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BIFURCATION ANALYSIS AND EXACT TRAVELING WAVE SOLUTIONS FOR A GENERIC TWO-DIMENSIONAL SINE-GORDON EQUATION IN NONLINEAR OPTICS*

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Abstract We focus on investigating a generic two-dimensional sine-Gordon equation in nonlinear optics. Based on a viable transformation, the bifurcation analysis of the equation is carried out in this paper. The phase portraits are given and different kinds of traveling wave solutions are obtained. The analytical results are also numerically simulated.

Keywords Two-dimensional sine-Gordon equation, bifurcation analysis, exact traveling wave solution, numerical simulation.

MSC(2010) 34C25, 34F10, 35C07, 35C08.

1. Introduction

The sine-Gordon equation, double sine-Gordon equation, (n+1)-dimensional sine-Gordon equation, some generalized sine-Gordon and fractional sine-Gordon equations are widely applied in physics and engineering. For examples, in a resonant five-fold degenerate medium, the propagation and creation of ultra-short optical pulses, the sine-Gordon and double sine-Gordon equations are usually used [1, 10, 19], the (n+1)-dimensional sine-Gordon equation is a model of fluxon dynamics in Josephson junctions, dislocation dynamics in crystal lattices, vortex states in spin systems with an anisotropy created by an external magnetic field [34, 35], some generalized sine-Gordon equations can also be used to model the propagation of magnetic flues in Josephson junctions, motion of dislocations in crystals, Bloch wall transmission of ferromagnetic waves [8, 20, 21, 24, 29, 32], and so on. Many analytic methods which include the dressing method [8], the bilinear method [24], the meshless finite point method [21], the approach of dynamical system [15, 16, 25, 28], the Painlevé analysis [30], the double elliptic equation method [27], the binary F-expansion method [31], the symbolic computation method [14] and the numerical simulation methods [3–7, 17, 18, 33] have been employed to investigate those equations. Meanwhile, different localized coherent structures of those equations, such

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as the solitary waves, breathers, kink and anti-kink waves, double kink and anti-kink waves, periodic waves, double periodic waves, peakons, ring solitons and kink solitons have been presented [3, 5–8, 14, 17, 18, 20, 21, 24, 27–33].

In 2010, Leblond and Mihalache [23] proposed a generic two-dimensional sine-Gordon equation in nonlinear optics

$$\begin{cases} V_{zt} = UV + V_{yy}, \\ U_t = -VW, \\ W_t = UV, \end{cases} \tag{1.1}$$

where $U = \rho_0/\rho_r, V = E_0/E_r$, and W is an inductive quantity which has no physical sense. The ρ_0 is the population and the E_0 is external electric field. The independent variable z is the propagation distance, y is the width of laser, and t is the time. Equation (1.1) can be used to describe the propagation of femtosecond spatiotemporal solitary waves in a system of two-level atoms, with the assumption that the characteristic optical frequency is much larger than the transition frequency of the two-level atoms [23]. Numerical simulation of (1.1) indicates that the few cycle pulses oscillating in both space and time may form from a transversely perturbed input plane wave, and reach stable state after a transitory stage in which the pulse radiates energy [23]. Besides, the Painlevé integrability of (1.1) has been investigated. Subsequently, kink-periodic, kink-soliton and kink-kink interactions have been found during the propagation of laser pulse by employing consistent Tanh expansion method [9]. Moreover, based on the bilinear method and truncated Painlevé expansion, multisoliton and quasi-periodic peakon solutions for (1.1) are derived in [26].

There are some interesting problems: How do the bifurcations of equation (1.1)? How do the traveling wave solutions of equation (1.1) depend on the parameters of the system? To our knowledge, these problems have not been considered in published literatures. In addition, are there other types of traveling wave solutions besides the solutions obtained in [9, 23, 26]? In this paper, we will consider the bifurcations of equation (1.1) in different regions of the parametric space. We will also give all possible exact parametric representations for traveling waves of equation (1.1). The results of this paper more completely answer the above problems and enrich the results of [9, 23, 26].

2. Preliminaries

Using a viable transformation [26]

$$\begin{cases} U = \alpha \mathrm{e}^{\phi} + \beta \mathrm{e}^{-\phi}, \\ V = i\phi_t, \\ W = i\left(\alpha \mathrm{e}^{\phi} - \beta \mathrm{e}^{-\phi}\right), \end{cases} \tag{2.1}$$

where $i^2 = -1, \phi$ is a analytical function of variables y, z and t, α, β are two real parameters and $(\alpha, \beta) \neq (0, 0)$, equation (1.1) can be transformed into the following equation:

$$\phi_{ztt} = \phi_t \left(\alpha \mathrm{e}^{\phi} + \beta \mathrm{e}^{-\phi}\right) + \phi_{yyt}. \tag{2.2}$$

Integrating (2.2) once with respect to t, we have

$$\alpha e^{\phi} - \beta e^{-\phi} + \phi_{yy} - \phi_{zt} + g = 0, \tag{2.3}$$

where g is the integral constant.

Letting

$$e^{\phi} = u(\xi) \equiv u, \ \xi = ay + bz + ct, \ a^2 \neq bc, \tag{2.4}$$

one gets

$$e^{-\phi} = \frac{1}{u}, \ \phi = \ln(u), \ \phi_{yy} = \frac{a^2\left(uu'' - (u')^2\right)}{u^2}, \ \phi_{zt} = \frac{bc\left(uu'' - (u')^2\right)}{u^2}, \tag{2.5}$$

where $u' = \frac{du}{d\xi}$, $u'' = \frac{d^2u}{d\xi^2}$, a, b and c are three undetermined constants.

Substituting (2.5) into (2.3), it yields equation

$$u'' = \frac{(\alpha u^2 + gu - \beta)u + (bc - a^2)(u')^2}{(bc - a^2)u}, \tag{2.6}$$

or equivalent to the following two-dimensional dynamic system:

$$\frac{du}{d\xi} = y, \quad \frac{dy}{d\xi} = \frac{(\alpha u^2 + gu - \beta)u + (bc - a^2)y^2}{(bc - a^2)u} \tag{2.7}$$

with the first integral

$$H(u, y) = \frac{y^2}{u^2} + \frac{2\left(\alpha u^2 + gu\ln(u) + \beta\right)}{(a^2 - bc)u}. \tag{2.8}$$

Remark 2.1. By using the He's semi-inverse method [11,12], we obtain the variational principle of system (2.7) as following:

$$J(u) = \int \left(\frac{1}{2}(bc - a^2)u(u')^2 + \frac{1}{4}\alpha u^4 + \frac{1}{3}gu^3 - \frac{1}{2}\beta u^2\right) d\xi.$$

Its Hamiltonian, therefore, can be written in the form

$$H = K + E + P = \frac{1}{2}(a^2 - bc)u(u')^2 + \frac{1}{4}\alpha u^4 + \frac{1}{3}gu^3 - \frac{1}{2}\beta u^2,$$

where $K = 0$ is the kinetic energy, $E = \frac{1}{2}(a^2 - bc)u(u')^2$ is the external energy and $P = \frac{1}{4}\alpha u^4 + \frac{1}{3}gu^3 - \frac{1}{2}\beta u^2$ is the potential energy. If $E = 0$, then according to He's energy balance method for nonlinear oscillations [13], we know that (2.7) should be stable and regular system. However the $E = \frac{1}{2}(a^2 - bc)u(u')^2 \neq 0$, so that the (2.7) is a unstable and singular system.

From [15, 16, 22, 25, 28, 36], we known that a solitary wave solution of equation (2.6) corresponds to a homoclinic orbit of system (2.7), a blow-up wave solution of equation (2.6) corresponds to a open curve of system (2.7), a periodic orbit of system (2.7) corresponds to a smooth periodic wave solution of equation (2.6) and a periodic blow-up wave solution of equation (2.6) corresponds to a open curve of system (2.7). Thus, to investigate all possible solitary wave, blow-up wave and periodic wave solutions of equation (2.6), we need to find all homoclinic orbit, open curve and periodic orbit of system (2.7) which depend on the system parameters $a, b, c, \alpha, \beta, g$.

3. Bifurcations and phase portraits of system (2.7)

In this section, we only investigate the bifurcations and phase portraits of system (2.7) when $a^2 - bc < 0$ for convenience. Another case can be considered similarly, we omit it here.

Using transformation $d\xi = (bc - a^2)ud\tau$, system (2.7) be carried into

$$\frac{du}{d\tau} = (bc - a^2)uy, \quad \frac{dy}{d\tau} = (\alpha u^2 + gu - \beta)u + (bc - a^2)y^2. \tag{3.1}$$

Clearly, system (3.1) is a Hamiltonian system with the Hamiltonian function as (2.8).

