Journal of Applied Analysis and Computation
Volume 8, Number 6, December 2018, 1727–1746 Website:http://jaac.ijournal.cn
DOI:10.11948/2018.1727

LIE SYMMETRY ANALYSIS, CONSERVATION LAWS AND EXACT SOLUTIONS OF FOURTH-ORDER TIME FRACTIONAL BURGERS EQUATION*

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Abstract In this paper, the fourth-order time fractional Burgers equation has been investigated, which can be used to describe gas dynamics and traffic flow. By employing the Lie group analysis method, the invariance properties of the equation are provided. With the aid of the sub-equation method, a new type of explicit solutions are well constructed with a detailed derivation. Furthermore, based on the power series theory, we investigate its approximate analytical solutions. Finally, its conservation laws with two kinds of independent variables are performed by making use of the nonlinear self-adjointness method.

Keywords Lie group analysis method, invariance properties, sub-equation method, power series theory, conservation laws.

MSC(2010) 35Q51, 35Q53, 35C99.

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*The authors were supported by the Jiangsu Province Natural Science Foundation of China under Grant No. BK20181351, the Postgraduate Research & Practice Program of Education & Teaching Reform of CUMT under Grant No. YJSJG_2018_036, the No. [2016] 22 supported by Ministry of Industry and Information Technology of China, the Qinglan Engineering project of Jiangsu Universities, the National Natural Science Foundation of China under Grant Nos. 51522902 and 11301527, the Fundamental Research Fund for the Central Universities under the grant Grant Nos. DUT17ZD233 and 2017XKQY101, and the General Financial Grant from the China Postdoctoral Science Foundation under Grant Nos. 2015M570498 and 2017T100413.

1. Introduction

It is well known that partial differential equations (PDEs) play an important and central role in many fields. Lie theory of symmetry group provides a systemic, general and efficient method to study PDEs. This theory is mainly used to study the similarity reductions, group invariant solutions and the conservation laws. It was first introduced by the Norwegian mathematician Sophus Lie in the early 19th century, which has been made great progress in PDEs [4,5,7–9,20,26,33,34,48,56].

Recently, Gazizov and his collaborators [16] proposed the symmetry analysis of fractional-order partial differential equations (FPDEs) and the fractional derivatives. The study of FPDEs through symmetries is quite interesting and significant [10,12,18,19,69]. In Ref. [31], the symmetry theory and conservation laws of differential equations requiring existence of Euler-Lagrange equations are connected by the famous Noether theorem. The fractional generalizations of Nother's theorem are proposed to find conservation laws of FPDEs [1,6,15,29,32]. However, it is always invalid for many FPDEs with fractional generalizations. On the basis of new conservation law theorem firstly proposed by Ibragimov [21], Lukashchuk provided the generalized fractional Noether operators and derived conservation laws for time fractional subdiffusion and diffusion-wave equations [25]. Lukashchuk made an important step forward obtaining conservation laws for FPDEs that do not possess fractional Lagrangians. In addition, to our knowledge, conservation laws of some FPDEs have been considered by making use of the generalized fractional Noether operators [17,68].

In this paper, we will consider the following fourth-order time fractional Burgers equation

$$D_t^\alpha u + bu_{4x} + 10bu_x u_{2x} + 4buu_{3x} + 12buu_x^2 + 6bu^2 u_{2x} + 4bu^3 u_x = 0, \tag{1.1}$$

where b is an arbitrary constant, and α $(0 < \alpha \leq 1)$ is a parameter describing the order of the fractional time-derivative. Taking $\alpha = 1$, the fourth-order Burgers equation has firstly been proposed in Ref. [57,58]. The function $u(x,t)$ is assumed to be a causal function of time and $D_t^\alpha u$ is the Riemann-Liouville fractional derivative defined by Jumarie [22]

$$D_t^\alpha u = \begin{cases} \dfrac{\partial^m u}{\partial t^m}; \alpha = m \in N, \\ \dfrac{1}{\Gamma(m-\alpha)} \dfrac{\partial^m}{\partial t^m} \displaystyle\int_0^t \dfrac{u(\tau,x)}{(t-\tau)^{\alpha+1-m}} d\tau; m-1 < \alpha < m, m \in N, \end{cases} \tag{1.2}$$

where the Euler gamma function $\Gamma(z)$ is defined by the integral $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$, which converges in the right half of the complex plane $Re(z) > 0$.

To the best of our knowledge, much research have been done for integer order of the fourth-order Burgers equation with integer order, but there is no work reported to solve Eq. (1.1) with $0 < \alpha < 1$. The main purpose of this paper is to study the Lie symmetry analysis, symmetry reduction and exact solutions of the fourth-order time fractional Burgers equation (1.1). Moreover, conservation laws of Eq. (1.1) are also constructed.

The structure of the paper is as follows. In Sec.2, some properties analyzing FPDEs are introduced by using Lie group method. We further study the symmetry group of Eq. (1.1). The associated symmetry reductions of Eq. (1.1) are investigated in Sec.3. In Sec.4, a new type of explicit solutions and power series solutions of

Eq. (1.1) are derived. Furthermore, conservation laws of Eq. (1.1) are constructed by making use of the new conservation laws theorem and the fractional Noether operators in Sec.5. Finally, some conclusions and discussions of the main results are presented.

2. Lie symmetry analysis

In this section, we briefly review the main procedure to deal with symmetries for FPDEs. First of all, let's consider the symmetry analysis for a FPDE of the form

$$\frac{\partial^\alpha u}{\partial^\alpha t} = F(x, t, u, u_x, u_{xx}, \ldots). \tag{2.1}$$

Eq. (2.1) is invariant under a one-parameter Lie group of point transformations

$$\begin{aligned}
&t^* = t + \varepsilon\tau(x, t, u) + O(\varepsilon^2),\\
&x^* = x + \varepsilon\xi(x, t, u) + O(\varepsilon^2),\\
&u^* = u + \varepsilon\eta(x, t, u) + O(\varepsilon^2),\\
&\frac{\partial^\alpha \overline{u}}{\partial \overline{t}_\alpha} = \frac{\partial^\alpha u}{\partial^\alpha t} + \varepsilon\eta^0_\alpha(x, t, u) + O(\varepsilon^2),\\
&\frac{\partial \overline{u}}{\partial \overline{x}} = \frac{\partial u}{\partial x} + \varepsilon\eta^x(x, t, u) + O(\varepsilon^2),\\
&\frac{\partial^2 \overline{u}}{\partial \overline{x}^2} = \frac{\partial^2 u}{\partial^2 x} + \varepsilon\eta^{xx}(x, t, u) + O(\varepsilon^2),\\
&\frac{\partial^3 \overline{u}}{\partial \overline{x}^3} = \frac{\partial^3 u}{\partial^3 x} + \varepsilon\eta^{xxx}(x, t, u) + O(\varepsilon^2),\\
&\frac{\partial^4 \overline{u}}{\partial \overline{x}^4} = \frac{\partial^4 u}{\partial^4 x} + \varepsilon\eta^{xxxx}(x, t, u) + O(\varepsilon^2),
\end{aligned} \tag{2.2}$$

