BIFURCATIONS AND EXACT SOLUTIONS OF NONLINEAR SCHRÖDINGER EQUATION WITH AN ANTI-CUBIC NONLINEARITY*

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Abstract In this paper, we consider the nonlinear Schrödinger equation with an anti-cubic nonlinearity. By using the method of dynamical systems, we obtain bifurcations of the phase portraits of the corresponding planar dynamical system under different parameter conditions. Corresponding to different level curves defined by the Hamiltonian, we derive all exact explicit parametric representations of the bounded solutions (including periodic peakon solutions, periodic solutions, homoclinic solutions, heteroclinic solutions and compacton solutions).

Keywords Periodic solution, periodic peakon, compacton solution, bifurcation, homoclinic solution, nonlinear Schrödinger equation with an anti-cubic nonlinearity.

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1. Introduction

Recently, Lü and Ma, et al. [9] stated that the Madelung fluid description theory (see Auletta [1]) has been successfully used to discuss families of generalized onedimensional nonlinear Schrödinger equations containing a sum of cubic, anti-cubic and quintic nonlinearities as

$$i\mu\Psi_t + a\Psi_{xx} = q_0|\Psi|^{-4}\Psi + q_1|\Psi|^2\Psi + q_2|\Psi|^4\Psi, \qquad (1.1)$$

with μ , a, q_0 , q_1 and q_2 as real constants. For this equation, the upper-shifted bright envelope soliton-like solution has been studied by Fedele, et al. [5].

In this paper, it is different from [9] and [5]. Similar to Ref. [3], we consider the solutions of equation (1.1) with the form

 $\Psi(x,t) = \phi(\xi) \exp(-kx + \omega t)i, \quad \xi = x - ct, \tag{1.2}$

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where c, k and ω are constant parameters. Substituting (1.2) into (1.1) and decomposing into real and imaginary parts one obtains the real part as:

$$a\phi'' - (\mu\omega + ak^2)\phi - \frac{q_0}{\phi^3} - q_1\phi^3 - q_2\phi^5 = 0$$
(1.3)

and the imaginary part as:

$$(c\mu + 2ak)\phi' = 0. (1.4)$$

From the imaginary part (1.4), upon setting the coefficient to zero, it gives

$$c = -\frac{2ak}{\mu}.\tag{1.5}$$

Write that $\alpha_1 = \frac{1}{a}(\mu\omega + ak^2)$, $\alpha_3 = \frac{q_1}{a}$, $\alpha_5 = \frac{q_2}{a}$, $b = \frac{q_0}{a}$. Then, equation (1.3) is equivalent to the following planar Hamiltonian system:

$$\frac{d\phi}{d\xi} = y, \quad \frac{dy}{d\xi} = \frac{b + \phi^4(\alpha_1 + \alpha_3\phi^2 + \alpha_5\phi^4)}{\phi^3} \tag{1.6}$$

with the first integral

$$H(\phi, y) = \frac{1}{2}y^2 - \frac{1}{2}\alpha_1\phi^2 - \frac{1}{4}\alpha_3\phi^4 - \frac{1}{6}\alpha_5\phi^6 + \frac{b}{2\phi^2} = h.$$
(1.7)

Clearly, system (1.6) is a singular traveling wave system of the first class defined by Li and Chen [8] (and see Li [7]) with the singular straight lines $\phi = 0$. In fact, the existence of the singular straight line leads to a dynamical behavior of solutions with two scales, i.e. there exist periodic peakon solutions and compactons (see [4], [11]).

Because Refs. [5] and [3] did not study the dynamical behavior of system (1.6) and did not obtain all exact explicit solutions of equation (1.1) under different parameter conditions. In this paper, we use the dynamical system method to investigate the solutions of equation (1.1) having the form (1.2).

The paper is organized as follows. In section 2, we discuss the bifurcations of phase portraits of system (1.6) depending on parameter group $(b, \alpha_1, \alpha_3, \alpha_5)$. In section 3 and section 4, we find all exact parametric representations of solutions of system (1.6) in the case of there exist three equilibrium points. In section 5, we calculate exact parametric representations of solutions of system (1.6) in the cases of there exist two equilibrium points and there exists one equilibrium point of system (1.6).

Our main conclusions are the following Theorem.

Theorem 1.1. For the nonlinear Schrödinger equation (1.1) with an anti-cubic nonlinearity, considering the solutions with the form (1.2), then we have

(1) The function $\phi(\xi)$ satisfies singular nonlinear traveling wave system (1.6). Under different parameter conditions stated in section 2, system (1.6) has the bifurcations of phase portraits shown in Fig.1-Fig.4.

(2) Equation (1.1) has 26 exact explicit solutions given by (1.2), where $\phi(\xi)$ are given by (3.2)-(5.9).

(3) System (1.6) has periodic solutions, periodic peakon solutions, solitary wave solutions, quasi-peakon solutions as well as kink wave solutions and bounded solutions, which are given by (3.2)-(5.9), respectively.

The proof of the results of this theorem can be seen in next sections.

2. Bifurcations of phase portraits of system (1.6)

We consider the associated regular system of system (1.6) as follows:

$$\frac{d\phi}{d\zeta} = \phi^3 y, \quad \frac{dy}{d\zeta} = b + \phi^4 (\alpha_1 + \alpha_3 \phi^2 + \alpha_5 \phi^4), \tag{2.1}$$

where $d\xi = \phi^3 d\zeta$. For $\phi \neq 0$, this system has the same first integral as system (1.6). The dynamics of system (2.1) and (1.6) are different in the neighborhood of the straight line $\phi = 0$. Specially, under some parameter conditions, the variable ζ is a fast variable while the variable ξ is a slow variable in the sense of the geometric singular perturbation theory.

Let $F(X) = b + X^2 f(X)$, $f(X) = \alpha_1 + \alpha_3 X + \alpha_5 X^2$, where $X = \phi^2$. $F'(X) = (2\alpha_1 + 3\alpha_3 X + 4\alpha_5 X^2)X$. F'(X) has three real zeros at $X = 0, X = X_1 = \frac{-3\alpha_3 - \sqrt{\Delta}}{8\alpha_5}$, $X = X_2 = \frac{-3\alpha_3 + \sqrt{\Delta}}{8\alpha_5}$, when $\Delta = 9\alpha_3^2 - 32\alpha_1\alpha_5 \ge 0$. We have that

$$F(X_1) = \frac{-128a_1^2a_5^2 + 144a_1a_3^2a_5 - 27a_3^4 + 512ba_5^3 + (32a_1a_3a_5 - 9a_3^3)\sqrt{\Delta}}{512a_5^3}, \quad (2.2)$$

$$F(X_2) = -\frac{128a_1^2a_5^2 - 144a_1a_3^2a_5 + 27a_3^4 - 512ba_5^3 + (32a_1a_3a_5 - 9a_3^3)\sqrt{\Delta}}{512a_5^3}.$$
 (2.3)

Clearly, let z_j is a positive real zero of function F(X), then, in the positive ϕ -axis of the phase plane, system (2.1) has an equilibrium point $E_j(\sqrt{z_j}, 0)$.

