Journal of Applied Analysis and Computation
Volume 10, Number 2, April 2020, 624–648
Website:http://jaac-online.com/
DOI:10.11948/20190043

UPPER BOUNDS ON THE NUMBER OF DETERMINING MODES, NODES, AND VOLUME ELEMENTS FOR A 3D MAGENETOHYDRODYNAMIC-α MODEL

Cung The Anh[1], Nguyen Thi Minh Toai[1] and Vu Manh Toi[2,†]

Abstract In this paper we give upper bounds on the number of determining Fourier modes, determining nodes, and determining volume elements for a 3D MHD-α model. Here the bounds are estimated explicitly in terms of flow parameters, such as viscosity, magnetic diffusivity, smoothing length, forcing and domain size.

Keywords MHD-α model, determining modes, determining nodes, determining volume elements.

MSC(2010) 76W05, 35Q35, 35B40.

1. Introduction

Let $\Omega = (0, L)^3,\ L > 0$, be a periodic box in $\mathbb{R}^3$. We consider the following 3D MHD-α model which was introduced by Linshiz and Titi in [27]

$$\partial_t v - \nu\Delta v - u \times (\nabla \times v) - (B \cdot \nabla)B + \nabla p + \frac{1}{2}\nabla|B|^2 = f \ \text{ in } \Omega \times (0, \infty), \quad (1.1)$$

$$\partial_t B - \eta\Delta B + (u \cdot \nabla)B - (B \cdot \nabla)u = 0 \ \text{ in } \Omega \times (0, \infty), \quad (1.2)$$

$$v = u - \alpha^2\Delta u \ \text{ in } \Omega \times (0, \infty), \quad (1.3)$$

$$\nabla \cdot u = \nabla \cdot v = \nabla \cdot B = 0 \ \text{ in } \Omega \times (0, \infty), \quad (1.4)$$

subject to the periodic boundary conditions and the initial conditions

$$u(0) = u^0, B(0) = B^0 \ \text{ in } \Omega. \quad (1.5)$$

Here $u = u(x, t)$ is the unknown velocity, $B = B(x, t)$ is the unknown magnetic field and $p = p(x, t)$ is the unknown pressure, $\nu > 0$ is the kinematic viscosity coefficient, $\eta > 0$ is the constant magnetic diffusivity and α is a length scale parameter. When $\alpha = 0$ we formally recover the 3D classical MHD equations in [30]. Notice that here we only filter the velocity field but not the magnetic field, and it contrasts with the so-called Lagrangian-averaged magnetohydrodynamic-α (LAMHD-α) model (also called hyperbolic MHD equations or MHD-Voigt model) in [15].

[†]the corresponding author. Email address: toivm@tlu.edu.vn (V. M. Toi)
[1]Department of Mathematics, Hanoi National University of Education, 136 Xuan Thuy, Cau Giay, Hanoi, Vietnam
[2]Faculty of Computer Science and Engineering, Thuyloi University, 175 Tay Son, Dong Da, Hanoi, Vietnam

The MHD-α model (1.1)-(1.4) involves couping Maxwell's equations governing the magnetic field and the Navier-Stokes-α equations (sometimes called the viscous Camassa-Holm equations). In recent years, the existence and long-time behavior of solutions to this MHD-α model has attracted the attention of many mathematicians. In [27], Linshiz and Titi proved the existence, uniqueness and regularity of solutions with periodic boundary conditions, while Fan and Ozawa [9] and Liu [28] achieved the same result in the whole space $\mathbb{R}^2$ for both cases ($\nu = 1, \eta = 0$) and ($\nu = 0, \eta = 1$). More recently, Zhou and Fan [34] also established the regularity criteria to guarantee the existence of smooth solutions for higher dimensional case. For the long-time behavior of solutions, the existence of a finite-dimensional global attractor was proved by Catania in [5] in the case of three-dimensional periodic box, and the time decay rate in $L^2(\mathbb{R}^3)$ of solutions was proved by Jiang and Fan in [19]. The Sobolev regularity and the Gevrey regularity of the global attractor was proved recently in [2]. When $B = 0$, the above MHD-α model reduces to the well-known Navier-Stokes-α equations, where the existence of a finite-dimensional global attractor was proved in [10,18] in the case of periodic boundary conditions, and the decay rate of solutions on the whole space was proved by Bjorland and Schonbek in [4] and improved recently in [3] by using the theory of decay characters. We also refer the interested reader to [6, 7, 16, 24, 26, 35, 36] for results related to other MHD-α models.

The conventional theory of turbulence asserts that turbulent flows are monitored by a finite number of degrees of freedom. The notions and results for the case of 2D Navier-Stokes equations on determining modes [11, 12], determining nodes [13,14,20] and determining volume elements [14,21] are rigorous attempts to identify those parameters that control turbulent flows. We refer the interested reader to [8] for a general unified framework for this issue of determining parameters and [17, 29, 33] for some recent related results. In recent years, upper bounds on the number of determining modes and nodes were established for some α-models, which were suggested as regularization models for the 3D Navier-Stokes equations when α is a small regularization parameter. More precisely, for the 3D Navier-Stokes-Voigt equations, Kalantarov and Titi in [23] proved a result on the determining modes. The results on the determining modes and determining nodes for some regularization models of 3D Navier-Stokes equations such as 3D Navier-Stokes-α, 3D Leray-α and 3D Navier-Stokes-ω equations were proved by Korn in [25]. For MHD-α models, to the best of our knowledge, there is only a result on the determining modes for the 3D MHD-Voigt equations in [6].

In this paper, we study the number of determining modes, determining nodes and determining volume elements for the MHD-α model (1.1)-(1.4). To do this, we follow the general lines of the approach used in [22] for 2D Navier-Stokes equations. We first estimate the large time asymptotics for the solutions. Then, we establish some inequalities related to the nodal in the three-dimensional case which are extension of that in the two-dimensional case in [22], and hence we can get an upper bound on the number of determining nodes. The determining volume elements is proved in a similar manner with the help of our new inequalities related to the finite volume elements. To obtain the bound on the number of determining modes, we need some technical estimates which are similar to that used to prove the determining nodes and determining volume elements. It is worthy noticing that in the present paper the number of determining nodes, modes and finite volume elements is estimated explicitly in terms of flow parameters, such as viscosity, magnetic diffusivity, smoothing length, forcing and domain size, and these estimates are global as they do not depend on an individual solution. It is also noticed that our arguments in the paper and our new technical estimates in three-dimensional case given in Lemmas 3.1 and 4.1 below can be used to study the degrees of freedom for some other 3D MHD-α models in [7, 16, 27].

The paper is organized as follows. In Section 2, we recall the functional setting of the 3D MHD-α model. Section 3 gives an upper bound on the number of determining nodes. The number of determining volume elements is studied in Section 4. We prove an upper bound on the number of determining modes in Section 5. For clarity of the presentation, the proof of some technical results used in the proof of main results is given in the Appendix.

2. Functional setting and preliminaries

Let $\mathcal{V}$ be the set of all vector valued trigonometric polynomials u defined in Ω such that $\nabla \cdot u = 0$ and $\int_\Omega u(x)dx = 0$. Denote by H and V the closures of $\mathcal{V}$ in $L^2(\Omega)^3$ and in $H^1(\Omega)^3$, respectively. We denote by $(\cdot,\cdot)$ and $|\cdot|$ the inner product and the norm in H, and by $((\cdot,\cdot)) = (\nabla\cdot,\nabla\cdot)$ and $\|\cdot\| = |\nabla\cdot|$ the inner product and norm in V.

Let P be the Helmholtz-Leray orthogonal projection in $L^2(\Omega)^3$ onto the space H. Following the notations for the MHD-α equations, we denote

$$\mathcal{B}(u,v) = P(u\cdot\nabla)v \text{ and } \widetilde{\mathcal{B}}(u,v) := -P(u\times(\nabla\times v)), \quad \forall u,v\in V.$$

Using the identity

$$(b\cdot\nabla)a + \sum_{j=1}^{3} a_j\nabla b_j = -b\times(\nabla\times a) + \nabla(a\cdot b),$$

one can easily show that

$$\left\langle \widetilde{\mathcal{B}}(u,v), w)\right\rangle_{V',V} = \langle \mathcal{B}(u,v),w\rangle_{V',V} - \langle \mathcal{B}(w,v),u\rangle_{V',V}. \tag{2.1}$$

Here, for a Banach space X, we have used the notation $\langle .,.\rangle_{X',X}$ to denote the dual pairing between X and its dual space X'.

We denote by $A = -P\Delta$ the Stokes operator with domain $D(A) = H^2(\Omega)^3\cap V$. Notice the fact that in the case of periodic boundary conditions, $A = -\Delta$ is a self-adjoint positive operator with compact inverse. Hence there exists a complete set of eigenfunctions $\{w_j\}_{j=1}^\infty$ which is orthonormal in H, and orthogonal in both V and $D(A)$ such that $Aw_j = \lambda_j w_j$ with

$$(2\pi/L)^2 = \lambda_1 \le \lambda_2 \le \cdots \le \lambda_j \sim j^{2/3}L^{-2} \le \cdots$$

We have the following Poincaré type inequalities

$$\|u\|^2 \ge \lambda_1|u|^2 \quad \text{for all } u\in V, \tag{2.2}$$

$$|Au|^2 \ge \lambda_1\|u\|^2 \quad \text{for all } u\in D(A). \tag{2.3}$$

Notice that

$$|u+\alpha^2Au|^2 = |u|^2 + 2\alpha^2\|u\|^2 + \alpha^4|Au|^2,$$

so

$$|u+\alpha^2 Au|^2 \geq 2^{3/2}\alpha^3\|u\|\,|Au|, \tag{2.4}$$

and

$$|Au|^2 \leq \frac{1}{\alpha^4}|u+\alpha^2 Au|^2. \tag{2.5}$$

From the definitions of $\mathcal{B}$ and $\widetilde{\mathcal{B}}$, we have

$$\langle \mathcal{B}(u,v),w\rangle_{V',V} = -\langle \mathcal{B}(u,w),v\rangle_{V',V} \qquad \text{for all } u,v,w\in V, \tag{2.6}$$

and in particular,

$$\langle \mathcal{B}(u,v),v\rangle_{V',V} = 0 \qquad \text{for all } u,v\in V. \tag{2.7}$$

Also, since (2.1) we have

$$\left\langle \widetilde{\mathcal{B}}(u,v),u\right\rangle_{V',V} = 0 \qquad \text{for all } u,v\in V. \tag{2.8}$$

We have the following estimates (see e.g. [31]):

$$|(\mathcal{B}(u,v),w)| \leq c_1\|u\|^{1/2}|Au|^{1/2}\|v\|\,|w|,\ \forall u\in D(A), v\in V, w\in H, \tag{2.9}$$

$$\left|\left(\widetilde{\mathcal{B}}(u,v),w\right)\right| \leq c_1\|u\|^{1/2}|Au|^{1/2}\|v\|\,|w|,\ \forall u\in D(A), v\in V, w\in H, \tag{2.10}$$

$$|(\mathcal{B}(u,v),w)| \leq c_2\|u\|\,\|v\|^{1/2}|Av|^{1/2}\,|w|,\ \forall u\in V, v\in D(A), w\in H, \tag{2.11}$$

$$\left|\langle \mathcal{B}(u,v),w\rangle_{V',V}\right| \leq c_3\|u\|^{1/2}|Au|^{1/2}|v|\,\|w\|,\ \forall u\in D(A), v\in H, w\in V, \tag{2.12}$$

$$\left|\left\langle \widetilde{\mathcal{B}}(u,v),w\right\rangle_{V',V}\right| \leq c_4\|u\|\,\|v\|\,|w|^{1/2}\|w\|^{1/2},\ \forall u,v,w\in V, \tag{2.13}$$

for some positive constants $c_i, i=1,\dots,4$.