where ε is the group parameter, and ξ, τ, η are infinitesimals and $\eta^{\alpha t}$, $\eta^{ix}(i = 1, 2, 3, 4)$ are extended infinitesimals. The explicit expressions of η^{ix} are given by

$$\begin{aligned}
&\eta^x = D_x(\eta) - u_x D_x(\xi) - u_t D_x(\tau),\\
&\eta^{2x} = D_x(\eta^x) - u_{xt} D_x(\tau) - u_{xx} D_x(\xi),\\
&\eta^{3x} = D_x(\eta^{xx}) - u_{xxt} D_x(\tau) - u_{xxx} D_x(\xi), \ldots,
\end{aligned} \tag{2.3}$$

where D_x denotes the total derivative operator which is defined by

$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + \ldots, \tag{2.4}$$

and the associated vector field of the form

$$V = \tau(x, t, u)\frac{\partial}{\partial t} + \xi(x, t, u)\frac{\partial}{\partial x} + \eta(x, t, u)\frac{\partial}{\partial u}, \tag{2.5}$$

with the coefficient functions $\xi(x, t, u)$, $\tau(x, t, u)$ and $\eta(x, t, u)$ of the vector field being determined later. The infinitesimal invariance criterion for Eq. (2.1) can be written as

$$Pr^{(n)} V(\Delta)|_{\Delta=0} = 0, \tag{2.6}$$

where $\Delta = \frac{\partial^\alpha u}{\partial t^\alpha} - F(x, t, u, u_x, u_{xx}, \ldots)$. Based on the transformations of the form (2.2) which conserve the structure of fractional derivative operator (1.2), in (1.2) the lower limit of the integral is fixed and it should be invariant with regard to transformation (2.2). The invariance condition yields

$$\tau(x,t,u)|_{t=0} = 0. \tag{2.7}$$

According to [16], the αth extended infinitesimal with the Riemann-Liouville fractional time derivative (2.7) has the following form

$$\eta^{\alpha t} = D_t^\alpha(\eta) + \xi D_t^\alpha(u_x) - D_t^\alpha(\xi u_x) + D_t^\alpha\left(D_t(\tau)u\right) - D_t^{\alpha+1}(\tau u) + \tau D_t^{\alpha+1}(u), \tag{2.8}$$

where D_t^α is the total fractional derivative operator. By using the generalized Leibnitz rule [30]

$$D_t^\alpha[f(t)g(t)] = \sum_{n=0}^{\infty} \binom{\alpha}{n} D_t^{\alpha-n} f(t) D_t^n g(t), \quad \alpha > 0, \tag{2.9}$$

where $\binom{\alpha}{n} = \frac{(-1)^{n-1}\alpha\Gamma(n-\alpha)}{\Gamma(1-\alpha)\Gamma(n+1)}$, (2.8) can be rewritten as

$$\eta^{\alpha t} = D_t^\alpha(\eta) - \alpha D_t(\tau)\frac{\partial^\alpha u}{\partial t^\alpha} - \sum_{n=1}^{\infty}\binom{\alpha}{n} D_t^{\alpha-n}(u_x)D_t^n(\xi) - \sum_{n=1}^{\infty}\binom{\alpha}{n+1} D_t^{\alpha-n}(u)D_t^{n+1}(\tau). \tag{2.10}$$

In view of the generalization of the chain rule for composite functions [35]

$$\frac{d^m f(x(t))}{dt^m} = \sum_{k=0}^{m}\sum_{r=0}^{k}\binom{k}{r}\frac{1}{k!}[-x(t)]^r \frac{d^m}{dt^m}[x(t)^{k-r}]\frac{d^k f(x)}{dx^k}. \tag{2.11}$$

On account of the chain rule (2.11) and the generalized Leibnitz rule (2.9) with $f(t) = 1$, one can obtain

$$D_t^\alpha(\eta) = \frac{\partial^\alpha \eta}{\partial t^\alpha} + \eta_u \frac{\partial^\alpha u}{\partial t^\alpha} - u\frac{\partial^\alpha \eta_u}{\partial t^\alpha} + \sum_{n=1}^{\infty}\binom{\alpha}{n}\frac{\partial^\alpha \eta_u}{\partial t^n} D_t^{\alpha-n}(u) + \mu, \tag{2.12}$$

where

$$\mu = \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\alpha}{n}\binom{n}{m}\binom{k}{r}\frac{1}{k!} \times \frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)}[-u]^r\frac{\partial^m}{\partial t^m}[u^{k-r}]\frac{\partial^{n-m+k}\eta}{\partial t^{n-m}\partial u^k}. \tag{2.13}$$

Therefore, (2.10) can be rewritten as

$$\begin{aligned} \eta^{\alpha t} =& \frac{\partial^\alpha \eta}{\partial t^\alpha} + (\eta_u - \alpha D_t(\tau))\frac{\partial^\alpha u}{\partial t^\alpha} - u\frac{\partial^\alpha \eta_u}{\partial t^\alpha} + \mu \\ &+ \sum_{n=1}^{\infty}\left[\binom{\alpha}{n}\frac{\partial^\alpha \eta_u}{\partial t^\alpha} - \binom{\alpha}{n+1} D_t^{n+1}(\tau)\right] D_t^{\alpha-n}(u) - \sum_{n=1}^{\infty}\binom{\alpha}{n} D_t^n(\xi) D_t^{\alpha-n}(u_x). \end{aligned} \tag{2.14}$$

By using the above Lie symmetry analysis method of FPDEs, we investigate the fourth-order time fractional Burgers equation (1.1) and get the following theorem.