It is easy to show the following conclusion.

(i) When $\alpha_1\alpha_5 > 0, \alpha_3\alpha_5 < 0$ and $\Delta > 0, F'(X)$ has two positive real zeros. Thus, if $b\alpha_5 < 0$, and $F(X_1)F(X_2) < 0$, then F(X) has three positive real zeros z_j (j = 1, 2, 3) satisfying $0 < z_1 < X_1 < z_2 < X_2 < z_3$ for $\alpha_5 > 0$ and $0 < z_1 < X_2 < z_2 < X_2 < z_3$ for $\alpha_5 > 0$ and $0 < z_1 < X_2 < z_2 < X_2 < z_3$ for $\alpha_5 > 0$.

If $b\alpha_5 < 0$, and $F(X_1) = 0$ or $F(X_2) = 0$, then F(X) has a double positive real zero and a simple positive real zero.

If $b\alpha_5 < 0$, and $F(X_1)F(X_2) > 0$, then F(X) has a positive real zero.

(ii) When $\alpha_1\alpha_5 > 0$, $\alpha_3\alpha_5 < 0$ and $\Delta > 0$, F'(X) has two positive real zeros. If $b\alpha_5 > 0$, and $F(X_1) > 0$, $F(X_2) < 0$, then F(X) has two positive real zeros z_j (j = 1, 2) satisfying $0 < X_1 < z_1 < X_2 < z_2$ for $\alpha_5 > 0$ and $0 < X_2 < z_1 < X_1 < z_2$ for $\alpha_5 < 0$.

If $b\alpha_5 > 0$, and $F(X_1) = 0$ or $F(X_2) = 0$, then F(X) has a double positive real zero.

(iii) When $\alpha_1\alpha_5 < 0$ and $\Delta > 0$, F'(X) has one positive real zero and one negative real zero. If $b\alpha_5 > 0$, and $F(X_1) > 0$, $F(X_2) < 0$, then F(X) has two positive real zeros. If $b\alpha_5 > 0$, and $F(X_1) = 0$ or $F(X_2) = 0$, then F(X) has a double positive real zero.

If $b\alpha_5 < 0$, then F(X) has a positive real zero.

(iv) When $\alpha_1\alpha_5 > 0$, $\alpha_3\alpha_5 > 0$ and $\Delta > 0$, F'(X) has two negative real zeros. If $b\alpha_5 < 0$, then F(X) has a positive real zero.

(v) When $\Delta \leq 0$, if $b\alpha_5 < 0$, then F(X) has a positive real zero.

Let $M(\phi_j, 0)$ be the coefficient matrix of the linearized system of system (8) at an equilibrium point $E_j(\phi_j, 0), \phi_j = \sqrt{z_j}$, and $J(\phi_j, 0) = \det M(\phi_j, 0)$. We have

$$J(\phi_j, 0) = -2\phi_j^4 F'(z_j).$$
(2.4)

Write $h_j = H(\phi_j, 0)$, where H is given by (1.7).

By the theory of planar dynamical systems (see Li [7]), for an equilibrium point of a planar integrable system, if J < 0, then the equilibrium point is a saddle point; if J > 0 and $(\text{Trace}M)^2 - 4J < 0(> 0)$, then it is a center point (a node point); if J = 0 and the Poincaré index of the equilibrium point is 0, then this equilibrium point is a cusp.

By using the above information to do qualitative analysis, we have the following bifurcations of the phase portraits of system (1.6) shown in Fig.1-Fig.4.

1. The case of there exist three equilibrium points (including double points) of system (1.6) in the positive ϕ -axis.



Figure 1. Bifurcations of phase portraits of system (1.6) when F(X) has three positive zeros and $\alpha_5 > 0$



Figure 2. Bifurcations of phase portraits of system (1.6) when F(X) has three positive zeros and $\alpha_5 < 0$



Figure 2. Bifurcations of phase portraits of system (1.6) when F(X) has three positive zeros and $\alpha_5 < 0$ (continued)

2. The case of there exist two equilibrium points (including double points) of system (1.6) in the positive ϕ -axis.



Figure 3. Bifurcations of phase portraits of system (1.6) when F(X) has two positive zeros

3. The case of there exists one equilibrium point of system (1.6) in the positive $\phi-{\rm axis.}$



Figure 4. Bifurcations of phase portraits of system (1.6) when F(X) has one positive zero

3. Explicit exact parametric representations of solutions of system (1.6) in the case of there exist three equilibrium points and $\alpha_5 > 0$

We are interested in the bounded solutions of system (1.6). It is known that for a given real h, the function $H(\phi, y) = h$ given by (1.7) defines level curves of system (1.6), which can have different branches. We see from (1.7) that $y^2 = \frac{1}{\phi^2}(-b + 2h\phi^2 + \alpha_1\phi^4 + \frac{1}{2}\alpha_3\phi^6 + \frac{1}{3}\alpha_5\phi^8)$. Hence, by using the first equation of system (1.6) we obtain

$$\xi = \int_{\phi_0}^{\phi} \frac{\phi d\phi}{\sqrt{-b + 2h\phi^2 + \alpha_1\phi^4 + \frac{1}{2}\alpha_3\phi^6 + \frac{1}{3}\alpha_5\phi^8}}$$
$$= \int_{X_0}^X \frac{dX}{2\sqrt{-b + 2hX + \alpha_1X^2 + \frac{1}{2}\alpha_3X^3 + \frac{1}{3}\alpha_5X^4}},$$
(3.1)

where $X = \phi^2$.

By using (3.1), we can calculate the parametric representations of the orbits of system (1.6).

3.1. The parametric representations of the bounded orbits given by Fig.1 (a).

In this case, in the right half-phase plane, as h is varied, the changes of level curves defined by $H(\phi, y) = h$ are shown in Fig.5 (a)-(f).