Obviously, system (3.1) has three equilibrium points at $(0,0), (u_1, 0), (u_2, 0)$ in u-axis when $\alpha \neq 0, \Delta > 0$, has two equilibrium points at $(0,0), (u_*, 0)$ in u-axis when $\alpha \neq 0, \Delta = 0$, has two equilibrium points at $(0,0), (u_\star, 0)$ in u-axis when $\alpha = 0, g \neq 0$, has only one equilibrium point at $(0,0)$ when $\alpha \neq 0, \Delta < 0$ or $\alpha = 0, g = 0$, where $u_{1,2} = \frac{-g \pm \sqrt{\Delta}}{2\alpha}, u_* = \frac{-g}{2\alpha}, u_\star = \frac{\beta}{g}, \Delta = g^2 + 4\alpha\beta$.

Let us set the Hamiltonian value h as $H(u, y) = h$, and define that $H(u_1, 0) = h_1, H(u_*, 0) = h_*, H(u_2, 0) = h_2, H(u_\star, 0) = h_\star$, then we have

$$h_1 = \frac{\left(g - \sqrt{\Delta}\right)^2 - 2g\left(g - \sqrt{\Delta}\right)\ln\left(\frac{-g+\sqrt{\Delta}}{2\alpha}\right) + 4\alpha\beta}{(bc - a^2)\left(g - \sqrt{\Delta}\right)}, \quad h_* = \frac{2g\ln\left(\frac{-g}{2\alpha}\right)}{a^2 - bc},$$

$$h_2 = \frac{\left(g + \sqrt{\Delta}\right)^2 - 2g\left(g + \sqrt{\Delta}\right)\ln\left(\frac{-g-\sqrt{\Delta}}{2\alpha}\right) + 4\alpha\beta}{(bc - a^2)\left(g + \sqrt{\Delta}\right)}, \quad h_\star = \frac{2g\left(1 + \ln\left(\frac{\beta}{g}\right)\right)}{a^2 - bc}.$$

The determinant of the Jacobian matrix of the linearized system of the system (3.1) is

$$\begin{aligned} J(u, y) &= \begin{vmatrix} (bc - a^2)y & (bc - a^2)u \\ 3\alpha u^2 + 2gu - \beta & 2(bc - a^2)y \end{vmatrix} \\ &= (bc - a^2)\left(2(bc - a^2)y^2 - (3\alpha u^2 + 2gu - \beta)u\right). \end{aligned}$$

At the equilibrium points $(0,0), (u_*, 0), (u_\star, 0), (u_1, 0)$ and $(u_2, 0)$, the values of the determinant are, respectively,

$$J(0,0) = J(u_*, 0) = 0, J(u_\star, 0) = \frac{(a^2 - bc)\beta^2}{g},$$

$$J(u_1, 0) = \frac{(bc - a^2)\left(-g + \sqrt{\Delta}\right)\left(g\left(-g + \sqrt{\Delta}\right) - 4\alpha\beta\right)}{4\alpha^2},$$

$$J(u_2, 0) = \frac{(bc - a^2)\left(g + \sqrt{\Delta}\right)\left(g\left(g + \sqrt{\Delta}\right) + 4\alpha\beta\right)}{4\alpha^2}.$$

Based on the above analysis and using the approach of dynamical systems [15, 16, 22, 25, 28, 36], the bifurcations and phase portraits of system (2.7) are presented in Figures (1)-(3) for the case of $a^2 - bc < 0$.

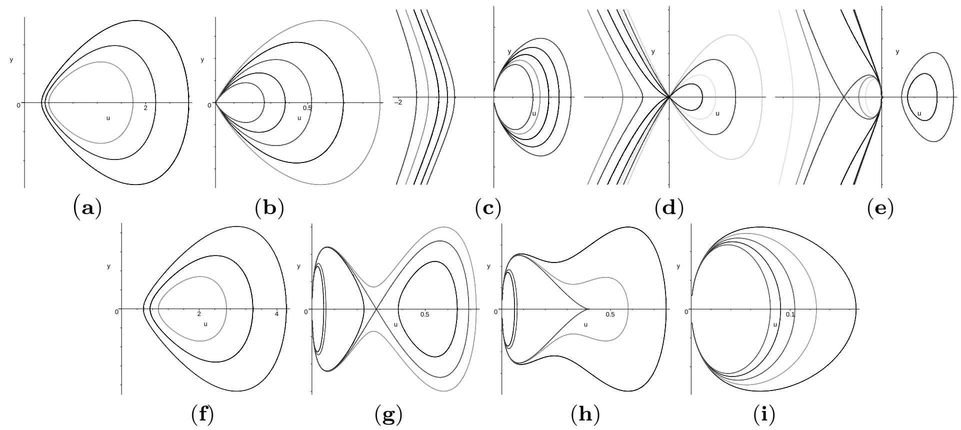

Figure 1. Bifurcations and phase portraits of system (2.7) when $a^2 - bc < 0, \alpha < 0$. Parameters: (a) $g < 0, \beta < 0$. (b) $g < 0, \beta \geq 0$. (c) $g = 0, \beta > 0$. (d) $g = 0, \beta = 0$. (e) $g = 0, \beta < 0$. (f) $g > 0, \beta \leq 0$. (g) $g > 0, 0 < \beta < -\frac{1}{4\alpha}g^2$. (h) $g > 0, \beta = -\frac{1}{4\alpha}g^2$. (i) $g > 0, \beta > -\frac{1}{4\alpha}g^2$.

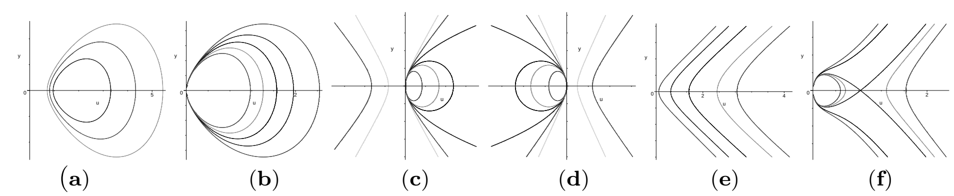

Figure 2. Bifurcations and phase portraits of system (2.7) when $a^2 - bc < 0, \alpha = 0$. Parameters: (a) $g < 0, \beta < 0$. (b) $g < 0, \beta > 0$. (c) $g = 0, \beta > 0$. (d) $g = 0, \beta < 0$. (e) $g > 0, \beta < 0$. (f) $g > 0, \beta > 0$.

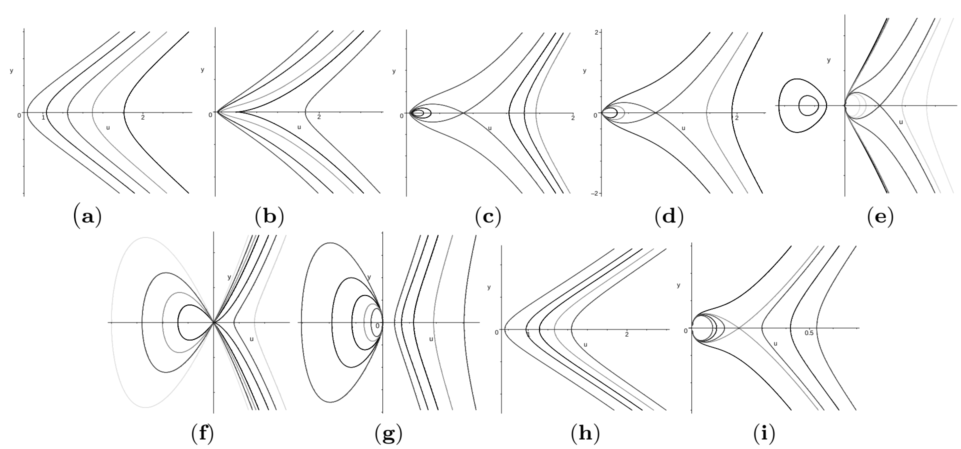

Figure 3. Bifurcations and phase portraits of system (2.7) when $a^2 - bc < 0, \alpha > 0$. Parameters: (a) $g < 0, \beta < -\frac{1}{4\alpha}g^2$. (b) $g < 0, \beta = -\frac{1}{4\alpha}g^2$. (c) $g < 0, -\frac{1}{4\alpha}g^2 < \beta < 0$. (d) $g < 0, \beta \geq 0$. (e) $g = 0, \beta > 0$. (f) $g = 0, \beta = 0$. (g) $g = 0, \beta < 0$. (h) $g > 0, \beta \leq 0$. (i) $g > 0, \beta > 0$.