Theorem 2.1. *The symmetry group of the fourth-order time fractional Burgers equation is spanned by the following vector fields*

$$V_1 = \frac{\partial}{\partial x}, \quad V_2 = x\frac{\partial}{\partial x} + \frac{4t}{\alpha}\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}. \tag{2.15}$$

Proof. Applying the fourth prolongation of the infinitesimal generator (2.5) to Eq. (1.1), one can get the determining equation

$$\begin{aligned}&\eta^{\alpha t} + (4bu_{3x} + 12bu_x^2 + 12buu_{2x} + 12bu^2u_x)\eta + (10bu_{2x} + 24buu_x + 4bu^3)\eta^x \\ &\quad + (10bu_x + 6bu^2)\eta^{2x} + 4bu\eta^{3x} + b\eta^{4x} = 0.\end{aligned} \tag{2.16}$$

Substituting (2.3) and (2.14) into (2.16), and equating the coefficients of the various monomials in partial derivatives with respect to x and various powers of u, one can find the determining equations for the symmetry group of Eq. (1.1) as follows

$$\begin{aligned}&\xi_t = \xi_u = \xi_{xx} = 0, \quad \tau_x = \tau_u = 0,\\ &\eta_x = \eta_{uu} = 0, \quad \alpha\tau_t - 4\xi_x = 0,\\ &(\alpha\tau_t + \eta_u - 2\xi_x)u + \eta = 0, \quad (\alpha\tau_t - 3\xi_x)u + \eta = 0,\\ &(\alpha\tau_t - 2\xi_x)u + 2\eta = 0, \quad (\alpha\tau_t - \xi_x)u + 3\eta = 0,\\ &\alpha\tau_t + \eta_u - 3\xi_x = 0,\\ &\binom{\alpha}{n}\frac{\partial^n \eta_u}{\partial t^n} - \binom{\alpha}{n+1} D_t^{n+1}(\tau) = 0, \quad \text{for} \quad n = 1, 2, \ldots.\end{aligned} \tag{2.17}$$

Solving above equations, we obtain the coefficient functions

$$\xi = c_0 x + c_1, \quad \tau = \frac{4c_0}{\alpha}t, \quad \eta = -c_0 u, \tag{2.18}$$

where c_0 and c_1 are two arbitrary constants. Thus, the explicit expression of the infinitesimal operator is given by

$$V = (c_0 x + c_1)\frac{\partial}{\partial x} + \frac{4c_0}{\alpha}t\frac{\partial}{\partial t} - c_0 u\frac{\partial}{\partial u}, \tag{2.19}$$

and the Lie algebra of Eq. (1.1) is spanned by the two vector fields (2.15). □

3. Similarity reductions

In this section, we derive the similarity reductions for the fourth-order time fractional Burgers equation from the corresponding vector fields.

3.1. For the symmetry V_1

The invariant solution is of the form

$$u(x,t) = h(t). \tag{3.1}$$

Substituting (3.1) into Eq. (1.1) yields the reduced fractional ordinary differential equation (ODE)

$$\frac{d^\alpha h(t)}{dt^\alpha} = 0. \tag{3.2}$$

Accordingly, we have the invariant solution by

$$u = ct^{\alpha-1}, \tag{3.3}$$

for arbitrary constant c.

3.2. For the symmetry V_2

For the symmetry of V_2, we get the characteristic equation

$$\frac{dx}{x} = \frac{\alpha dt}{4t} = -\frac{du}{u} \tag{3.4}$$

and the corresponding invariants are

$$\xi = xt^{-\frac{\alpha}{4}}, \quad u = t^{-\frac{\alpha}{4}} g(\xi). \tag{3.5}$$

Through the above discussion, we can find that (1.1) can be reduced to a nonlinear ODE of fractional order with a new independent variable $\xi = xt^{-\frac{\alpha}{4}}$. Consequently, one can get the following theorem.

Theorem 3.1. *Under the transformation* (3.5)*, Eq.* (1.1) *can be reduced to the following nonlinear ODE of fractional order*

$$\left(P_{\frac{4}{\alpha}}^{1-\frac{5\alpha}{4},n-\alpha} g\right)(\xi) + bg_{4\xi} + 10bg_\xi g_{2\xi} + 4bgg_{3\xi} + 12bgg_\xi^2 + 6bg^2 g_{2\xi} + 4bg^3 g_\xi = 0, \tag{3.6}$$

with the Erdélyi-Kober fractional differential operator $P_\beta^{\tau,\alpha}$ *of order*

$$\left(P_\beta^{\tau,\alpha} g\right) := \Pi_{j=0}^{n-1} \left(\tau + j - \frac{1}{\beta}\xi\frac{d}{d\xi}\right)\left(K_\beta^{\tau+\alpha,n-\alpha} g\right)(\xi), \tag{3.7}$$

where

$$\left(K_\beta^{\tau,\alpha} g\right)(\xi) := \begin{cases} \frac{1}{\Gamma(\alpha)} \int_1^\infty (u-1)^{\alpha-1} u^{-(\tau+\alpha)} g(\xi u^{\frac{1}{\beta}}) du, & \alpha > 0, \\ g(\xi), \quad \alpha = 0 \end{cases} \tag{3.8}$$

is the Erdélyi-Kober fractional integral operator, and

$$n = \begin{cases} [\alpha] + 1, & \alpha \in N, \\ \alpha, & \alpha \notin N. \end{cases} \tag{3.9}$$

Proof. Let $n - 1 < \alpha < n$, $n = 1, 2, 3, \ldots$. Based on the Riemann-Liouville fractional derivative, one can have

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^n}{\partial t^n}\left[\frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} s^{-\frac{\alpha}{4}} g\left(xs^{\frac{-\alpha}{4}}\right) ds\right]. \tag{3.10}$$

Letting $v = \frac{t}{s}$, one can get $ds = \frac{-t}{v^2} dv$, therefore (3.10) can be written as

$$\begin{aligned} \frac{\partial^\alpha u}{\partial t^\alpha} &= \frac{\partial^n}{\partial t^n}\left[t^{n-\frac{5\alpha}{4}} \frac{1}{\Gamma(n-\alpha)} \int_1^\infty (v-1)^{n-\alpha-1} v^{-(n+1-\frac{5\alpha}{4})} g\left(\xi v^{\frac{\alpha}{4}}\right) dv\right] \\ &= \frac{\partial^n}{\partial t^n}\left[t^{n-\frac{5\alpha}{4}} \left(K_{\frac{4}{\alpha}}^{1-\frac{\alpha}{4},n-\alpha} g\right)(\xi)\right]. \end{aligned} \tag{3.11}$$

Considering the relation $\xi = xt^{\frac{-\alpha}{4}}$, we can obtain

$$t\frac{\partial}{\partial t}\phi(\xi) = tx\left(-\frac{\alpha}{4}\right) t^{-\frac{\alpha}{4}-1} \phi'(\xi) = -\frac{\alpha}{4}\xi\frac{d}{d\xi}\phi(\xi). \tag{3.12}$$

