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ON Ψ-PROJECTIVE EXPANSION, QUASI PARTIAL METRICS AGGREGATION WITH AN APPLICATION

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Abstract In the present article, the notion of expansion between quasi partial metric spaces through aggregation is defined. With the help of aggregation functions, the concept of projective Ψ-expansion is introduced and some fixed point results are obtained through this notion. Furthermore, sufficient conditions are provided to characterize aggregation function and to ensure the existence and uniqueness of fixed point. All the results presented in this paper are new and an application to asymptotic complexity analysis is also given after the results.

Keywords Fixed point, quasi partial metric space, aggregation function, projective expansion.

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1. Introduction and Preliminaries

Borsik and Doboš [3] investigated the problem of aggregation for a collection of metrics (which need not be finite). They studied the properties of those functions that permit a collection of metrics to be merged in a single one.

Throughout the paper, $\mathbb{N}, \mathbb{R}$ and $\mathbb{R}^+$ will denote the set of natural numbers, the set of real numbers and the set of non-negative real numbers respectively.

Definition 1.1. A function $\Phi : \mathbb{R}_+^n \to \mathbb{R}_+$ is said to be homogeneous if $\Phi(\alpha x) = \alpha\Phi(x)$ for each $x \in \mathbb{R}_+^n$ and $\alpha \in \mathbb{R}_+$.

Borsik and Doboš [3] defined the notion of metric aggregation function as follows:

Definition 1.2 ([3]). A function $\Phi : \mathbb{R}_+^2 \to \mathbb{R}_+$ is a metric aggregation function provided that the function $d_\Phi : X \times X \to \mathbb{R}_+$ is a metric for every pair of metric spaces (X_1, d_1) and (X_2, d_2), where $X = X_1 \times X_2$ and

$$d_\Phi((x, y), (z, w)) = \Phi(d_1(x, z), d_2(y, w))$$

for all $(x, y),\ (z, w) \in X$.

The authors of [3] defined the monotonicity and sub-additivity of Φ as:

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Definition 1.3 ([3]). A function $\Phi : \mathbb{R}_+^n \to \mathbb{R}_+$ is said to be monotone if $x \preceq y \Rightarrow \Phi(x) \le \Phi(y)$ for all $x, y \in \mathbb{R}_+^n$ and sub-additive if $\Phi(x+y) \le \Phi(x) + \Phi(y)$ for all $x, y \in \mathbb{R}_+^n$ where $\preceq$ stands for the following pointwise order relation on $\mathbb{R}_+^n$:

$$x \preceq y \quad \Leftrightarrow \quad x_i \le y_i \quad ; \quad i = 1, \ldots n.$$

In 1994, Matthews [12, 13] introduced partial metric spaces with an application in denotational semantics and program verification. Many authors worked with this notion afterwards such as Heckmann [7] defined weak partial metric following Matthews' notion, Romaguera *et al.* [16] presented Scott topology based on complete partial metric space, Romaguera and Valero [17] introduced a quantitative computational model for partial metric spaces with formal balls. After that many authors worked in the similar directions, some of them are [6, 18, 19].

Massanet and Valero [11] were motivated by the application of partial metrics to computer science and the fact that many partial metrics used in computer science can be obtained via aggregation.

In 2013, Karapinar [8] presented a generalized version of Matthews's work by introducing quasi partial metric spaces by removing the symmetry axiom.

Definition 1.4 ([8]). For a nonempty set X, a mapping $q : X \times X \to \mathbb{R}^+$ is said to be a quasi partial metric if the following conditions hold:

$(q1)$ *if* $0 \le q(x,x) = q(x,y) = q(y,y)$, then $x = y$;
$(q2)$ $q(x,x) \le q(x,y)$;
$(q3)$ $q(x,x) \le q(y,x)$;
$(q4)$ $q(x,z) \le q(x,y) + q(y,z) - q(y,y)$

for all $x, y \in X$. Then the pair (X, q) is called a quasi partial metric space (QPMS).

If $q(y,x) = q(x,y)$ for each $x, y \in X$, then (X, q) reduces to partial metric space (PMS). Also, for a quasi-partial metric q on X, the mapping $d_q : X \times X \to \mathbb{R}_+$ defined by

$$d_q(x,y) = q(x,y) + q(y,x) - q(x,x) - q(y,y)$$

is a (usual) metric on X.

Karapinar [8] defined the concept of convergence and completeness in quasi partial metric space in the following way:

Definition 1.5 ([8]). Let (X, q) be a quasi partial metric space (QPMS). Then

1. a sequence $\{x_n\} \subset X$ is called a Cauchy sequence iff $\lim_{m,n\to\infty} q(x_m, x_n)$ and $\lim_{n,m\to\infty} q(x_n, x_m)$ exist and are finite;
2. the quasi partial metric space (X, q) is said to be complete if every Cauchy sequence $\{x_n\} \subset X$ converges to some $x \in X$ such that $q(x,x) = \lim_{m,n\to\infty} q(x_m, x_n) = \lim_{n,m\to\infty} q(x_n, x_m)$.

Definition 1.6 ([8]). Let (X, q) be a quasi partial metric space (QPMS). Then a Cauchy sequence $\{x_n\} \subset X$ converges to $x \in X$ iff $q(x,x) = \lim_{n\to\infty} q(x, x_n) = \lim_{n\to\infty} q(x_n, x)$.

Following are some relevant results in the field of quasi partial metric spaces presented by Karapinar [8]:

Lemma 1.1 ([8]). *Let (X,q) be a QPMS. Let (X,p_q) be the corresponding PMS and let (X,d_{p_q}) be the corresponding metric space. The following statements are equivalent:*

1. *The sequence $\{x_n\}$ is Cauchy in (X,q).*
2. *The sequence $\{x_n\}$ is Cauchy in (X,p_q).*
3. *The sequence $\{x_n\}$ is Cauchy in (X,d_{p_q}).*

Lemma 1.2 ([8]). *Let (X,q) be a QPMS. Let (X,p_q) be the corresponding PMS and let (X,d_{p_q}) be the corresponding metric space. The following statements are equivalent:*

1. *(X,q) is complete.*
2. *(X,p_q) is complete.*
3. *(X,d_{p_q}) is complete.*

Moreover,

$$\begin{aligned}\lim_{n\to\infty} d_{p_q}(x,x_n) &= 0 \Leftrightarrow p_q(x,x) = \lim_{n\to\infty} p_q(x,x_n) = \lim_{n,m\to\infty} p_q(x_n,x_m)\\ &\Leftrightarrow q(x,x) = \lim_{n\to\infty} q(x,x_n) = \lim_{n,m\to\infty} q(x_n,x_m)\\ &= \lim_{n\to\infty} q(x_n,x) = \lim_{m,n\to\infty} q(x_m,x_n).\end{aligned}$$

On the other hand, Wang *et al.* introduced the notion of expansion mappings for a metric space in [20].

