Journal of Applied Analysis and Computation
Volume 10, Number 3, June 2020, 935–945

Website:http://jaac-online.com/
DOI:10.11948/20190162

LUMPS AND THEIR INTERACTION SOLUTIONS OF A (2+1)-DIMENSIONAL GENERALIZED POTENTIAL KADOMTSEV-PETVIASHVILI EQUATION*

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Abstract A (2+1)-dimensional generalized potential Kadomtsev-Petviashvili (gpKP) equation which possesses a Hirota bilinear form is constructed. The lump waves are derived by using a positive quadratic function solution. By combining an exponential function with a quadratic function, an interaction solution between a lump and a one-kink soliton is obtained. Furthermore, an interaction solution between a lump and a two-kink soliton is presented by mixing two exponential functions with a quadratic function. This type of lump wave just appears to a line $k_2x + k_3y + k_4t + k_5 \sim 0$. We call this kind of lump wave is a special rogue wave. Some visual figures are depicted to explain the propagation phenomena of these interaction solutions.

Keywords Generalized potential Kadomtsev-Petviashvili equation, Hirota bilinear form, lump wave, lump-soliton.

MSC(2010) 35Q51, 35Q53, 37K40.

1. Introduction

In nonlinear science field, solitary waves of nonlinear partial differential equations are significantly important role in a variety of science and engineering applications [2, 5, 10, 35–38]. Among these exact solutions, lump waves admit a kind of rational solutions that are localized in all direction of spaces, historically found for nonlinear integrable equations [34]. Lumps have excited a growing amount of attention on both experimental observations and theoretical predictions [13]. It has been investigated in various fields, including fluids [8, 13], plasmas [33], and optic media [24]. The Darboux transformation [6, 7, 12] and the Hirota bilinear method [15, 16, 18–22, 29, 39, 41–44] are effective direct methods to construct lump solutions. Lots of integrable systems admit lump solutions, such as the Kadomtsev-Petviashvili equation [18, 19], the Sawada-Kotera equation [16] and the Caudrey-Dodd-Gibbon-Kotera-Sawada equation [39], etc [15, 20–22, 29, 41–44]. To describe complex physical phenomena, the interaction between solitons and other kinds of complicated waves [9, 26–28], interaction between lumps and other kinds of compli-

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* The authors were supported by National Natural Science Foundation of China (Nos. 11775146, 11835011, 11472177,11975156).

cated waves [1, 4, 11, 45] are widely studied by combining of variable functions. Interaction between solitons and other kinds of complicated waves are studied by the localization procedure related with the nonlocal symmetry and the consistent tanh expansion method [9,26–28]. Interaction between lump waves and solitons of the nonlinear evolution equations have been studied by the Hirota bilinear method [1,11,25,30,31,45]. In this paper, we shall focus on lumps and their interaction solutions of a (2+1)-dimensional generalized potential Kadomtsev-Petviashvili (gpKP) equation.

This paper is organized as follows. In Section 2, we construct a (2+1)-dimensional gpKP equation which possesses a Hirota bilinear form. Lump solutions are constructed by solving a Hirota bilinear form of the gpKP equation. In Section 3, by adding an exponential and two exponential terms with a quadratic function, the interaction solutions between a lump and one-kink soliton, and between a lump and two-kink soliton are obtained, respectively. The last section is a simple summary and discussion.

2. Study lump waves from a bilinear form

We consider a (2+1)-dimensional gpKP equation

$$u_{xt} + \frac{3}{2}u_x u_{xx} + \frac{1}{4}u_{xxxx} + \delta_1 u_{yy} + \delta_2 u_{xy} + \delta_3 u_{xx} = 0, \tag{2.1}$$

where δ_1, δ_2 and δ_3 are arbitrary constants. (2.1) will become a usual (2+1)-dimensional (pKP) equation with $\delta_2 = \delta_3 = 0$. The pKP equation describes the dynamics of a wave with small and finite amplitude. The periodic kink wave and the group-invariant solutions of the pKP equation were found by using extended homoclinic test technique [3] and the Lie symmetry approach [14]. The nonlocal symmetry and interaction solutions of the pKP equation were obtained by the localization procedure related with the nonlocal symmetry [32].