4. Main results and their proofs

Remark 4.1. Clearly, (2.8) can be rewritten as

$$\frac{du}{\sqrt{hu^2-\frac{2(\alpha u^2+gu\ln(u)+\beta)u}{a^2-bc}}}=\pm d\xi. \tag{4.1}$$

From Figures 1-3, we get some results as follows. Equation (2.6) has a family of smooth periodic wave solutions when $a^2-bc<0, \alpha<0, g<0$, has two solitary wave and three family of smooth periodic wave solutions when $a^2-bc<0, \alpha<0, g>0, 0<\beta<-\frac{1}{4\alpha}g^2$, has one solitary wave and two family of smooth periodic wave solutions when $a^2-bc<0, \alpha<0, g>0, \beta=-\frac{1}{4\alpha}g^2$, has one solitary wave, one blow-up wave, a family of smooth periodic wave and a family of periodic blow-up wave solutions when $a^2-bc<0, \alpha=0, g>0, \beta>0$ (or $a^2-bc<0, \alpha>0, g<0, \beta>-\frac{1}{4\alpha}g^2$, and or $a^2-bc<0, \alpha>0, g>0, \beta>0$), etc. But we can not present the exact parametric representations of above nonlinear waves because we can not solve the equation (4.1) when $g\neq 0$.

In this section, we give all possible exact traveling wave solutions of equation (2.6) when $g=0$. Main results and their proofs as follows.

4.1. Main results

Theorem 4.1. *If $a^2-bc<0, \alpha\neq 0, g=0, \beta=0$, then when $h>0$, equation (2.6) has infinite many solitary wave solutions*

$$u=u_M sech^2(\omega_1\xi), \tag{4.2}$$

and infinite many blow-up wave solutions

$$u=-u_M csch^2(\omega_1\xi), \tag{4.3}$$

when $h=0$, equation (2.6) has a blow-up wave solution

$$u=-\frac{2(a^2-bc)}{\alpha\xi^2}, \tag{4.4}$$

when $h<0$, equation (2.6) has infinite many periodic blow-up wave solutions

$$u=u_M csc^2(\omega_2\xi), \tag{4.5}$$

where $u_M=\frac{h(a^2-bc)}{2\alpha}, \omega_1=\sqrt{\frac{\alpha u_M}{2(a^2-bc)}}, \omega_2=\sqrt{-\frac{\alpha u_M}{2(a^2-bc)}}$.

Theorem 4.2. *If $a^2-bc<0, \alpha\beta>0, g=0$, then when $h=\frac{4\sqrt{\alpha\beta}}{a^2-bc}$, equation (2.6) has a solitary wave solution*

$$u=\frac{\sqrt{\alpha\beta}}{\alpha}\tanh^2(\omega_3\xi), \tag{4.6}$$

and a blow-up wave solution

$$u=\frac{\sqrt{\alpha\beta}}{\alpha}\coth^2(\omega_3\xi), \tag{4.7}$$

when $h < \frac{4\sqrt{\alpha\beta}}{a^2-bc}$*, equation (2.6) has infinite many smooth periodic wave solutions*

$$u = \gamma_1 sn^2(\omega_4\xi, k_1), \tag{4.8}$$

and infinite many periodic blow-up wave solutions

$$u = \gamma_2 ns^2(\omega_4\xi, k_1), \tag{4.9}$$

when $h = -\frac{4\sqrt{\alpha\beta}}{a^2-bc}$*, equation (2.6) has a periodic blow-up wave solution*

$$u = \frac{\sqrt{\alpha\beta}}{\alpha}\cot^2(\omega_3\xi), \tag{4.10}$$

when $h > -\frac{4\sqrt{\alpha\beta}}{a^2-bc}$*, equation (2.6) has infinite many smooth periodic wave solutions*

$$u = \gamma_1 - (\gamma_1 - \gamma_2) sn^2(\omega_5\xi, k_2), \tag{4.11}$$

and infinite many periodic blow-up wave solutions

$$u = \gamma_1\left(1 - ns^2(\omega_5\xi, k_2)\right), \tag{4.12}$$

where $\omega_3 = \sqrt{-\frac{\sqrt{\alpha\beta}}{2(a^2-bc)}}, \omega_4 = \sqrt{-\frac{\alpha\gamma_2}{2(a^2-bc)}}, \omega_5 = \sqrt{\frac{\alpha\gamma_1}{2(a^2-bc)}}, k_1 = \sqrt{\frac{\gamma_1}{\gamma_2}}, k_2 = \sqrt{\frac{\gamma_1-\gamma_2}{\gamma_1}}, \gamma_{1,2} = \frac{h(a^2-bc)\mp\sqrt{h^2(a^2-bc)^2-16\alpha\beta}}{4\alpha}$*,* $sn(\cdot,\cdot)$ *and* $ns(\cdot,\cdot)$ *are the Jacobian elliptic functions [2].*

If $a^2 - bc < 0, \alpha < 0, g = 0, \beta < 0$*, then when* $\frac{4\sqrt{\alpha\beta}}{a^2-bc} < h < -\frac{4\sqrt{\alpha\beta}}{a^2-bc}$*, equation (2.6) has infinite many periodic blow-up wave solutions*

$$u = -\frac{A\left(1 + cn(\omega_6\xi, k_3)\right)}{1 - cn(\omega_6\xi, k_3)}, \tag{4.13}$$

if $a^2 - bc < 0, \alpha > 0, g = 0, \beta > 0$*, then when* $\frac{4\sqrt{\alpha\beta}}{a^2-bc} < h < -\frac{4\sqrt{\alpha\beta}}{a^2-bc}$*, equation (2.6) has infinite many periodic blow-up wave solutions*

$$u = \frac{A\left(1 + cn(\bar{\omega}_6\xi, \bar{k}_3)\right)}{1 - cn(\bar{\omega}_6\xi, \bar{k}_3)}, \tag{4.14}$$

where $\omega_6 = \sqrt{\frac{2\alpha A}{a^2-bc}}, \bar{\omega}_6 = \sqrt{-\frac{2\alpha A}{a^2-bc}}, k_3 = \sqrt{\frac{A-b_1}{2A}}, \bar{k}_3 = \sqrt{\frac{A+b_1}{2A}}, A = \sqrt{a_1^2+b_1^2},\ a_1 = \sqrt{\frac{\beta}{\alpha} - \frac{h^2(a^2-bc)^2}{16\alpha^2}}, b_1 = \frac{h(a^2-bc)}{4\alpha}$ *and* $cn(\cdot,\cdot)$ *is the Jacobian elliptic functions [2].*

Theorem 4.3. *If* $a^2 - bc < 0, \alpha\beta < 0, g = 0$*, then for* $h \in \mathbb{R}$*, equation (2.6) has infinite many smooth periodic wave solutions*

$$u = \gamma_2 cn^2(\omega_7\xi, k_4), \tag{4.15}$$

and infinite many periodic blow-up wave solutions

$$u = \gamma_2 - (\gamma_2 - \gamma_1) ns^2(\omega_7\xi, k_4), \tag{4.16}$$

where $\omega_7 = \sqrt{\frac{\alpha(\gamma_2-\gamma_1)}{2(a^2-bc)}}, k_4 = \sqrt{\frac{\gamma_2}{\gamma_2-\gamma_1}}, \gamma_{1,2} = \frac{h(a^2-bc)\pm\sqrt{h^2(a^2-bc)^2-16\alpha\beta}}{4\alpha}$*,* $cn(\cdot,\cdot)$ *and* $ns(\cdot,\cdot)$ *are the Jacobian elliptic functions [2].*

Figure 4. The level curves given by $H(u,y)=h$ when $h>0$. Parameters: (**a**) $a^2-bc<0,\alpha<0,g=0,\beta=0$. (**b**) $a^2-bc<0,\alpha>0,g=0,\beta=0$.

Theorem 4.4. *If $a^2-bc<0,\alpha=0,g=0,\beta\neq 0$, then when $h<0$, equation (2.6) has infinite many smooth periodic wave solutions*

$$u=u_m\cos^2(\omega_8\xi), \tag{4.17}$$

when $h=0$, equation (2.6) has a blow-up wave solution

$$u=-\frac{\beta\xi^2}{2(a^2-bc)}, \tag{4.18}$$

when $h>0$, equation (2.6) has infinite many blow-up wave solutions

$$u=-u_m\sinh^2(\omega_9\xi), \tag{4.19}$$

$$u=u_m\cosh^2(\omega_9\xi), \tag{4.20}$$

where $u_m=\frac{2\beta}{h(a^2-bc)},\omega_8=\frac{1}{2}\sqrt{-h},\omega_9=\frac{1}{2}\sqrt{h}$.