Theorem 1.1 ([20]). *Let $T : X \to X$ be an onto mapping defined on a complete metric space (X,d) satisfying the condition*

$$d(Ta,Tb) \geq cd(a,b) \quad \forall a,b \in X.$$

where $c > 1$. Then T has a unique fixed point in X.

Later on, various authors extended this result by using more generalized expansion conditions.

Recently, Dhawan *et al.* [5] defined expansion in quasi partial metric spaces and proved some fixed point theorems stated below:

Lemma 1.3 ([5]). *Let (X,q) be a quasi partial metric space and $\{x_n\}$ be a sequence of points of X. If there exists a number $k \in (0,1)$ such that*

$$q(x_{n+1},x_n) \leq kq(x_n,x_{n-1}); \quad n = 1,2,...$$

Then $\{x_n\}$ is a Cauchy sequence in X.

Theorem 1.2 ([5]). *Let (X,q) be a complete quasi partial metric space and $\mathfrak{D} : X \to X$ be a bijective mapping defined on X. Suppose that there exists $c_1,c_2,c_3 \geq 0$ such that $c_1+c_2+c_3 > 1$ and*

$$q(\mathfrak{D}x,\mathfrak{D}y) \geq c_1q(x,y) + c_2q(x,\mathfrak{D}x) + c_3q(y,\mathfrak{D}y), \quad \forall x,y \in X.$$

Then $\mathfrak{D}$ has a fixed point in X.

Proof. For more details, the reader can refer to [5]. □

Corollary 1.1 ([5]). *Let (X, q) be a complete quasi partial metric space and $\mathfrak{D} : X \to X$ be a bijective mapping. Suppose that there exists a constant $c > 1$ such that*

$$q(\mathfrak{D}x, \mathfrak{D}y) \geq cq(x, y), \quad \forall x, y \in X.$$

Then $\mathfrak{D}$ has a unique fixed point in X.

After that many authors worked on aggregation functions, some of them are [2, 4, 11, 14, 15].

The main focus of this paper is to introduce the notion of expansion mappings in quasi partial metric spaces with the involvement of aggregation functions in such a way that the previous results existing in literature can be retrieved as a particular case of our new ones.

The manuscript is organized as follows: Section 2 presents quasi partial metric aggregation with some properties and conditions required to characterize aggregation functions. In section 3, Projective Ψ-expansion is introduced via these notions and some fixed point results are obtained through it. Section 4 presents some useful examples. In section 5, an application to computer science is presented.

2. Quasi Partial Metric Aggregation

Inspired by the notions of metric aggregation functions due to Massanet and Valero [11] and quasi partial metric due to Karapinar [8], the new notion of quasi partial metric aggregation functions is presented in this section.

Definition 2.1. A function $\Psi : \mathbb{R}_+^n \to \mathbb{R}_+$ is said to be a quasi partial metric aggregation function provided that the function $Q_\Psi : X \times X \to [0, +\infty[$ is a quasi partial metric for every collection of Quasi partial metric spaces $\{(X_i, q_i)\}_{i=1}^n$, where $X = X_1 \times X_2 ... \times X_n$ and

$$Q_\Psi(x, y) = \Psi(q_1(x_1, y_1), ..., q_n(x_n, y_n))$$

for all $x = (x_1, ...x_n) \in X,\ y = (y_1, ...y_n) \in X$.

Following results will help us to characterize quasi partial metric aggregation functions.

Proposition 2.1. *Let $\Psi : \mathbb{R}_+^n \to \mathbb{R}_+$ be a quasi partial metric aggregation function, then Ψ is monotone.*

Proof. Consider the quasi partial metric $q : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ defined as $q(x, y) = max\{(x-y), (y-x)\}+x \ \forall \ x, y \in \mathbb{R}_+$. Since Ψ is a quasi partial metric aggregation function, therefore the function $Q_\Psi : \mathbb{R}_+^n \times \mathbb{R}_+^n \to \mathbb{R}_+$ defined by

$$Q_\Psi(x, y) = \Psi(q(x_1, y_1), ..., q(x_n, y_n))$$

for all $x = (x_1, ...x_n),\ y = (y_1, ...y_n) \in \mathbb{R}_+^n$ is a quasi partial metric.

Consider $x, y \in \mathbb{R}_+^n$ where $x \preceq y$. Then

$$\begin{aligned} \Psi(x) &= \Psi(x_1, ..., x_n) \\ &= \Psi(q(x_1, x_1), ..., q(x_n, x_n)) \\ &= Q_\Psi(x, x) \\ &\leq Q_\Psi(x, y) \\ &= \Psi(q(x_1, y_1), ..., q(x_n, y_n)) \\ &= \Psi(y_1, ..., y_n) \\ &= \Psi(y) \\ \Rightarrow \Psi(x) &\leq \Psi(y). \end{aligned}$$

□

Proposition 2.2. *Let $\Psi : \mathbb{R}_+^n \to \mathbb{R}_+$ be a quasi partial metric aggregation function, If $\Psi(x) = 0$ for some $x \in \mathbb{R}_+^n$, then $x = \bar{0}$ where $\bar{0} = (0, ..., 0) \in \mathbb{R}_+^n$.*

Proof. Let us assume that $\Psi(x) = 0$ for some $x \in \mathbb{R}_+^n$. Since Ψ is a quasi partial metric aggregation function, therefore, by Proposition 2.1, Ψ is monotone and thus $\Psi\left(\frac{x}{3}\right) \leq \Psi(x)$ which implies that $\Psi\left(\frac{x}{3}\right) = 0$.

Consider the quasi partial metric $Q_\Psi : \mathbb{R}_+^n \times \mathbb{R}_+^n \to \mathbb{R}_+$ introduced in Proposition 2.1. Now

$$\begin{aligned} Q_\Psi\left(\frac{x}{3}, x\right) &= \Psi\left(q\left(\frac{x_1}{3}, x_1\right), ..., q\left(\frac{x_n}{3}, x_n\right)\right) \\ &= \Psi(x) = 0, \\ Q_\Psi(x, x) &= \Psi(q(x_1, x_1), ..., q(x_n, x_n)) \\ &= \Psi(x) = 0, \\ Q_\Psi\left(\frac{x}{3}, \frac{x}{3}\right) &= \Psi\left(q\left(\frac{x_1}{3}, \frac{x_1}{3}\right), ..., q\left(\frac{x_n}{3}, \frac{x_n}{3}\right)\right) \\ &= \Psi\left(\frac{x}{3}\right) = 0. \end{aligned}$$

Thus, by definition of quasi partial metric, $\frac{x}{3} = x$ and therefore, $x = \bar{0}$. □