Based on the Painlevé analysis [40], the Painlevé-Bäcklund transformation of the gpKP equation reads

$$u = \frac{u_0}{\phi} + u_1, \tag{2.2}$$

where ϕ is an arbitrary function of variables x, y and t, and the function of u_1 is also a solution of the gpKP equation. By substituting (2.2) into (2.1) and balancing the coefficient ϕ^{-5}, we get

$$u_0 = 2\phi_x. \tag{2.3}$$

By substituting (2.3) and a seed solution $u_1 = 0$ into (2.2), we get

$$u = \frac{2\phi_x}{\phi}. \tag{2.4}$$

A bilinear form of (2.1) is yielded

$$\begin{aligned} &2\phi\phi_{xt} - 2\phi_t\phi_x + \frac{1}{2}\phi\phi_{xxxx} - 2\phi_x\phi_{xxx} + \frac{3}{2}\phi_{xx}^2 + 2\delta_1(\phi\phi_{yy} - \phi_y^2) \\ &+2\delta_2(\phi\phi_{xy} - \phi_x\phi_y) + 2\delta_3(\phi\phi_{xx} - \phi_x^2) = 0. \end{aligned} \tag{2.5}$$

A bilinear equation (2.5) has a following equivalent formula

$$(D_xD_t + \frac{1}{4}D_x^4 + \delta_1 D_y^2 + \delta_2 D_x D_y + \delta_3 D_x^2) f \cdot f = 0, \tag{2.6}$$

with the D-operator defined by

$$\begin{aligned} &D_x^l D_y^n D_t^m f(x,y,t)\cdot g(x',y',t') \\ =&\Big(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\Big)^l \Big(\frac{\partial}{\partial y}-\frac{\partial}{\partial y'}\Big)^n \Big(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\Big)^m f(x,y,t)\cdot g(x',y',t')|_{x=x',y=y',t=t'}. \end{aligned} \tag{2.7}$$

To get lump solutions of the gpKP equation, a quadratic function ϕ supposes

$$\begin{aligned} \phi &= g^2 + h^2 + a_9, \\ g &= a_1x + a_2y + a_3t + a_4, \\ h &= a_5x + a_6y + a_7t + a_8. \end{aligned} \tag{2.8}$$

By substituting (2.8) into (2.6) and balancing the different powers of x, y and t, we get the relations

$$\begin{aligned} a_3 &= -\frac{\delta_1(a_1a_2^2 + 2a_2a_5a_6 - a_1a_6^2)}{a_1^2 + a_5^2} - \delta_2 a_2 - \delta_3 a_1, \qquad a_9 = -\frac{3(a_1^2+a_5^2)^3}{4\delta_1(a_1a_6 - a_2a_5)^2}, \\ a_7 &= -\frac{\delta_1(a_5a_6^2 + 2a_1a_2a_6 - a_2a_5^2)}{a_1^2 + a_5^2} - \delta_2 a_6 - \delta_3 a_5, \end{aligned} \tag{2.9}$$

which should satisfy the conditions

$$a_5 \neq 0, \qquad \delta_1 < 0, \qquad a_1a_6 - a_2a_5 \neq 0. \tag{2.10}$$

Then, the solution of u is localized in all directions in the (x, y)-plane. A class of lump waves of a gpKP equation is thus generated

$$u = \frac{4a_1g + 4a_5h}{\phi}, \tag{2.11}$$

where

$$\begin{aligned} \phi &= g^2 + h^2 - \frac{3(a_1^2+a_5^2)^3}{4\delta_1(a_1a_6 - a_2a_5)^2}, \\ g &= a_1x + a_2y - \Big(\frac{\delta_1(a_1a_2^2 + 2a_2a_5a_6 - a_1a_6^2)}{a_1^2+a_5^2} + \delta_2a_2 + \delta_3a_1\Big)t + a_4, \\ h &= a_5x + a_6y - \Big(\frac{\delta_1(a_5a_6^2 + 2a_1a_2a_6 - a_2a_5^2)}{a_1^2+a_5^2} + \delta_2a_6 + \delta_3a_5\Big)t + a_8. \end{aligned} \tag{2.12}$$

