^{v_h} x^2 y\, dx}{\int_{u_h}^{v_h} y\, dx} = \theta^2(h),$$

where $\theta(h)$ is between u_h and v_h. When $h \to -a^2+0$, Γ_h shrinks to the equilibrium point $O(0,0)$, hence $(u_h, v_h) \to (0,0)$, and $\theta(h) \to 0$. This means that $P(-a^2) = 0$.

When $h \to a^2-0$, Γ_h expands to the heteroclinic loop $\tilde{\Gamma}$, which has the equation $y = \pm\sqrt{2}\,a\sqrt{1+\cos x}$, $x \in (-\pi, \pi)$. Hence

$$P(a^2) = \frac{\int_0^\pi x^2\sqrt{1+\cos x}\,dx}{\int_0^\pi \sqrt{1+\cos x}\,dx} = \pi^2 - 8.$$

We use Lemma 3.2 to show the monotonicity of $P(h)$. The Hamiltonian function is given in (3.2) with $\Phi(x) = -a^2\cos x$, satisfying $\Phi'(x)x = a^2 x\sin x > 0$ for $x \in (-\pi,0)\cup(0,\pi)$. For any $x \in (-\pi, 0)$ there is a unique $\tilde{x} = -x \in (0,\pi)$ such that $\Phi(x) = \Phi(\tilde{x})$. Now $f_1(x) = 1$ and $f_2(x) = x^2$, from (3.4) we have

$$\xi(x) = \left.\frac{x^2\Phi'(\tilde{x}) - \tilde{x}^2\Phi'(x)}{\Phi'(\tilde{x}) - \Phi'(x)}\right|_{\tilde{x}=-x} = x^2.$$

Hence $\xi'(x) = 2x < 0$ for $x \in (-\pi, 0)$. By Lemma 3.2 we obtain $P'(h) > 0$ for $h \in (-a^2, a^2)$, and (3.6) is verified.

From (1.4) and (1.5) we can express Γ_h by the function $y = y(x;h)$, and along Γ_h, $\frac{\partial y}{\partial h} = \frac{1}{y}$, hence $I_0'(h) = \oint_{\Gamma_h} \frac{1}{y}dx$. By the first equation of (1.4) $I_0'(h)$ is the period $T(h)$ of the motion along Γ_h. Thus, for any fixed $h \in (-a^2, a^2)$, if $I(h) = 0$, i.e. $h = P^{-1}(\lambda)$ by (3.5) and (3.6), then $I'(h) = I_0(h)P'(h) > 0$, because $I_0'(h) = T(h)$ is finite.

Hence, by (3.5), (3.6) and Lemma 3.1 for any fixed $\lambda \in (0, \pi^2-8)$ there are unique $h_\lambda = P^{-1}(\lambda) \in (-a^2, a^2)$ and a $\mu_\lambda > 0$, such that for $\mu \in (0, \mu_\lambda)$ system (1.2) has a unique limit cycle $\gamma_{\mu,\lambda}$, bifurcating from Γ_{h_λ}. By Theorem 1.1, $\gamma_{\mu,\lambda}$ is the unique periodic orbit globally, it is stable and hyperbolic, and $\gamma_{\mu,\lambda}$ tends to Γ_{h_λ} as $\mu \to 0$. Especially, for small $\mu > 0$, if $\lambda \sim 0^+$ then $\gamma_{\mu,\lambda}$ is near the equilibrium point $O(0,0)$; and if $\lambda \sim (\pi^2-8)^-$ then $\gamma_{\mu,\lambda}$ is close to the heteroclinic loop $\tilde{\Gamma}$ of system (1.3).

The proof of Theorem 1.2 is finished. □

Remark 3.3. For $\mu > 0$ and $\lambda \sim 0$ a Hopf bifurcation of order one happens at the equilibrium point $(0,0)$. In fact, the eigenvalues of the linear part of system (1.2) at $(0,0)$ are $\frac{\mu}{2}\lambda \pm \frac{1}{2}\sqrt{4a^2 - (\mu\lambda)^2}\,i$ where $\lambda > 0$ small, they become $\pm a\,i$ when $\lambda = 0$. By using the formula (2.34) in Chapter 3 of [3], we find the first Lyapunov constant is $\mathrm{Re}(C_1) = -\frac{1}{8a^2}\mu < 0$ when $\lambda = 0$. Hence a Hopf bifurcation of order 1 happens: there is a $\sigma > 0$ such that system (1.2) has a unique limit cycle for $\lambda \in (0,\sigma)$, and the limit cycle is stable, because the equilibrium point $(0,0)$ is unstable for $\lambda > 0$ and $\mu > 0$.

Remark 3.4. If $\mu = 0$ then system (1.2) becomes (1.3), which is a Hamiltonian system and has a heteroclinic loop $\tilde{\Gamma}$, connecting the two saddles $(-\pi, 0)$ and $(\pi, 0)$. In general, the number of zeros of Abelian integral can not control the number of limit cycles, bifurcating from the heteroclinic loop, see [1] for example. But we can use Theorem 2.6 of [7] to study the heteroclinic bifurcation. Note that we use μ and λ respectively instead of the parameters α and δ in [7]. Since $\tilde{\Gamma}$ is symmetry with respect to the x-axis and it has the equation $y = y(x) = \sqrt{2}\,a\sqrt{1+\cos x}$ for $y > 0$, $x \in (-\pi, \pi)$, the function $M(\lambda)$ in [7] can be calculated as follows:

$$2\int_{-\infty}^{+\infty}(\lambda - x^2(t))y^2(t)\mathrm{d}t = 2\int_{-\pi}^{+\pi}(\lambda - x^2)y(x)\mathrm{d}x = 16a[\lambda - (\pi^2-8)].$$

Hence $M(\pi^2 - 8) = 0$, $M'(\pi^2 - 8) \neq 0$. On the other hand, $\Delta_i(\mu, \lambda) = \mu(\lambda - \pi^2)$ for $i = 1, 2$, implying

$$\Delta_{1\mu}(0, \pi^2 - 8) + \Delta_{2\mu}(0, \pi^2 - 8) = -16 \neq 0.$$

Thus, by Theorem 2.6 of [7], a heteroclinic bifurcation of order one happens near $\tilde{\Gamma}$. The same conclusion can be obtained by using Theorem 1.3 of [8]. See the recent book [6] for more details.

Remark 3.5. Theorem 1.1 is useful to control globally the number of limit cycle of system (1.2) , and Theorem 1.2 shows that the pendulum with the small damping in (1.1) can have periodic oscillation with the maximum angular displacement between 0^+ and π^- depending on the value of λ between 0^+ and $(\pi^2 - 8)^-$, which is independent of the value of a^2. This gives some nature of the pendulum.

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