(i) Corresponding to the level curves defined by $H(\phi, y) = h, h \in (-\infty, h_2)$, there exist two families of open orbits of system (1.6) (see Fig.5 (a)). Because there is a singular straight line $\phi = 0$ of system (1.6), by using the theorem of finite time interval given by Li and Chen [8], the open orbit family near the straight line $\phi = 0$ gives rise to a family of compacton solutions. Now, (3.1) can be written as $-2\sqrt{\frac{\alpha_5}{3}}\xi = \int_X^{r_2} \frac{dX}{\sqrt{(r_1-X)(r_2-X)[(X-b_1)^2+a_1^2]}}$. Therefore, we obtain the parametric representation of the family of compacton solutions (see Fig.6 (a)) of system (1.6)



Figure 5. The changes of level curves defined by $H(\phi, y) = h$ in Fig.1 (a)

as follows:

$$\phi(\xi) = \left(\frac{r_1\tilde{A} - r_2\tilde{B} - (r_2\tilde{B} + r_1\tilde{A})\mathrm{cn}(\Omega_0\xi, k)}{(\tilde{A} - \tilde{B}) - (\tilde{A} + \tilde{B})\mathrm{cn}(\Omega_0\xi, k)}\right)^{\frac{1}{2}}, \quad \xi \in (-\xi_{01}, \xi_{01}), \tag{3.2}$$

where $\tilde{A}^2 = (r_2 + b_1)^2 + a_1^2$, $\tilde{B}^2 = (r_1 + b_1)^2 + a_1^2$, $k^2 = \frac{(\tilde{A} + \tilde{B})^2 - (r_1 - r_2)^2}{4\tilde{A}\tilde{B}}$, $\Omega_0 = 2\sqrt{\frac{1}{3}\alpha_5\tilde{A}\tilde{B}}$, $\xi_{01} = \frac{1}{\Omega_0} \text{cn}^{-1} \left(\frac{r_1\tilde{A} - r_2\tilde{B}}{r_2\tilde{B} + r_1\tilde{A}}, k\right)$, $\text{cn}(\cdot, k)$, $\text{sn}(\cdot, k)$, $\text{dn}(\cdot, k)$ are Jacobian elliptical elliptic for k and k are $\frac{1}{2}$. tic functions (see [2]).

(ii) Corresponding to the level curves defined by $H(\phi, y) = h, h \in (h_2, h_1)$, there exist a family of periodic orbits and two families of open orbits of system (1.6) (see Fig.5 (b)). For the family of periodic orbits, (3.1) can be written as $2\sqrt{\frac{\alpha_5}{3}}\xi$ = $\int_{r_3}^X \frac{dX}{\sqrt{(r_1 - X)(r_2 - X)(X - r_3)(X - r_4)}}.$ Thus, we obtain the parametric representation of the family of periodic solutions of system (1.6) as follows:

$$\phi(\xi) = \left(r_4 + \frac{r_3 - r_4}{1 - \tilde{\alpha}_1^2 \mathrm{sn}^2(\Omega_1 \xi, k)}\right)^{\frac{1}{2}},\tag{3.3}$$

where $\tilde{\alpha}_1^2 = \frac{r_2 - r_3}{r_2 - r_4}, k^2 = \frac{\tilde{\alpha}_1^2(r_1 - r_4)}{r_1 - r_3}, \Omega_1 = \sqrt{\frac{1}{3}\alpha_5(r_1 - r_3)(r_2 - r_4)}.$ For the family of open orbits which tend to the singular straight line $\phi = 0$ when $|y| \to +\infty$, (3.1) can be written as $-2\sqrt{\frac{\alpha_5}{3}}\xi = \int_X^{r_4} \frac{dX}{\sqrt{(r_1 - X)(r_2 - X)(r_3 - X)(r_4 - X)}}.$ Hence, we have the following compacton solution family of system (1.6):

$$\phi(\xi) = \left(r_3 - \frac{r_3 - r_4}{1 - \tilde{\alpha}_2^2 \mathrm{sn}^2(\Omega_1 \xi, k)}\right)^{\frac{1}{2}}, \quad \xi \in (-\xi_{02}, \xi_{02}), \tag{3.4}$$

where
$$\tilde{\alpha}_2^2 = \frac{r_1 - r_4}{r_1 - r_3}, k^2 = \frac{\tilde{\alpha}_2^2(r_2 - r_3)}{r_2 - r_4}, \xi_{02} = \frac{1}{\Omega_1} \operatorname{sn}^{-1} \left(\sqrt{\frac{r_4}{r_3 \tilde{\alpha}_2^2}}, k \right).$$

(iii) The level curves defined by $H(\phi, y) = h_1$ are a homoclinic orbit to the equilibrium point $(\phi_1, 0)$, a stable manifold and an unstable manifold of the saddle point $(\phi_1, 0)$, which tend to the singular straight line $\phi = 0$ when $|y| \to +\infty$ (see Fig.5 (c)). For the homoclinic orbit enclosing the equilibrium point $(\phi_2, 0)$, (3.1) can be written as $-2\sqrt{\frac{\alpha_5}{3}}\xi = \int_X^{X_M} \frac{dX}{(X-\phi_1^2)\sqrt{(X_a-X)(X_M-X)}}$. It follows the parametric representations of the bright envelope soliton solution of system (1.6):

$$\phi(\xi) = \left(\phi_1^2 + \frac{2(X_a - \phi_1^2)(X_M - \phi_1^2)}{(X_a - X_M)\cosh(\omega_0\xi) + (X_a + X_M - 2\phi_1^2))}\right)^{\frac{1}{2}}, \quad (3.5)$$

where $\omega_0 = 2\sqrt{\frac{1}{3}\alpha_5(X_a - \phi_1^2)(X_M - \phi_1^2)}$. This is so call bright envelope soliton solution of system (1.6).

For the stable manifold of the saddle point $(\phi_1, 0)$ which tends to the singular

straight line $\phi = 0$ when $|y| \to +\infty$, now (3.1) becomes $2\sqrt{\frac{\alpha_5}{3}}\xi = \int_0^X \frac{dX}{(\phi_1^2 - X)\sqrt{(X_a - X)(X_M - X)}} = -\int_{\phi_1^2}^u \frac{du}{u\sqrt{F(u)}}$, where $\tilde{F}(u) = (X_a - \phi_1^2)(X_M - \phi_1^2) + (X_a + X_M - 2\phi_1^2)u + u^2$. Thus, we have the following bounded solution of system (1.6):

$$\phi(\xi) = \left(\phi_1^2 - \frac{4(X_a - \phi_1^2)(X_M - \phi_1^2)P_1}{P_1^2 e^{\omega_1\xi} + (X_a - X_M)^2 e^{-\omega_1\xi} - 2(X_a + X_M - 2\phi_1^2)P_1}\right)^{\frac{1}{2}}, \quad \xi \in (0, \infty).$$
(3.6)

where $\omega_1 = 2\sqrt{\frac{1}{3}\alpha_5(X_a - \phi_1^2)(X_M - \phi_1^2)}, P_1 = \frac{1}{\phi_\tau^2}$ $\left[2\sqrt{(X_a-\phi_1^2)(X_M-\phi_1^2)X_aX_M}+(X_a+X_M-2\phi_1^2)\phi_1^2+2(X_a-\phi_1^2)(X_M-\phi_1^2)\right].$ (iv) The level curves defined by $H(\phi, y) = h, h \in (h_1, h_3)$ are two open curve

families, for which one curve family tends to the singular straight line $\phi = 0$ when $|y| \rightarrow +\infty$ (see Fig.5 (d)). It gives rise to a compacton solution family (see Fig.6 (b)) having the same parametric representation as (3.2).