Therefore, one can get

$$\frac{\partial^n}{\partial t^n}\left[t^{n-\frac{5\alpha}{4}}\left(K_{\frac{4}{\alpha}}^{1-\frac{\alpha}{4},n-\alpha}g\right)(\xi)\right]=\frac{\partial^{n-1}}{\partial t^{n-1}}\left[\frac{\partial}{\partial t}\left(t^{n-\frac{5\alpha}{4}}\left(K_{\frac{4}{\alpha}}^{1-\frac{\alpha}{4},n-\alpha}g\right)(\xi)\right)\right]$$

$$=\frac{\partial^{n-1}}{\partial t^{n-1}}\left[t^{n-\frac{5\alpha}{4}}\left(n-\frac{5\alpha}{4}-\frac{\alpha}{4}\xi\frac{d}{d\xi}\times\left(K_{\frac{4}{\alpha}}^{1-\frac{\alpha}{4},n-\alpha}g\right)(\xi)\right)\right]. \tag{3.13}$$

Repeating the similar procedure as above for $n-1$ times, one can obtain

$$\frac{\partial^n}{\partial t^n}\left[t^{n-\frac{5\alpha}{4}}\left(K_{\frac{4}{\alpha}}^{1-\frac{\alpha}{4},n-\alpha}g\right)(\xi)\right]=\frac{\partial^{n-1}}{\partial t^{n-1}}\left[\frac{\partial}{\partial t}\left(t^{n-\frac{5\alpha}{4}}\left(K_{\frac{4}{\alpha}}^{1-\frac{\alpha}{4},n-\alpha}g\right)(\xi)\right)\right]$$

$$=\frac{\partial^{n-1}}{\partial t^{n-1}}\left[t^{n-\frac{5\alpha}{4}}\left(n-\frac{5\alpha}{4}-\frac{\alpha}{4}\xi\frac{d}{d\xi}\left(K_{\frac{4}{\alpha}}^{1-\frac{\alpha}{4},n-\alpha}g\right)(\xi)\right)\right]$$

$$=\cdots=t^{-\frac{5\alpha}{4}}\Pi_{j=0}^{n-1}\left(1-\frac{5\alpha}{4}+j-\frac{\alpha}{4}\xi\frac{d}{d\xi}\right)\left(K_{\frac{4}{\alpha}}^{1-\frac{\alpha}{4},n-\alpha}g\right)(\xi). \tag{3.14}$$

Now using (3.7), we get

$$\frac{\partial^n}{\partial t^n}\left[t^{n-\frac{5\alpha}{4}}\left(K_{\frac{4}{\alpha}}^{1-\frac{\alpha}{4},n-\alpha}g\right)(\xi)\right]=t^{-\frac{5\alpha}{4}}\left(P_{\frac{4}{\alpha}}^{1-\frac{5\alpha}{4},n-\alpha}g\right)(\xi). \tag{3.15}$$

Substituting (3.15) into (3.11), one can get

$$\frac{\partial^\alpha u}{\partial t^\alpha}=t^{-\frac{5\alpha}{4}}\left(P_{\frac{4}{\alpha}}^{1-\frac{5\alpha}{4},n-\alpha}g\right)(\xi). \tag{3.16}$$

Thus, the fourth-order time fractional Burgers equation can be reduced into a fractional ordinary differential equation

$$\left(P_{\frac{4}{\alpha}}^{1-\frac{5\alpha}{4},n-\alpha}g\right)(\xi)+bg_{4\xi}+10bg_\xi g_{2\xi}+4bgg_{3\xi}+12bgg_\xi^2+6bg^2g_{2\xi}+4bg^3g_\xi=0. \tag{3.17}$$

This complete the proof. □

4. Explicit solutions and power series solutions

As is well known to us, the fractional sub-equation approach is widely used to construct the explicit solutions of FPDEs. In this section, based on the fractional sub-equation approach and the power series method [3,60,69,73], explicit solutions and a kind of power series solutions for Eq. (1.1) are well constructed with a detailed derivation.

4.1. Explicit solutions

In this section, in order to deal with the fourth-order time fractional Burgers equation (1.1), we will apply the fractional sub-equation method. According to the steps in Ref. [69], first we introduce the following transformation

$$u(x,t)=u(\xi),\quad \xi=x+ct+\xi_0, \tag{4.1}$$

in which c is a constant. Substituting (4.1) into (1.1), then (1.1) can be reduced to the following nonlinear fractional ordinary differential equation (NFODE)

$$c^\alpha D_\xi^\alpha u+bu_{4\xi}+10bu_\xi u_{2\xi}+4buu_{3\xi}+12buu_\xi^2+6bu^2u_{2\xi}+4bu^3u_\xi=0. \tag{4.2}$$

We assume that Eq.(4.2) has the following solution

$$u(\xi) = a_0 + \sum_{i=1}^{n} a_i(\phi)^i, \tag{4.3}$$

in which $a_i(i = 1 \dots n)$ are constants to be determined later. Balancing the highest order derivative terms with nonlinear terms in Eq. (4.2), one has $n = 1$. Then, Eq. (4.2) has the following formal solution

$$u(\xi) = a_0 + a_1(\phi). \tag{4.4}$$

Inserting (4.4) along with the fractional Riccati eqaution $D_\xi^\alpha \phi(\xi) = \sigma + \phi^2(\xi)$ into (4.2) and then taking the coefficients of $(\phi)^i$ to be zero, one can get a series of algebraic equations about c, a_0, a_1. Solving the algebraic equations by Maple, we obtain

$$\begin{aligned} &b = b, \quad c = (4ba_o\sigma - 4ba_0^3)^{\frac{1}{\alpha}}, \\ &\alpha = \alpha, \quad \sigma = \sigma, \quad a_0 = a_0, \quad a_1 = -1. \end{aligned} \tag{4.5}$$

In view of (4.5), we can get new types of explicit solutions of Eq. (1.1) as follows

$$\begin{aligned} u_1 &= a_0 + \sqrt{-\sigma}\tanh(-\sqrt{-\sigma}\xi, \alpha), \quad \sigma < 0, \\ u_2 &= a_0 + \sqrt{-\sigma}\coth(-\sqrt{-\sigma}\xi, \alpha), \quad \sigma < 0, \\ u_3 &= a_0 - \sqrt{\sigma}\tan(\sqrt{\sigma}\xi, \alpha), \quad \sigma > 0, \\ u_4 &= a_0 + \sqrt{\sigma}\cot(\sqrt{\sigma}\xi, \alpha), \quad \sigma > 0, \\ u_5 &= a_0 + \frac{\Gamma(1+\alpha)}{\xi^\alpha + \omega}, \quad \sigma = 0, \end{aligned} \tag{4.6}$$

in which $\xi = x + ct + \xi_0$, with c given by (4.5).