4.2. Proofs of the main results

Proof of theorem 4.1. For given $h>0$ in Figure 1(d) and Figure 3(f), the level curves are shown in Figure 4(a) and (b), respectively. From Figure 4(a) and (b), we see that there are a family of homoclinic orbits and a family of open curves of system (2.7) defined by $H(u,y)=h$ passing through the higher-order equilibrium point $(0,0)$ when $a^2-bc<0,\alpha\neq 0,g=0,\beta=0,h>0$. When $a^2-bc<0,\alpha<0,g=0,\beta=0,h>0$, the homoclinic orbits and the open curves are defined by the following algebraic equations, respectively,

$$y=\pm u\sqrt{\frac{2\alpha}{a^2-bc}(u_M-u)},\quad 0<u\leq u_M, \tag{4.21}$$

$$y=\pm u\sqrt{\frac{2\alpha}{a^2-bc}(u_M-u)},\quad -\infty<u<0, \tag{4.22}$$

when $a^2-bc<0,\alpha>0,g=0,\beta=0,h>0$, the homoclinic orbits and the open curves are defined by the following algebraic equations, respectively,

$$y=\pm u\sqrt{-\frac{2\alpha}{a^2-bc}(u-u_M)},\quad u_M\leq u<0, \tag{4.23}$$

$$y=\pm u\sqrt{-\frac{2\alpha}{a^2-bc}(u-u_M)},\quad 0<u<+\infty, \tag{4.24}$$

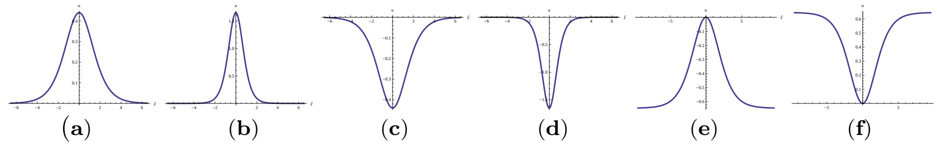

Figure 5. Solitary waves of (2.6). Parameters: (**a**) $a^2 - bc = -1.1, \alpha = -1.5, h = 1.2$. (**b**) $a^2 - bc = -1.1, \alpha = -1.5, h = 4.5$. (**c**) $a^2 - bc = -1.1, \alpha = 1.5, h = 1.2$. (**d**) $a^2 - bc = -1.1, \alpha = 1.5, h = 4.5$. (**e**) $a^2 - bc = -2.5, \alpha = -1.2, \beta = -0.5$. (**f**) $a^2 - bc = -2.5, \alpha = 1.2, \beta = 0.5$.

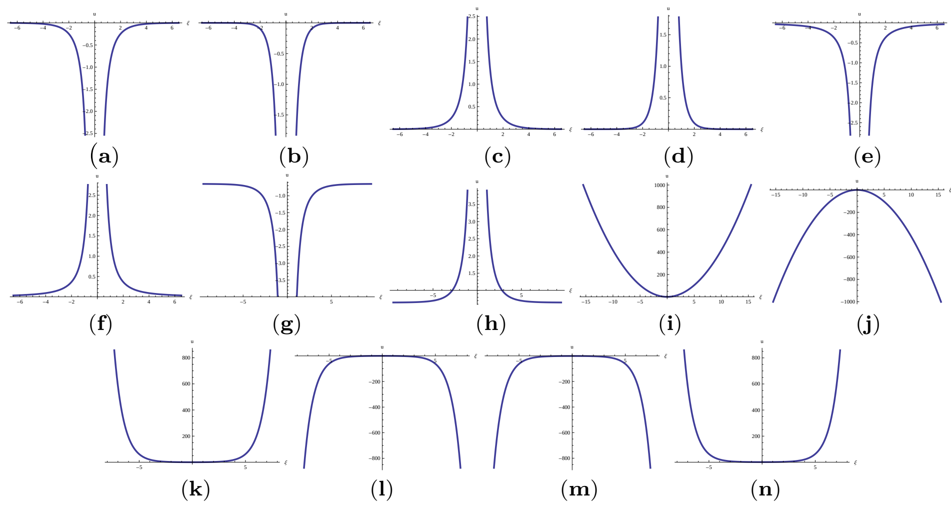

Figure 6. Blow-up waves of (2.6). Parameters: (**a**) $a^2 - bc = -1.1, \alpha = -1.5, h = 1.2$. (**b**) $a^2 - bc = -1.1, \alpha = -1.5, h = 4.5$. (**c**) $a^2 - bc = -1.1, \alpha = 1.5, h = 1.2$. (**d**) $a^2 - bc = -1.1, \alpha = 1.5, h = 4.5$. (**e**) $a^2 - bc = -1.1, \alpha = -1.5$. (**f**) $a^2 - bc = -1.1, \alpha = 1.5$. (**g**) $a^2 - bc = -2.5, \alpha = -1.2, \beta = -0.5$. (**h**) $a^2 - bc = -2.5, \alpha = 1.2, \beta = 0.5$. (**i**) $a^2 - bc = -0.15, \beta = 1.25$. (**j**) $a^2 - bc = -0.15, \beta = -1.25$. (**k**) $a^2 - bc = -3.0, \beta = 1.75, h = 1.25$. (**l**) $a^2 - bc = -3.0, \beta = -1.75, h = 1.25$. (**m**) $a^2 - bc = -3.0, \beta = 1.75, h = 1.25$. (**n**) $a^2 - bc = -3.0, \beta = -1.75, h = 1.25$.

where $u_M = \frac{h(a^2-bc)}{2\alpha}$. Corresponding to (4.21) and (4.23), (4.22) and (4.24), respectively, we have infinite many solitary wave and blow-up wave solutions of (2.6) as (4.2) and (4.3), respectively, when $a^2 - bc < 0, \alpha \neq 0, g = 0, \beta = 0, h > 0$. The profiles of (4.2) and (4.3) are shown in Figure 5(a)-(d) and Figure 6(a)-(d), respectively.

For given $h = 0$ in Figure 1(d) and Figure 3(f), the level curves are shown in Figure 7(a) and (b), respectively. From Figure 7(a) and (b), we see that there is a open curve of system (2.7) defined by $H(u, y) = 0$ connecting with the higher-order equilibrium point $(0, 0)$ when $a^2 - bc < 0, \alpha \neq 0, g = 0, \beta = 0, h = 0$. When $a^2 - bc < 0, \alpha < 0, g = 0, \beta = 0, h = 0$, the open curve is defined by the following algebraic equation

$$y = \pm u\sqrt{-\frac{2\alpha}{a^2 - bc}u}, \quad -\infty < u < 0, \tag{4.25}$$

when $a^2 - bc < 0, \alpha > 0, g = 0, \beta = 0, h = 0$, the open curve is defined by the following algebraic equation

$$y = \pm u\sqrt{-\frac{2\alpha}{a^2 - bc}u}, \quad 0 < u < +\infty. \tag{4.26}$$

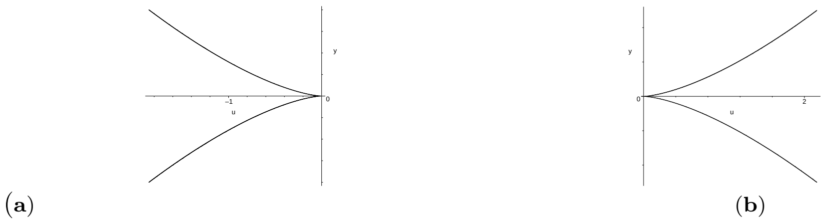

Figure 7. The level curves given by $H(u,y) = 0$. Parameters: (**a**) $a^2 - bc < 0, \alpha < 0, g = 0, \beta = 0$. (**b**) $a^2 - bc < 0, \alpha > 0, g = 0, \beta = 0$.

Figure 8. The level curves given by $H(u,y) = h$ when $h < 0$. Parameters: (**a**) $a^2 - bc < 0, \alpha < 0, g = 0, \beta = 0$. (**b**) $a^2 - bc < 0, \alpha > 0, g = 0, \beta = 0$.

Corresponding to (4.25) and (4.26), we have a blow-up wave solution of (2.6) as (4.4) when $a^2 - bc < 0, \alpha \neq 0, g = 0, \beta = 0, h = 0$. The profiles of (4.4) are shown in Figure 6(e) and (f).

For given $h < 0$ in Figure 1(d) and Figure 3(f), the level curves are shown in Figure 8(a) and (b), respectively. From Figure 8(a) and (b), we see that there are a family of open curves of system (2.7) defined by $H(u,y) = h$ when $a^2 - bc < 0, \alpha \neq 0, g = 0, \beta = 0, h < 0$. When $a^2 - bc < 0, \alpha < 0, g = 0, \beta = 0, h < 0$, the open curves are defined by the following algebraic equation

$$y = \pm u\sqrt{\frac{2\alpha}{a^2 - bc}\,(u_M - u)}, \quad -\infty < u \leq u_M, \tag{4.27}$$

when $a^2 - bc < 0, \alpha > 0, g = 0, \beta = 0, h < 0$, the open curves are defined by the following algebraic equation

$$y = \pm u\sqrt{-\frac{2\alpha}{a^2 - bc}\,(u - u_M)}, \quad u_M \leq u < +\infty, \tag{4.28}$$

where $u_M = \frac{h(a^2-bc)}{2\alpha}$. Corresponding to (4.27) and (4.28), we have infinite many periodic blow-up wave solutions of (2.6) as (4.5) when $a^2 - bc < 0, \alpha \neq 0, g = 0, \beta = 0, h < 0$. The profiles of (4.5) are shown in Figure 9(a)-(d).