Figure 6. Two families of compacton solutions defined by $H(\phi, y) = h$ in Fig.1 (a)

(v) The level curves defined by $H(\phi, y) = h_3$ are two stable manifolds and two unstable manifolds of the saddle point $(\phi_3, 0)$, for which two manifolds tend to the singular straight line $\phi = 0$ when $|y| \to +\infty$ (see Fig.5 (e)). For the left stable manifold, now (3.1) becomes $2\sqrt{\frac{\alpha_5}{3}}\xi = \int_0^X \frac{dX}{(\phi_3^2 - X)\sqrt{(X+a_3)(X+b_3)}} = -\int_{\phi_3^2}^u \frac{du}{u\sqrt{\tilde{F}_1(u)}}$, where $\tilde{F}_1(u) = (\phi_3^2 + a_3)(\phi_3^2 + b_3) - (a_3 + b_3 + 2\phi_3^2)u + u^2$. Thus, we have the following bounded solution of system (1.6):

$$\phi(\xi) = \left(\phi_3^2 - \frac{4(\phi_3^2 + a_3)(\phi_3^2 + b_3)P_2}{P_2^2 e^{\omega_2\xi} + (a_3 - b_3)^2 e^{-\omega_2\xi} + 2(a_3 + b_3 + 2\phi_3^2)P_2}\right)^{\frac{1}{2}}, \quad \xi \in (0, \infty),$$
(3.7)

where
$$\omega_2 = 2\sqrt{\frac{1}{3}\alpha_5(a_3+\phi_3^2)(b_3+\phi_3^2)},$$

 $P_2 = \frac{1}{\phi_3^2} \left[2\sqrt{(\phi_3^2+a_3)(\phi_3^2+b_3)a_3b_3} - (a_3+b_3+2\phi_3^2)\phi_3^2 + 2(a_3+\phi_3^2)(b_3+\phi_3^2) \right].$

3.2. The parametric representations of the bounded orbits given by Fig.1 (b).

Now, we have $h_1 = h_3$.

(i) Corresponding to the level curves defined by $H(\phi, y) = h, h \in (h_2, h_1)$, there exist a family of periodic orbits and two families of open orbits of system (1.6). The family of periodic orbits has the same parametric representations as (3.3).

The open curve family tending to the singular straight line $\phi = 0$ when $|y| \to +\infty$ has the same parametric representations as (3.4).

(ii) The level curves defined by $H(\phi, y) = h_1 = h_3$ are two heteroclinic orbits to the equilibrium points $(\phi_1, 0)$ and $(\phi_3, 0)$, and the stable manifolds and unstable manifolds of these saddle points. For the above heteroclinic orbit enclosing the equilibrium point $(\phi_2, 0)$, (3.1) can be written as $2\sqrt{\frac{\alpha_5}{3}}\xi = \int_{\phi_2^2}^X \frac{dX}{(\phi_3^2 - X)(X - \phi_1^2)}$. It follows the parametric representations of the the envelope kink wave solution of system (1.6):

$$\phi(\xi) = \left(\phi_3^2 - \frac{\phi_3^2 - \phi_1^2}{1 + m_0 e^{\omega_3 \xi}}\right)^{\frac{1}{2}},\tag{3.8}$$

where $m_0 = \frac{\phi_2^2 - \phi_1^2}{\phi_3^2 - \phi_2^2}$, $\omega_3 = 2\sqrt{\frac{1}{3}\alpha_5}(\phi_3^2 - \phi_1^2)$. For the stable manifold of the saddle point $(\phi_1, 0)$ which tends to the singular straight line $\phi = 0$ when $|y| \to +\infty$, now (3.1) becomes $2\sqrt{\frac{\alpha_5}{3}}\xi = \int_0^X \frac{dX}{(\phi_1^2 - X)(\phi_2^2 - X)}$. Thus, we have the following bounded solution of system (1.6):

$$\phi(\xi) = \left(\phi_1^2 + \frac{\phi_3^2 - \phi_1^2}{1 - m_1 e^{\omega_3 \xi}}\right)^{\frac{1}{2}}, \quad \xi \in (0, \infty),$$
(3.9)

where $m_1 = \frac{\phi_3^2}{\phi_1^2}, \omega_3 = 2\sqrt{\frac{1}{3}\alpha_5}(\phi_3^2 - \phi_1^2).$

3.3. The parametric representations of the bounded orbits given by Fig.1 (c).

(i) Corresponding to the level curves defined by $H(\phi, y) = h, h \in (h_2, h_3)$, there exist a family of periodic orbits and two families of open orbits of system (1.6). The family of periodic orbits has the same parametric representations as (3.3).

The open curve family tending to the singular straight line $\phi = 0$ when $|y| \to +\infty$ has the same parametric representations as (3.4).

(ii) The level curves defined by $H(\phi, y) = h_3$ are a homoclinic orbit to the equilibrium point $(\phi_3, 0)$, a stable manifold and an unstable manifold of the saddle point $(\phi_3, 0)$, and a open orbit which tend to the singular straight line $\phi = 0$ when $|y| \to +\infty$ and passes through the point $(\phi_c, 0)$. For the homoclinic orbit enclosing the equilibrium point $(\phi_2, 0)$, (3.1) can be written as $2\sqrt{\frac{\alpha_5}{3}}\xi = \int_{X_m}^X \frac{dX}{(\phi_3^2 - X)\sqrt{(X - X_c)(X - X_m)}}$, where $X_c = \phi_c^2, X_m = \phi_m^2$. It follows the parametric representations of the dark envelope soliton solution of system (1.6):

$$\phi(\xi) = \left(\phi_3^2 - \frac{2(\phi_3^2 - \phi_m^2)(\phi_3^2 - \phi_c^2)}{(\phi_m^2 - \phi_c^2)\cosh(\omega_4\xi) + (2\phi_3^2 - \phi_m^2 - \phi_c^2))}\right)^{\frac{1}{2}}, \quad (3.10)$$

where $\omega_4 = 2\sqrt{\frac{1}{3}\alpha_5(\phi_3^2 - \phi_m^2)(\phi_3^2 - \phi_c^2)}$.

3.4. The parametric representations of the bounded orbits given by Fig.1 (d).

In this case, $h_1 = h_2, \phi_1 = \phi_2$.

(i) Corresponding to the level curves defined by $H(\phi, y) = h, h \in (-\infty, h_1)$, in the right self-pase plane, there exist two families of open orbits of system (1.6). The open orbit family near the straight line $\phi = 0$ gives rise to a family of compacton solutions, which has the same parametric representation as (3.2).