We apply the projection P to (1.1)-(1.5) to obtain the equivalent system of equations

$$\frac{dv}{dt} + \nu Av + \widetilde{\mathcal{B}}(u,v) - \mathcal{B}(B,B) = Pf, \tag{2.14}$$

$$v = u + \alpha^2 Au, \tag{2.15}$$

$$\frac{dB}{dt} + \eta AB + \mathcal{B}(u,B) - \mathcal{B}(B,u) = 0, \tag{2.16}$$

with the initial datum

$$u(0)=u^0,\ B(0)=B^0. \tag{2.17}$$

Definition 2.1 ([27])**.** Let $T>0$ and given $(u^0,B^0)\in V\times H$ and $f\in L^\infty(0,T;H)$. A weak solution of (2.14)-(2.17) on the interval $[0,T]$ is a pair of functions (u,B) such that

$$u\in C([0,T];V)\cap L^2(0,T;D(A)) \text{ with } \frac{du}{dt}\in L^2(0,T;H)$$

(or equivalently $v\in C([0,T];V')\cap L^2(0,T;H)$ with $dv/dt\in L^2(0,T;D(A)')$ and

$$B\in C([0,T];H)\cap L^2(0,T;V) \text{ with } \frac{dB}{dt}\in L^2(0,T;V'),$$

satisfying $u(0) = u^0$, $B(0) = B^0$, and

$$\begin{aligned}&\langle \frac{dv}{dt}(t), w\rangle_{D(A)', D(A)} + \nu(v(t), Aw) + \left\langle \widetilde{\mathcal{B}}(u(t), v(t)), w\right\rangle_{D(A)', D(A)}\\ &- \langle \mathcal{B}(B(t), B(t)), w\rangle_{V', V} = (f(t), w),\end{aligned}$$

and

$$\begin{aligned}&\langle \frac{dB}{dt}(t), \varphi\rangle_{V', V} + \eta(B(t), A\varphi) + \langle \mathcal{B}(u(t), B(t)), \varphi\rangle_{V, V'}\\ &- \langle \mathcal{B}(B(t), u(t)), \varphi\rangle_{V, V'} = 0,\end{aligned}$$

for every $w \in D(A), \varphi \in V$ and for almost every $t \in [0, T]$.

When $(u^0, B^0) \in D(A) \times V$, we call a strong solution of (2.14)-(2.17) in the interval $[0, T]$ the solution that satisfies

$$B \in C([0, T]; V) \cap L^2(0, T; D(A)), \quad u \in C([0, T]; D(A)) \cap L^2(0, T; D(A^{3/2}))$$

(or equivalently $v \in C([0, T]; H) \cap L^2(0, T; V)$).

We define the generalized three-dimensional Grashof number Gr as follows

$$Gr = \frac{1}{\mu^2 \lambda_1^{3/4}} \limsup_{t\to\infty} |f(t)|, \text{ for } f \in L^\infty(0, \infty; H),$$

where $\mu = \min\{\nu, \eta\}$.

Remark 2.1. It is noticing that that for any f_1 and f_2 belonging to $L^\infty(0, \infty; H)$ such that $\lim\limits_{t\to\infty} |f_1(t) - f_2(t)| = 0$, then the generalized Grashof number Gr which defines on f_1 is equivalent to the one which defines on f_2.

To prove our main results, we will use the following well-posedness and large-time asymptotic result, whose proof will be postponed in the Appendix.

Theorem 2.1. *Let $(u^0, B^0) \in V \times H$ and $f \in L^\infty(0, \infty; H)$, then problem (2.14)-(2.17) has a unique global weak solution (u, B) such that*

$$\limsup_{t\to\infty} \frac{1}{T} \int_t^{t+T} \left(|u(\tau)|^2 + \alpha^2\|u(\tau)\|^2 + |B(\tau)|^2\right) d\tau \leq \frac{\mu^3 Gr^2}{\nu \lambda_1^{1/2}}, \tag{2.18}$$

$$\limsup_{t\to\infty} \frac{1}{T} \int_t^{t+T} \|u(\tau)\|^4 d\tau \leq \frac{\mu^6 Gr^4}{\nu^2 \lambda_1 \alpha^4}, \tag{2.19}$$

$$\limsup_{t\to\infty} \frac{1}{T} \int_t^{t+T} \left(\|u(\tau)\|^2 + \alpha^2|Au(\tau)|^2 + \|B(\tau)\|^2\right) d\tau \leq \frac{2\mu^3 \lambda_1^{1/2} Gr^2}{\nu}, \tag{2.20}$$

$$\begin{aligned}&\limsup_{t\to\infty} \frac{1}{T} \int_t^{t+T} \|B(\tau)\|^4 d\tau\\ \leq &\frac{32\mu^6 \lambda_1^3 Gr^4}{\nu^2} + 2\left(\frac{216 c_4^4(\lambda_1^{-1} + \alpha^2)^4}{\nu^6 \alpha^{10}} + \frac{16 c_3^4}{\nu^5 \eta} + \frac{27(c_1 + c_2)^4}{\nu^3 \eta^3}\right)^2 \frac{\mu^{16} Gr^{12}}{\lambda_1 \alpha^8}\end{aligned} \tag{2.21}$$

and

$$\begin{aligned}&\limsup_{t\to\infty} \frac{1}{T} \int_t^{t+T} \left(|Au(\tau)|^2 + \alpha^2\|Au(\tau)\|^2 + |AB(\tau)|^2\right) d\tau\\ \leq &\frac{6\mu^3 \lambda_1^{3/2} Gr^2}{\nu} + \left(\frac{432 c_4^4(\lambda_1^{-1} + \alpha^2)^4}{\nu^6 \alpha^{10}} + \frac{32 c_3^4}{\nu^5 \eta} + \frac{54(c_1 + c_2)^4}{\nu^3 \eta^3}\right) \frac{\mu^8 Gr^6}{\lambda_1^{1/2} \alpha^4},\end{aligned} \tag{2.22}$$

where $T = (\mu\lambda_1)^{-1}$.

We also need the following generalized Gronwall inequality.

Lemma 2.1 ([11,20]). *Suppose that $\phi(t)$ is an absolutely continuous non-negative function on $[0,\infty)$ that satisfies the following inequality*

$$\frac{d\phi}{dt} + \beta\phi \leq \gamma, \ \ a.e. \ \ on \ [0,\infty),$$

where β and γ are locally integrable real-valued functions on $[0,\infty)$ that satisfy the following conditions for some $T > 0$

$$\liminf_{t\to\infty} \frac{1}{T}\int_t^{t+T} \beta(\tau)d\tau > 0, \quad \limsup_{t\to\infty} \frac{1}{T}\int_t^{t+T} \beta^-(\tau)d\tau < \infty,$$

and

$$\limsup_{t\to\infty} \frac{1}{T}\int_t^{t+T} \gamma^+(\tau)d\tau = 0,$$

with $\beta^- := \max\{-\beta, 0\}, \gamma^+ := \max\{\gamma, 0\}$. Then it follows that $\lim_{t\to\infty} \phi(t) = 0$.

3. Determining nodes

A finite set of points $\mathcal{N} := \{x_1, x_2, \ldots, x_N\} \subset \Omega$ is called a set of determining nodes if for any two solutions (u_1, B_1) and (u_2, B_2) of (2.14)-(2.17) with the initial data $(u_1(0), B_1(0)), (u_2(0), B_2(0))$ and forcings $f_1, f_2 \in L^\infty(0,\infty; H)$, respectively, the assumptions

$$\lim_{t\to\infty} |(u_1, B_1)(x_j, t) - (u_2, B_2)(x_j, t)| = 0 \text{ for } x_j \in \mathcal{N}, \ j = 1, \ldots, N,$$

and

$$\lim_{t\to\infty} |f_1(t) - f_2(t)| = 0,$$

imply that

$$\lim_{t\to\infty} \left(|u_1(t) - u_2(t)|^2 + |B_1(t) - B_2(t)|^2\right) = 0.$$

We divide the domain Ω into N equal squares $\Omega_j, j = 1, \ldots, N$, where Ω_j is the j-th cubic with edge $h = L/\sqrt[3]{N}$. Furthermore, we place the point $x_j \in \Omega_j, \ j = 1, \ldots, N$.

To estimate the number of determining nodes, we need the following lemma whose proof is given in the Appendix.

Lemma 3.1. *For every $w \in D(A)$, there exist some positive constants c_5, c_6, c_7, c_8 such that*

$$|w|^2 \leq 4L^3\vartheta^2(w) + \frac{c_5 L^4}{N^{4/3}}|Aw|^2, \tag{3.1}$$

$$\|w\|^2 \leq c_6 L N^{2/3}\vartheta^2(w) + \frac{c_7 L^2}{N^{2/3}}|Aw|^2, \tag{3.2}$$

$$\|w\|^2_{L^\infty(\Omega)} \leq c_7 L N \vartheta^2(w) + \frac{c_8 L}{N^{1/3}}|Aw|^2, \tag{3.3}$$

where

$$\vartheta(w) = \max_{1\leq j\leq N} |w(x_j)|.$$

The main result in this section is the following theorem.