The proof of theorem 4.1 is completed. □

Proof of theorem 4.2. For given $h = \frac{4\sqrt{\alpha\beta}}{a^2-bc}$ in Figure 1(e) and Figure 3(e), the level curves are shown in Figure 10(a) and (b), respectively. From Figure 10(a) and (b), we see that there are a homoclinic orbit and a open curve of system (2.7) defined by $H(u,y) = \frac{4\sqrt{\alpha\beta}}{a^2-bc}$ passing through the saddle point $\left(\frac{\sqrt{\alpha\beta}}{\alpha}, 0\right)$, and the homoclinic orbit also connecting with the higher-order equilibrium point $(0,0)$ when $a^2 - bc <$

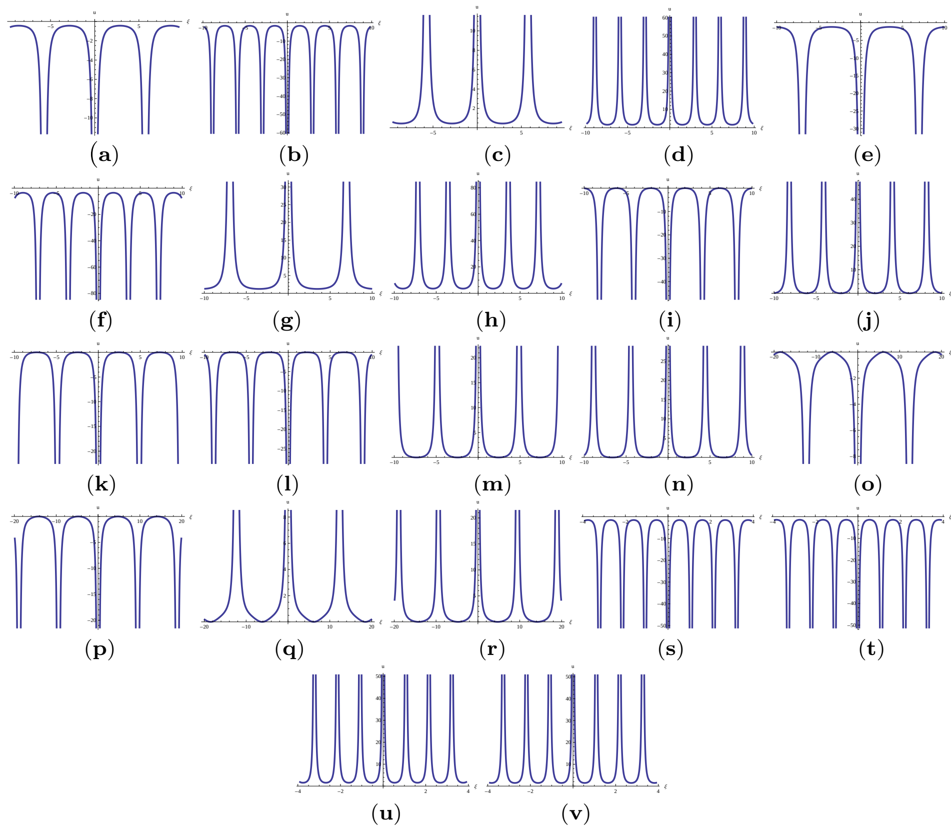

Figure 9. Periodic blow-up waves of (2.6). Parameters: (**a**) $a^2 - bc = -1.1, \alpha = -1.5, h = -1.2$. (**b**) $a^2 - bc = -1.1, \alpha = -1.5, h = -4.5$. (**c**) $a^2 - bc = -1.1, \alpha = 1.5, h = -1.2$. (**d**) $a^2 - bc = -1.1, \alpha = 1.5, h = -4.5$. (**e**) $a^2 - bc = -2.5, \alpha = -1.2, \beta = -0.5, h = -1.5$. (**f**) $a^2 - bc = -2.5, \alpha = -1.2, \beta = -0.5, h = -3.5$. (**g**) $a^2 - bc = -2.5, \alpha = 1.2, \beta = 0.5, h = -1.5$. (**h**) $a^2 - bc = -2.5, \alpha = 1.2, \beta = 0.5, h = -1.5$. (**i**) $a^2 - bc = -1.2, \alpha = -1.0, \beta = -2.0$. (**j**) $a^2 - bc = -1.2, \alpha = -1.0, \beta = 2.0$. (**k**) $a^2 - bc = -1.5, \alpha = -2.0, \beta = -1.2, h = 4.5$. (**l**) $a^2 - bc = -1.5, \alpha = -2.0, \beta = -1.2, h = 8.0$. (**m**) $a^2 - bc = -1.5, \alpha = 2.0, \beta = 1.2, h = 4.5$. (**n**) $a^2 - bc = -1.5, \alpha = 2.0, \beta = 1.2, h = 8.0$. (**o**) $a^2 - bc = -2.5, \alpha = -1.2, \beta = -0.5, h = -0.6$. (**p**) $a^2 - bc = -2.5, \alpha = -1.2, \beta = -0.5, h = 0.6$. (**q**) $a^2 - bc = -2.5, \alpha = 1.2, \beta = 0.5, h = -0.6$. (**r**) $a^2 - bc = -2.5, \alpha = 1.2, \beta = 0.5, h = 0.6$. (**s**) $a^2 - bc = -0.1, \alpha = -1.0, \beta = 2.2, h = -2.0$. (**t**) $a^2 - bc = -0.1, \alpha = -1.0, \beta = 2.2, h = 2.0$. (**u**) $a^2 - bc = -0.1, \alpha = 1.0, \beta = -2.2, h = -2.0$. (**v**) $a^2 - bc = -0.1, \alpha = 1.0, \beta = -2.2, h = 2.0$.

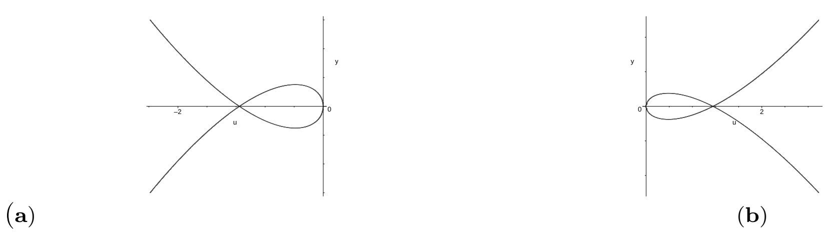

Figure 10. The level curves given by $H(u, y) = \frac{4\sqrt{\alpha\beta}}{a^2 - bc}$. Parameters: (**a**) $a^2 - bc < 0, \alpha < 0, g = 0, \beta < 0$. (**b**) $a^2 - bc < 0, \alpha > 0, g = 0, \beta > 0$.

(**a**) (**b**)

Figure 11. The level curves given by $H(u,y)=h$ when $h<\frac{4\sqrt{\alpha\beta}}{a^2-bc}$. Parameters: (**a**) $a^2-bc<0, \alpha<0, g=0, \beta<0$. (**b**) $a^2-bc<0, \alpha>0, g=0, \beta>0$.

$0, \alpha\beta>0, g=0, h=\frac{4\sqrt{\alpha\beta}}{a^2-bc}$. When $a^2-bc<0, \alpha<0, g=0, \beta<0, h=\frac{4\sqrt{\alpha\beta}}{a^2-bc}$, their expressions are, respectively,

$$y=\pm\left(u-\frac{\sqrt{\alpha\beta}}{\alpha}\right)\sqrt{-\frac{2\alpha}{a^2-bc}u}, \quad \frac{\sqrt{\alpha\beta}}{\alpha}<u\le 0, \tag{4.29}$$

$$y=\pm\left(\frac{\sqrt{\alpha\beta}}{\alpha}-u\right)\sqrt{-\frac{2\alpha}{a^2-bc}u}, \quad -\infty<u<\frac{\sqrt{\alpha\beta}}{\alpha}, \tag{4.30}$$

when $a^2-bc<0, \alpha>0, g=0, \beta>0, h=\frac{4\sqrt{\alpha\beta}}{a^2-bc}$, their expressions are, respectively,

$$y=\pm\left(\frac{\sqrt{\alpha\beta}}{\alpha}-u\right)\sqrt{-\frac{2\alpha}{a^2-bc}u}, \quad 0\le u<\frac{\sqrt{\alpha\beta}}{\alpha}, \tag{4.31}$$

$$y=\pm\left(u-\frac{\sqrt{\alpha\beta}}{\alpha}\right)\sqrt{-\frac{2\alpha}{a^2-bc}u}, \quad \frac{\sqrt{\alpha\beta}}{\alpha}<u<+\infty. \tag{4.32}$$

Corresponding to (4.29) and (4.31), (4.30) and (4.32), respectively, we have a solitary wave and a blow-up wave solutions of (2.6) as (4.6) and (4.7), respectively, when $a^2-bc<0, \alpha\beta>0, g=0, h=\frac{4\sqrt{\alpha\beta}}{a^2-bc}$. The profiles of (4.6) and (4.7) are shown in Figure 5(e) and (f), Figure 6(g) and (h), respectively.