(ii) Corresponding to the level curves defined by $H(\phi, y) = h_1$, in the right self-pase plane, there exist a stable manifold of the cusp point $(\phi_1, 0)$ and a open curve which gives rise a unbounded solution. For the stable manifold, (3.1) can be written as $2\sqrt{\frac{\alpha_5}{3}}\xi = \int_0^X \frac{dX}{(\phi_1^2 - X)\sqrt{(X_a - X)(\phi_1^2 - X)}}$, where $X_a = \phi_a^2$. It gives rise to the following result:

$$\phi(\xi) = \left(\phi_1^2 - \frac{\phi_a^2 - \phi_1^2}{\left(\omega_5\xi + \frac{\phi_a}{\phi_1}\right)^2 - 1}\right)^{\frac{1}{2}},\tag{3.11}$$

where $\omega_{5} = \sqrt{\frac{1}{3}\alpha_{5}}(\phi_{a}^{2} - \phi_{1}^{2}).$

(iii) The level curves defined by $H(\phi, y) = h, h \in (h_1, h_3)$ are two open curve families, for which one curve family tends to the singular straight line $\phi = 0$ when $|y| \to +\infty$. It gives rise to a compacton solution family having the same parametric representation as (3.2).

(iv) The level curves defined by $H(\phi, y) = h_3$ are two stable manifolds and two unstable manifolds of the saddle point $(\phi_3, 0)$, for which two manifolds tend to the singular straight line $\phi = 0$ when $|y| \to +\infty$ (see Fig.5 (e)). The left stable manifold has the same parametric representation as (3.7).

4. Explicit exact parametric representations of solutions of system (1.6) in the case of there exist three equilibrium points and $\alpha_5 < 0$

We consider the case $\alpha_5 < 0$ in this section.

4.1. The parametric representations of the bounded orbits given by Fig.2 (a).

(i) Corresponding to the level curves defined by $H(\phi, y) = h, h \in (h_3, h_1)$, there exists a family of periodic orbits enclosing the equilibrium point $(\phi_3, 0)$ of system (1.6). Now, (3.1) can be written as $2\sqrt{\frac{|\alpha_5|}{3}}\xi = \int_{r_2}^{X} \frac{dX}{\sqrt{(r_1 - X)(X - r_2)[(X - b_1)^2 + a_1^2]}}$. Thus, we obtain the parametric representation of the family of periodic solutions of system (1.6) as follows:

$$\phi(\xi) = \left(\frac{(r_1\tilde{B} + r_2\tilde{A}) - (r_1\tilde{B} - r_2\tilde{A})\mathrm{cn}(\Omega_2\xi, k)}{(\tilde{A} + \tilde{B}) + (\tilde{A} - \tilde{B})\mathrm{cn}(\Omega_2\xi, k)}\right)^{\frac{1}{2}},\tag{4.1}$$

where $\tilde{A}^2 = (r_1 - b_1)^2 + a_1^2$, $\tilde{B} = (r_2 - b_1)^2 + a_1^2$, $k^2 = \frac{(r_1 - r_2)^2 - (\tilde{A} - \tilde{B})^2}{4\tilde{A}\tilde{B}}$, $\Omega_2 = 2\sqrt{\frac{1}{3}|\alpha_5|\tilde{A}\tilde{B}}$.

(ii) Corresponding to the level curves defined by $H(\phi, y) = h, h \in (h_1, h_2)$, there exist two families of periodic orbits of system (1.6), enclosing the equilibrium point $(\phi_3, 0)$ and $(\phi_1, 0)$, respectively. Now, for the left family of periodic orbits enclosing the center $(\phi_1, 0), (3.1)$ can be written as $2\sqrt{\frac{|\alpha_5|}{3}}\xi = \int_{r_4}^X \frac{dX}{\sqrt{(r_1 - X)(r_2 - X)(r_3 - X)(X - r_4)}}$. Thus, we obtain the parametric representation of this family of periodic solutions of system (1.6) as follows:

$$\phi(\xi) = \left(r_1 - \frac{r_1 - r_4}{1 - \tilde{\alpha}_3^2 \mathrm{sn}^2(\Omega_3 \xi, k)}\right)^{\frac{1}{2}},\tag{4.2}$$

where $k^2 = \frac{(r_1 - r_2)(r_3 - r_4)}{(r_1 - r_3)(r_2 - r_4)}$, $\tilde{\alpha}_3^2 = \frac{r_4 - r_3}{r_1 - r_3}$, $\Omega_3 = \sqrt{\frac{1}{3}|\alpha_5|(r_1 - r_3)(r_2 - r_4)}$. We notice that when *h* approaches *h*₂, the periodic orbit in the left family

We notice that when h approaches h_2 , the periodic orbit in the left family defined by $H(\phi, y) = h$ tends to the left homoclinic loop which is close to the singular straight line $\phi = 0$. Therefore, the left homoclinic orbit gives rise to an envelope quasi-peakon solution and periodic orbit family gives rise to a periodic peakon family (see Fig.6 (a), (b).)

For the right family of periodic orbits enclosing the center $(\phi_3, 0)$, (3.1) can be written as $2\sqrt{\frac{|\alpha_5|}{3}}\xi = \int_{r_2}^{X} \frac{dX}{\sqrt{(r_1-X)(X-r_2)(X-r_3)(X-r_4)}}$. Hence, we obtain the parametric representation of this family of periodic solutions of system (1.6) as follows (see Fig.6 (d)):

$$\phi(\xi) = \left(r_3 + \frac{r_2 - r_3}{1 - \tilde{\alpha}_4^2 \mathrm{sn}^2(\Omega_3 \xi, k)}\right)^{\frac{1}{2}},\tag{4.3}$$

where $k^2 = \frac{(r_1 - r_2)(r_3 - r_4)}{(r_1 - r_3)(r_2 - r_4)}, \tilde{\alpha}_4^2 = \frac{r_1 - r_2}{r_1 - r_3}, \Omega_3 = \sqrt{\frac{1}{3}|\alpha_5|(r_1 - r_3)(r_2 - r_4)}.$

(iii) Corresponding to the level curves defined by $H(\phi, y) = h_2$, there exist two homoclinic orbits of system (1.6) to the saddle point $(\phi_2, 0)$, enclosing the equilibrium point $(\phi_3, 0)$ and $(\phi_1, 0)$, respectively. For the right homoclinic orbit, (3.1) can be written as $-2\sqrt{\frac{|\alpha_5|}{3}}\xi = \int_X^{X_M} \frac{dX}{(X-\phi_2^2)\sqrt{(X_M-X)(X-X_m)}}$, where $X_M = \phi_M^2$. It follows the parametric representation of the bright envelope soliton solution of system (1.6) as follows:

$$\phi(\xi) = \left(\phi_2^2 + \frac{2(\phi_M^2 - \phi_2^2)(\phi_2^2 - \phi_m^2)}{(\phi_M^2 - \phi_m^2)\cosh(\omega_6\xi) - (\phi_M^2 + \phi_m^2 - 2\phi_2^2)}\right)^{\frac{1}{2}}, \qquad (4.4)$$

where $\omega_6 = 2\sqrt{\frac{1}{3}|\alpha_5|(\phi_M^2 - \phi_2^2)(\phi_2^2 - \phi_m^2)}$.