In order to help us analyze the properties of the explicit solution well, the graphic of the explicit solutions (4.6) are plotted by choosing the appropriate parameters (see Figs.1-4).

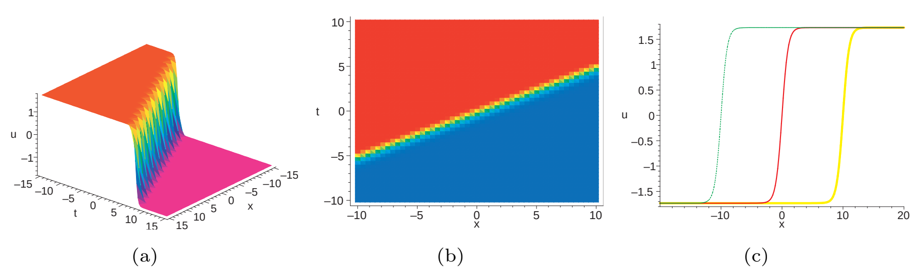
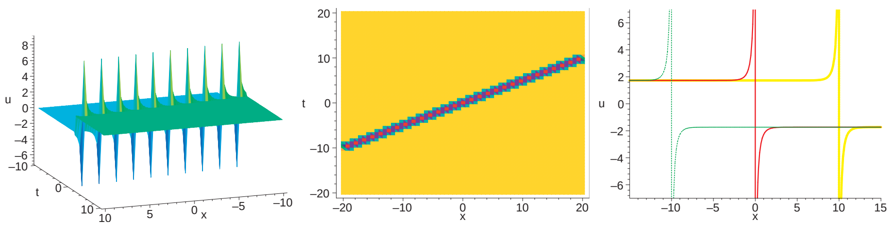
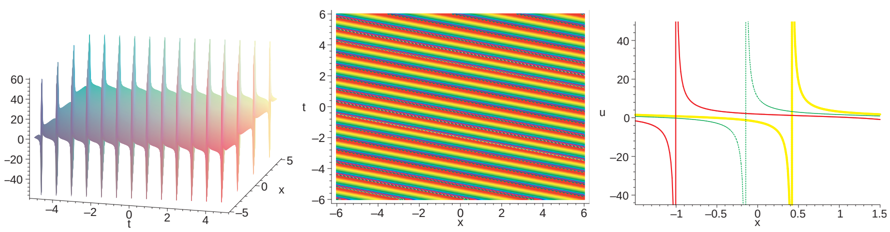


Figure 1. (Color online) Explicit solution u_1 in system (4.6) for Eq. (1.1) with suitable parameters: $a_0 = 1$, $b = 1$, $\sigma = -1$, $\alpha = 1$. (a)Perspective view of the real part of explicit solution. (b) The overhead view of the solution. (c) The wave propagation pattern of the wave along the x-axis with $t = -5$ (superposed green line), $t = 0$ (solid red line), $t = 5$ (dashed yellow line).



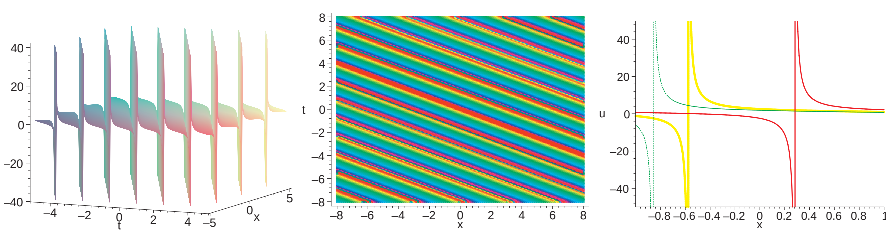
(a) (b) (c)

Figure 2. (Color online) Explicit solution u_2 in system (4.6) for Eq. (1.1) with suitable parameters: $a_0 = 1$, $b = 1$, $\sigma = -1$, $\alpha = 1$. (a) Perspective view of the real part of explicit solution. (b) The overhead view of the solution. (c) The wave propagation pattern of the wave along the x-axis with $t = -5$ (superposed green line), $t = 0$ (solid red line), $t = 5$ (dashed yellow line).



(a) (b) (c)

Figure 3. (Color online) Explicit solution u_3 in system (4.6) for Eq. (1.1) with suitable parameters: $a_0 = 1$, $b = 1$, $\sigma = 2$, $\alpha = 1$. (a) Perspective view of the real part of explicit solution. (b) The overhead view of the solution. (c) The wave propagation pattern of the wave along the x-axis with $t = 2$ (superposed green line), $t = 3$ (solid red line), $t = 5$ (dashed yellow line).



(a) (b) (c)

Figure 4. (Color online) Explicit solution u_4 in system (4.6) for Eq. (1.1) with suitable parameters: $a_0 = 1$, $b = 1$, $\sigma = 2$, $\alpha = 1$. (a) Perspective view of the real part of explicit solution. (b) The overhead view of the solution. (c) The wave propagation pattern of the wave along the x-axis with $t = 2$ (superposed green line), $t = 3$ (solid red line), $t = 5$ (dashed yellow line).

4.2. Power series solutions

Based on the power series method and symbolic computations [11,13,14,27,28,36–47,49–54,59,61–67,70–72], we will construct the power series solutions of Eq. (1.1). We first introduce a very important transformation

$$u(x,t) = u(\xi), \qquad \xi = kx - \frac{\omega t^{\alpha}}{\Gamma(1+\alpha)}, \tag{4.7}$$

where k, ω are arbitrary constants. Substituting (4.7) into (1.1), Eq. (1.1) can be transformed into the following nonlinear ODE

$$-\omega u_\xi+k^4bu_{4\xi}+10k^3bu_\xi u_{2\xi}+4k^3buu_{3\xi}+12k^2buu_\xi^2+6k^2bu^2u_{2\xi}+4kbu^3u_\xi=0. \quad (4.8)$$

Integrating Eq.(4.8) with respect to ξ, we obtain

$$-\omega u+k^4bu_{3\xi}+4k^3buu_{2\xi}+3k^3bu_\xi^2+6k^2bu^2u_\xi+kbu^4+\delta=0, \quad (4.9)$$

in which δ is an integration constant. Then we suppose that the solution of Eq. (4.9) has following form

$$u=\sum_{n=0}^{\infty}s_n\xi^n, \quad (4.10)$$

where $s_n(n=0,1,2,\ldots)$ are constants to be known later. Substituting (4.10) into (4.9), we obtain

$$\begin{aligned}&-\omega\sum_{n=0}^{\infty}s_n\xi^n+k^4b\sum_{n=0}^{\infty}(n+1)(n+2)(n+3)s_{n+3}\xi^n\\&+4k^3b\sum_{n=0}^{\infty}\sum_{k=0}^{n}(n+1-k)(n+2-k)s_ks_{n+2-k}\xi^n+3k^3b\\&\sum_{n=0}^{\infty}\sum_{k=0}^{n}(n-k+1)(k+1)s_{k+1}s_{n-k+1}\xi^n\\&+6k^2b\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{j=0}^{k}(n+1-k)s_js_{k-j}s_{n+1-k}\xi^n\\&+kb\left(\sum_{n=0}^{\infty}s_n\xi^n\right)^4+\delta=0. \quad (4.11)\end{aligned}$$