To catch the moving path of lump waves (2.11), the critical point of the lump waves is calculated by taking $\phi_x = \phi_y = 0$. The approximate moving path of the lump waves is read

$$\begin{aligned} x = x(t) &= \frac{(a_2a_7 - a_3a_6)t - (a_2a_8 - a_4a_6)}{a_1a_6 - a_2a_5}, \\ y = y(t) &= \frac{(a_1a_7 - a_3a_5)t - (a_1a_8 - a_4a_5)}{a_1a_6 - a_2a_5}, \end{aligned} \tag{2.13}$$

which can describe the traveling path of the lump waves along a straight line

$$y = \frac{a_3a_5 - a_1a_7}{a_2a_7 - a_3a_6}x + \frac{a_3a_8 - a_4a_7}{a_2a_7 - a_3a_6}, \tag{2.14}$$

with a_3, a_7 and a_9 satisfy (2.9). The parameters select $a_1 = 1, a_2 = 2, a_4 = 4, a_5 = 1, a_6 = -2, a_8 = 2, \delta_1 = -1, \delta_2 = 2, \delta_3 = 3$. A lump wave of u is presented in figure 1. The spatial structure of a lump wave is described in Fig. 1(a). From Fig. 1(a), we can easily know that the lump wave has a localized characteristic at $t = 0$. Fig. 1(b) represents the density plot of the lump wave. Fig. 1(c) displays the contour plot of lump wave at $t = -3, t = 0, t = 3$, and the blue line is the relevant moving progress (2.14), i.e., $y = \frac{2}{7}x + \frac{5}{14}$.

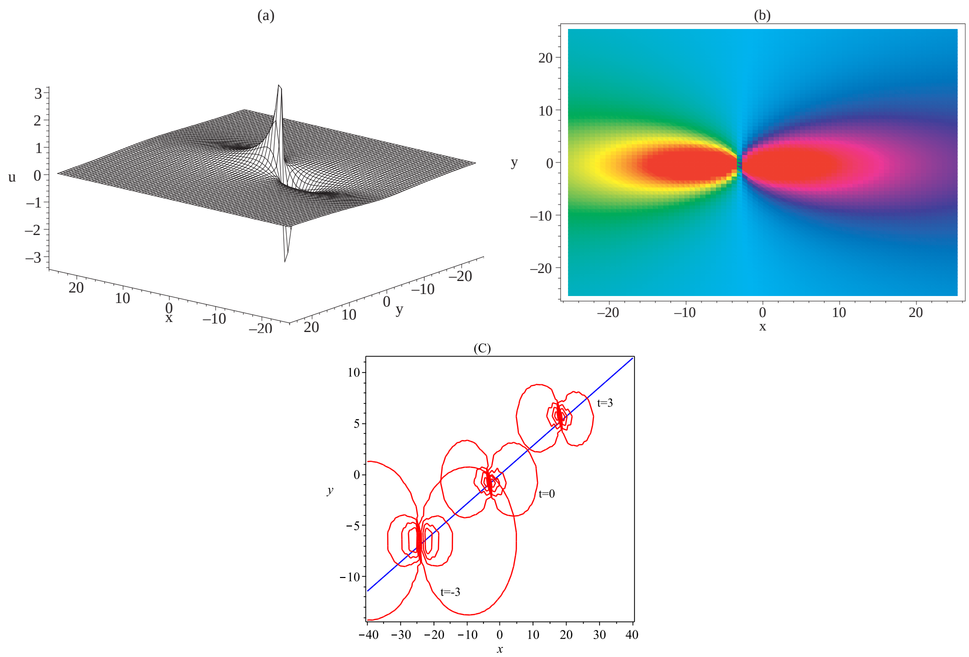

Figure 1. Profile of the solution (2.11). (a) 3-dimensional plot with the time $t = 0$, (b) the corresponding density plot, (c) the contour plot about the moving path described by the straight line (2.14), i.e., $y = \frac{2}{7}x + \frac{5}{14}$.