Lemma 2.1. *For every $u, v, w, t \in \mathbb{R}_+$ such that $u \leq w + t - v$ where $v \leq w$ and $v \leq t$, there exists $x, y, z \in \mathbb{R}_+^2$ for which $\tilde{q}(x, y) = w + t - v$, $\tilde{q}(x, z) = w$, $\tilde{q}(z, y) = t$ and $\tilde{q}(z, z) = v$ where $\tilde{q} : \mathbb{R}_+^2 \times \mathbb{R}_+^2 \to \mathbb{R}_+$ is the quasi partial metric defined by $\tilde{q}(x, y) = max\{x_1, y_1\} + max\{x_2, y_2\} + max\{(y_1 - x_1), 0\}$ for every $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{R}_+^2$.*

Proof. It can be easily seen that $\tilde{q}$ is the quasi partial metric defined on $\mathbb{R}_+^2$. Furthermore, the following points of $\mathbb{R}_+^2$ satisfy the required conditions:

$$x = \left(w - \frac{v}{2}, \frac{v}{2}\right), \quad y = \left(\frac{v}{2}, t - \frac{v}{2}\right), \quad z = \left(\frac{v}{2}, \frac{v}{2}\right).$$

□

Lemma 2.2. *For every $u, v, w \in \mathbb{R}_+$ such that $u \geq v$ and $u \geq w$, there exists $x, y \in I(\mathbb{R})$ for which $\hat{q}(x, y) = u$, $\hat{q}(x, x) = v$ and $\hat{q}(y, y) = w$ where $\hat{q} : I(\mathbb{R}) \times I(\mathbb{R}) \to \mathbb{R}_+$ is the quasi partial metric defined as $\hat{q}(x, y) = max\{x_2, y_2\} - min\{x_1, y_1\} + max\{(y_1 - x_1), 0\}$ for every $x = [x_1, x_2]$, $y = [y_1, y_2] \in I(\mathbb{R})$ where $I(\mathbb{R})$ denotes an interval in $\mathbb{R}_+$.*

Proof. It can be easily seen that $\hat{q}$ is the quasi partial metric defined on $I(\mathbb{R})$. Furthermore, the following elements of $I(\mathbb{R})$ fulfill the required conditions:

$$x = [-v, 0], \quad y = [-u, -u + w].$$

□

Theorem 2.1. $\Psi : \mathbb{R}_+^n \to \mathbb{R}_+$ *is a quasi partial metric aggregation function if and only if for each* $x, y, z, w \in \mathbb{R}_+^n$, *we have*

(1) $\Psi(x) + \Psi(y) \leq \Psi(z) + \Psi(w)$ *whenever* $x + y \preceq z + w, \quad y \preceq z, \quad y \preceq w.$

(2) $\Psi(x) = \Psi(y) = \Psi(z) \Rightarrow x = y = z$ *whenever* $y \preceq x, \quad z \preceq x.$

Proof. Suppose that Ψ is a quasi partial metric aggregation function. Let $x, y, z, w \in \mathbb{R}_+^n$ where $x + y \preceq z + w, \quad y \preceq z, \quad y \preceq w$. Then by Lemma 2.1, there exists $\hat{x_i}, \hat{y_i}, \hat{z_i} \in \mathbb{R}_+^2$ such that $\tilde{q}(\hat{x_i}, \hat{y_i}) = z_i + w_i - y_i, \quad \tilde{q}(\hat{x_i}, \hat{z_i}) = z_i, \quad \tilde{q}(\hat{z_i}, \hat{y_i}) = w_i$ and $\tilde{q}(\hat{z_i}, \hat{z_i}) = y_i$ for all $i = 1, ..., n$.

Let $X = \prod_{i=1}^n \mathbb{R}_+^2$ and $\hat{x} = (\hat{x_1}, ..., \hat{x_n}), \quad \hat{y} = (\hat{y_1}, ..., \hat{y_n}), \quad \hat{z} = (\hat{z_1}, ..., \hat{z_n})$. Then $\hat{x}, \hat{y}, \hat{z} \in X$. Consider the quasi partial metric Q_Ψ defined on X by

$$Q_\Psi(\hat{x}, \hat{y}) = \Psi(\tilde{q}(\hat{x_1}, \hat{y_1}), ..., \tilde{q}(\hat{x_n}, \hat{y_n}))$$

for every $\hat{x}, \hat{y} \in X$. Then

$$\begin{aligned} \Psi(z + w - y) = Q_\Psi(\hat{x}, \hat{y}) &\leq Q_\Psi(\hat{x}, \hat{z}) + Q_\Psi(\hat{z}, \hat{y}) - Q_\Psi(\hat{z}, \hat{z}) \\ &= \Psi(z) + \Psi(w) - \Psi(y) \\ \Rightarrow \Psi(z + w - y) &\leq \Psi(z) + \Psi(w) - \Psi(y). \end{aligned}$$

Also, monotonicity of Ψ implies that

$$\begin{aligned} \Psi(x) &\leq \Psi(z + w - y) \leq \Psi(z) + \Psi(w) - \Psi(y) \\ \Rightarrow \Psi(x) &+ \Psi(y) \leq \Psi(z) + \Psi(w). \end{aligned}$$

Thus, condition (1) is proved.

Now, let $x, y, z \in \mathbb{R}_+^n$ where $y \preceq x, \quad z \preceq x$. Then by Lemma 2.2, there exists $\hat{x}, \hat{y} \in I(\mathbb{R})$ such that $\hat{q}(\hat{x_i}, \hat{x_i}) = y_i, \quad \hat{q}(\hat{y_i}, \hat{y_i}) = z_i$, and $\hat{q}(\hat{x_i}, \hat{y_i}) = x_i$ for each $i = 1, ..., n$. Let $X = \prod_{i=1}^n I(\mathbb{R})$ and $\hat{x} = (\hat{x_1}, ..., \hat{x_n}), \hat{y} = (\hat{y_1}, ..., \hat{y_n}), \hat{z} = (\hat{z_1}, ..., \hat{z_n})$. Then $\hat{x}, \hat{y}, \hat{z} \in X$.

Consider the quasi partial metric Q_Ψ on X defined by

$$Q_\Psi(\hat{x}, \hat{y}) = \Psi(\tilde{q}(\hat{x_1}, \hat{y_1}), ..., \tilde{q}(\hat{x_n}, \hat{y_n}))$$

for every $\hat{x}, \hat{y} \in X$.

Let $\Psi(\hat{x}) = \Psi(\hat{y}) = \Psi(\hat{z})$ for $\hat{x}, \hat{y}, \hat{z} \in \mathbb{R}_+^n$. It is easy to see that $Q_\Psi(\hat{x}, \hat{y}) = Q_\Psi(\hat{x}, \hat{x}) = Q_\Psi(\hat{y}, \hat{y})$ and therefore, by definition of quasi partial metric, $\hat{x} = \hat{y}$. By Lemma 2.2, $\hat{x} = [-y, 0]$ and $\hat{y} = [-x, -x + z]$ and Consequently, $x = y = z$. Hence, condition (2) is proved.