For the left homoclinic orbit, (3.1) can be written as $2\sqrt{\frac{|\alpha_5|}{3}}\xi = \int_{x_m}^X \frac{dX}{(\phi_2^2 - X)\sqrt{(X_M - X)(X - X_m)}}$. It gives rise to the following parametric representation of the dark envelope soliton solution of system (1.6):

$$\phi(\xi) = \left(\phi_2^2 - \frac{2(\phi_M^2 - \phi_2^2)(\phi_2^2 - \phi_m^2)}{(\phi_M^2 - \phi_m^2)\cosh(\omega_6\xi) + (\phi_M^2 + \phi_m^2 - 2\phi_2^2)}\right)^{\frac{1}{2}}.$$
 (4.5)

(iv) Corresponding to the level curves defined by $H(\phi, y) = h, h \in (h_2, \infty)$, there exists a larger family of periodic orbits enclosing three equilibrium point $(\phi_j, 0), j = 1, 2, 3$ of system (1.6). In this case, this family has the same parametric representation as (4.1). Because there exists a segment of every periodic orbit in this periodic family for which it is very close to the singular straight line $\phi = 0$. So that, this larger periodic orbit loop gives rise to a periodic peakon solution of system (1.6) (see Fig.6 (c)).



Figure 7. The wave profiles of quasi-peakon and periodic peakons in Fig.2 (a)

4.2. The parametric representations of the bounded orbits given by Fig.2 (b).

In this case, we have $h_1 = h_3$.

(i) Corresponding to the level curves defined by $H(\phi, y) = h, h \in (h_1, h_2)$, there exist two families of periodic orbits of system (1.6), enclosing the equilibrium point $(\phi_3, 0)$ and $(\phi_1, 0)$, respectively. Now, for the left family of periodic orbits enclosing the center $(\phi_1, 0), (3.1)$ can be written as $2\sqrt{\frac{|\alpha_5|}{3}}\xi = \int_{r_4}^X \frac{dX}{\sqrt{(r_1-X)(r_2-X)(r_3-X)(X-r_4)}}$.

By the symmetry of the phase portrait in Fig.2 (b), we have $r_1 - r_2 = r_3 - r_4$. Thus, we obtain the parametric representation of this family of periodic orbits of system (1.6) as follows:

$$\phi(\xi) = \left(r_1 - \frac{r_1 - r_4}{1 - \tilde{\alpha}_3^2 \mathrm{sn}^2(\Omega_4 \xi, k)}\right)^{\frac{1}{2}},\tag{4.6}$$

where $k = \frac{r_1 - r_2}{r_1 - r_3}$, $\tilde{\alpha}_3^2 = \frac{r_4 - r_3}{r_1 - r_3}$, $\Omega_4 = \sqrt{\frac{1}{3}|\alpha_5|}(r_1 - r_3)$. For the right family of periodic orbits enclosing the center $(\phi_3, 0)$, we obtain the parametric representation of this family of periodic orbits of system (1.6) as follows:

$$\phi(\xi) = \left(r_3 + \frac{r_2 - r_3}{1 - \tilde{\alpha}_4^2 \mathrm{sn}^2(\Omega_4 \xi, k)}\right)^{\frac{1}{2}},\tag{4.7}$$

where $k = \frac{r_1 - r_2}{r_1 - r_3}$, $\tilde{\alpha}_4^2 = \frac{r_1 - r_2}{r_1 - r_3}$, $\Omega_4 = \sqrt{\frac{1}{3}|\alpha_5|}(r_1 - r_3)$.

(ii) Corresponding to the level curves defined by $H(\phi, y) = h_2$, there exist two symetric homoclinic orbits of system (1.6) to the saddle point $(\phi_2, 0)$, enclosing the equilibrium point $(\phi_3, 0)$ and $(\phi_1, 0)$, respectively. These orbits have the same parametric representations as (4.4) and (4.5), where $\phi_M - \phi_2 = \phi_2 - \phi_m$.

(iii) Corresponding to the level curves defined by $H(\phi, y) = h, h \in (h_2, \infty)$, there exists a larger family of periodic orbits enclosing three equilibrium point $(\phi_j, 0), j = 1, 2, 3$ of system (1.6). In this case, this family has the same parametric representation as (4.1).

Similarly, we can study the parametric representations for the orbits shown in Fig.2 (c).

4.3. The parametric representations of the bounded orbits given by Fig.2 (e).

In this case, we have $h_1 = h_2 > h_3$.

(i) Corresponding to the level curves defined by $H(\phi, y) = h, h \in (h_3, h_1)$, there exists a family of periodic orbits of system (1.6), enclosing the equilibrium point $(\phi_3, 0)$. This family has the same parametric representation as (4.1).

(ii) Corresponding to the level curves defined by $H(\phi, y) = h_1 = h_2$, there is a homoclinic orbit to the cusp point $(\phi_1, 0) = (\phi_2, 0)$, enclosing the equilibrium point $(\phi_3, 0)$. Now, (3.1) can be written as $-2\sqrt{\frac{|\alpha_5|}{3}}\xi = \int_X^{X_M} \frac{dX}{(X-\phi_1^2)\sqrt{(X_M-X)(X-\phi_1^2)}}$, where $X_M = \phi_M^2$. It follows the parametric representation of the bright envelope soliton solution of system (1.6):

$$\phi(\xi) = \left(\phi_1^2 + \frac{\phi_M^2 - \phi_1^2}{1 + (\omega_7 \xi)^2}\right)^{\frac{1}{2}},\tag{4.8}$$

where $\omega_7 = \sqrt{\frac{1}{3}|\alpha_5|}(\phi_M^2 - \phi_1^2).$

(iii) Corresponding to the level curves defined by $H(\phi, y) = h, h \in (h_1, \infty)$, there exists a larger family of periodic orbits of system (1.6), enclosing the equilibrium point $(\phi_3, 0)$ and cusp point $(\phi_1, 0)$. This family has the same parametric representation as (4.1).

5. Explicit exact parametric representations of solutions of system (1.6) in the case of system (1.6)has one or two equilibrium points

We first consider the exact parametric representations of solutions of system (1.6) in the case of system (1.6) has two equilibrium points in the positive ϕ -axis (see Fig.3).