Theorem 3.1. *Let Ω be divided into N squares with the points $\mathcal{N} := \{x_1, x_2, \dots, x_N\}$ distributed one in each square. Then $\mathcal{N}$ is a set of determining nodes provided*

$$N \geq L^3 \left(\frac{4c_7}{3\mu}\right)^{3/2} \left(\rho^{MHD-\alpha}\right)^{3/2},$$

where

$$\begin{aligned}
&\rho^{MHD-\alpha}\\
&= \Big\{ \frac{\sqrt{2}}{\nu^2\lambda_1^{3/2}\alpha^3} + \frac{6\sqrt{2}\lambda_1^{1/2}}{\nu^2\alpha^3} + \frac{\sqrt{2}}{\nu\eta\lambda_1^{1/2}\alpha^3} + \frac{\sqrt{2}}{\nu^2\lambda_1^{1/2}\alpha^3} + \frac{2\sqrt{2}\lambda_1^{1/2}}{\nu\eta\alpha}\\
&\quad + \frac{2\sqrt{2}\lambda_1^{1/2}}{\nu^2\alpha} + \frac{2\sqrt{2}}{\nu\eta\lambda_1^{1/2}\alpha^3} + \frac{24\lambda_1^{1/2}}{\nu^2} + \frac{24}{\nu\eta\lambda_1\alpha^4}\Big\} c_1^2\mu^3 Gr^2\\
&\quad + \left(\frac{108}{\nu^2\eta^3\lambda_1\alpha^4} + \frac{512\lambda_1^3}{\nu^4\eta}\right) c_1^4\mu^6 Gr^4\\
&\quad + \left(\frac{\sqrt{2}}{\nu\alpha^3} + \frac{4}{\nu} + \frac{4}{\eta\lambda_1^2\alpha^4}\right)\left(\frac{432c_4^4(\lambda_1^{-1}+\alpha^2)^4}{\nu^6\alpha^{10}} + \frac{32c_3^4}{\nu^5\eta} + \frac{54(c_1+c_2)^4}{\nu^3\eta^3}\right)\frac{c_1^2\mu^8 Gr^6}{\lambda_1^{3/2}\alpha^4}\\
&\quad + \left(\frac{216c_4^4(\lambda_1^{-1}+\alpha^2)^4}{\nu^6\alpha^{10}} + \frac{16c_3^4}{\nu^5\eta} + \frac{27(c_1+c_2)^4}{\nu^3\eta^3}\right)^2 \frac{32c_1^4\mu^{16}Gr^{12}}{\nu^2\eta\lambda_1\alpha^8}.
\end{aligned}$$

Proof. Let $\widetilde{u} = u_1 - u_2, \widetilde{B} = B_1 - B_2$. Then $\widetilde{u}$ and $\widetilde{B}$ satisfy

$$\frac{d\widetilde{v}}{dt} + \nu A\widetilde{v} + \widetilde{\mathcal{B}}(\widetilde{u}, v_2) + \widetilde{\mathcal{B}}(u_1, \widetilde{v}) - \mathcal{B}(\widetilde{B}, B_2) - \mathcal{B}(B_1, \widetilde{B}) = P(f_1 - f_2), \tag{3.4}$$

$$\widetilde{v} = \widetilde{u} + \alpha^2 A\widetilde{u},$$

$$\frac{d\widetilde{B}}{dt} + \eta A\widetilde{B} + \mathcal{B}(u_1, \widetilde{B}) + \mathcal{B}(\widetilde{u}, B_2) - \mathcal{B}(B_1, \widetilde{u}) - \mathcal{B}(\widetilde{B}, u_2) = 0. \tag{3.5}$$

Multiplying (3.4) by $A\widetilde{v}$ and (3.5) by $A\widetilde{B}$, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\widetilde{v}\|^2 + \|\widetilde{B}\|^2\right) + \nu|A\widetilde{v}|^2 + \eta|A\widetilde{B}|^2\\
&= -\left(\widetilde{\mathcal{B}}(\widetilde{u}, v_2), A\widetilde{v}\right) - \left(\widetilde{\mathcal{B}}(u_1, \widetilde{v}), A\widetilde{v}\right)\\
&\quad + \left(\mathcal{B}(\widetilde{B}, B_2), A\widetilde{v}\right) + \left(\mathcal{B}(B_1, \widetilde{B}), A\widetilde{v}\right) + (f_1 - f_2, A\widetilde{v})\\
&\quad - \left(\mathcal{B}(u_1, \widetilde{B}), A\widetilde{B}\right) - \left(\mathcal{B}(\widetilde{u}, B_2), A\widetilde{B}\right) + \left(\mathcal{B}(B_1, \widetilde{u}), A\widetilde{B}\right) + \left(\mathcal{B}(\widetilde{B}, u_2), A\widetilde{B}\right).
\end{aligned}$$

We now estimate the terms on the right-hand side. First, we have

$$|(f_1 - f_2, A\widetilde{v})| \leq \frac{2}{\nu}|f_1 - f_2|^2 + \frac{\nu}{8}|A\widetilde{v}|^2. \tag{3.6}$$

Using (2.10), the Cauchy inequality and the Poincaré inequality (2.2), we have

$$\begin{aligned}
\left|\left(\widetilde{\mathcal{B}}(\widetilde{u}, v_2), A\widetilde{v}\right)\right| &\leq c_1\|\widetilde{u}\|^{1/2}|A\widetilde{u}|^{1/2}\|v_2\|\,|A\widetilde{v}|\\
&\leq \frac{c_1^2\lambda_1^{-1}}{\sqrt{2}\nu\alpha^3}\|v_2\|^2\|\widetilde{v}\|^2 + \frac{\nu}{8}|A\widetilde{v}|^2,
\end{aligned} \tag{3.7}$$

where we have used (2.4). Analogously, we derive

$$\begin{aligned}\left|\left(\widetilde{\mathcal{B}}(u_1,\widetilde{v}),A\widetilde{v}\right)\right| &\le c_1\|u_1\|^{1/2}|Au_1|^{1/2}\|\widetilde{v}\|\,|A\widetilde{v}| \\ &\le \frac{c_1^2}{\sqrt{2}\nu\alpha^3}|v_1|^2\|\widetilde{v}\|^2+\frac{\nu}{8}|A\widetilde{v}|^2.\end{aligned} \tag{3.8}$$

Now, using (2.9) and the Cauchy inequality, we obtain

$$\begin{aligned}\left|\left(\mathcal{B}(\widetilde{B},B_2),A\widetilde{v}\right)\right| &\le c_1\|\widetilde{B}\|^{1/2}|A\widetilde{B}|^{1/2}\|B_2\|\,|A\widetilde{v}| \\ &\le \frac{2c_1^2}{\nu}\|\widetilde{B}\|\,|A\widetilde{B}|\,\|B_2\|^2+\frac{\nu}{8}|A\widetilde{v}|^2 \\ &\le \frac{8c_1^4}{\nu^2\eta}\|B_2\|^4\|\widetilde{B}\|^2+\frac{\eta}{8}|A\widetilde{B}|^2+\frac{\nu}{8}|A\widetilde{v}|^2.\end{aligned} \tag{3.9}$$

By (2.9), the Cauchy inequality and the Poincaré inequality (2.3), we have

$$\begin{aligned}\left|\left(\mathcal{B}(B_1,\widetilde{B}),A\widetilde{v}\right)\right| &\le c_1\|B_1\|^{1/2}|AB_1|^{1/2}\|\widetilde{B}\|\,|A\widetilde{v}| \\ &\le \frac{2c_1^2\lambda_1^{-1}}{\nu}|AB_1|^2\,\|\widetilde{B}\|^2+\frac{\nu}{8}|A\widetilde{v}|^2.\end{aligned} \tag{3.10}$$

Using (2.9) and the Cauchy inequality, we deduce that

$$\begin{aligned}\left|\left(\mathcal{B}(u_1,\widetilde{B}),A\widetilde{B}\right)\right| &\le c_1\|u_1\|^{1/2}|Au_1|^{1/2}\,\|\widetilde{B}\|\,|A\widetilde{B}| \\ &\le \frac{c_1^2}{\sqrt{2}\eta\alpha^3}|v_1|^2\|\widetilde{B}\|^2+\frac{\eta}{8}|A\widetilde{B}|^2,\end{aligned} \tag{3.11}$$

where we have used (2.4). Analogously with using the Poincaré inequality (2.2), we arrive at

$$\begin{aligned}\left|\left(\mathcal{B}(\widetilde{u},B_2),A\widetilde{B}\right)\right| &\le c_1\|\widetilde{u}\|^{1/2}|A\widetilde{u}|^{1/2}\|B_2\|\,|A\widetilde{B}| \\ &\le \frac{c_1^2\lambda_1^{-1}}{\sqrt{2}\eta\alpha^3}\|B_2\|^2\|\widetilde{v}\|^2+\frac{\eta}{8}|A\widetilde{B}|^2.\end{aligned} \tag{3.12}$$

Using (2.9), the Cauchy inequality and the Poincaré inequality (2.3) yields

$$\begin{aligned}\left|\left(\mathcal{B}(B_1,\widetilde{u}),A\widetilde{B}\right)\right| &\le c_1\|B_1\|^{1/2}|AB_1|^{1/2}\|\widetilde{u}\|\,|A\widetilde{B}| \\ &\le \frac{2c_1^2\lambda_1^{-3/2}}{\eta}|AB_1|^2|A\widetilde{u}|^2+\frac{\eta}{8}|A\widetilde{B}|^2 \\ &\le \frac{2c_1^2\lambda_1^{-5/2}}{\eta\alpha^4}|AB_1|^2\|\widetilde{v}\|^2+\frac{\eta}{8}|A\widetilde{B}|^2,\end{aligned} \tag{3.13}$$

where we have used (2.5). Using (2.9) and the Young inequality, we have

$$\begin{aligned}\left|\left(\mathcal{B}(\widetilde{B},u_2),A\widetilde{B}\right)\right| &\le c_1\|\widetilde{B}\|^{1/2}\|u_2\|\,|A\widetilde{B}|^{3/2} \\ &\le \frac{54c_1^4}{\eta^3}\|u_2\|^4\|\widetilde{B}\|^2+\frac{\eta}{8}|A\widetilde{B}|^2.\end{aligned} \tag{3.14}$$

From (3.6)-(3.14), we get

$$\frac{d}{dt}\left(\|\widetilde{v}\|^2+\|\widetilde{B}\|^2\right)+\frac{3}{4}\left(\nu|A\widetilde{v}|^2+\eta|A\widetilde{B}|^2\right)\leq K_1(t)\|\widetilde{v}\|^2+K_2(t)\|\widetilde{B}\|^2+\frac{4}{\nu}|f_1-f_2|^2, \quad (3.15)$$

where

$$K_1(t)=2\left(\frac{c_1^2\lambda_1^{-1}}{\sqrt{2}\nu\alpha^3}\|v_2\|^2+\frac{c_1^2}{\sqrt{2}\nu\alpha^3}|v_1|^2+\frac{c_1^2\lambda_1^{-1}}{\sqrt{2}\eta\alpha^3}\|B_2\|^2+\frac{2c_1^2\lambda_1^{-5/2}}{\eta\alpha^4}|AB_1|^2\right),$$
$$K_2(t)=2\left(\frac{8c_1^4}{\nu^2\eta}\|B_2\|^4+\frac{2c_1^2\lambda_1^{-1}}{\nu}|AB_1|^2+\frac{c_1^2}{\sqrt{2}\eta\alpha^3}|v_1|^2+\frac{54c_1^4}{\eta^3}\|u_2\|^4\right). \quad (3.16)$$

Using (3.2) then (3.15) becomes

$$\frac{d}{dt}\left(\|\widetilde{v}\|^2+\|\widetilde{B}\|^2\right)+\left(\frac{3\mu N^{2/3}}{4c_7L^2}-(K_1(t)+K_2(t))\right)\left(\|\widetilde{v}\|^2+\|\widetilde{B}\|^2\right)$$
$$\leq\frac{3c_6N^{4/3}}{4c_7L}\left(\nu\vartheta^2(\widetilde{v})+\eta\vartheta^2(\widetilde{B})\right)+\frac{4}{\nu}|f_1-f_2|^2,$$

where $\mu=\min\{\nu,\eta\}$. We rewrite the last inequality in the form

$$\frac{d}{dt}\left(\|\widetilde{v}\|^2+\|\widetilde{B}\|^2\right)+\beta(t)\left(\|\widetilde{v}\|^2+\|\widetilde{B}\|^2\right)\leq\gamma(t),$$

with

$$\beta(t)=\frac{3\mu N^{2/3}}{4c_7L^2}-(K_1(t)+K_2(t)),$$

and

$$\gamma(t)=\frac{3c_6N^{4/3}}{4c_7L}\left(\nu\vartheta^2(\widetilde{v})+\eta\vartheta^2(\widetilde{B})\right)+\frac{4}{\nu}|f_1(t)-f_2(t)|^2.$$

Hence, to complete the proof, we will show that $\beta(t)$ and $\gamma(t)$ fulfill the requirements of Lemma 2.1.