3. Interaction between lumps and soliton solutions

3.1. A lump and one-kink soliton solution

The interaction between lumps and other type solutions can be obtained by combining a quadratic function with other type functions. In order to find the interaction between lump waves and one soliton, we assume the interaction solution as a quadratic function and an exponential function

$$\begin{aligned}\phi &= g^2 + h^2 + a_9 + k_1 \exp(k_2x + k_3y + k_4t + k_5), \\ g &= a_1x + a_2y + a_3t + a_4, \\ h &= a_5x + a_6y + a_7t + a_8,\end{aligned} \tag{3.1}$$

where k_i $(i = 1, 2, \cdots, 5)$ being five undetermined real parameters. By substituting (3.1) into (2.5) and vanishing the different powers of x, y and t, we obtain the following set of constraining relations for the parameters

$$\begin{aligned}
a_3 =& -\frac{\delta_1(a_1a_2^2 + 2a_2a_5a_6 - a_1a_6^2)}{a_1^2 + a_5^2} - \delta_2 a_2 - \delta_3 a_1, \quad a_9 = -\frac{3(a_1^2 + a_5^2)^3}{4\delta_1(a_1a_6 - a_2a_5)^2},\\
a_7 =& -\delta_2 a_6 - \delta_3 a_5 - \frac{\delta_1\big(a_5a_6^2 + 2a_1a_2a_6 - a_5a_2^2\big)}{a_1^2 + a_5^2}, \quad k_2 = \frac{2\delta\sqrt{-3\delta_1}(a_1a_6 - a_2a_5)}{3(a_1^2 + a_5^2)},\\
k_3 =& \frac{2\delta\sqrt{-3\delta_1}(a_1a_2 + a_5a_6)(a_1a_6 - a_2a_5)}{3(a_1 + a_5^2)^2},\\
k_4 =& \frac{2\delta_1(a_1a_6 - a_2a_5)}{\sqrt{-3\delta_1}(a_1^2 + a_5^2)}\Big(\delta_3 + \frac{\delta_2(a_1a_2 + a_5a_6)}{a_1^2 + a_5^2}\\
& + \frac{\delta_1(3a_1^2a_2^2 - a_1^2a_6^2 + 8a_1a_2a_5a_6 - a_2^2a_5^2 + 3a_5^2a_6^2)}{3(a_1^2 + a_5^2)^2}\Big),
\end{aligned} \tag{3.2}$$

with $\delta^2 = 1$. The constraint conditions have

$$a_5 \neq 0, \qquad \delta_1 < 0, \qquad a_1a_6 - a_2a_5 \neq 0. \tag{3.3}$$

Then, the corresponding solution of u is localized in all directions in the (x, y)-plane. By substituting (3.1) into (2.4) and combining the parameters relations (3.2), we get the following interaction solution of the gpKP equation (2.1)

$$u = \frac{4a_1 g + 4a_5 h + \frac{4\delta^2\sqrt{-3\delta_1}k_1(a_1a_6 - a_2a_5)}{3(a_1^2 + a_5^2)}\exp(f)}{\phi}, \tag{3.4}$$