When $n=0$, by comparing coefficients of ξ, we obtain

$$s_3=\frac{1}{6k^4b}\left(\omega s_0-8k^3bs_0s_2-3k^3bs_1^2-6k^2bs_0^2s_1-kbs_0^4-\delta\right). \quad (4.12)$$

When $n\geq 1$, we have

$$\begin{aligned}s_{n+3}=&\frac{1}{k^4b(n+1)(n+2)(n+3)}\Bigg[\omega s_n-4k^3b\sum_{k=0}^{n}(n+1-k)(n+2-k)s_ks_{n+2-k}-3k^3b\\&\times\sum_{k=0}^{n}(n-k+1)(k+1)s_{k+1}s_{n-k+1}-6k^2b\sum_{k=0}^{n}\sum_{j=0}^{k}(n+1-k)s_js_{k-j}s_{n+1-k}\\&-kb\sum_{n_1+n_2+n_3+n_4=n}s_{n_1}s_{n_2}s_{n_3}s_{n_4}\Bigg]. \quad (4.13)\end{aligned}$$

Thus, any coefficient $s_n(n\geq 3)$ of Eq. (4.10) are determined by the arbitrary constants s_0,s_1,s_2,ω,k,b. It implies that there is a power series solution for Eq. (4.9), and its coefficients rely on (4.12) and (4.13). Furthermore, we find it easy to prove the convergence of the power series solution (4.10) with the coefficients depend on (4.12) and (4.13). Therefore, it is clear that the solution to Eq. (4.9) is a power series solution of Eq. (1.1).

So the power series solution for Eq. (4.9) can be rewritten as follows

$$
\begin{aligned}
u(\xi) =& s_0 + s_1\xi + s_2\xi^2 + s_3\xi^3 + \sum_{n=1}^{\infty} s_{n+3}\xi^{n+3} \\
=& s_0+s_1\xi+s_2\xi^2+\frac{1}{6k^4b}\left(\omega s_0-8k^3bs_0s_2-3k^3bs_1^2-6k^2bs_0^2s_1-kbs_0^4-\delta\right)\xi^3 \\
&+\sum_{n=1}^{\infty}\frac{1}{k^4b(n+1)(n+2)(n+3)} \\
&\times\Bigg[\omega s_n-4k^3b\sum_{k=0}^{n}(n+1-k)(n+2-k)s_ks_{n+2-k} \\
&-3k^3b\sum_{k=0}^{n}(n-k+1)(k+1)s_{k+1}s_{n-k+1} \\
&-6k^2b\sum_{k=0}^{n}\sum_{j=0}^{k}(n+1-k)s_js_{k-j}s_{n+1-k}-kb\sum_{n_1+n_2+n_3+n_4=n}s_{n_1}s_{n_2}s_{n_3}s_{n_4}\Bigg]\xi^{n+3}.
\end{aligned} \tag{4.14}
$$

Thus, we obtain the power series solution for Eq. (1.1) as follows

$$
\begin{aligned}
u(\xi) =& s_0 + s_1\left(kx-\frac{\omega t^{\alpha}}{\Gamma(1+\alpha)}\right)+s_2\left(kx-\frac{\omega t^{\alpha}}{\Gamma(1+\alpha)}\right)^2 \\
&+\frac{1}{6k^4b}\left(\omega s_0-8k^3bs_0s_2-3k^3bs_1^2-6k^2bs_0^2s_1-kbs_0^4-\delta\right)\left(kx-\frac{\omega t^{\alpha}}{\Gamma(1+\alpha)}\right)^3 \\
&+\sum_{n=1}^{\infty}\frac{1}{k^4b(n+1)(n+2)(n+3)}\Bigg(\omega s_n-4k^3b\sum_{k=0}^{n}(n+1-k)(n+2-k)s_ks_{n+2-k}-3k^3b \\
&\times\sum_{k=0}^{n}(n-k+1)(k+1)s_{k+1}s_{n-k+1}-6k^2b\sum_{k=0}^{n}\sum_{j=0}^{k}(n+1-k)s_js_{k-j}s_{n+1-k} \\
&-kb\sum_{n_1+n_2+n_3+n_4=n}s_{n_1}s_{n_2}s_{n_3}s_{n_4}\Bigg)\left(kx-\frac{\omega t^{\alpha}}{\Gamma(1+\alpha)}\right)^{n+3},
\end{aligned} \tag{4.15}
$$

in which s_0,s_1, s_2, k, b and $\omega \neq 0$ are arbitrary constants, the other coefficients $s_n(n \geq 3)$ depend on (4.12) and (4.13). Based on the above analysis, the following assertion is easily established.

Theorem 4.1. *Eq.* (1.1) *admits the following power series solution*

$$
u(x,t) = \sum_{n=0}^{\infty} s_n\left(kx-\frac{\omega t^{\alpha}}{\Gamma(1+\alpha)}\right)^n, \tag{4.16}
$$

where s_0,s_1, s_2, k, b *and* $\omega \neq 0$ *are arbitrary constants, the other coefficients* $s_n(n \geq 3)$ *rely on* (4.12) *and* (4.13).

The graphical representation of the power series solutions are plotted in Figs.5-7 by choosing the appropriate parameters. Figure 5 shows the power series solution in system (4.15) when $n = 0$. Figure 6 shows the power series solution in system (4.15) when $n = 1$. Figure 7 shows the power series solution in system (4.15) when $n = 4$.

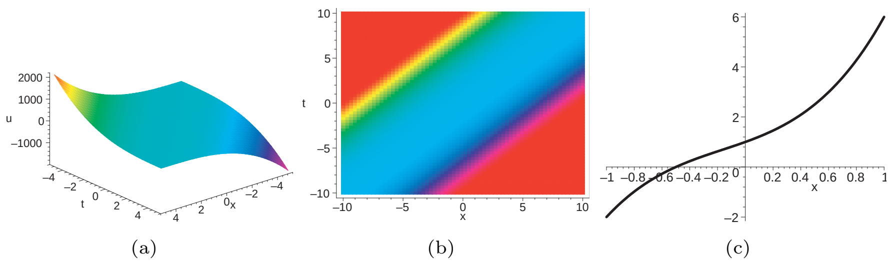


Figure 5. (Color online) Power series solution u in system (4.15) for Eq. (1.1) by choosing suitable parameters: $s_0 = 1, s_1 = 2, s_2 = 1, s_3 = 2, k = 1, \omega = 1, \Gamma = 1, \alpha = 1$. (a) Perspective view of the real part of explicit solution. (b) The overhead view of the solution. (c) The wave propagation pattern of the wave along the x axis.