First, since the assumption of f_1, f_2 and ϑ one sees that

$$\limsup_{t\to\infty}\frac{1}{T}\int_t^{t+T}\gamma^+(\tau)d\tau=0. \quad (3.17)$$

Now, from the definitions of $K_1(t)$ and $K_2(t)$ we have

$$\begin{aligned}&\liminf_{t\to\infty}\frac{1}{T}\int_t^{t+T}(K_1(\tau)+K_2(\tau))\,d\tau\\
&\leq\frac{\sqrt{2}c_1^2}{\nu\lambda_1\alpha^3}\limsup_{t\to\infty}\frac{1}{T}\int_t^{t+T}\|v_2(\tau)\|^2d\tau+\frac{\sqrt{2}c_1^2(\nu+\eta)}{\nu\eta\alpha^3}\limsup_{t\to\infty}\frac{1}{T}\int_t^{t+T}|v_1(\tau)|^2d\tau\\
&\quad+\frac{\sqrt{2}c_1^2}{\eta\lambda_1\alpha^3}\limsup_{t\to\infty}\frac{1}{T}\int_t^{t+T}\|B_2(\tau)\|^2d\tau+\frac{16c_1^4}{\nu^2\eta}\limsup_{t\to\infty}\frac{1}{T}\int_t^{t+T}\|B_2(\tau)\|^4d\tau\\
&\quad+4c_1^2\left(\frac{1}{\nu\lambda_1}+\frac{1}{\eta\lambda_1^{5/2}\alpha^4}\right)\limsup_{t\to\infty}\frac{1}{T}\int_t^{t+T}|AB_1(\tau)|^2d\tau\\
&\quad+\frac{108c_1^4}{\eta^3}\limsup_{t\to\infty}\frac{1}{T}\int_t^{t+T}\|u_2(\tau)\|^4d\tau. \end{aligned} \quad (3.18)$$

By using the facts that

$$|v_1|^2 = |u_1|^2 + \alpha^2\|u_1\|^2 + \alpha^4|Au_1|^2,$$

and

$$\|v_2\|^2 = \|u_2\|^2 + \alpha^2|Au_2|^2 + \alpha^4\|Au_2\|^2,$$

then (3.18) becomes

$$\begin{aligned}
&\liminf_{t\to\infty}\frac{1}{T}\int_t^{t+T}(K_1(\tau)+K_2(\tau))\,d\tau \\
&\le \frac{\sqrt{2}c_1^2}{\nu\lambda_1\alpha^3}\limsup_{t\to\infty}\frac{1}{T}\int_t^{t+T}\left(\|u_2(\tau)\|^2+\alpha^2\left[|Au_2(\tau)|^2+\alpha^2\|Au_2(\tau)\|^2\right]\right)d\tau \\
&\quad+\frac{\sqrt{2}c_1^2(\nu+\eta)}{\nu\eta\alpha^3}\limsup_{t\to\infty}\frac{1}{T}\int_t^{t+T}\left(|u_1(\tau)|^2+\alpha^2\|u_1(\tau)\|^2+\alpha^4|Au_1(\tau)|^2\right)d\tau \\
&\quad+\frac{\sqrt{2}c_1^2}{\eta\lambda_1\alpha^3}\limsup_{t\to\infty}\frac{1}{T}\int_t^{t+T}\|B_2(\tau)\|^2d\tau+\frac{16c_1^4}{\nu^2\eta}\limsup_{t\to\infty}\frac{1}{T}\int_t^{t+T}\|B_2(\tau)\|^4d\tau \\
&\quad+4c_1^2\left(\frac{1}{\nu\lambda_1}+\frac{1}{\eta\lambda_1^{5/2}\alpha^4}\right)\limsup_{t\to\infty}\frac{1}{T}\int_t^{t+T}|AB_1(\tau)|^2d\tau \\
&\quad+\frac{108c_1^4}{\eta^3}\limsup_{t\to\infty}\frac{1}{T}\int_t^{t+T}\|u_2(\tau)\|^4d\tau.
\end{aligned}$$

So, using the large-time asymptotic estimates (2.18)-(2.22) together Remark 2.1, we deduce with the choice $T=(\mu\lambda_1)^{-1}$ that

$$\begin{aligned}
&\liminf_{t\to\infty}\frac{1}{T}\int_t^{t+T}(K_1(\tau)+K_2(\tau))\,d\tau \\
&\le\Big\{\frac{\sqrt{2}}{\nu^2\lambda_1^{3/2}\alpha^3}+\frac{6\sqrt{2}\lambda_1^{1/2}}{\nu^2\alpha^3}+\frac{\sqrt{2}}{\nu\eta\lambda_1^{1/2}\alpha^3}+\frac{\sqrt{2}}{\nu^2\lambda_1^{1/2}\alpha^3}+\frac{2\sqrt{2}\lambda_1^{1/2}}{\nu\eta\alpha} \\
&\quad+\frac{2\sqrt{2}\lambda_1^{1/2}}{\nu^2\alpha}+\frac{2\sqrt{2}}{\nu\eta\lambda_1^{1/2}\alpha^3}+\frac{24\lambda_1^{1/2}}{\nu^2}+\frac{24}{\nu\eta\lambda_1\alpha^4}\Big\}c_1^2\mu^3Gr^2 \\
&\quad+\left(\frac{108}{\nu^2\eta^3\lambda_1\alpha^4}+\frac{512\lambda_1^3}{\nu^4\eta}\right)c_1^4\mu^6Gr^4 \\
&\quad+\left(\frac{\sqrt{2}}{\nu\alpha^3}+\frac{4}{\nu}+\frac{4}{\eta\lambda_1^2\alpha^4}\right)\left(\frac{432c_4^4(\lambda_1^{-1}+\alpha^2)^4}{\nu^6\alpha^{10}}+\frac{32c_3^4}{\nu^5\eta}+\frac{54(c_1+c_2)^4}{\nu^3\eta^3}\right)\frac{c_1^2\mu^8Gr^6}{\lambda_1^{3/2}\alpha^4} \\
&\quad+\left(\frac{216c_4^4(\lambda_1^{-1}+\alpha^2)^4}{\nu^6\alpha^{10}}+\frac{16c_3^4}{\nu^5\eta}+\frac{27(c_1+c_2)^4}{\nu^3\eta^3}\right)^2\frac{32c_1^4\mu^{16}Gr^{12}}{\nu^2\eta\lambda_1\alpha^8}.
\end{aligned}\tag{3.19}$$

So,

$$\limsup_{t\to\infty}\frac{1}{T}\int_t^{t+T}\beta^-(\tau)d\tau<\infty\quad\text{and}\quad\liminf_{t\to\infty}\frac{1}{T}\int_t^{t+T}\beta(\tau)d\tau>0\tag{3.20}$$

hold provided $N\ge L^3\left(\frac{4c_7}{3\mu}\right)^{3/2}\left(\rho^{MHD-\alpha}\right)^{3/2}$. From (3.17) and (3.20), applying Lemma 2.1, we get

$$\lim_{t\to\infty}\left(\|\widetilde{v}(t)\|^2+\|\widetilde{B}(t)\|^2\right)=0.$$

This completes the proof. □

4. Determining volume elements

We divide the domain Ω into N equal squares $\Omega_j, j = 1, \dots, N$, where Ω_j is the j-th cubic with edge $h = L/\sqrt[3]{N}$, and so the volume of Ω_j is $|\Omega_j| = L^3/N$. The local average of φ in Ω_j defined by

$$\langle \varphi \rangle_{\Omega_j} = \frac{1}{|\Omega_j|} \int_{\Omega_j} \varphi(x) dx.$$

Let f_1, f_2 belong to $L^\infty(0, \infty; H)$ satisfying

$$\lim_{t\to\infty} |f_1(t) - f_2(t)| = 0.$$

A set of volume elements is said to be determining if for any two solutions (u_1, B_1) and (u_2, B_2) corresponding to external forces f_1 and f_2 satisfying

$$\lim_{t\to\infty} \left(\langle u_1 \rangle_{\Omega_j} - \langle u_2 \rangle_{\Omega_j} \right) = \lim_{t\to\infty} \left(\langle B_1 \rangle_{\Omega_j} - \langle B_2 \rangle_{\Omega_j} \right) = 0,$$

we have

$$\lim_{t\to\infty} \left(|u_1(t) - u_2(t)|^2 + |B_1(t) - B_2(t)|^2 \right) = 0.$$

To establish the result on the number of determining volume elements, we need the following lemma, which will be proved in the Appendix.

Lemma 4.1. *For every $w \in D(A)$, we have the following estimates for some positive constants $c_9, c_{10}, c_{11}, c_{12}, c_{13}$,*

$$|w|^2 \le L^3 \chi^2(w) + \frac{L^2}{6N^{2/3}} \|w\|^2, \tag{4.1}$$

$$|w|^2 \le 4L^3 \chi^2(w) + \frac{c_9 L^4}{N^{4/3}} |Aw|^2, \tag{4.2}$$

$$\|w\|^2 \le c_{10} L N^{2/3} \chi^2(w) + \frac{c_{11} L^2}{N^{2/3}} |Aw|^2, \tag{4.3}$$

$$\|w\|^2_{L^\infty(\Omega)} \le c_{12} L N \chi^2(w) + \frac{c_{13} L}{N^{1/3}} |Aw|^2, \tag{4.4}$$

where

$$\chi(w) = \max_{1\le j\le N} \left| \langle w \rangle_{\Omega_j} \right|.$$

Theorem 4.1. *Let Ω be divided into N squares Ω_j. Suppose*

$$\lim_{t\to\infty} \left(\langle u_1 \rangle_{\Omega_j} - \langle u_2 \rangle_{\Omega_j} \right) = \lim_{t\to\infty} \left(\langle B_1 \rangle_{\Omega_j} - \langle B_2 \rangle_{\Omega_j} \right) = 0,$$

for $j = 1, \dots, N$. Then the volume elements are determining, that is,

$$\lim_{t\to\infty} \left(|u_1(t) - u_2(t)|^2 + |B_1(t) - B_2(t)|^2 \right) = 0,$$

provided that

$$N \ge L^3 \left(\frac{4c_{11}}{3\mu} \right)^{3/2} \left(\rho^{MHD-\alpha} \right)^{3/2},$$

where $\rho^{MHD-\alpha}$ is defined in Theorem 3.1.