where

$$\begin{aligned}
\phi =& g^2 + h^2 + a_9 + k_1\exp(f),\\
g =& a_1x + a_2y - \Big(\frac{\delta_1(a_1a_2^2 + 2a_2a_5a_6 - a_1a_6^2)}{a_1^2 + a_5^2} + \delta_2a_2 + \delta_3a_1\Big)t + a_4,\\
h =& a_5x + a_6y - \Big(\frac{\delta_1(a_5a_6^2 + 2a_1a_2a_6 - a_5a_2^2)}{a_1^2 + a_5^2} + \delta_2a_6 + \delta_3a_5\Big)t + a_8,\\
f =& \frac{2\delta\sqrt{-3\delta_1}(a_1a_6 - a_2a_5)}{3(a_1^2 + a_5^2)}x + \frac{2\delta\sqrt{-3\delta_1}(a_1a_2 + a_5a_6)(a_1a_6 - a_2a_5)}{3(a_1 + a_5^2)^2}y\\
& + \frac{2\delta_1(a_1a_6 - a_2a_5)}{\sqrt{-3\delta_1}(a_1^2 + a_5^2)}\\
& \Big(\delta_3 + \frac{\delta_2(a_1a_2 + a_5a_6)}{a_1^2 + a_5^2} + \frac{\delta_1(3a_1^2a_2^2 - a_1^2a_6^2 + 8a_1a_2a_5a_6 - a_2^2a_5^2 + 3a_5^2a_6^2)}{3(a_1^2 + a_5^2)^2}\Big)t + k_5.
\end{aligned} \tag{3.5}$$

The parameters are selected as $a_1 = 1, a_2 = 3, a_4 = 2, a_5 = 5, a_6 = 1, a_8 = 4, k_1 = 2, k_5 = 3, \delta = 1, \delta_1 = -1, \delta_2 = 2, \delta_3 = 1$. The interaction solution between a lump and one-kink soliton of u is presented in Fig. 2(a) at $t = 0$. Fig. 2(b) displays the corresponding density plot of the lump-kink wave. Fig. 2(c) represents the homologous contour plot at time $t = -25, t = 0, t = 25$ and the blue line is the relevant moving progress of the lump wave (3.4), i.e., $y = x + \frac{2}{7}$.

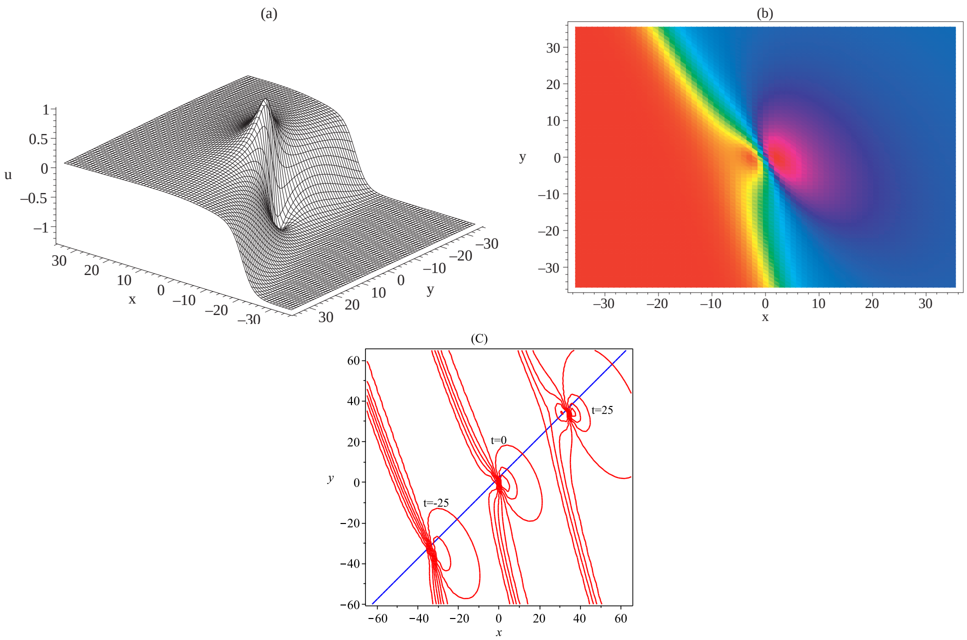

Figure 2. Profile of the solution (3.4). (a) 3-dimensional plot with the time $t = 0$, (b) the corresponding density plot, (c) the contour plot about the moving path described by the straight line (2.14), i.e., $y = x + \frac{2}{7}$.