For given $h<\frac{4\sqrt{\alpha\beta}}{a^2-bc}$ in Figure 1(e) and Figure 3(e), the level curves are shown in Figure 11(a) and (b), respectively. From Figure 11(a) and (b), we see that there are a family of periodic orbits and a family of open curves of system (2.7) defined by $H(u,y)=h$ when $a^2-bc<0, \alpha\beta>0, g=0, h<\frac{4\sqrt{\alpha\beta}}{a^2-bc}$. When $a^2-bc<0, \alpha<0, g=0, \beta<0, h<\frac{4\sqrt{\alpha\beta}}{a^2-bc}$, their expressions are, respectively,

$$y=\pm\sqrt{-\frac{2\alpha}{a^2-bc}u(u-\gamma_1)(u-\gamma_2)}, \quad \gamma_1\le u\le 0, \tag{4.33}$$

$$y=\pm\sqrt{-\frac{2\alpha}{a^2-bc}u(\gamma_1-u)(\gamma_2-u)}, \quad -\infty<u\le\gamma_2, \tag{4.34}$$

when $a^2-bc<0, \alpha>0, g=0, \beta>0, h<\frac{4\sqrt{\alpha\beta}}{a^2-bc}$, their expressions are, respectively,

$$y=\pm\sqrt{-\frac{2\alpha}{a^2-bc}u(\gamma_2-u)(\gamma_1-u)}, \quad 0\le u\le\gamma_1, \tag{4.35}$$

$$y=\pm\sqrt{-\frac{2\alpha}{a^2-bc}u(u-\gamma_2)(u-\gamma_1)}, \quad \gamma_2\le u<+\infty, \tag{4.36}$$

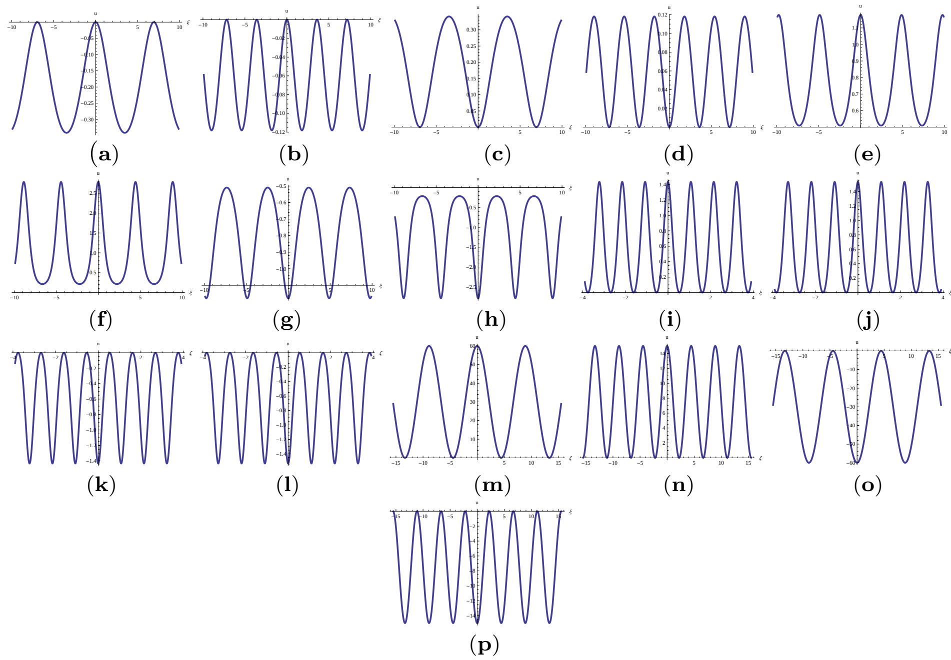

Figure 12. Smooth periodic waves of (2.6). Parameters: (**a**) $a^2 - bc = -2.5, \alpha = -1.2, \beta = -0.5, h = -1.5$. (**b**) $a^2 - bc = -2.5, \alpha = -1.2, \beta = -0.5, h = -3.5$. (**c**) $a^2 - bc = -2.5, \alpha = 1.2, \beta = 0.5, h = -1.5$. (**d**) $a^2 - bc = -2.5, \alpha = 1.2, \beta = 0.5, h = -1.5$. (**e**) $a^2 - bc = -1.5, \alpha = -2.0, \beta = -1.2, h = 4.5$. (**f**) $a^2 - bc = -1.5, \alpha = -2.0, \beta = -1.2, h = 8.0$. (**g**) $a^2 - bc = -1.5, \alpha = 2.0, \beta = 1.2, h = 4.5$. (**h**) $a^2 - bc = -1.5, \alpha = 2.0, \beta = 1.2, h = 8.0$. (**i**) $a^2 - bc = -0.1, \alpha = -1.0, \beta = 2.2, h = -2.0$. (**j**) $a^2 - bc = -0.1, \alpha = -1.0, \beta = 2.2, h = 2.0$. (**k**) $a^2 - bc = -0.1, \alpha = 1.0, \beta = -2.2, h = -2.0$. (**l**) $a^2 - bc = -0.1, \alpha = 1.0, \beta = -2.2, h = 2.0$. (**m**) $a^2 - bc = -0.2, \beta = 3.0, h = -0.5$. (**n**) $a^2 - bc = -0.2, \beta = 3.0, h = -2.0$. (**o**) $a^2 - bc = -0.2, \beta = -3.0, h = -0.5$. (**p**) $a^2 - bc = -0.2, \beta = -3.0, h = -2.0$.

where $\gamma_{1,2} = \frac{h(a^2-bc)\mp\sqrt{h^2(a^2-bc)^2-16\alpha\beta}}{4\alpha}$. Corresponding to (4.33) and (4.35), (4.34) and (4.36), respectively, we have infinite many smooth periodic wave and periodic blow-up wave solutions of (2.6) as (4.8) and (4.9), respectively, when $a^2 - bc < 0, \alpha\beta > 0, g = 0, h < \frac{4\sqrt{\alpha\beta}}{a^2-bc}$. The profiles of (4.8) and (4.9) are shown in Figure 12(a)-(d) and Figure 9(e)-(h), respectively.

For given $h = -\frac{4\sqrt{\alpha\beta}}{a^2-bc}$ in Figure 1(e) and Figure 3(e), the level curves are shown in Figure 13(a) and (b), respectively. From Figure 13(a) and (b), we see that there is a open curve of system (2.7) defined by $H(u, y) = -\frac{4\sqrt{\alpha\beta}}{a^2-bc}$ connecting with the higher-order equilibrium point $(0, 0)$ when $a^2 - bc < 0, \alpha\beta > 0, g = 0, h = -\frac{4\sqrt{\alpha\beta}}{a^2-bc}$. When $a^2 - bc < 0, \alpha < 0, g = 0, \beta < 0, h = -\frac{4\sqrt{\alpha\beta}}{a^2-bc}$, the open curve is defined by the following algebraic equation

$$y = \pm\left(\frac{\sqrt{\alpha\beta}}{\alpha} + u\right)\sqrt{-\frac{2\alpha}{a^2-bc}u}, \quad -\infty < u \le 0, \tag{4.37}$$

when $a^2 - bc < 0, \alpha > 0, g = 0, \beta > 0, h = -\frac{4\sqrt{\alpha\beta}}{a^2-bc}$, the open curve is defined by the following algebraic equation

$$y = \pm\left(u + \frac{\sqrt{\alpha\beta}}{\alpha}\right)\sqrt{-\frac{2\alpha}{a^2-bc}u}, \quad 0 \le u < +\infty. \tag{4.38}$$

(a) (b)

Figure 13. The level curves given by $H(u,y) = -\frac{4\sqrt{\alpha\beta}}{a^2-bc}$. Parameters: (**a**) $a^2 - bc < 0, \alpha < 0, g = 0, \beta < 0$. (**b**) $a^2 - bc < 0, \alpha > 0, g = 0, \beta > 0$.