Conversely, let us assume that conditions (1) and (2) hold. We will show that Ψ is a quasi partial metric aggregation function. Let $\{(X_i, q_i)\}_{i=1}^n$ be a family of quasi partial metric spaces and $X = \prod_{i=1}^n X_i$. Let $x, y \in X$ and $Q_\Psi(x, y) = Q_\Psi(x, x) = Q_\Psi(y, y)$. Then, by condition (2) and Lemma 2.1, we have $q_i(x_i, y_i) = q_i(x_i, x_i) = q_i(y_i, y_i) \;\; \forall\, i = 1, ..., n$. It follows that $x_i = y_i \;\; \forall\, i = 1, ..., n$ and thus, $x = y$.

Now, set $y = w = \bar{0}$ in condition (1). Since Ψ is monotone and $q_i(x_i, x_i) \leq q_i(x_i, y_i)$, $q_i(x_i, x_i) \leq q_i(y_i, x_i)$ $\forall\, i = 1, ..., n$, we obtain

$$\begin{aligned} Q_\Psi(x, x) &= \Psi(q_1(x_1, x_1), ..., q_n(x_n, x_n)) \\ &\leq \Psi(q_1(x_1, y_1), ..., q_n(x_n, y_n)) \\ &= Q_\Psi(x, y). \end{aligned}$$

Similarly, $Q_\Psi(x, x) \leq Q_\Psi(y, x)$.

Also, by (1), for each $x, y, z \in X$,

$$\begin{aligned} Q_\Psi(x, z) &= \Psi(q_i(x_i, z_i)) \\ &\leq \Psi(q_i(x_i, y_i)) + \Psi(q_i(y_i, z_i)) - \Psi(q_i(y_i, y_i)) \\ &= Q_\Psi(x, y) + Q_\Psi(y, z) - Q_\Psi(y, y). \end{aligned}$$

Thus, all the axioms are satisfied for quasi partial metric Q_Ψ induced through aggregation of quasi partial metrics q_i where $i = 1, ..., n$. □

3. Projective expansion and Quasi Partial Metric Aggregation

This section presents some fixed point theorems in quasi partial metric spaces through aggregation. For this, the notions of expansion and completeness in the aforesaid context are firstly introduced.

Remark 3.1. Let $\{X_i\}_{i=1}^n$ be a collection of nonempty sets and $X = \prod_{i=1}^n X_i$. Let $\mathfrak{D}$ be a self mapping defined on X with coordinate functions $\mathfrak{D}_\mathfrak{i} : X \to X_i,\ i = 1, ...n$ such that

$$\mathfrak{D}(x) = (\mathfrak{D}_\mathfrak{1}(x), ..., \mathfrak{D}_\mathfrak{n}(x)) \ \text{ for all } \ x \in X.$$

Definition 3.1. Let $\{(X_i, q_i)\}_{i=1}^n$ be a family of quasi partial metric spaces and $X = \prod_{i=1}^n X_i$. Let $\Psi : \mathbb{R}_+^n \to \mathbb{R}_+$ be a quasi partial metric aggregation function. Then the mapping $\mathfrak{D} : X \to X$ is called a projective Ψ-expansion from (X, Q_Ψ) into itself, if there exists n constants $\lambda_1, ..., \lambda_n > 1$ such that

$$q_i(\mathfrak{D}_i(x), \mathfrak{D}_i(y)) \geq \lambda_i \Psi(q_1(x_1, y_1), ...q_n(x_n, y_n))$$

for all $x, y \in X$, where Q_Ψ is the quasi partial metric induced by aggregation of the collection of quasi partial metric spaces $\{(X_i, q_i)\}_{i=1}^n$ through aggregation function Ψ.

Note that if we put $n = 1$ and Ψ an identity function in Definition 3.1, then, the notion given by Wang *et al.* becomes a particular case of Ψ-projective expansion (see Theorem 1.1).

Example 3.1. Let $X_i = [0, 1]$; $\ i = 1, 2$ and q be the quasi partial metric defined as $q(x, y) = max\{(x-y), (y-x)\} + x$ for all $(x, y) \in [0, 1] \times [0, 1]$. Let $\{(X_i, q_i)\}_{i=1}^2$ be the complete quasi partial metric spaces where $q_1 = q_2 = q$ and $\Psi : \mathbb{R}_+^2 \to \mathbb{R}_+$ be the function defined as $\Psi((x_1, x_2)) = \frac{x_1+x_2}{2} + \frac{1}{2}$ for all $x \in \mathbb{R}_+^2$.

It can be easily verified that conditions (1) and (2) hold in the statement of Theorem 2.1 and therefore, it is a quasi partial metric aggregation function.

Define the mapping $\mathfrak{D} : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ by $\mathfrak{D}(x) = (2, 2)$ for each $x = (x_1, x_2) \in \mathbb{R}^2_+$. It is not hard to see that $\mathfrak{D}$ is a projective Ψ- expansion as for $x = (1, 0), \ \ y = (0, 0)$, we obtain by definition of q,

$$\begin{aligned} q(\mathfrak{D}_i(x), \mathfrak{D}_i(y)) &= q(2, 2) \\ &= 2 \geq \frac{4}{3}\Psi(q_1(1, 0), q_2(0, 0)) \end{aligned}$$

for all $x, y \in \mathbb{R}^2_+$.

Example 3.2. Let $X_i = \mathbb{R}_+; \ \ i = 1, 2$ and q be the quasi partial metric defined as $q(x, y) = max\{(x - y), (y - x)\} + x$ for all $(x, y) \in \mathbb{R}_+ \times \mathbb{R}_+$. Let $\{(X_i, q_i)\}^2_{i=1}$ be the complete quasi partial metric spaces where $q_1 = q_2 = q$ and $\Psi : \mathbb{R}^2_+ \to \mathbb{R}_+$ be the function defined as $\Psi((x_1, x_2)) = (x_1 + x_2)$ for all $x \in \mathbb{R}^2_+$.

It can be easily verified that conditions (1) and (2) hold in the statement of Theorem 2.1 and therefore, it is a quasi partial metric aggregation function.