3.2. A lump and a pair of kink soliton

For the interaction between the lumps and two soliton, we use a quadratic function with two exponential functions. In order to find the interaction between a lump wave and two-soliton, the interaction solution of (2.5) is defined by

$$\begin{aligned} \phi &= g^2 + h^2 + a_9 + k_1 \exp(f) + k_6 \exp(-f), \\ g &= a_1 x + a_2 y + a_3 t + a_4, \\ h &= a_5 x + a_6 y + a_7 t + a_8, \\ f &= k_2 x + k_3 y + k_4 t + k_5. \end{aligned} \tag{3.6}$$

By substituting (3.6) into (2.5) and collecting the coefficients of x, y and t, the following set of constraining relations for the parameters yields by solving the algebraic equations.

$$\begin{aligned} a_3 &= \frac{\delta_1 k_3(a_1k_3 - 2a_2k_2)}{k_2^2} - \delta_2 a_2 - \delta_3 a_1 - \frac{3a_1k_2^2}{4}, \quad a_5 = \frac{2\delta\sqrt{-3\delta_1}(a_1k_3 - a_2k_2)}{3k_2^2}, \\ a_6 &= \frac{3a_1k_2}{2\delta\sqrt{-3\delta_1}} - \frac{2\delta_1k_3(a_1k_3 - a_2k_2)}{\delta k_2^3\sqrt{-3\delta_1}}, \qquad k_4 = -\delta_2k_3 - \delta_3k_2 - \frac{k_2^3}{4} - \frac{\delta_1k_3^2}{k_2}, \\ a_7 &= -\frac{6\delta_1(a_1k_3 - a_2k_2)}{k_2^2\sqrt{-3\delta_1}}\left(\delta_3 + \frac{\delta_2k_3}{k_2} + \frac{\delta_1k_3^2}{k_2^2}\right) + \frac{9\delta_1(a_1k_3 + a_2k_2)}{2\sqrt{-3\delta_1}} + \frac{9\delta_2a_1k_2}{2\sqrt{-3\delta_1}}, \\ a_9 &= \frac{24\delta_1a_1^2k_2^4(a_1k_3 - a_2k_2)^2 - 16\delta_1^2(a_1k_3 - a_2)^4 - 9k_2^8(k_1k_6k_2^4 + a_1^4)}{3k_2^6(\delta_1(a_1k_3 - a_2k_2)^2 - 3a_1^2k_2^4)}, \end{aligned} \tag{3.7}$$

with $\delta^2 = 1$. The constraint conditions are

$$k_2 \neq 0, \qquad \delta_1 < 0, \qquad a_1^4 + k_1 k_6 k_2^4 > 0. \tag{3.8}$$

Then, the corresponding solution of u is localized in all directions in the (x, y)-plane. By substituting (3.6) into (2.4), a class of interaction solution reads

$$u = \frac{2\Big(2a_1 g - \frac{4\delta^2\sqrt{-3\delta_1}(a_1k_3 - a_2k_2)}{3k_2^2} h + k_1 k_2 \exp(f) - k_2 k_6 \exp(-f)\Big)}{\phi}, \tag{3.9}$$