Proof. Let $\widetilde{u} = u_1 - u_2, \widetilde{B} = B_1 - B_2$. Similarly to the proof of inequality (3.15) in Theorem 3.1, we have the following estimate

$$\frac{d}{dt}\left(\|\widetilde{v}\|^2+\|\widetilde{B}\|^2\right)+\frac{3}{4}\left(\nu|A\widetilde{v}|^2+\eta|A\widetilde{B}|^2\right) \leq K_1(t)\|\widetilde{v}\|^2+K_2(t)\|\widetilde{B}\|^2+\frac{4}{\nu}|f_1-f_2|^2,$$

where $K_1(t)$ and $K_2(t)$ are as in (3.16). Using (4.3) we have

$$\begin{aligned} &\frac{d}{dt}\left(\|\widetilde{v}\|^2 + \|\widetilde{B}\|^2\right) + \left(\frac{3\mu N^{2/3}}{4c_{11}L^2} - (K_1(t) + K_2(t))\right)\left(\|\widetilde{v}\|^2 + \|\widetilde{B}\|^2\right) \\ &\leq \frac{3c_{10}N^{4/3}}{4c_{11}L}\left(\nu\chi^2(\widetilde{v}) + \eta\chi^2(\widetilde{B})\right) + \frac{4}{\nu}|f_1 - f_2|^2. \end{aligned} \tag{4.5}$$

Hence, to complete the proof, we rewrite (4.5) in the form

$$\frac{d}{dt}\left(\|\widetilde{v}\|^2 + \|\widetilde{B}\|^2\right) + \beta(t)\left(\|\widetilde{v}\|^2 + \|\widetilde{B}\|^2\right) \leq \gamma(t),$$

with

$$\beta(t) = \frac{3\mu N^{2/3}}{4c_{11}L^2} - (K_1(t) + K_2(t)),$$

and

$$\gamma(t) = \frac{3c_{10}N^{4/3}}{4c_{11}L}\left(\nu\chi^2(\widetilde{v}) + \eta\chi^2(\widetilde{B})\right) + \frac{4}{\nu}|f_1(t) - f_2(t)|^2.$$

Firstly, since the assumptions of f_1, f_2 and χ one has

$$\limsup_{t\to\infty}\frac{1}{T}\int_t^{t+T}\gamma^+(\tau)d\tau = 0.$$

Using (3.19) then

$$\limsup_{t\to\infty}\frac{1}{T}\int_t^{t+T}\beta^-(\tau)d\tau < \infty \text{ and } \liminf_{t\to\infty}\frac{1}{T}\int_t^{t+T}\beta(\tau)d\tau > 0$$

hold provided $N \geq L^3\left(\frac{4c_{11}}{3\mu}\right)^{3/2}\left(\rho^{MHD-\alpha}\right)^{3/2}$. Hence, applying Lemma 2.1, we get

$$\lim_{t\to\infty}\left(\|\widetilde{v}(t)\|^2 + \|\widetilde{B}(t)\|^2\right) = 0.$$

This completes the proof. □

5. Determining modes

Let $\{w_1, \ldots, w_m\}$ be the first m eigenfunctions of the Stokes operator A. We denote by P_m the orthogonal projection onto span$\{w_1, \ldots, w_m\}$, and $Q_m = I - P_m$. Let (u_1, B_1) and (u_2, B_2) be two solutions of (2.14)-(2.17) with the forcings f_1 and f_2 given in $L^\infty(0, \infty; H)$, respectively.

A set modes $\{w_j\}_{j=1}^m$ is called determining if we have

$$\lim_{t\to\infty}\left(|u_1(t) - u_2(t)|^2 + |B_1(t) - B_2(t)|^2\right) = 0$$

whenever

$$\lim_{t\to\infty} |f_1(t) - f_2(t)| = 0,$$

and

$$\lim_{t\to\infty} \left(|P_m(u_1(t) - u_2(t))|^2 + |P_m(B_1(t) - B_2(t))|^2\right) = 0.$$

Theorem 5.1. *With $\rho^{MHD-\alpha}$ as in Theorem 3.1, suppose that m satisfies*

$$\lambda_{m+1} \geq \frac{4}{3\mu}\rho^{MHD-\alpha}.$$

Then the number of determining modes is not larger than m.

Proof. Let $\widetilde{u} = u_1 - u_2, \widetilde{B} = B_1 - B_2$. Then $\widetilde{u}$ and $\widetilde{B}$ satisfy

$$\frac{d\widetilde{v}}{dt} + \nu A\widetilde{v} + \widetilde{\mathcal{B}}(\widetilde{u}, v_2) + \widetilde{\mathcal{B}}(u_1, \widetilde{v}) - \mathcal{B}(\widetilde{B}, B_2) - \mathcal{B}(B_1, \widetilde{B}) = P(f_1 - f_2), \tag{5.1}$$

$$\widetilde{v} = \widetilde{u} + \alpha^2 A\widetilde{u},$$

$$\frac{d\widetilde{B}}{dt} + \eta A\widetilde{B} + \mathcal{B}(u_1, \widetilde{B}) + \mathcal{B}(\widetilde{u}, B_2) - \mathcal{B}(B_1, \widetilde{u}) - \mathcal{B}(\widetilde{B}, u_2) = 0. \tag{5.2}$$

Multiplying (5.1) by $Q_m A\widetilde{v}$ and (5.2) by $Q_m A\widetilde{B}$, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|Q_m\widetilde{v}\|^2 + \|Q_m\widetilde{B}\|^2\right) + \nu|Q_m A\widetilde{v}|^2 + \eta|Q_m A\widetilde{B}|^2 \\
&= (f_1 - f_2, Q_m A\widetilde{v}) - \left(\widetilde{\mathcal{B}}(\widetilde{u}, v_2), Q_m A\widetilde{v}\right) - \left(\widetilde{\mathcal{B}}(u_1, \widetilde{v}), Q_m A\widetilde{v}\right) \\
&\quad + \left(\mathcal{B}(\widetilde{B}, B_2), Q_m A\widetilde{v}\right) + \left(\mathcal{B}(B_1, \widetilde{B}), Q_m A\widetilde{v}\right) \\
&\quad - \left(\mathcal{B}(u_1, \widetilde{B}), Q_m A\widetilde{B}\right) - \left(\mathcal{B}(\widetilde{u}, B_2), Q_m A\widetilde{B}\right) \\
&\quad + \left(\mathcal{B}(B_1, \widetilde{u}), Q_m A\widetilde{B}\right) + \left(\mathcal{B}(\widetilde{B}, u_2), Q_m A\widetilde{B}\right).
\end{aligned}$$

We now estimate the terms on the right-hand side. First, we have

$$|(f_1 - f_2, Q_m A\widetilde{v})| \leq \frac{2}{\nu}|f_1 - f_2|^2 + \frac{\nu}{8}|Q_m A\widetilde{v}|^2. \tag{5.3}$$

Similarly to (3.7)-(3.15) with note that $\widetilde{v} = P_m\widetilde{v} + Q_m\widetilde{v}$ and $\widetilde{B} = P_m\widetilde{B} + Q_m\widetilde{B}$, we have the following estimates

$$\left|\left(\widetilde{\mathcal{B}}(\widetilde{u}, v_2), Q_m A\widetilde{v}\right)\right| \leq \frac{c_1^2\lambda_1^{-1}}{\sqrt{2}\nu\alpha^3}\|v_2\|^2\left(\|P_m\widetilde{v}\|^2 + \|Q_m\widetilde{v}\|^2\right) + \frac{\nu}{8}|Q_m A\widetilde{v}|^2, \tag{5.4}$$

$$\left|\left(\widetilde{\mathcal{B}}(u_1, \widetilde{v}), Q_m A\widetilde{v}\right)\right| \leq \frac{c_1^2}{\sqrt{2}\nu\alpha^3}|v_1|^2\left(\|P_m\widetilde{v}\|^2 + \|Q_m\widetilde{v}\|^2\right) + \frac{\nu}{8}|Q_m A\widetilde{v}|^2, \tag{5.5}$$

$$\begin{aligned}
\left|\left(\mathcal{B}(\widetilde{B}, B_2), Q_m A\widetilde{v}\right)\right| \leq{}& \frac{8c_1^4}{\nu^2\eta}\|B_2\|^4\left(\|P_m\widetilde{B}\|^2 + \|Q_m\widetilde{B}\|^2\right) \\
&+ \frac{\eta}{8}|P_m A\widetilde{B}|^2 + \frac{\eta}{8}|Q_m A\widetilde{B}|^2 + \frac{\nu}{8}|Q_m A\widetilde{v}|^2,
\end{aligned} \tag{5.6}$$

$$\left|\left(\mathcal{B}(B_1, \widetilde{B}), Q_m A\widetilde{v}\right)\right| \leq \frac{2c_1^2\lambda_1^{-1}}{\nu}|AB_1|^2\left(\|P_m\widetilde{B}\|^2 + \|Q_m\widetilde{B}\|^2\right) + \frac{\nu}{8}|Q_m A\widetilde{v}|^2, \tag{5.7}$$

$$\left|\left(\mathcal{B}(u_1,\widetilde{B}),Q_mA\widetilde{B}\right)\right| \le \frac{c_1^2}{\sqrt{2}\eta\alpha^3}|v_1|^2\left(\|P_m\widetilde{B}\|^2+\|Q_m\widetilde{B}\|^2\right)+\frac{\eta}{8}|Q_mA\widetilde{B}|^2, \quad (5.8)$$

$$\left|\left(\mathcal{B}(\widetilde{u},B_2),Q_mA\widetilde{B}\right)\right| \le \frac{c_1^2\lambda_1^{-1}}{\sqrt{2}\eta\alpha^3}\|B_2\|^2\left(\|P_m\widetilde{v}\|^2+\|Q_m\widetilde{v}\|^2\right)+\frac{\eta}{8}|Q_mA\widetilde{B}|^2, \quad (5.9)$$

$$\left|\left(\mathcal{B}(B_1,\widetilde{u}),Q_mA\widetilde{B}\right)\right| \le \frac{2c_1^2\lambda_1^{-5/2}}{\eta\alpha^4}|AB_1|^2\left(\|P_m\widetilde{v}\|^2+\|Q_m\widetilde{v}\|^2\right)+\frac{\eta}{8}|Q_mA\widetilde{B}|^2, \quad (5.10)$$

$$\left|\left(\mathcal{B}(\widetilde{B},u_2),Q_mA\widetilde{B}\right)\right| \le \frac{54c_1^4}{\eta^3}\|u_2\|^4\left(\|P_m\widetilde{B}\|^2+\|Q_m\widetilde{B}\|^2\right)+\frac{\eta}{8}|Q_mA\widetilde{B}|^2. \quad (5.11)$$

From (5.3)-(5.11), we get

$$\begin{aligned}
&\frac{d}{dt}\left(\|Q_m\widetilde{v}\|^2+\|Q_m\widetilde{B}\|^2\right)+\frac{3}{4}\left(\nu|Q_mA\widetilde{v}|^2+\eta|Q_mA\widetilde{B}|^2\right)\\
\le& K_1(t)\left(\|Q_m\widetilde{v}\|^2+\|P_m\widetilde{v}\|^2\right)+K_2(t)\left(\|Q_m\widetilde{B}\|^2+\|P_m\widetilde{B}\|^2\right)\\
&+\frac{\eta}{4}|P_mA\widetilde{B}|^2+\frac{4}{\nu}|f_1-f_2|^2,
\end{aligned} \quad (5.12)$$

where $K_1(t)$ and $K_2(t)$ are as in (3.16).