Define the mapping $\mathfrak{D} : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ by $\mathfrak{D}(x) = (4(x_1 + x_2), 4(x_1 + x_2))$ for each $x = (x_1, x_2) \in \mathbb{R}^2_+$. It is not hard to see that $\mathfrak{D}$ is a projective Ψ- expansion as

Case I: For $x \succeq y$, we obtain by definition of q,

$$\begin{aligned} q(\mathfrak{D}_i(x), \mathfrak{D}_i(y)) &= q(4(x_1 + x_2), 4(y_1 + y_2)) \\ &= 8(x_1 + x_2) - 4(y_1 + y_2) \\ &\geq 3[2(x_1 + x_2) - (y_1 + y_2)] \\ &= \lambda\Psi(q_1(x_1, y_1)q_2(x_2, y_2)) \end{aligned}$$

for all $x, y \in \mathbb{R}^2_+$ where $\lambda = 3 > 1$.

Case II: For $x \preceq y$, we obtain by definition of q,

$$\begin{aligned} q(\mathfrak{D}_i(x), \mathfrak{D}_i(y)) &= q(4(x_1 + x_2), 4(y_1 + y_2)) \\ &= 4(y_1 + y_2) \\ &\geq 3(y_1 + y_2) \\ &= \lambda\Psi(q_1(x_1, y_1)q_2(x_2, y_2)) \end{aligned}$$

for all $x, y \in \mathbb{R}^2_+$ where $\lambda = 3 > 1$.

The next result will be crucial in order to prove the existence and uniqueness of fixed point in quasi partial metric spaces considered via aggregation.

We will set $1_i = (0, ..., 0, \overbrace{1}^{\text{i}}, 0, ..., 0)$ for all $i = 1, ..., n$.

Lemma 3.1. *Let $\Psi : \mathbb{R}^n_+ \to \mathbb{R}_+$ be a homogeneous quasi partial metric aggregation function such that $\Psi(1, ..., 1) = 1 = \Psi(1_i)$ for all $i = 1, ...n$. Let $\{(X_i, q_i)\}^n_{i=1}$ be a family of complete quasi partial metric spaces and $X = \prod_{i=1}^n X_i$. Then the quasi partial metric space (X, Q_Ψ) is complete, where Q_Ψ is quasi partial metric aggregation induced by Ψ.*

Proof. Let $\{x^p\}_{p\in\mathbb{N}}$ be a Cauchy sequence in (X, Q_Ψ). Then there exists $l \in \mathbb{R}_+$ such that $\lim_{p,r} Q_\Psi(x^p, x^r) = l$ and for given $\varepsilon > 0$, there exists $p_0 \in \mathbb{N}$ such that $Q_\Psi(x^p, x^r) < \varepsilon + l$ for all $p, r \geq p_0$.
This implies that

$$\Psi(q_1(x^p_1, x^r_1), ..., q_1(x^p_n, x^r_n)) < \varepsilon + l.$$

Since Ψ is a quasi partial metric aggregation, therefore Ψ is monotone and thus we have

$$\Psi(q_i(x_i^p, x_i^r).1_i) \leq \Psi(q_i(x_i^p, x_i^r)) < \varepsilon + l \quad \text{for all} \quad i = 1, ...n,$$

and as Ψ is homogeneous, it follows that

$$q_i(x_i^p, x_i^r) = q_i(x_i^p, x_i^r)\Psi(1_i) = \Psi(q_i(x_i^p, x_i^r).1_i) < \varepsilon + l$$

for all $i = 1, ...n$ and for all $p, r \geq p_0$.

This shows that there exists $x_i \in X_i$ such that $\lim_p x_i^p = x_i$ and $\lim_{p,r} q_i(x_i^p, x_i^r) = q_i(x_i, x_i) = \lim_p q_i(x_i, x_i^p) = \lim_p q_i(x_i^p, x_i) = l$ for all $i = 1, ...n$. Also, since (X_i, q_i) is complete quasi partial metric space for all $i = 1, ...n$; therefore for given $\varepsilon > 0$, there exists $m_0 \in \mathbb{N}$ such that $q_i(x_j, x_j^m) - q_i(x_j, x_j) < \frac{\varepsilon}{3}$ for all $m \geq m_0$ and for all $i = 1, ..., n$.

By Theorem 2.1, we obtain

$$\begin{aligned} Q_\Psi(x, x^m) - Q_\Psi(x, x) &= \Psi(q_1(x_1, x_1^m), ..., q_1(x_n, x_n^m)) - \Psi(q_1(x_1, x_1), ..., q_1(x_n, x_n)) \\ &\leq \Psi\left(\frac{\varepsilon}{3}, ..., \frac{\varepsilon}{3}\right) \\ &= \frac{\varepsilon}{3}\Psi(1, ..., 1) < \varepsilon, \end{aligned}$$

as Ψ is homogeneous and $\Psi(1, ...1) = 1$.

Thus, $Q_\Psi(x, x_m) - Q_\Psi(x, x) < \varepsilon$ for all $m \geq m_0$ and $\lim_m Q_\Psi(x, x_m) = Q_\Psi(x, x)$. Similarly, we can show that $\lim_m Q_\Psi(x_m, x) = Q_\Psi(x, x)$.

Also, $Q_\Psi(x, x) = \Psi(q_1(x_1, x_1), ..., q_1(x_n, x_n)) = \Psi(l, ...l) = l\Psi(1, ...1) = l$. Hence the quasi partial metric space (X, Q_Ψ) is complete. □

In the next theorem, we shall show that every projective Ψ-expansion satisfying the condition $\Psi(1, ..., 1) \geq 1$, is an expansion.

Theorem 3.1. *Let $\{(X_i, q_i)\}_{i=1}^n$ be a family of quasi partial metric spaces with $X = \prod_{i=1}^n X_i$. Let Ψ be a homogeneous quasi partial metric aggregation function such that $\Psi(1, ..., 1) \geq 1$ and $\mathfrak{D}$ is a projective Ψ-expansion. Then, $\mathfrak{D}$ is an expansion from the quasi partial metric space (X, Q_Ψ) to itself where Q_Ψ is the quasi partial metric induced by aggregation.*

Proof. It follows from Proposition 2.1 that Ψ is monotone. For $x, y \in X$, monotonicity of Ψ and nature of mapping $\mathfrak{D}$ implies

$$\begin{aligned} Q_\Psi(\mathfrak{D}(x), \mathfrak{D}(y)) &= \Psi(q_1(\mathfrak{D}_1(x), \mathfrak{D}_1(y)), ..., q_n(\mathfrak{D}_n(x), \mathfrak{D}_n(y))) \\ &\geq \Psi(\lambda_1\Psi(q_1(x_1, y_1), ..., q_n(x_n, y_n)), \\ &\quad ...\ \lambda_n\Psi(q_1(x_1, y_1), ..., q_n(x_n, y_n))) \\ &\geq \Psi(\lambda\Psi(q_1(x_1, y_1), ..., q_n(x_n, y_n)), ...\lambda\Psi(q_1(x_1, y_1), ..., q_n(x_n, y_n))) \end{aligned}$$

where $\lambda = min\{\lambda_1, ..., \lambda_n\}$. Homogeneity of Ψ yields

$$\begin{aligned} &\Psi(\lambda\Psi(q_1(x_1, y_1), ..., q_n(x_n, y_n)), ...\lambda\Psi(q_1(x_1, y_1), ..., q_n(x_n, y_n))) \\ &= \lambda\Psi(1, ..., 1)\Psi(q_1(x_1, y_1), ..., q_n(x_n, y_n)). \end{aligned}$$

Thus, above inequality becomes

$$\begin{aligned} Q_\Psi(\mathfrak{D}(x), \mathfrak{D}(y)) &\geq \lambda\Psi(1, ..., 1)Q_\Psi(x, y) \\ &\geq \lambda Q_\Psi(x, y). \end{aligned}$$

Hence, the mapping $\mathfrak{D}$ is an expansion from the quasi partial metric space (X, Q_Ψ) to itself. $\square$

Next we shall show the existence and uniqueness of fixed point.