where

$$\begin{aligned}
\phi &= g^2 + h^2 + a_9 + k_1 \exp(f) + k_6 \exp(-f), \\
g &= a_1 x + a_2 y + \Big(\frac{\delta_1 k_3(a_1k_3 - 2a_2k_2)}{k_2^2} - \delta_2 a_2 - \delta_3 a_1 - \frac{3a_1k_2^2}{4}\Big)t + a_4, \\
h &= \frac{2\delta\sqrt{-3\delta_1}(a_1k_3 - a_2k_2)}{3k_2^2} x + \Big(\frac{3a_1k_2}{2\delta\sqrt{-3\delta_1}} - \frac{2\delta_1 k_3(a_1k_3 - a_2k_2)}{\delta k_2^3\sqrt{-3\delta_1}}\Big) y \\
&\quad + \Big[\frac{6\delta_1(a_2k_2 - a_1k_3)}{k_2^2\sqrt{-3\delta_1}}\Big(\delta_3 + \frac{\delta_2 k_3}{k_2} + \frac{\delta_1 k_3^2}{k_2^2}\Big) + \frac{9\delta_1(a_1k_3 + a_2k_2)}{2\sqrt{-3\delta_1}} + \frac{9\delta_2 a_1 k_2}{2\sqrt{-3\delta_1}}\Big] t + a_8, \\
f &= k_2 x + k_3 y - \Big(\delta_2 k_3 + \delta_3 k_2 + \frac{k_2^3}{4} + \frac{\delta_1 k_3^2}{k_2}\Big) t + k_5.
\end{aligned} \tag{3.10}$$

The solution (3.6), which includes a quadratic function and a pair of exponential functions, is called a special rogue waves [17]. As a special rogue waves, lump waves are invisible

$$k_2 x + k_3 y + k_4 t + k_5 < 0, \qquad \text{or} \qquad k_2 x + k_3 y + k_4 t + k_5 > 0, \tag{3.11}$$

while lump waves just move to a line

$$k_2 x + k_3 y + k_4 t + k_5 \sim 0. \tag{3.12}$$

By combining the traveling path of lump waves (2.13) with the center line

$$k_2 x + k_3 y + k_4 t + k_5 = 0, \tag{3.13}$$

the time and place of the special rogue waves are expressed

$$\begin{aligned}
t &= 4k_2 A\big(k_2(a_2a_8 - a_4a_6) + k_3(a_4a_5 - a_1a_8) + k_2(a_1a_6 - a_2a_5)\big), \\
x &= A\Big((a_2a_8 - a_4a_6)(k_2^4 + 4\delta_1 k_3^2 + 4\delta_3 k_2^2) + 4k_2k_3(\delta_2 a_2 a_8 - \delta_2 a_4 a_6 + a_3 a_8 - a_4 a_7)\Big), \\
y &= A\Big((a_4a_5 - a_1a_8)(k_2^4 + 4\delta_1 k_3^2 + 4\delta_2 k_2 k_3) + 4k_2^2(\delta_3 a_4 a_5 + a_4 a_7 - \delta_3 a_1 a_8 - a_3 a_8) \\
&\quad + 4k_2 k_5(a_3 a_5 - a_1 a_7)\Big), \\
A &= \Big((k_2^4 + 4\delta_1 k_3^2)(a_1a_6 - a_2a_5) + 4k_2^2(\delta_3 a_1 a_6 + a_3 a_6 - \delta_3 a_2 a_5 - a_2 a_7) \\
&\quad + 4k_2k_3(\delta_2 a_1 a_6 + a_1 a_7 - \delta_2 a_2 a_5 - a_3 a_5)\Big)^{-1},
\end{aligned} \tag{3.14}$$

where a_3, a_5, a_6, a_7 and a_9 satisfy (3.7). In order to describe this special rogue waves, we select the parameters as $a_1 = 1, a_2 = 2, a_4 = 3, a_8 = 1, k_1 = 1/4, k_2 = 1/3, k_3 = 1/8, k_5 = 2, k_6 = 1/8, \delta = 1, \delta_1 = -1, \delta_2 = 2, \delta_3 = 3$. The interaction solution between a lump and a two-kink soliton of u is shown by Fig. 3(a). A lump wave catches with a middle of one-kink soliton. Fig. 1(b) shows the corresponding density plot of a lump and a two-kink soliton. The Fig. 3(c) displays the homologous contour plot at time $t = -20, t = 0, t = 20$ and the blue line is the relevant moving progress of lump waves (2.14), i.e., $y = -\frac{3}{16}x - \frac{291}{169} - \frac{25\sqrt{3}}{169}$.