5.1. The parametric representations of the bounded orbits given by Fig.3 (b).

(i) Corresponding to the level curves defined by $H(\phi, y) = h, h \in (-\infty, h_2)$, there exist a family of open orbits of system (1.6) which tends to the singular straight line $\phi = 0$ when $|y| \to +\infty$. Now, (3.1) can be written as $-2\sqrt{\frac{|\alpha_5|}{3}}\xi = \int_{0}^{10} \int_{0}^{10} |x|^{-1} dx$

 $\int_X^{r_1} \frac{dX}{\sqrt{(r_1 - X)(X + r_2)[(X - b_1)^2 + a_1^2]}}$. Thus, we obtain the parametric representation of the family of compacton solutions of system (1.6) as follows:

$$\phi(\xi) = \left(\frac{(r_1\tilde{A} - r_2\tilde{B}) + (r_1\tilde{A} + r_2\tilde{B})\mathrm{cn}(\Omega_5\xi, k)}{(\tilde{A} + \tilde{B}) + (\tilde{A} - \tilde{B})\mathrm{cn}(\Omega_5\xi, k)}\right)^{\frac{1}{2}}, \quad \xi \in (-\xi_{03}, \xi_{03}), \tag{5.1}$$

where $\tilde{A}^2 = (r_2 - b_1)^2 + a_1^2$, $\tilde{B} = (r_1 + b_1)^2 + a_1^2$, $k^2 = \frac{(r_1 + r_2)^2 - (\tilde{A} - \tilde{B})^2}{4\tilde{A}\tilde{B}}$, $\Omega_5 = 2\sqrt{\frac{1}{3}|\alpha_5|\tilde{A}\tilde{B}}$, $\xi_{03} = \frac{1}{\Omega_5} \operatorname{cn}^{-1}\left(\frac{r_2\tilde{B} - r_1\tilde{A}}{r_1\tilde{A} + r_2\tilde{B}}, k\right)$. (ii) Corresponding to the level curves defined by $H(\phi, y) = h, h \in (h_2, h_1)$, there

(ii) Corresponding to the level curves defined by $H(\phi, y) = h, h \in (h_2, h_1)$, there exist a family of periodic orbits enclosing the equilibrium point $(\phi_2, 0)$ of system (1.6) and an open curve family which tends to the singular straight line $\phi = 0$ when $|y| \to +\infty$. For the periodic family. Now, (3.1) can be written as $2\sqrt{\frac{|\alpha_5|}{3}}\xi = \int_{r_2}^{X} \frac{dX}{\sqrt{(r_1-X)(X-r_2)(X-r_3)(X+r_4)}}$. Thus, we obtain the parametric representation of the family of periodic solutions of system (1.6) as follows:

$$\phi(\xi) = \left(r_3 + \frac{r_2 - r_3}{1 - \tilde{\alpha}_5^2 \mathrm{sn}^2(\Omega_6 \xi, k)}\right)^{\frac{1}{2}},\tag{5.2}$$

where $\tilde{\alpha}_5^2 = \frac{r_1 - r_2}{r_1 - r_3}, k^2 = \frac{\tilde{\alpha}_5^2(r_3 + r_4)}{r_2 + r_4}, \Omega_6 = \sqrt{\frac{|\alpha_5|}{3}(r_1 - r_3)(r_2 + r_4)}.$

For the open curve family, (3.1) can be written as $-2\sqrt{\frac{|\alpha_5|}{3}}\xi = \int_X^{r_3} \frac{dX}{\sqrt{(r_1-X)(r_2-X)(r_3-X)(X+r_4)}}$. Hence, we have the parametric representation of the compacton solution family of system (1.6) as follows:

$$\phi(\xi) = \left(r_2 - \frac{r_2 - r_3}{1 - \tilde{\alpha}_6^2 \mathrm{sn}^2(\Omega_6 \xi, k)}\right)^{\frac{1}{2}}, \xi \in (-\xi_{04}, \xi_{04}), \tag{5.3}$$

where $\tilde{\alpha}_6^2 = \frac{r_3 + r_4}{r_2 + r_4}, k^2 = \frac{\tilde{\alpha}_6^2(r_1 - r_2)}{r_1 - r_3}, \Omega_6 = \sqrt{\frac{|\alpha_5|}{3}(r_1 - r_3)(r_2 + r_4)}, \xi_{04} = \frac{1}{\Omega_6} \mathrm{sn}^{-1} \left(\sqrt{\frac{r_3}{\tilde{\alpha}_6^2 r_2}}, k\right).$

(iii) Corresponding to the level curves defined by $H(\phi, y) = h_1$, there exist a homoclinic orbit of system (1.6) to the saddle point $(\phi_1, 0)$, enclosing the equilibrium point $(\phi_2, 0)$ and a stable manifold, an unstable manifold of the saddle point $(\phi_1, 0)$ which tend to the singular straight line $\phi = 0$ when $|y| \to +\infty$. For the homoclinic orbit, (3.1) can be written as $-2\sqrt{\frac{|\alpha_5|}{3}}\xi = \int_X^{X_M} \frac{dX}{(X-\phi_1^2)\sqrt{(\phi_M^2-X)})(X+r_{04})}$. It gives rise to the the parametric representation of the bright envelope soliton solution of system (1.6):

$$\phi(\xi) = \left(\phi_1^2 + \frac{2(\phi_M^2 - \phi_1^2)(\phi_1^2 + r_{04})}{(\phi_M^2 + r_{04})\cosh(\omega_8\xi) - (\phi_M^2 - r_{04} - 2\phi_1^2)}\right)^{\frac{1}{2}},\tag{5.4}$$

where $\omega_8 = 2\sqrt{\frac{|\alpha_5|}{3}}(\phi_M^2 - \phi_1^2)(\phi_1^2 + r_{04})$. For the stable manifold of the saddle point $(\phi_1, 0)$ which tends to the singular

For the stable manifold of the saddle point $(\phi_1, 0)$ which tends to the singular straight line $\phi = 0$ when $|y| \to +\infty$, (3.1) becomes $2\sqrt{\frac{|\alpha_5|}{3}}\xi = \int_0^X \frac{dX}{(\phi_1^2 - X)\sqrt{(r_1 - X)(X + r_2)}}$. Thus, we have the following bounded solution of system (1.6):

$$\phi(\xi) = \left(\phi_1^2 - \frac{4(r_1 - \phi_1^2)(\phi_1^2 + r_2)P_3}{P_3^2 e^{\omega_9\xi} + (r_1 + r_2)^2 e^{-\omega_9\xi} - 2(\phi_1^2 + r_2 - r_1)P_4}\right)^{\frac{1}{2}}, \quad \xi \in (0, \infty),$$
(5.5)

where $\omega_9 = -2\sqrt{\frac{1}{3}|\alpha_5|(r_1 - \phi_1^2)(\phi_1^2 + r_2)},$ $P_3 = \frac{1}{\phi_1^2} \left[2\sqrt{(r_1 - \phi_1^2)(\phi_1^2 + r_2)r_1r_2} + 2(r_1 - \phi_1^2)(\phi_1^2 + r_2) + \phi_1^2(2\phi_1^2 + r_1 - r_2) \right].$ (iv) Corresponding to the level curves defined by $H(\phi, y) = h, h \in (h_1, +\infty),$

(iv) Corresponding to the level curves defined by $H(\phi, y) = h, h \in (h_1, +\infty)$, there exists an open curve family of system (1.6) which tends to the singular straight line $\phi = 0$ when $|y| \to +\infty$. It gives rise to a compacton solution family having the same parametric representation as (5.1).