(a) (b)

Figure 14. The level curves given by $H(u,y) = h$ when $h > -\frac{4\sqrt{\alpha\beta}}{a^2-bc}$. Parameters: (**a**) $a^2 - bc < 0, \alpha < 0, g = 0, \beta < 0$. (**b**) $a^2 - bc < 0, \alpha > 0, g = 0, \beta > 0$.

Corresponding to (4.37) and (4.38), we have a periodic blow-up wave solution of (2.6) as (4.10) when $a^2 - bc < 0, \alpha\beta > 0, g = 0, h = -\frac{4\sqrt{\alpha\beta}}{a^2-bc}$. The profiles of (4.10) are shown in Figure 9(i) and (j).

For given $h > -\frac{4\sqrt{\alpha\beta}}{a^2-bc}$ in Figure 1(e) and Figure 3(e), the level curves are shown in Figure 14(a) and (b), respectively. From Figure 14(a) and (b), we see that there are a family of periodic orbits and a family of open curves of system (2.7) defined by $H(u,y) = h$ when $a^2 - bc < 0, \alpha\beta > 0, g = 0, h > -\frac{4\sqrt{\alpha\beta}}{a^2-bc}$. When $a^2 - bc < 0, \alpha < 0, g = 0, \beta < 0, h > -\frac{4\sqrt{\alpha\beta}}{a^2-bc}$, their expressions are, respectively,

$$y = \pm\sqrt{\frac{2\alpha}{a^2-bc}(\gamma_1 - u)(u - \gamma_2)u}, \quad \gamma_2 \le u \le \gamma_1, \tag{4.39}$$

$$y = \pm\sqrt{-\frac{2\alpha}{a^2-bc}(\gamma_1 - u)(\gamma_2 - u)u}, \quad -\infty < u \le 0, \tag{4.40}$$

when $a^2 - bc < 0, \alpha > 0, g = 0, \beta > 0, h > -\frac{4\sqrt{\alpha\beta}}{a^2-bc}$, their expressions are, respectively,

$$y = \pm\sqrt{\frac{2\alpha}{a^2-bc}(\gamma_2 - u)(u - \gamma_1)u}, \quad \gamma_1 \le u \le \gamma_2, \tag{4.41}$$

$$y = \pm\sqrt{-\frac{2\alpha}{a^2-bc}u(u - \gamma_2)(u - \gamma_1)}, \quad 0 \le u < +\infty, \tag{4.42}$$

where $\gamma_{1,2} = \frac{h(a^2-bc)\mp\sqrt{h^2(a^2-bc)^2-16\alpha\beta}}{4\alpha}$. Corresponding to (4.39) and (4.41), (4.40) and (4.42), respectively, we have infinite many smooth periodic wave and periodic blow-up wave solutions of (2.6) as (4.11) and (4.12), respectively, when $a^2 - bc < 0, \alpha\beta > 0, g = 0, h > -\frac{4\sqrt{\alpha\beta}}{a^2-bc}$. The profiles of (4.11) and (4.12) are shown in Figure 12(e)-(h) and Figure 9(k)-(n), respectively.

(**a**) (**b**)

Figure 15. The level curves given by $H(u, y) = h$ when $\frac{4\sqrt{\alpha\beta}}{a^2-bc} < h < -\frac{4\sqrt{\alpha\beta}}{a^2-bc}$. Parameters: (**a**) $a^2 - bc < 0, \alpha < 0, g = 0, \beta < 0$. (**b**) $a^2 - bc < 0, \alpha > 0, g = 0, \beta > 0$.

(**a**) (**b**)

Figure 16. The level curves given by $H(u, y) = h$ when $h \in \mathbb{R}$. Parameters: (**a**) $a^2 - bc < 0, \alpha < 0, g = 0, \beta > 0$. (**b**) $a^2 - bc < 0, \alpha > 0, g = 0, \beta < 0$.

For given $\frac{4\sqrt{\alpha\beta}}{a^2-bc} < h < -\frac{4\sqrt{\alpha\beta}}{a^2-bc}$ in Figure 1(e) and Figure 3(e), the level curves are shown in Figure 15(a) and (b), respectively. From Figure 15(a) and (b), we see that there are a family of open curves of system (2.7) defined by $H(u, y) = h$ connecting with the higher-order equilibrium point $(0, 0)$ when $a^2 - bc < 0, \alpha\beta > 0, g = 0, \frac{4\sqrt{\alpha\beta}}{a^2-bc} < h < -\frac{4\sqrt{\alpha\beta}}{a^2-bc}$. When $a^2 - bc < 0, \alpha < 0, g = 0, \beta < 0, \frac{4\sqrt{\alpha\beta}}{a^2-bc} < h < -\frac{4\sqrt{\alpha\beta}}{a^2-bc}$, the open curves are defined by the following algebraic equation

$$y = \pm\sqrt{-\frac{2\alpha}{a^2 - bc}u\left((u - b_1)^2 + a_1^2\right)}, \quad -\infty < u \le 0, \tag{4.43}$$

when $a^2 - bc < 0, \alpha > 0, g = 0, \beta > 0, \frac{4\sqrt{\alpha\beta}}{a^2-bc} < h < -\frac{4\sqrt{\alpha\beta}}{a^2-bc}$, the open curves are defined by the following algebraic equation

$$y = \pm\sqrt{-\frac{2\alpha}{a^2 - bc}u\left((u - b_1)^2 + a_1^2\right)}, \quad 0 \le u < +\infty, \tag{4.44}$$

where $a_1 = \sqrt{\frac{\beta}{\alpha} - \frac{h^2(a^2-bc)^2}{16\alpha^2}}, b_1 = \frac{h(a^2-bc)}{4\alpha}$. Corresponding to (4.43), we have infinite many periodic blow-up wave solutions of (2.6) as (4.13) when $a^2 - bc < 0, \alpha < 0, g = 0, \beta < 0, \frac{4\sqrt{\alpha\beta}}{a^2-bc} < h < -\frac{4\sqrt{\alpha\beta}}{a^2-bc}$. Corresponding to (4.44), we have infinite many periodic blow-up wave solutions of (2.6) as (4.14) when $a^2 - bc < 0, \alpha > 0, g = 0, \beta > 0, \frac{4\sqrt{\alpha\beta}}{a^2-bc} < h < -\frac{4\sqrt{\alpha\beta}}{a^2-bc}$. The profiles of (4.13) and (4.14) are shown in Figure 9(o) and (p), (q) and (r), respectively.

The proof of theorem 4.2 is completed. □

Proof of theorem 4.3. For given h ($h \in \mathbb{R}$) in Figure 1(c) and Figure 3(g), the level curves are shown in Figure 16(a) and (b), respectively. From Figure 16(a) and (b), we see that there are a family of periodic orbits and a family of open curves of

Figure 17. The level curves given by $H(u, y) = h$ when $h < 0$. Parameters: (a) $a^2 - bc < 0, \alpha = 0, g = 0, \beta > 0$. (b) $a^2 - bc < 0, \alpha = 0, g = 0, \beta < 0$.

system (2.7) defined by $H(u, y) = h$ when $a^2 - bc < 0, \alpha\beta < 0, g = 0, h \in \mathbb{R}$. When $a^2 - bc < 0, \alpha < 0, g = 0, \beta > 0, h \in \mathbb{R}$, their expressions are, respectively,

$$y = \pm\sqrt{\frac{2\alpha}{a^2 - bc}(\gamma_2 - u)(u - \gamma_1)u}, \quad 0 \le u \le \gamma_2, \tag{4.45}$$

$$y = \pm\sqrt{-\frac{2\alpha}{a^2 - bc}(\gamma_2 - u)(\gamma_1 - u)u}, \quad -\infty < u \le \gamma_1, \tag{4.46}$$

when $a^2 - bc < 0, \alpha > 0, g = 0, \beta < 0, h \in \mathbb{R}$, their expressions are, respectively,

$$y = \pm\sqrt{\frac{2\alpha}{a^2 - bc}(\gamma_1 - u)(u - \gamma_2)u}, \quad \gamma_2 \le u \le 0, \tag{4.47}$$

$$y = \pm\sqrt{-\frac{2\alpha}{a^2 - bc}(u - \gamma_1)(u - \gamma_2)u}, \quad \gamma_1 \le u < +\infty, \tag{4.48}$$

where $\gamma_{1,2} = \frac{h(a^2-bc)\pm\sqrt{h^2(a^2-bc)^2-16\alpha\beta}}{4\alpha}$. Corresponding to (4.45) and (4.47), (4.46) and (4.48), respectively, we have infinite many smooth periodic wave and periodic blow-up wave solutions of (2.6) as (4.15) and (4.16), respectively, when $a^2 - bc < 0, \alpha\beta < 0, g = 0, h \in \mathbb{R}$. The profiles of (4.15) and (4.16) are shown in Figure 12(i)-(l) and Figure 9(s)-(v), respectively.