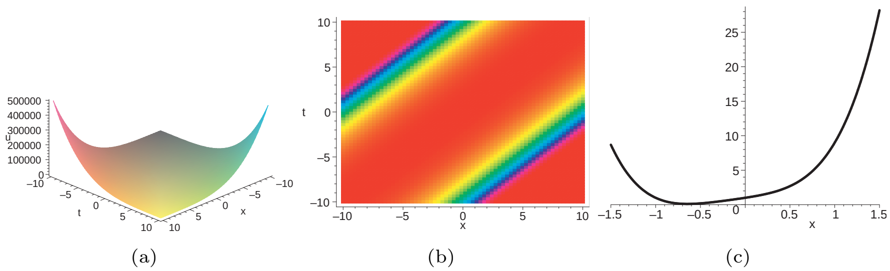


Figure 6. (Color online) Power series solution u in system (4.15) for Eq. (1.1) by choosing suitable parameters: $s_0 = 1, s_1 = 2, s_2 = 1, s_3 = 2, s_4 = 3, k = 1, \omega = 1, \Gamma = 1, \alpha = 1$. (a) Perspective view of the real part of explicit solution. (b) The overhead view of the solution. (c) The wave propagation pattern of the wave along the x axis.

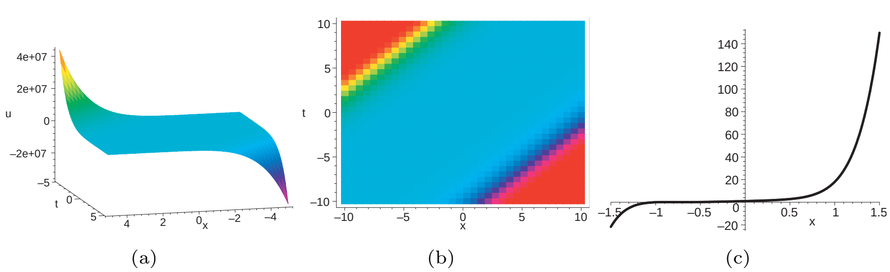


Figure 7. (Color online) Power series solution u in system (4.15) for Eq. (1.1) by choosing suitable parameters: $s_0 = 1, s_1 = 2, s_2 = 1, s_3 = 2, s_4 = 3, s_5 = 1, s_6 = 4, s_7 = 4, k = 1, \omega = 1, \Gamma = 1, \alpha = 1$. (a) Perspective view of the real part of explicit solution. (b) The overhead view of the solution. (c) The wave propagation pattern of the wave along the x axis.

5. Conservations Laws

In this section, conservation laws with two kinds of independent variables of the fourth-order time fractional Burgers equation (1.1) are derived by means of the Lie symmetry.

5.1. Preliminaries of Conservations Laws

Based on the Riemann-Liouville left-sided time-fractional derivative [2, 23, 24, 55], we get

$$ {}_0D_t^{\alpha} u = D_t^n({}_0I_t^{n-\alpha} u), \tag{5.1} $$

where D_t is the operator concerning t of differentiation, $n = [\alpha] + 1$. In addition, the definition of ${}_0I_t^{n-\alpha}u$ is given as follows.

Definition 5.1. The left-sided time-fractional integral of order ${}_0D_t^{n-\alpha}u$ is defined by

$$ ({}_0I_t^{n-\alpha} u)(x,t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{u(\theta, x)}{(t-\theta)^{1-n+\alpha}} d\theta, \tag{5.2} $$

where $\Gamma(z)$ is the Gamma function.

Definition 5.2. Assume that a vector $C = (C^t, C^x)$ admits the following conservation equation

$$ D_t(C^t) + D_x(C^x)|_{(1.1)} = 0, \tag{5.3} $$

where $C^t = C^t(t, x, u, \ldots)$ and $C^x = C^x(t, x, u, \ldots)$. The vector $C = (C^t, C^x)$ is called a conserved vector for Eq. (1.1). Firstly, a formal Lagrangian for Eq. (1.1) can be written in the following form

$$ L = \nu(x,t)[u_t^{\alpha} + bu_{4x} + 10bu_x u_{2x} + 4buu_{3x} + 12buu_x^2 + 6bu^2 u_{2x} + 4bu^3 u_x], \tag{5.4} $$

where $\nu(x, t)$ is a new dependent variable. Considering the previous Lagrangian, we get an action integral as follows

$$ \int_0^T \int_{\Omega} L(x, t, u, \nu, D_t^{\alpha}(u), u_x, \ldots) dxdt. \tag{5.5} $$

Similar to the case of integral-order nonlinear differential equations [16], the adjoint equation is available. So we have adjoint equation to Eq. (1.1) as Euler-Lagrange equation

$$ \frac{\delta L}{\delta u} = 0. \tag{5.6} $$

The definition of the Euler-Lagrange operator is obtained as follows.

Definition 5.3. The Euler-Lagrange operator is defined by [16]

$$ \frac{\delta}{\delta u} = \frac{\partial}{\partial u} + (D_t^{\alpha})^* \frac{\partial}{\partial D_t^{\alpha} u} - D_x \frac{\partial}{\partial u_x} + D_x^2 \frac{\partial}{\partial u_{2x}} - D_x^3 \frac{\partial}{\partial u_{3x}} + D_x^4 \frac{\partial}{\partial u_{4x}}, \tag{5.7} $$

where $(D_t^{\alpha})^*$ denotes the adjoint operator of (D_t^{α}).