Corollary 3.1. *Let $\{(X_i, q_i)\}_{i=1}^n$ be a family of quasi partial metric spaces with complete metrics q_i; $i = 1, ..., n$ and $X = \prod_{i=1}^n X_i$. Let Ψ be a homogeneous quasi partial metric aggregation function such that $\Psi(1, ..., 1) = \Psi(1_i) = 1$; $i = 1, ..., n$ and $\mathfrak{D}$ is an onto projective Ψ-expansion. Then $\mathfrak{D}$ has a unique fixed point x^*.*

Proof. By Lemma 3.1, it follows that the quasi partial metric space (X, Q_Ψ) is complete and by Theorem 3.1 shows that $\mathfrak{D}$ is an expansion from the quasi partial metric space (X, Q_Ψ) to itself. By Corollary 1.1, we see that $\mathfrak{D}$ has a unique fixed point x^* in X. $\square$

According to these results, every projective Ψ-expansion is an expansion but does the converse hold? Example 4.1 gives an answer to this query *i.e.* every expansion mapping need not be a Ψ-projective expansion.

4. Examples

Example 4.1. Let $X_i = \mathbb{R}_+$; $i = 1, 2$ and q be the quasi partial metric defined as $q(x, y) = max\{(x - y), (y - x)\} + x$ for all $(x, y) \in \mathbb{R}_+ \times \mathbb{R}_+$. Let $\{(X_i, q_i)\}_{i=1}^2$ be the collection of complete quasi partial metric spaces where $q_1 = q_2 = q$ and $\Psi : \mathbb{R}_+^2 \to \mathbb{R}_+$ be the function defined as $\Psi((x_1, x_2)) = (x_1 + x_2)$ for all $x \in \mathbb{R}_+^2$.

It can be easily verified that conditions (1) and (2) hold in the statement of Theorem 2.1 and therefore, it is a quasi partial metric aggregation function. Moreover, $\Psi(1, 1) = 2 \geq 1$. Define the mapping $\mathfrak{D} : \mathbb{R}_+^2 \to \mathbb{R}_+^2$ by $\mathfrak{D}(x) = (2(x_1 + x_2), 2(x_1 + x_2))$ for each $x = (x_1, x_2) \in \mathbb{R}_+^2$. It is not hard to see that $\mathfrak{D}$ is an expansion as

$$\begin{aligned} Q_\Psi(\mathfrak{D}(x), \mathfrak{D}(y)) &= Q_\Psi((2(x_1 + x_2), 2(x_1 + x_2)), (2(y_1 + y_2), 2(y_1 + y_2))) \\ &= \Psi(q_1(2(x_1 + x_2), 2(y_1 + y_2)), q_2(2(x_1 + x_2), 2(y_1 + y_2))). \end{aligned}$$

Case I: For $x \succeq y$, we obtain by definition of q,

$$\begin{aligned} Q_\Psi(\mathfrak{D}(x), \mathfrak{D}(y)) &= 8(x_1 + x_2) - 4(y_1 + y_2) \\ &\geq 2[2(x_1 + x_2) - (y_1 + y_2)] \\ &= 2\Psi(2x_1 - y_1, 2x_2 - y_2) \\ &= 2\Psi(q_1(x_1, y_1), q_2(x_2, y_2)) \\ \Rightarrow Q_\Psi(\mathfrak{D}(x), \mathfrak{D}(y)) &\geq \lambda Q_\Psi(x, y) \end{aligned}$$

for all $x, y \in \mathbb{R}_+^2$ where $\lambda = 2 > 1$.

Case II: For $x \preceq y$, we obtain by definition of q,

$$\begin{aligned} Q_\Psi(\mathfrak{D}(x), \mathfrak{D}(y)) &= 4(y_1 + y_2) \\ &\geq 2(y_1 + y_2) \\ &= 2\Psi(q_1(x_1, y_1), q_2(x_2, y_2)) \\ \Rightarrow Q_\Psi(\mathfrak{D}(x), \mathfrak{D}(y)) &\geq \lambda Q_\Psi(x, y) \end{aligned}$$

for all $x, y \in \mathbb{R}_+^2$ where $\lambda = 2 > 1$.

It follows that $\mathfrak{D}$ is an expansion from quasi partial metric space $(\mathbb{R}_+^2, Q_\Psi)$ into itself. Next we show that $\mathfrak{D}$ is not a projective Ψ-expansion.

Consider $x, y \in \mathbb{R}_+^2$ with $x = (0,1)$ and $y = (1,0)$. Then $q(\mathfrak{D}_i(x), \mathfrak{D}_i(y)) = q(2,2) = max\{0,0\} + 2 = 2$ and $\lambda\Psi(q_1(x_1,y_1), q_2(x_2,y_2)) = \lambda\Psi(q_1(0,1), q_2(1,0)) = \lambda\Psi(1,2) = 3\lambda$.

Thus, $q(\mathfrak{D}_i(x), \mathfrak{D}_i(y)) \ngeq \lambda\Psi(q_1(x_1,y_1), q_2(x_2,y_2))$ as $\lambda > 1$.

The next example shows that the assumption 'Ψ is homogeneous' cannot be omitted in the statement of Theorem 3.1.