Hence from the facts that $|Q_mAw|^2 \ge \lambda_{m+1}\|Q_mw\|^2$, $|P_mAw|^2 \le \lambda_m\|P_mw\|^2 \le \lambda_m^2|P_mw|^2$, and using $\widetilde{v}=\widetilde{u}+\alpha^2A\widetilde{u}$, we have from (5.12) that

$$\frac{d}{dt}\left(\|Q_m\widetilde{v}\|^2+\|Q_m\widetilde{B}\|^2\right)+\beta(t)\left(\|Q_m\widetilde{v}\|^2+\|Q_m\widetilde{B}\|^2\right)\le\gamma(t),$$

where

$$\beta(t)=\frac{3\mu}{4}\lambda_{m+1}-K_1(t)-K_2(t),$$

and

$$\gamma(t)=K_1(t)\lambda_m(1+\alpha^2\lambda_m)^2|P_m\widetilde{u}|^2+K_2(t)\lambda_m|P_m\widetilde{B}|^2+\frac{\eta}{4}\lambda_m^2|P_m\widetilde{B}|^2+\frac{4}{\nu}|f_1-f_2|^2.$$

By using Theorem 2.1 and assumptions on $|P_m\widetilde{v}(t)|, |P_m\widetilde{B}(t)|$ and $|f_1(t)-f_2(t)|$, we get

$$\limsup_{t\to\infty}\frac{1}{T}\int_t^{t+T}\gamma^+(\tau)d\tau=0.$$

Using (3.19) then

$$\limsup_{t\to\infty}\frac{1}{T}\int_t^{t+T}\beta^-(\tau)d\tau<\infty \text{ and } \liminf_{t\to\infty}\frac{1}{T}\int_t^{t+T}\beta(\tau)d\tau>0$$

hold provided $\lambda_{m+1}\ge\frac{4}{3\mu}\rho^{MHD-\alpha}$. Hence, applying Lemma 2.1, we get

$$\lim_{t\to\infty}\left(\|\widetilde{v}(t)\|^2+\|\widetilde{B}(t)\|^2\right)=0.$$

This completes the proof. $\square$

6. Appendix

We now give the proofs of some technical results that have been used in the previous sections.

6.1. Proof of Theorem 2.1

The existence and uniqueness of a weak solution to (2.14)-(2.17) can be proved similarly as in the case the external force $f \equiv 0$ in [27]. We only prove the estimates (2.18)-(2.22). Multiplying (2.14) by u and (2.16) by B, then integrating over Ω and using (2.6), (2.7), (2.8), we have

$$\frac{1}{2}\frac{d}{dt}\left(|u|^2+\alpha^2\|u\|^2\right)+\nu(\|u\|^2+\alpha^2|Au|^2)=(\mathcal{B}(B,B),u)+(f,u),$$

and

$$\frac{1}{2}\frac{d}{dt}|B|^2+\eta\|B\|^2=-(\mathcal{B}(B,B),u).$$

Summing up these equalities and using the Cauchy inequality we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(|u|^2+\alpha^2\|u\|^2+|B|^2\right)+\nu(\|u\|^2+\alpha^2|Au|^2)+\eta\|B\|^2\\ =&(f,u)\le\frac{1}{2\nu\lambda_1}|f|^2+\frac{\nu\lambda_1}{2}|u|^2.\end{aligned}$$

By using the Poincaré inequality, we have

$$\frac{d}{dt}\left(|u|^2+\alpha^2\|u\|^2+|B|^2\right)+\mu\left(\|u\|^2+\alpha^2|Au|^2+\|B\|^2\right)\le\frac{1}{\nu\lambda_1}|f|^2, \tag{6.1}$$

where $\mu=\min\{\nu,\eta\}$. By using the Poincaré inequality once again, we deduce from (6.1) that

$$\frac{d}{dt}\left(|u|^2+\alpha^2\|u\|^2+|B|^2\right)+\lambda_1\mu\left(|u|^2+\alpha^2\|u\|^2+|B|^2\right)\le\frac{1}{\nu\lambda_1}|f|^2.$$

Applying the Gronwall inequality, we infer that

$$|u(t)|^2+\alpha^2\|u(t)\|^2+|B(t)|^2\le e^{-\lambda_1\mu t}I_0+\frac{e^{-\lambda_1\mu t}}{\nu\lambda_1}\int_0^t e^{\lambda_1\mu s}|f(s)|^2ds,\ \ \forall t\ge 0,$$

where

$$I_0:=|u^0|^2+\alpha^2\|u^0\|^2+|B^0|^2.$$

Hence,

$$\limsup_{t\to\infty}\left(|u(t)|^2+\alpha^2\|u(t)\|^2+|B(t)|^2\right)\le\frac{1}{\mu\nu\lambda_1^2}\limsup_{t\to\infty}|f(t)|^2.$$

Using the definition of Gr, we get

$$\limsup_{t\to\infty}\left(|u(t)|^2+\alpha^2\|u(t)\|^2+|B(t)|^2\right)\le\frac{\mu^3Gr^2}{\nu\lambda_1^{1/2}}. \tag{6.2}$$

So we get (2.18) and (2.19) from (6.2) for some $T > 0$. From now on, we choose $T = (\lambda_1 \mu)^{-1}$.

Integrating (6.1) over $(t, t+T)$ we have

$$\mu \int_t^{t+T} \left(\|u(\tau)\|^2 + \alpha^2 |Au(\tau)|^2 + \|B(\tau)\|^2\right) d\tau \leq |u(t)|^2 + \alpha^2 \|u(t)\|^2 + |B(t)|^2 + \frac{1}{\nu\lambda_1} \int_t^{t+T} |f(\tau)|^2 d\tau.$$

Hence

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{T} \int_t^{t+T} \left(\|u(\tau)\|^2 + \alpha^2 |Au(\tau)|^2 + \|B(\tau)\|^2\right) d\tau \\
\leq &\lambda_1 \limsup_{t\to\infty} \left(|u(t)|^2 + \alpha^2 \|u(t)\|^2 + |B(t)|^2\right) + \frac{1}{\mu\nu\lambda_1} \limsup_{t\to\infty} |f(t)|^2.
\end{aligned} \tag{6.3}$$

Substituting (6.2) into (6.3) and using the definition of Gr we deduce (2.20).

Taking the inner product of (2.14) with Au, the inner product of (2.16) with AB and summing up, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt} \left(\|u\|^2 + \alpha^2 |Au|^2 + \|B\|^2\right) + \nu \left(|Au|^2 + \alpha^2 \|Au\|^2\right) + \eta |AB|^2 \\
=&(f, Au) + (\mathcal{B}(B,B), Au) - \left(\widetilde{\mathcal{B}}(u,v), Au\right) + (\mathcal{B}(B,u), AB) - (\mathcal{B}(u,B), AB).
\end{aligned} \tag{6.4}$$

By the Cauchy inequality, we have

$$|(f, Au)| \leq \frac{|f|^2}{\nu} + \frac{\nu}{4} |Au|^2. \tag{6.5}$$

Using (2.12) and the Cauchy inequality, we have

$$\begin{aligned}
|(\mathcal{B}(B,B), Au)| &\leq c_3 \|B\|^{1/2} |AB|^{1/2} |B|\ \|Au\| \\
&\leq \frac{4c_3^4}{\nu^2 \eta \alpha^4} \|B\|^2 |B|^4 + \frac{\nu\alpha^2}{8} \|Au\|^2 + \frac{\eta}{4} |AB|^2.
\end{aligned} \tag{6.6}$$

Now, by using (2.13) and the Young inequality, we get

$$\begin{aligned}
\left|\left(\widetilde{\mathcal{B}}(u,v), Au\right)\right| &\leq c_4 \|u\|\ \|v\|\ |Au|^{1/2} \|Au\|^{1/2} \\
&\leq c_4 (\lambda_1^{-1} + \alpha^2) \|u\|\ |Au|^{1/2} \|Au\|^{3/2} \\
&\leq \frac{54 c_4^4 (\lambda_1^{-1} + \alpha^2)^4}{(\nu\alpha^2)^3} \|u\|^4 |Au|^2 + \frac{\nu\alpha^2}{8} \|Au\|^2.
\end{aligned} \tag{6.7}$$

Using (2.9), (2.11) and the Young inequality yields

$$\begin{aligned}
|(\mathcal{B}(B,u), AB)| + |(\mathcal{B}(u,B), AB)| &\leq (c_1 + c_2) \|B\|^{1/2} \|u\|\ |AB|^{3/2} \\
&\leq \frac{27 (c_1 + c_2)^4}{4\eta^3} \|B\|^2 \|u\|^4 + \frac{\eta}{4} |AB|^2.
\end{aligned} \tag{6.8}$$

Substituting (6.5), (6.6), (6.7) and (6.8) into (6.4) we deduce that

$$\begin{aligned}
&\frac{d}{dt}\left(\|u\|^2+\alpha^2|Au|^2+\|B\|^2\right)+\mu\left(|Au|^2+\alpha^2\|Au\|^2+|AB|^2\right)\\
&\le \frac{2}{\nu}|f|^2+\frac{8c_3^4}{\nu^2\eta\alpha^4}\|B\|^2|B|^4+\frac{54c_4^4(\lambda_1^{-1}+\alpha^2)^4}{(\nu\alpha^2)^3}\|u\|^4|Au|^2+\frac{27(c_1+c_2)^4}{2\eta^3}\|B\|^2\|u\|^4.
\end{aligned} \tag{6.9}$$

Integrating the inequality (6.9) from t to $t+T$ we get

$$\begin{aligned}
&\frac{1}{\mu}\left(\|u(t)\|^2+\alpha^2|Au(t)|^2+\|B(t)\|^2\right)+\int_s^t\left(|Au(\tau)|^2+\alpha^2\|Au(\tau)\|^2+|AB(\tau)|^2\right)d\tau\\
&\le\frac{1}{\mu}\left(\|u(s)\|^2+\alpha^2|Au(s)|^2+\|B(s)\|^2\right)\\
&\quad+\frac{2}{\mu\nu}\int_s^t|f(\tau)|^2d\tau+\frac{8c_3^4}{\mu\nu^2\eta\alpha^4}\int_s^t\|B(\tau)\|^2|B(\tau)|^4d\tau\\
&\quad+\frac{108c_4^4(\lambda_1^{-1}+\alpha^2)^4}{\mu(\nu\alpha^2)^3}\int_s^t\|u(\tau)\|^4|Au(\tau)|^2d\tau+\frac{27(c_1+c_2)^4}{2\mu\eta^3}\int_s^t\|B(\tau)\|^2\|u(\tau)\|^4d\tau.
\end{aligned} \tag{6.10}$$

From (6.2), we have

$$\limsup_{t\to\infty}|B(t)|^4\le\frac{\mu^6Gr^4}{\nu^2\lambda_1}\text{ and }\limsup_{t\to\infty}\|u(t)\|^4\le\frac{\mu^6Gr^4}{\nu^2\lambda_1\alpha^4}. \tag{6.11}$$