5.2. The parametric representations of the bounded orbits given by Fig.3 (a).

(i) Corresponding to the level curves defined by $H(\phi, y) = h, h \in (h_1, h_2)$, there exist a family of periodic orbits enclosing the equilibrium point $(\phi_1, 0)$ of system (1.6) and an unbounded open orbit. For the periodic family, (3.1) can be written as $2\sqrt{\frac{\alpha_5}{3}}\xi =$ $\int_{r_3}^X \frac{dX}{\sqrt{(r_1-X)(r_2-X)(X-r_3)(X+r_4)}}$. Thus, we obtain the parametric representation of the family of periodic orbits of system (1.6) as follows:

$$\phi(\xi) = \left(-r_4 + \frac{r_3 + r_4}{1 - \tilde{\alpha}_7^2 \mathrm{sn}^2(\Omega_7 \xi, k)}\right)^{\frac{1}{2}},\tag{5.6}$$

where $\tilde{\alpha}_7^2 = \frac{r_2 - r_3}{r_2 + r_4}, k^2 = \frac{\tilde{\alpha}_7^2(r_1 + r_4)}{r_1 - r_3}, \Omega_7 = \sqrt{\frac{\alpha_5}{3}(r_1 - r_3)(r_2 + r_4)}.$ (ii) Corresponding to the level curves defined by $H(\phi, y) = h_2$, there exists a

(ii) Corresponding to the level curves defined by $H(\phi, y) = h_2$, there exists a homoclinic orbit of system (1.6) to the saddle point $(\phi_2, 0)$, enclosing the equilibrium point $(\phi_1, 0)$. (3.1) can be written as $2\sqrt{\frac{\alpha_5}{3}}\xi = \int_{r_m}^X \frac{dX}{(\phi_2^2 - X)\sqrt{(X - r_m)(X + r_2)}}$. It gives rise to the the parametric representation of the dark envelope soliton solution of

system (1.6):

$$\phi(\xi) = \left(\phi_2^2 - \frac{2(\phi_2^2 - r_m)(\phi_2^2 + r_2)}{(r_m + r_2)\cosh(\omega_{10}\xi) - (r_m - r_2 - 2\phi_2^2)}\right)^{\frac{1}{2}},$$
(5.7)

where $\omega_{10} = 2\sqrt{\frac{\alpha_5}{3}(\phi_2^2 - r_m)(\phi_2^2 + r_2)}.$

5.3. The parametric representations of the bounded orbits given by Fig.3 (d).

(i) Corresponding to the level curves defined by $H(\phi, y) = h_1$, there exist a stable manifold and an unstable manifold of the cusp point $(\phi_1, 0)$ which tends to the singular straight line $\phi = 0$ when $|y| \to +\infty$. (3.1) can be written as $2\sqrt{\frac{|\alpha_5|}{3}}\xi = \int_0^X \frac{dX}{\sqrt{(\phi_1^2 - X)^3(X + r_4)}}$. Thus, we obtain the parametric representation of a bounded solution of system (1.6) as follows:

$$\phi(\xi) = \left(\frac{\phi_1^2 + r_4}{1 + (\sqrt{\frac{r_4}{\phi_1^2}} + \omega_{11}\xi)^2}\right)^{\frac{1}{2}},\tag{5.8}$$

where $\omega_{11} = \sqrt{\frac{|\alpha_5|}{3}} (\phi_1^2 + r_4).$

(ii) Corresponding to the level curves defined by $H(\phi, y) = h, h \in (-\infty, h_1)$ and $h \in (h_1, \infty)$, respatively, there exist two families of open orbits of system (1.6). They have the same parametric representation of the family of compacton solutions as (5.1).

Finally, we consider the exact parametric representations of solutions of system (1.6) in the case of system (1.6) has only one equilibrium point in the positive ϕ -axis (see Fig.4).

5.4. The parametric representations of the bounded orbits given by Fig.4 (a).

(i) Corresponding to the level curves defined by $H(\phi, y) = h, h \in (-\infty, h_1)$, there exist two families of open orbits of system (1.6). The open orbit family near the straight line $\phi = 0$ gives rise to a family of compacton solutions of system (1.6). Now, (3.1) can be written as $2\sqrt{\frac{\alpha_5}{3}}\xi = \int_X^{r_2} \frac{dX}{\sqrt{(r_1-X)(r_2-X)[(X-b_1)^2+a_1^2]}}$. We obtain the same parametric representation of the family of compacton solutions of system (1.6) as (3.2).

(ii) Corresponding to the level curves defined by $H(\phi, y) = h_1$, there exist two stable manifolds and two unstable manifolds of the saddle point $(\phi_1, 0)$ of system (1.6). For the left stable manifold, (3.1) can be written as $2\sqrt{\frac{\alpha_5}{3}}\xi = \int_0^X \frac{dX}{(\phi_1^2 - X)\sqrt{(X-b_1)^2 + a_1^2}}$. It gives rise to the the parametric representation of the homoclinic orbit of system (1.6):

$$\phi(\xi) = \left(\phi_1^2 - \frac{4((\phi_1^2 - b_1)^2 + a_1^2)P_4}{P_4^2 e^{\omega_{12}\xi} - 4a_1^2 e^{-\omega_{12}\xi} + 4(\phi_1^2 - b_1)P_4}\right)^{\frac{1}{2}}, \quad \xi \in (0, \infty), \tag{5.9}$$

where $P_4 = \frac{1}{\phi_1^2} \left[2\sqrt{((\phi_1^2 - b_1)^2 + a_1^2)(a_1^2 + b_1^2)} + 2(a_1^2 + b_1^2) - 2b_1\phi_1^2 \right],$ $\omega_{12} = 2\sqrt{\frac{1}{3}\alpha_5((\phi_1^2 - b_1)^2 + a_1^2)}.$

5.5. The parametric representations of the bounded orbits given by Fig.4 (b).

Corresponding to the level curves defined by $H(\phi, y) = h, h \in (h_1, +\infty)$, there exists a family of periodic orbits enclosing the equilibrium point $(\phi_1, 0)$ of system (1.6). Now, (3.1) can be written as $2\sqrt{\frac{|\alpha_5|}{3}}\xi = \int_{r_2}^{X} \frac{dX}{\sqrt{(r_1-X)(X-r_2)[(X-b_1)^2+a_1^2]}}$. Thus, we obtain the same parametric representation of the family of periodic orbits of system (1.6) as (4.1).

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