Example 4.2. Let $([0,1], q)$ be the complete quasi partial metric space such that q denotes the restriction of the quasi partial metric introduced in Proposition 2.1 to $[0,1]$. Consider the family of complete quasi partial metric spaces $\{([0,1], q_i)\}_{i=1,2}$ such that $q_1 = q_2 = q$. Define the function $\Psi : \mathbb{R}_+^2 \to \mathbb{R}_+$ by $\Psi(x) = x_1 + x_2 + \frac{1}{4}$ for all $x \in \mathbb{R}_+^2$. It is easy to see that for the function Ψ assertions (1) and (2) hold in the statement of Theorem 2.1 and, thus, it is a quasi partial metric aggregation function. Moreover, it is clear that $\Psi(1,1) \geq 1$. However, Ψ is not homogeneous. Indeed,

$$\Psi(2,2) = \frac{17}{4} \neq \frac{18}{4} = 2\Psi(1,1).$$

Next, consider the mapping $\mathfrak{D} : [0,1]^2 \to [0,1]^2$ defined by $\mathfrak{D}(x) = (0,0)$ for all $x \in [0,1]^2$. It is clear that $\mathfrak{D}$ is a projective Ψ-expansion. Nevertheless, $\mathfrak{D}$ is not an expansion from $([0,1]^2, Q_\Psi)$ into itself, where Q_Ψ is the quasi partial metric induced by aggregation of the family of quasi partial metric spaces $\{([0,1], q_i)\}_{i=1,2}$., through Ψ. Indeed,

$$Q_\Psi(\mathfrak{D}(0,0), \mathfrak{D}(0,0)) = Q_\Psi((0,0),(0,0)) = \Psi(0,0) = \frac{1}{4}.$$

Therefore, there does not exist $\lambda > 1$ such that

$$Q_\Psi(\mathfrak{D}(0,0), \mathfrak{D}(0,0)) \geq \lambda Q_\Psi((0,0),(0,0)).$$

In the next example, we show that the assumption $\Psi(1,...,1) \geq 1$ cannot be omitted in the statement of Theorem 3.1 in order to guarantee that a projective Ψ-expansion is also an expansion from (X, Q_Ψ) into itself.

Example 4.3. Let $\{([0,1], q_i)\}_i = 1,2.$ be the family of complete quasi partial metric spaces such that $q_1 = q_2 = q$. Define the function $\Psi : \mathbb{R}_+^2 \to \mathbb{R}_+$ by $\Psi(x) = \frac{x_1+x_2}{3}$ for all $x \in \mathbb{R}_+^2$. It is easy to see that Ψ is a homogeneous quasi partial metric aggregation function. Nevertheless, $\Psi(1,1) = \frac{2}{3} < 1$.

Next, consider the mapping $\mathfrak{D} : [0,1]^2 \to [0,1]^2$ defined by $\mathfrak{D}(x) = (2(x_1 + x_2), 2(x_1 + x_2))$ for all $x \in [0,1]^2$. Then we have for $x \succeq y$,

$$\begin{aligned} q(\mathfrak{D}_i(x), \mathfrak{D}_i(y)) &= q(2(x_1+x_2), 2(y_1+y_2)) \\ &= 4(x_1+x_2) - 2(y_1+y_2) \\ &\geq 3\left(\frac{2}{3}(x_1+x_2) - \frac{1}{3}(y_1+y_2)\right) \\ &= \lambda\Psi(q(x_1,y_1), q(x_2,y_2)) \text{ with } \lambda = 3 > 1 \end{aligned}$$

for all $x, y \in [0,1]^2$ and for $i = 1,2$. Similar is the case for $x \preceq y$. So, $\mathfrak{D}$ is a projective Ψ-expansion. However, $\mathfrak{D}$ is not an expansion from the quasi partial metric space $([0,1]^2, Q_\Psi)$ into itself where Q_Ψ is the quasi partial metric induced by aggregation of the family of quasi partial metric spaces $\{([0,1], q_i)\}_{i=1,2}$., through Ψ. Indeed, take $x, y \in [0,1]^2$ given by $x = (0,0)$ and $y = (0,1)$. Then there does not exist $\lambda > 1$ such that

$$Q_\Psi(\mathfrak{D}(0,0), \mathfrak{D}(0,1)) \geq \lambda Q_\Psi((0,0),(2,2))$$

Since

$$Q_\Psi(\mathfrak{D}(0,0), \mathfrak{D}(0,1)) = Q_\Psi((0,0),(2,2)) = \Psi(2,2) = \frac{4}{3}.$$

5. Application

In the field of computer science, the objective of complexity analysis is to assess which of the algorithm is most suitable or in other words, the algorithm which takes minimum running time with minimum space even with large inputs and other suitable resources. This is usually done by means of asymptotic analysis where the running time of an algorithm A is denoted by a mapping $T_A : \mathbb{N} \to (0, \infty)$. The time or space taken by an algorithm to solve the problem under consideration is denoted by $T_A(n)$ where $n \in \mathbb{N}$ represents the size of input data to be processed. Let $S(T_A)$ denotes the set of all functions from $\mathbb{N}$ to $(0, \infty)$.

When the complexity analysis of an algorithm has to be determined, one approaches to asymptotic complexity analysis rather than exact analysis. So, they try to find such an algorithm that takes "approximately" minimum running time, minimum space even with large inputs and other suitable resources.

Let $f \in S(T_A)$ denote the running time or space taken by an algorithm. Then, we can define an asymptotic upper bound for f in the following way:

If there exists $n_0 \in \mathbb{N}$, $k \in \mathbb{R}^+$ and a function $g \in S(T_A)$ such that $f(n) \leq kg(n)$ for all $n \in \mathbb{N}$ such that $n_0 \leq n$. Then, g gives an asymptotic upper bound of f, and represents an "approximate" information of the algorithm. We write it as $f \in \mathcal{U}(g)$. Similarly, we can also define an asymptotic lower bound for the algorithm. The notation $f \in \mathcal{L}(g)$ means that there exists $n_0 \in \mathbb{N}$, $k \in \mathbb{R}^+$ and a function $g \in S(T_A)$ such that $kg(n) \leq f(n)$ for all $n \in \mathbb{N}$ such that $n_0 \leq n$. The best situation is the case when we can find such a function f which satisfy the condition $f \in \mho(g)$ where $\mho(g) = \mathcal{U}(g) \cap \mathcal{L}(g)$. In this case, the function f represents a 'tight' asymptotic bound of algorithm *i.e.* it represents the total asymptotic information about the most suitable resources to solve the problem under consideration.

Let the pair $(\mathcal{C}, d^c)$ represents the complexity space, where

$$\mathcal{C} = \left\{ f \in S(T_A) : \sum_{n=1}^{\infty} 2^{-n} \frac{1}{f(n)} < \infty \right\}$$

and d^c is the complete quasi partial metric on $\mathcal{C}$ defined by

$$d^c(f,g) = \sum_{n=1}^{\infty} 2^{-n} max \left\{ \frac{1}{f(n)} - \frac{1}{g(n)}, \frac{1}{g(n)} - \frac{1}{f(n)} \right\} + \frac{1}{f(n)}.$$

The members of $\mathcal{C}$ are called complexity functions and $d^c(f,g)$ represents the complexity distance from f to g. Then $d^c(f,g) = 0$ means 'f is as efficient as g'.