Moreover we get from (6.10) that

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{1}{T}\int_t^{t+T}\left(|Au(\tau)|^2+\alpha^2\|Au(\tau)\|^2+|AB(\tau)|^2\right)d\tau\\
&\le\frac{1}{T\mu}\limsup_{t\to\infty}\left(\|u(t)\|^2+\alpha^2|Au(t)|^2+\|B(t)\|^2\right)+\frac{2}{\mu\nu}\limsup_{t\to\infty}|f(t)|^2\\
&\quad+\frac{108c_4^4(\lambda_1^{-1}+\alpha^2)^4}{\mu(\nu\alpha^2)^3}\limsup_{t\to\infty}\|u(t)\|^4\limsup_{t\to\infty}\frac{1}{T}\int_t^{t+T}|Au(\tau)|^2d\tau\\
&\quad+\frac{8c_3^4}{\mu\nu^2\eta\alpha^4}\limsup_{t\to\infty}|B(t)|^4\limsup_{t\to\infty}\frac{1}{T}\int_t^{t+T}\|B(\tau)\|^2d\tau\\
&\quad+\frac{27(c_1+c_2)^4}{2\mu\eta^3}\limsup_{t\to\infty}\|u(t)\|^4\limsup_{t\to\infty}\frac{1}{T}\int_t^{t+T}\|B(\tau)\|^2d\tau.
\end{aligned} \tag{6.12}$$

So, using (2.20), (6.11) and the definition of Gr then (6.12) becomes

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{1}{T}\int_t^{t+T}\left(|Au(\tau)|^2+\alpha^2\|Au(\tau)\|^2+|AB(\tau)|^2\right)d\tau\\
&\le\lambda_1\limsup_{t\to\infty}\left(\|u(t)\|^2+\alpha^2|Au(t)|^2+\|B(t)\|^2\right)+\frac{2\mu^3\lambda_1^{3/2}Gr^2}{\nu}\\
&\quad+\left(\frac{216c_4^4(\lambda_1^{-1}+\alpha^2)^4}{\nu^6\alpha^{10}}+\frac{16c_3^4}{\nu^5\eta}+\frac{27(c_1+c_2)^4}{\nu^3\eta^3}\right)\frac{\mu^8Gr^6}{\lambda_1^{1/2}\alpha^4}.
\end{aligned} \tag{6.13}$$

Now, we consider

$$\lambda_1\limsup_{t\to\infty}\left(\|u(t)\|^2+\alpha^2|Au(t)|^2+\|B(t)\|^2\right).$$

For any $0 < s < t$, integrating the inequality (6.9) from s to t we get

$$\begin{aligned}\|u(t)\|^2 + \alpha^2|Au(t)|^2 + \|B(t)\|^2 \le{}& \|u(s)\|^2 + \alpha^2|Au(s)|^2 + \|B(s)\|^2\\ &+\frac{2}{\nu}\int_s^t |f(\tau)|^2 d\tau + \frac{8c_3^4}{\nu^2\eta\alpha^4}\int_s^t \|B(\tau)\|^2|B(\tau)|^4 d\tau\\ &+ \frac{108c_4^4(\lambda_1^{-1}+\alpha^2)^4}{(\nu\alpha^2)^3}\int_s^t \|u(\tau)\|^4|Au(\tau)|^2 d\tau\\ &+ \frac{27(c_1+c_2)^4}{2\eta^3}\int_s^t \|B(\tau)\|^2\|u(\tau)\|^4 d\tau.\end{aligned}$$

Integrating the last inequality with respect to s over the interval $(t-\frac{1}{\mu\lambda_1}, t)$ we get

$$\begin{aligned}&\frac{1}{\mu\lambda_1}\left(\|u(t)\|^2 + \alpha^2|Au(t)|^2 + \|B(t)\|^2\right)\\ \le& \int_{t-\frac{1}{\mu\lambda_1}}^t \left(\|u(s)\|^2 + \alpha^2|Au(s)|^2 + \|B(s)\|^2\right) ds + \frac{2}{\nu}\int_{t-\frac{1}{\mu\lambda_1}}^t\int_s^t |f(\tau)|^2 d\tau ds\\ &+ \frac{8c_3^4}{\nu^2\eta\alpha^4}\int_{t-\frac{1}{\mu\lambda_1}}^t\int_s^t \|B(\tau)\|^2|B(\tau)|^4 d\tau ds\\ &+ \frac{108c_4^4(\lambda_1^{-1}+\alpha^2)^4}{(\nu\alpha^2)^3}\int_{t-\frac{1}{\mu\lambda_1}}^t\int_s^t \|u(\tau)\|^4|Au(\tau)|^2 d\tau ds\\ &+ \frac{27(c_1+c_2)^4}{2\eta^3}\int_{t-\frac{1}{\mu\lambda_1}}^t\int_s^t \|B(\tau)\|^2\|u(\tau)\|^4 d\tau ds.\end{aligned}\tag{6.14}$$

Using (2.20), we have

$$\begin{aligned}&\limsup_{t\to\infty}\int_{t-\frac{1}{\mu\lambda_1}}^t \left(\|u(s)\|^2 + \alpha^2|Au(s)|^2 + \|B(s)\|^2\right) ds\\ =&T\limsup_{t\to\infty}\frac{1}{T}\int_t^{t+T} \left(\|u(\tau)\|^2 + \alpha^2|Au(\tau)|^2 + \|B(\tau)\|^2\right) d\tau\\ \le&\frac{2\mu^2 Gr^2}{\nu\lambda_1^{1/2}}.\end{aligned}\tag{6.15}$$

Using the fact that

$$\int_{t-\frac{1}{\mu\lambda_1}}^t\int_s^t |f(\tau)|^2 d\tau ds \le \int_{t-\frac{1}{\mu\lambda_1}}^t\int_{t-\frac{1}{\mu\lambda_1}}^t |f(\tau)|^2 d\tau ds = \frac{1}{\mu\lambda_1}\int_{t-\frac{1}{\mu\lambda_1}}^t |f(\tau)|^2 d\tau,$$

then

$$\limsup_{t\to\infty}\int_{t-\frac{1}{\mu\lambda_1}}^t\int_s^t |f(\tau)|^2 d\tau ds \le \frac{1}{\mu^2\lambda_1^2}\limsup_{t\to\infty}|f(t)|^2 = \frac{\mu^2 Gr^2}{\lambda_1^{1/2}}.\tag{6.16}$$

We also have the fact that

$$\begin{aligned}\int_{t-\frac{1}{\mu\lambda_1}}^t\int_s^t \|B(\tau)\|^2|B(\tau)|^4 d\tau ds \le{}& \int_{t-\frac{1}{\mu\lambda_1}}^t\int_{t-\frac{1}{\mu\lambda_1}}^t \|B(\tau)\|^2|B(\tau)|^4 d\tau ds\\ ={}& \frac{1}{\mu\lambda_1}\int_{t-\frac{1}{\mu\lambda_1}}^t \|B(\tau)\|^2|B(\tau)|^4 d\tau,\end{aligned}$$

then

$$\begin{aligned}
&\limsup_{t\to\infty}\int_{t-\frac{1}{\mu\lambda_1}}^{t}\int_s^t \|B(\tau)\|^2|B(\tau)|^4 d\tau ds\\
&\le \limsup_{t\to\infty}\frac{1}{\mu\lambda_1}\int_{t-\frac{1}{\mu\lambda_1}}^{t}\|B(\tau)\|^2|B(\tau)|^4 d\tau\\
&\le \frac{1}{\mu\lambda_1}\limsup_{t\to\infty}|B(t)|^4\ \limsup_{t\to\infty}\int_{t-\frac{1}{\mu\lambda_1}}^{t}\|B(\tau)\|^2 d\tau\\
&= \frac{1}{\mu^2\lambda_1^2}\limsup_{t\to\infty}|B(t)|^4\ \limsup_{t\to\infty}\frac{1}{T}\int_t^{t+T}\|B(\tau)\|^2 d\tau. \qquad (6.17)
\end{aligned}$$

So, by using (6.2) and (2.20) we get from (6.17) that

$$\limsup_{t\to\infty}\int_{t-\frac{1}{\mu\lambda_1}}^{t}\int_s^t \|B(\tau)\|^2|B(\tau)|^4 d\tau ds \le \frac{2\mu^7 Gr^6}{\nu^3\lambda_1^{5/2}}. \qquad (6.18)$$

By the same as (6.18), we deduce that

$$\limsup_{t\to\infty}\int_{t-\frac{1}{\mu\lambda_1}}^{t}\int_s^t \|u(\tau)\|^4|Au(\tau)|^2 d\tau ds \le \frac{2\mu^7 Gr^6}{\nu^3\lambda_1^{5/2}\alpha^6}, \qquad (6.19)$$

and

$$\int_{t-\frac{1}{\mu\lambda_1}}^{t}\int_s^t \|B(\tau)\|^2\|u(\tau)\|^4 d\tau ds \le \frac{2\mu^7 Gr^6}{\nu^3\lambda_1^{5/2}\alpha^4}. \qquad (6.20)$$

Substituting (6.15), (6.16), (6.18), (6.19) and (6.20) into (6.14), we conclude that

$$\begin{aligned}
&\lambda_1\limsup_{t\to\infty}\left(\|u(t)\|^2+\alpha^2|Au(t)|^2+\|B(t)\|^2\right)\\
&\le \frac{4\mu^3\lambda_1^{3/2}Gr^2}{\nu}+\left(\frac{216c_4^4(\lambda_1^{-1}+\alpha^2)^4}{\nu^6\alpha^{10}}+\frac{16c_3^4}{\nu^5\eta}+\frac{27(c_1+c_2)^4}{\nu^3\eta^3}\right)\frac{\mu^8 Gr^6}{\lambda_1^{1/2}\alpha^4}. \qquad (6.21)
\end{aligned}$$

Combining (6.21) and (6.13), we deduce that

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{1}{T}\int_t^{t+T}\left(|Au(\tau)|^2+\alpha^2\|Au(\tau)\|^2+|AB(\tau)|^2\right)d\tau\\
&\le \frac{6\mu^3\lambda_1^{3/2}Gr^2}{\nu}+\left(\frac{432c_4^4(\lambda_1^{-1}+\alpha^2)^4}{\nu^6\alpha^{10}}+\frac{32c_3^4}{\nu^5\eta}+\frac{54(c_1+c_2)^4}{\nu^3\eta^3}\right)\frac{\mu^8 Gr^6}{\lambda_1^{1/2}\alpha^4}.
\end{aligned}$$

This is the estimate (2.22). Moreover, we get (2.21) by using (6.21) with noting that

$$\limsup_{t\to\infty}\frac{1}{T}\int_t^{t+T}\|B(\tau)\|^4 d\tau \le \limsup_{t\to\infty}\|B(t)\|^4 \le \left(\limsup_{t\to\infty}\|B(t)\|^2\right)^2.$$

6.2. Proof of Lemma 3.1

We have

$$|w|^2=\sum_{j=1}^N\int_{\Omega_j}|w(x)|^2dx=\sum_{j=1}^N\int_{\Omega}|w(x)|^2 1_{\Omega_j}(x)dx.$$

where 1_{Ω_j} is the characteristic function of Ω_j. Consider Ω_j for j fixed, but arbitrary. Choose $z \in \Omega_j$ such that z is in the line of the intersection of two planes: the plane which contains the point x and is parallel to xy-plane and the plane which contains the point x_j and is parallel to the xz-plane in three dimensions. In other words, if x and x_j are such that $x = (\xi_1, \xi_2, \xi_3)$ and $x_j = (\eta_1, \eta_2, \eta_3)$, then $z = (\tau_1, \eta_2, \xi_3)$. Therefore,

$$\begin{aligned}|w(x) - w(x_j)|^2 &\leq (|w(x) - w(z)| + |w(z) - w(x_j)|)^2 \\ &\leq 2\left(|w(x) - w(z)|^2 + |w(z) - w(x_j)|^2\right).\end{aligned}$$