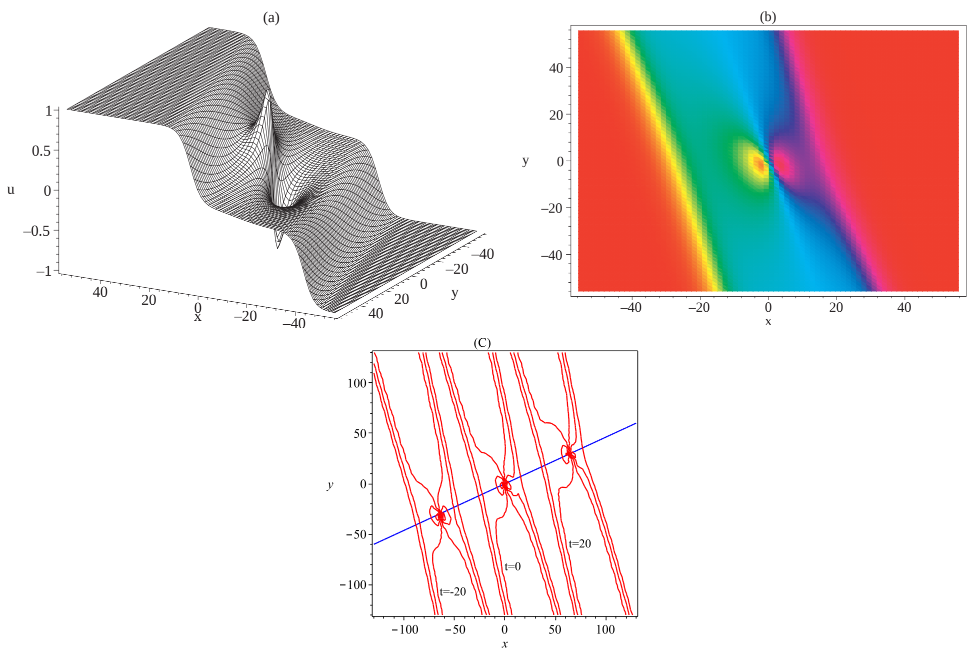

Figure 3. Profile of the solution (3.9). (a) 3-dimensional plot with the time $t = 0$, (b) the corresponding density plot, (c) the contour plot about the moving path described by the straight line (2.14), i.e., $y = -\frac{6}{13}x - \frac{291}{169} - \frac{25\sqrt{3}}{169}$.

4. Conclusion

In summary, some interaction solutions among the lump waves and the kink-soliton solitons of a gpKP equation are studied in this paper. We construct a bilinear form of the gpKP equation by the truncated Painlevé analysis. A positive quadratic function is used to find the lump waves. By solving a bilinear form of the gpKP equation, interaction between a lump and one-kink soliton, and between a lump and two-kink soliton are given by introducing additional exponential functions. Relevant propagation behaviors have been displayed by graphical simulations.

In addtion, the generalized bilinear operators are defined by the linear superposition principle [23]. A new nonlinear partial differential system can be constructed by using the generalized bilinear operators. According to the generalized bilinear operators, new gpKP-like equations have the following forms by selecting the prime numbers $p = 3, 5$

$$(D_{3,x}D_{3,t} + \frac{1}{4}D_{3,xxxx} + \delta_1 D_{3,yy} + \delta_2 D_{3,x}D_{3,y} + \delta_3 D_{3,xx})f \cdot f = 0, \tag{4.1}$$

$$(D_{5,x}D_{5,t} + \frac{1}{4}D_{5,xxxx} + \delta_1 D_{5,yy} + \delta_2 D_{5,x}D_{5,y} + \delta_3 D_{5,xx})f \cdot f = 0. \tag{4.2}$$

The lump waves and their interaction solutions of these two systems (4.1) and (4.2) are worthy of further study.

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