We will solve the problem by using Divide and Conquer method given in [16]. In this procedure, we will split the problem into subproblems (depending upon different resources) and solving them separately using same algorithm to find the suitable solution. After obtaining the solutions of the subproblems, we will aggregate all subproblems to obtain a global solution to the original problem which will represent an algorithm with all approximately suitable resources.

The next result explores the significance of above theory.

Proposition 5.1. *Let* $\mathfrak{D}$ *be an onto self mapping defined on* $\mathcal{C}$ *with coordinate functions* $\mathfrak{D}_{\mathfrak{i}} : \mathcal{C} \to \mathcal{C}_i,\ i = 1, ...n$ *such that*

$$\mathfrak{D}(f)(n) = (\mathfrak{D}_{\mathfrak{1}}(f)(n), ..., \mathfrak{D}_{\mathfrak{n}}(f)(n)) \quad for \quad each \quad f \in \mathcal{C} \quad and\ n \in \mathbb{N}.$$

satisfying the expansion inequality

$$d_i^c(\mathfrak{D}_i(f)(n), \mathfrak{D}_i(g)(n)) \geq \lambda_i \Psi(d_1^c(f_1, g_1), ...d_n^c(f_n, g_n))$$

for all $f_i, g_i \in \mathcal{C}$*,* $i = 1, 2, ..., n$ *and* $\lambda_1, ..., \lambda_n > 1$*. Then* $\mathfrak{D} \in \mho(g)$*.*

Proof. We construct the members $\mathcal{C}_i, \quad i = 1, 2, ...n;\ n \in \mathbb{N}$ of complexity class $\mathcal{C}$ in such a way that they will be based on different resources such as time, space, data etc. and $\mathcal{C} = \prod_{i=1}^{n} \mathcal{C}_i$. It is easy to see that $(\mathcal{C}_i, d_i^c)$ is a collection of complete quasi partial metric spaces. Let $\Psi : \mathbb{R}_+^n \to \mathbb{R}_+$ be the function used to aggregate these members and it is defined in such a way that $\Psi(1, 1, ..., 1) \geq 1$. Let $\mathfrak{D}$ be an onto self mapping defined on $\mathcal{C}$ with coordinate functions $\mathfrak{D}_{\mathfrak{i}} : \mathcal{C} \to \mathcal{C}_i,\ i = 1, ...n$ such that

$$\mathfrak{D}(f)(n) = (\mathfrak{D}_{\mathfrak{1}}(f)(n), ..., \mathfrak{D}_{\mathfrak{n}}(f)(n)) \quad \text{for} \quad \text{each} \quad f \in \mathcal{C} \quad \text{and } n \in \mathbb{N}.$$

satisfying the expansion inequality

$$d_i^c(\mathfrak{D}_i(f)(n), \mathfrak{D}_i(g)(n)) \geq \lambda_i \Psi(d_1^c(f_1, g_1), ...d_n^c(f_n, g_n))$$

for all $f_i, g_i \in \mathcal{C}$, $i = 1, 2, ..., n$ and $\lambda_1, ..., \lambda_n > 1$.

Thus, all the conditions of Corollary 3.1 are fulfilled and therefore, $\mathfrak{D}$ has a fixed point f^* *i.e.* $\mathfrak{D} \in \mathcal{L}(g) \cap \mathcal{U}(g)$. It follows that $\mathfrak{D} \in \mho(g)$. □

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References

[1] M. A. Alghamdi, N. Shahzad and O. Valero, *Fixed point theorems in generalized metric spaces with applications to computer science*, Fixed Point Theory and Applications, 2013, 118, 1–20.

[2] M. A. Alghamdi, N. Shahzad and O. Valero, *Projective contractions, generalized metrics, and fixed points*, Fixed Point Theory and Applications, 2015, 181, 1–13.

[3] J. Borsik and J. Doboš, *On a product of metric spaces*, Mathemaica Slovaca, 1981, 31, 193–205.

[4] M. J. Campión, R.G. Catalán, E. Induráin, I. Lizasoain, A. R. Pujol and O.Valero, *Geometrical aggregation of finite fuzzy sets*, International Journal of Approximate Reasoning, 2018, 103, 248–266.

[5] P. Dhawan and J. Kaur, *Fixed Points of Expansive Mappings in Quasi Partial Metric Spaces*, International Journal of Advanced Research in Science and Engineering, 2016, 5, 145–151.

[6] V. Gupta, R. Deep and A. K. Tripathi, *Existence of coincidence point for weakly increasing mappings satisfies (ψ, ϕ)-weakly contractive condition in partially ordered metric spaces*, International Journal of Computing Science and Mathematics, 2016, 7(6), 495–508.

[7] R. Heckmann, *Approximation of metric spaces by partial metric spaces*, Applied Categorical Structures, 1999, 7, 71–83.

[8] E. Karapinar, *Fixed point theorems on quasi-partial metric spaces*, Mathematical and Computer Modelling, 2013, 57, 2442–2448.

[9] W. Kirk and N. Shahzad, *Fixed point theory in distance spaces*, Cham: Springer, 2014.

[10] J. Martin, G. Mayor and O. Valero, *Aggregation of asymmetric distances in Computer Science*, Information Sciences, 2010, 180, 803–812.

[11] S. Massanet and O. Valero, *New results on metric aggregation*, Proceedings of the 17th Spanish Conference on Fuzzy Technology and Fuzzy Logic(Estylf2012), 2012, 558–563.

[12] S. G. Matthews, *Partial metric topology*, Annals of the New York Academy of Sciences, 1994, 728, 183–197.

[13] S. G. Matthews, *An extension treatment of lazy data flow deadlock*, Theoretical Computer Science, 1995, 151, 195–205.

[14] G. Mayor and O. Valero, *Metric aggregation functions revisited*, European Journal of Combinatorics, 2019, 80, 390–400.

[15] J. J. Minana, *An overview on transformations on generalized metrics*, Workshop on Applied Topological Structures, Valencia, 2017, 95–102.

[16] S. Romaguera, P. Tirado and O. Valero, *Complete partial metric spaces have partially metrizable computational models*, Internation Journal of Computer Mathematics, 2012, 89, 284–290.

[17] S. Romaguera and O. Valero, *A quantitative computational model for complete partial metric spaces via formal balls*, Mathematical Structures in Computer Science, 2009, 19, 541–563.

[18] W. Shatanawi and K. Abodayeh, *Common Fixed Point under Nonlinear Contractions on Quasi Metric Spaces*, Mathematics, 2019, 7, 453.

[19] W. Shatanawi, M. Salmi, M. D. Noorani, H. Alsamir and A. Bataihah, *Fixed and common fixed point theorems in partially ordered quasi-metric spaces*, Journal of Mathematics and Computer Science, 2016, 16, 516–528.

[20] S. Wang, B. Li, Z. Gao and K. Iseki, *Some fixed point theorems on expansion mappings*, Mathematica Japonica, 1984, 29, 631–636.