The proof of theorem 4.3 is completed. □

Proof of theorem 4.4. For given h ($h < 0$) in Figure 2(c) and (d), the level curves are shown in Figure 17(a) and (b), respectively. From Figure 17(a) and (b), we see that there are a family of periodic orbits of system (2.7) defined by $H(u, y) = h$ when $a^2 - bc < 0, \alpha = 0, g = 0, \beta \neq 0, h < 0$. When $a^2 - bc < 0, \alpha = 0, g = 0, \beta > 0, h < 0$, the periodic orbits are defined by the following algebraic equation

$$y = \pm\sqrt{-h(u_m - u)u}, \quad 0 \le u \le u_m, \tag{4.49}$$

when $a^2 - bc < 0, \alpha = 0, g = 0, \beta < 0, h < 0$, the periodic orbits are defined by the following algebraic equation

$$y = \pm\sqrt{h(u - u_m)u}, \quad u_m \le u \le 0, \tag{4.50}$$

where $u_m = \frac{2\beta}{h(a^2-bc)}$. Corresponding to (4.49) and (4.50), we have infinite many smooth periodic wave solutions of (2.6) as (4.17) when $a^2 - bc < 0, \alpha = 0, g = 0, \beta \neq 0, h < 0$. The profiles of (4.17) are shown in Figure 12(m)-(p).

(a) (b)

Figure 18. The level curves given by $H(u, y) = 0$ Parameters: (a) $a^2 - bc < 0, \alpha = 0, g = 0, \beta > 0$. (b) $a^2 - bc < 0, \alpha = 0, g = 0, \beta < 0$.

(a) (b)

Figure 19. The level curves given by $H(u, y) = h$ when $h > 0$. Parameters: (a) $a^2 - bc < 0, \alpha = 0, g = 0, \beta > 0$. (b) $a^2 - bc < 0, \alpha = 0, g = 0, \beta < 0$.

For given $h = 0$ in Figure 2(c) and (d), the level curves are shown in Figure 18(a) and (b), respectively. From Figure 18(a) and (b), we see that there is a open curve of system (2.7) defined by $H(u, y) = 0$ when $a^2 - bc < 0, \alpha = 0, g = 0, \beta \neq 0$. When $a^2 - bc < 0, \alpha = 0, g = 0, \beta > 0, h = 0$, their expressions are

$$y = \pm\sqrt{-\frac{2\beta}{a^2 - bc}u}, \quad 0 \leq u < +\infty, \tag{4.51}$$

when $a^2 - bc < 0, \alpha = 0, g = 0, \beta < 0, h = 0$, their expressions are

$$y = \pm\sqrt{-\frac{2\beta}{a^2 - bc}u}, \quad -\infty < u \leq 0. \tag{4.52}$$

Corresponding to (4.51) and (4.52), we have a blow-up wave solution of (2.6) as (4.18) when $a^2 - bc < 0, \alpha = 0, g = 0, \beta \neq 0, h = 0$. The profiles of (4.18) are shown in Figure 6(i) and (j).

For given h ($h > 0$) in Figure 2(c) and (d), the level curves are shown in Figure 19(a) and (b), respectively. From Figure 19(a) and (b), we see that there are two family of open curves of system (2.7) defined by $H(u, y) = h$ when $a^2 - bc < 0, \alpha = 0, g = 0, \beta \neq 0, h > 0$. When $a^2 - bc < 0, \alpha = 0, g = 0, \beta > 0, h > 0$, their expressions are, respectively,

$$y = \pm\sqrt{hu(u - u_m)}, \quad 0 \leq u < +\infty, \tag{4.53}$$

$$y = \pm\sqrt{-hu(u_m - u)}, \quad -\infty < u \leq u_m, \tag{4.54}$$

when $a^2 - bc < 0, \alpha = 0, g = 0, \beta < 0, h > 0$, their expressions are, respectively,

$$y = \pm\sqrt{-hu(u_m - u)}, \quad -\infty < u \leq 0, \tag{4.55}$$

$$y = \pm\sqrt{hu(u - u_m)}, \quad u_m \leq u < +\infty, \tag{4.56}$$

where $u_m = \frac{2\beta}{h(a^2-bc)}$. Corresponding to (4.53) and (4.55), (4.54) and (4.56), respectively, we have infinite many blow-up wave solutions of (2.6) as (4.19) and (4.20), respectively, when $a^2 - bc < 0, \alpha = 0, g = 0, \beta \neq 0, h > 0$. The profiles of (4.19) and (4.20) are shown in Figure 6(k) and (m), (l) and (n), respectively.

The proof of theorem 4.4 is completed. □

Remark 4.2. Using $\phi = \ln(u(\xi))$, $\xi = ay + bz + ct$, $a^2 \neq bc$, (2.1) and Theorem 4.1-Theorem 4.4, we can present the exact traveling wave solutions of equation (1.1). For examples, when $a^2 - bc < 0, \alpha \neq 0, h > 0$, equation (1.1) has exact traveling wave solutions

$$\begin{cases} U = \alpha u_M \operatorname{sech}^2(\omega_1(ay+bz+ct)), \\ V = i(-2c\omega_1 \tanh(\omega_1(ay+bz+ct))), \\ W = i\left(\alpha u_M \operatorname{sech}^2(\omega_1(ay+bz+ct))\right), \end{cases} \tag{4.57}$$

where $u_M = \frac{h(a^2-bc)}{2\alpha}, \omega_1 = \sqrt{\frac{\alpha u_M}{2(a^2-bc)}}$. When $a^2 - bc < 0, \alpha\beta > 0$, equation (1.1) has exact traveling wave solution

$$\begin{cases} U = \sqrt{\alpha\beta}\left(\tanh^2(\omega_3(ay+bz+ct)) + \coth^2(\omega_3(ay+bz+ct))\right), \\ V = i(4c\omega_3 \operatorname{csch}(2\omega_3(ay+bz+ct))), \\ W = i\left(\sqrt{\alpha\beta}\left(\tanh^2(\omega_3(ay+bz+ct)) - \coth^2(\omega_3(ay+bz+ct))\right)\right), \end{cases} \tag{4.58}$$

where $\omega_3 = \sqrt{-\frac{\sqrt{\alpha\beta}}{2(a^2-bc)}}$. When $a^2 - bc < 0, \alpha\beta < 0, h \in \mathbb{R}$, equation (1.1) has exact traveling wave solutions

$$\begin{cases} U = \alpha\gamma_2 \operatorname{cn}^2(\omega_7(ay+bz+ct), k_4) + \frac{\beta}{\gamma_2}\operatorname{nc}^2(\omega_7(ay+bz+ct), k_4), \\ V = i(-2c\omega_7 \operatorname{dn}(\omega_7(ay+bz+ct), k_4)\operatorname{sc}(\omega_7(ay+bz+ct), k_4)), \\ W = i\left(\alpha\gamma_2 \operatorname{cn}^2(\omega_7(ay+bz+ct), k_4) - \frac{\beta}{\gamma_2}\operatorname{nc}^2(\omega_7(ay+bz+ct), k_4)\right), \end{cases} \tag{4.59}$$

where $\omega_7 = \sqrt{\frac{\alpha(\gamma_2-\gamma_1)}{2(a^2-bc)}}, k_4 = \sqrt{\frac{\gamma_2}{\gamma_2-\gamma_1}}, \gamma_{1,2} = \frac{h(a^2-bc)\pm\sqrt{h^2(a^2-bc)^2-16\alpha\beta}}{4\alpha}$, $\operatorname{cn}(\cdot,\cdot)$, $\operatorname{nc}(\cdot,\cdot), \operatorname{dn}(\cdot,\cdot)$ and $\operatorname{sc}(\cdot,\cdot)$ are the Jacobian elliptic functions [2]. When $a^2 - bc < 0, \beta \neq 0, h < 0$, equation (1.1) has exact traveling wave solutions

$$\begin{cases} U = \frac{\beta}{u_m}\sec^2(\omega_8(ay+bz+ct)), \\ V = i(-2c\omega_8 \tan(\omega_8(ay+bz+ct))), \\ W = i\left(-\frac{\beta}{u_m}\sec^2(\omega_8(ay+bz+ct))\right), \end{cases} \tag{4.60}$$

where $u_m = \frac{2\beta}{h(a^2-bc)}, \omega_8 = \frac{1}{2}\sqrt{-h}$.

We omit other exact traveling wave solutions of equation (1.1) here for convenience.

5. Conclusion

In this paper, we investigate the bifurcations and present some exact traveling wave solutions of equation (1.1). Compared with the published references, we obtained some new results. Actually, there are some interesting and important problems for equation (1.1) to be further studied. For examples, How do the dynamical behaviors of multiple solitary wave and double periodic wave solutions for equation (1.1)? How do the Lie symmetry analysis and are there other similarity solutions of equation (1.1)? We will study equation (1.1) further in the coming papers.

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