Following the results which were proved in [1, Lemma 4], then for every $w \in H^2(\Omega_j)$, we have

$$|w(x) - w(z)|^2 \leq \frac{2}{h}\left(4\|\nabla w\|^2_{L^2(\Omega_j)} + h^2\left\|\frac{\partial^2 w}{\partial x \partial y}\right\|^2_{L^2(\Omega_j)}\right),$$

and

$$|w(z) - w(x_j)|^2 \leq \frac{2}{h}\left(4\|\nabla w\|^2_{L^2(\Omega_j)} + h^2\left\|\frac{\partial^2 w}{\partial x \partial z}\right\|^2_{L^2(\Omega_j)}\right).$$

Hence

$$|w(x) - w(x_j)|^2 \leq \frac{4}{h}\left(8\|\nabla w\|^2_{L^2(\Omega_j)} + h^2\left\|\frac{\partial^2 w}{\partial x \partial y}\right\|^2_{L^2(\Omega_j)} + h^2\left\|\frac{\partial^2 w}{\partial x \partial z}\right\|^2_{L^2(\Omega_j)}\right).$$

This implies that

$$\begin{aligned}|w(x)|^2 &\leq 2|w(x_j)|^2 + 2|w(x) - w(x_j)|^2 \\ &\leq 2|w(x_j)|^2 + \frac{8}{h}\left(8\|\nabla w\|^2_{L^2(\Omega_j)} + h^2\left\|\frac{\partial^2 w}{\partial x \partial y}\right\|^2_{L^2(\Omega_j)} + h^2\left\|\frac{\partial^2 w}{\partial x \partial z}\right\|^2_{L^2(\Omega_j)}\right).\end{aligned}$$

Hence,

$$\begin{aligned}|w|^2 \leq& 2\sum_{j=1}^{N}\int_{\Omega_j}|w(x_j)|^2 1_{\Omega_j}(x)dx \\ &+ \sum_{j=1}^{N}\int_{\Omega}\frac{8}{h}\left(8\|\nabla w\|^2_{L^2(\Omega_j)} + h^2\left\|\frac{\partial^2 w}{\partial x \partial y}\right\|^2_{L^2(\Omega_j)} + h^2\left\|\frac{\partial^2 w}{\partial x \partial z}\right\|^2_{L^2(\Omega_j)}\right)1_{\Omega_j}(x)dx. \\ \leq& 2\vartheta^2(w)\sum_{j=1}^{N}\int_{\Omega_j}1_{\Omega_j}(x)dx + \sum_{j=1}^{N}\frac{64}{h}\|\nabla w\|^2_{L^2(\Omega_j)}\int_{\Omega}1_{\Omega_j}(x)dx \\ &+ 8h\sum_{j=1}^{N}\left\|\frac{\partial^2 w}{\partial x \partial y}\right\|^2_{L^2(\Omega_j)}\int_{\Omega}1_{\Omega_j}(x)dx + 8h\sum_{j=1}^{N}\left\|\frac{\partial^2 w}{\partial x \partial z}\right\|^2_{L^2(\Omega_j)}1_{\Omega_j}(x)dx \\ =& 2\vartheta^2(w)Nh^3 + 64h^2\|\nabla w\|^2_{L^2(\Omega)} + 8h^4\left\|\frac{\partial^2 w}{\partial x \partial y}\right\|^2_{L^2(\Omega)} + 8h\left\|\frac{\partial^2 w}{\partial x \partial z}\right\|^2_{L^2(\Omega)},\end{aligned}$$

where we have used the fact that $\int_{\Omega} 1_{\Omega_j}(x)dx = |\Omega_j| = h^3$ for all $j = 1, \ldots, N$. So, we obtain

$$|w|^2 \leq 2L^3\vartheta^2(w) + 64\frac{L^2}{N^{2/3}}\|w\|^2 + 8c\frac{L^4}{N^{4/3}}|Aw|^2. \tag{6.22}$$

Since

$$\|w\|^2 \le |w|\,|Aw|,$$

then by the Cauchy inequality, we get

$$64\frac{L^2}{N^{2/3}}\|w\|^2 \le 64\frac{L^2}{N^{2/3}}|w|\,|Aw| \le \frac{1}{2}|w|^2 + \frac{64^2}{2}\frac{L^4}{N^{4/3}}|Aw|^2. \tag{6.23}$$

Substituting (6.23) into (6.22) we get

$$|w|^2 \le 4L^3\vartheta^2(w) + \left(64^2 + 16c\right)\frac{L^4}{N^{4/3}}|Aw|^2.$$

So we get (3.1) with $c_5 = 64^2 + 16c$.

By the inequality (6.2), using (3.1) and the Cauchy inequality, we get (3.2). Moreover, by the Agmon inequality in the case of three dimensions: $\|w\|^2_{L^\infty(\Omega)} \le c\|w\||Aw|$, then using (3.2) and the Cauchy inequality, we deduce (3.3). The proof of Lemma 3.1 is complete.

6.3. Proof of Lemma 4.1

We first prove the estimates for the domain $(0,\ell)^3 := (0,\ell)\times(0,\ell)\times(0,\ell)$ for any $\ell > 0$. Following the proof in the one-dimensional and two-dimensional cases in [20, Appendix], we have the following estimates:

- In the case of one dimension: for all $w \in C_0^\infty(\mathbb{R})$,

$$\int_0^\ell |w(x)|^2 dx \le \ell|\langle w\rangle|^2 + \frac{\ell^2}{6}\int_0^\ell |w_x|^2 dx, \tag{6.24}$$

  where

$$\langle w\rangle = \frac{1}{\ell}\int_0^\ell w(x)dx.$$

- In the case of two dimensions: for all $w(x_1,x_2) \in C_0^\infty(\mathbb{R}^2)$,

$$\int_0^\ell\int_0^\ell |w(x_1,x_2)|^2 dx_1dx_2 \le \ell^2|\langle w\rangle|^2 + \frac{\ell^2}{6}\int_0^\ell\int_0^\ell\left(\left|\frac{\partial w}{\partial x_1}\right|^2 + \left|\frac{\partial w}{\partial x_2}\right|^2\right)dx_1dx_2, \tag{6.25}$$

  where

$$\langle w\rangle = \frac{1}{\ell^2}\int_0^\ell\int_0^\ell w(x_1,x_2)dx_1dx_2.$$

Applying the two-dimensional estimate (6.25) to $w(x_1,x_2,x_3)$ holding x_3 fixed, we have

$$\begin{aligned}\int_0^\ell\int_0^\ell |w(x_1,x_2,x_3)|^2dx_1dx_2 \le& \frac{1}{\ell^2}\left(\int_0^\ell\int_0^\ell w(x_1,x_2,x_3)dx_1dx_2\right)^2\\ &+ \frac{\ell^2}{6}\int_0^\ell\int_0^\ell\left(\left|\frac{\partial w}{\partial x_1}\right|^2 + \left|\frac{\partial w}{\partial x_2}\right|^2\right)dx_1dx_2.\end{aligned}$$

Integrating this inequality with respect to x_3 from 0 to ℓ to obtain

$$\begin{aligned}&\int_0^\ell\int_0^\ell\int_0^\ell |w(x_1,x_2,x_3)|^2 dx_1dx_2dx_3\\ \leq&\frac{1}{\ell^2}\int_0^\ell\left(\int_0^\ell\int_0^\ell w(x_1,x_2,x_3)dx_1dx_2\right)^2 dx_3\\ &+\frac{\ell^2}{6}\int_0^\ell\int_0^\ell\int_0^\ell\left(\left|\frac{\partial w}{\partial x_1}\right|^2+\left|\frac{\partial w}{\partial x_2}\right|^2\right)dx_1dx_2dx_3.\end{aligned}\tag{6.26}$$

Now, applying the estimate (6.24) to $\varphi(x_3)=\int_0^\ell\int_0^\ell w(x_1,x_2,x_3)dx_1dx_2$, we have

$$\begin{aligned}&\int_0^\ell\left(\int_0^\ell\int_0^\ell w(x_1,x_2,x_3)dx_1dx_2\right)^2 dx_3\\ \leq&\frac{1}{\ell}\left(\int_0^\ell\int_0^\ell\int_0^\ell w(x_1,x_2,x_3)dx_1dx_2dx_3\right)^2\\ &+\frac{\ell^2}{6}\int_0^\ell\left(\int_0^\ell\int_0^\ell\frac{\partial w}{\partial x_3}dx_1dx_2\right)^2 dx_3.\end{aligned}$$

Using the Buniakovsky-Schwarz inequality twice, we have

$$\int_0^\ell\left(\int_0^\ell\int_0^\ell\frac{\partial w}{\partial x_3}dx_1dx_2\right)^2 dx_3\leq\ell^2\int_0^\ell\int_0^\ell\int_0^\ell\left|\frac{\partial w}{\partial x_3}\right|^2 dx_1dx_2dx_3.$$

So

$$\begin{aligned}&\int_0^\ell\left(\int_0^\ell\int_0^\ell w(x_1,x_2,x_3)dx_1dx_2\right)^2 dx_3\\ \leq&\frac{1}{\ell}\left(\int_0^\ell\int_0^\ell\int_0^\ell w(x_1,x_2,x_3)dx_1dx_2dx_3\right)^2\\ &+\frac{\ell^4}{6}\int_0^\ell\int_0^\ell\int_0^\ell\left|\frac{\partial w}{\partial x_3}\right|^2 dx_1dx_2x_3.\end{aligned}\tag{6.27}$$

Substituting (6.27) into (6.26), we deduce that

$$\begin{aligned}&\int_0^\ell\int_0^\ell\int_0^\ell |w(x_1,x_2,x_3)|^2 dx_1dx_2dx_3\\ \leq&\ell^3\left(\frac{1}{\ell^3}\int_0^\ell\int_0^\ell\int_0^\ell w(x_1,x_2,x_3)dx_1dx_2dx_3\right)^2\\ &+\frac{\ell^2}{6}\int_0^\ell\int_0^\ell\int_0^\ell\left(\left|\frac{\partial w}{\partial x_1}\right|^2+\left|\frac{\partial w}{\partial x_2}\right|^2+\left|\frac{\partial w}{\partial x_3}\right|^2\right)dx_1dx_2dx_3.\end{aligned}$$

Hence, in any box Ω_j, we have

$$\int_{\Omega_j}|w(x)|^2dx\leq\frac{L^3}{N}|\left\langle w\right\rangle_{\Omega_j}|^2+\frac{L^2}{6N^{2/3}}\int_{\Omega_j}|\nabla w(x)|^2dx.$$

Summing in j from 1 to N we get

$$|w|^2 \leq L^3\chi^2(w) + \frac{L^2}{6N^{2/3}}\|w\|^2.$$

This is the inequality (4.1).

Using the following interpolation inequality

$$\|w\|^2 \leq |w||Aw| \tag{6.28}$$

and the Cauchy inequality, we get (4.2). Moreover, by using (4.1) and (6.28) once again and the Cauchy inequality, we get (4.3). Finally, by using the Agmon inequality in three dimensions, namely

$$\|w\|^2_{L^\infty(\Omega)} \leq c\|w\|\,|Aw|,$$

we get (4.4).

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