Let us consider the Riemann-Liouville differential operators

$$(D_t^\alpha)^* = (-1)^n I_T^{n-\alpha}(D_t^n) = (D_T^\alpha)_t^I, \tag{5.8}$$

where $I_T^{n-\alpha}$ is the right-sided operator of fractional integration of order $n-\alpha$, $(D_T^\alpha)_t^I$ is the right-sided Caputo operator, and $I_T^{n-\alpha}$ is defined by

$$I_T^{n-\alpha} f(t,x) = \frac{1}{\Gamma(n-\alpha)} \int_t^T \frac{f(\tau,x)}{(\tau-t)^{1+\alpha-n}} d\tau, \quad n = [\alpha]+1. \tag{5.9}$$

Taking into account of the case of variables t, x, and $u(x,t)$, we get

$$\overline{X} + D_t(\tau)l + D_x(\xi)l = W\frac{\delta}{\delta u} + D_t N^t + D_x N^x, \tag{5.10}$$

where l denotes the identity operator, $\frac{\delta}{\delta u}$ is the Euler-Lagrange operator, and the Noether operators are provided by N^t, N^x respectively. In addition, $\overline{X}$ is defined by

$$\overline{X} = \tau\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial u} + \eta_\alpha^0 \frac{\partial}{\partial D_t^\alpha u} + \eta^x \frac{\partial}{\partial u_x} + \eta^{2x}\frac{\partial}{\partial u_{2x}} + \eta^{3x}\frac{\partial}{\partial u_{3x}} + \eta^{4x}\frac{\partial}{\partial u_{4x}}, \tag{5.11}$$

and W is defined by

$$W = \eta - \tau u_t - \xi u_x. \tag{5.12}$$

Considering the above conditions and introducing Riemann-Liouville time-fractional derivative into Eq. (1.1), we get the following operator N^t in Refs. [16]

$$N^t = \tau l + \sum_{k=0}^{n-1} (-1)^k {}_0D_t^{\alpha-1-k}(W) D_t^k \frac{\partial}{\partial({}_0D_t^\alpha u)} - (-1)^n J\left(W, D_t^n \frac{\partial}{\partial({}_0D_t^\alpha u)}\right), \tag{5.13}$$

where the integral J [16] reads

$$J(f,g) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \int_t^T \frac{f(\tau,x)g(\mu,x)}{(\mu-\tau)^{\alpha+1-n}} d\mu dt. \tag{5.14}$$

The operator N^x is defined by

$$\begin{aligned} N^x =& \xi l + W\left(\frac{\partial}{\partial u_x} - D_x\frac{\partial}{\partial u_{2x}} + D_x^2\frac{\partial}{\partial u_{3x}} - D_x^3\frac{\partial}{\partial u_{4x}}\right) \\ &+ D_x(W)\left(\frac{\partial}{\partial u_{2x}} - D_x\frac{\partial}{\partial u_{3x}} + D_x^2\frac{\partial}{\partial u_{4x}}\right) \\ &+ D_x^2(W)\left(\frac{\partial}{\partial u_{3x}} - D_x\frac{\partial}{\partial u_{4x}}\right) + D_x^3(W)\frac{\partial}{\partial u_{4x}}, \end{aligned} \tag{5.15}$$

where D_x is the total derivative operator defined as

$$D_x = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{2x}\frac{\partial}{\partial u_x} + \cdots. \tag{5.16}$$

For any solutions and generator V of Eq. (1.1), we obtain

$$(\overline{X}L + D_t(\tau)L + D_x(\xi)L)|_{(1.1)} = 0. \tag{5.17}$$

Considering Eq. (1.1), we find that the Euler-Lagrange Eq. (5.6) is zero obviously. Hence, the conservation law resulted by the right-hand side of the equality is given by

$$D_t(N^t L) + D_x(N^x L) = 0. \tag{5.18}$$

Comparing (5.3) and (5.18), it is obvious that the following components of conserved vectors with Lie point symmetry of Eq. (1.1) are always valid

$$C^t = N^t L, \quad C^x = N^x L. \tag{5.19}$$

5.2. Conservation laws of Eq.(1.1)

In the previous subsection, we gave some basic definitions. In this subsection, we will present the conservation laws of Eq. (1.1).

Using (5.13) and (5.15), the components of conserved vectors of the forth-order time fractional Burgers equation (1.1) with $\alpha \in (0,1)$ are given by

$$\begin{aligned} C_i^t &= \tau L + (-1)^0 {}_0D_t^{\alpha-1}(W_i) D_t^0 \frac{\partial L}{\partial({}_0D_t^{\alpha-1}u)} - (-1)^1 J\left(W_i, D_t^1 \frac{\partial L}{\partial({}_0D_t^{\alpha}u)}\right) \\ &= \nu_0 D_t^{\alpha-1}(W_i) + J(W_i, \nu_t), \end{aligned} \tag{5.20}$$

$$\begin{aligned} C_i^x =& \xi L + W_i\left(\frac{\partial L}{\partial u_x} - D_x\frac{\partial L}{\partial u_{2x}} + D_x^2\frac{\partial L}{\partial u_{3x}} - D_x^3\frac{\partial L}{\partial u_{4x}}\right) \\ &+ D_x(W_i)\left(\frac{\partial L}{\partial u_{2x}} - D_x\frac{\partial L}{\partial u_{3x}} + D_x^2\frac{\partial L}{\partial u_{4x}}\right) \\ &+ D_x^2(W_i)\left(\frac{\partial L}{\partial u_{3x}} - D_x\frac{\partial L}{\partial u_{4x}}\right) + D_x^3(W_i)\frac{\partial L}{\partial u_{4x}} \\ =& W_i\left\{\nu\left(10bu_{2x}+24buu_x+4bu^3\right) - D_x\left[\nu(10bu_x+6bu^2)\right] + D_x^2(4bu\nu) - D_x^3(b\nu)\right\} \\ &+ D_x(W_i)\left[\nu(10bu_x + 6bu^2) - D_x(4bu\nu) + D_x^2(b\nu)\right] \\ &+ D_x^2(W_i)\left[4bu\nu - D_x(b\nu)\right] + D_x^3(W_i)(b\nu), \end{aligned} \tag{5.21}$$

in which $i = 1, 2$ and functions W_i are

$$W_1 = -u_x, \quad W_2 = -u - \frac{4t}{\alpha}u_t - xu_x. \tag{5.22}$$

6. Conclusions and discussions

In this work, we have studied the efficiency of the classical Lie symmetry group analysis to FPDEs. The fractional Lie symmetries method has been considered for the application to the fourth-order time fractional Burgers equation with Riemann-Liouville derivative. We have obtained the Lie point symmetries and performed symmetry reductions. It implies that under the Lie point symmetries, Eq. (1.1) can be reduced to a nonlinear ODE of fractional order with a new independent variables. Furthermore, we have constructed the explicit solutions for Eq. (1.1) by using sub-equation method with a detailed derivation and based on the power series theory. The approximate analytical solutions of the equation have been also constructed. At the end of the paper, the conservation laws of Eq. (1.1) have been constructed. Our results show that the extended Lie group analysis and the fractional sub-equation method are very effective and powerful technique for investigating FPDEs in mathematical physics.

Acknowledgements

The authors would like to thank the editor and the referees for their valuable comments and suggestions.

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