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SOLVABILITY FOR RIEMANN-STIELTJES INTEGRAL BOUNDARY VALUE PROBLEMS OF BAGLEY-TORVIK EQUATIONS AT RESONANCE

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Abstract In this paper, we study the solvability for Riemann-Stieltjes integral boundary value problems of Bagley-Torvik equations with fractional derivative under resonant conditions. Firstly, the kernel function is presented through the Laplace transform and the properties of the kernel function are obtained. And then, some new results on the solvability for the boundary value problem are established by using Mawhin's coincidence degree theory. Finally, two examples are presented to illustrate the applicability of our main results.

Keywords Bagley-Torvik equation, Caputo derivative, Riemann-Stieltjes integral, boundary value problem, resonant condition, Mawhin's coincidence degree theory.

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1. Introduction

Since the viscoelastic medium damping could not be well described through the forced vibration equation of integer order, many researchers use fractional integral or fractional derivative to describe the properties of viscoelastic materials. Therefore, fractional differential equation is playing an important role in describing viscous damping model, see [13, 14, 20, 22, 25]. In [25], Torvik and Bagley established generalized constitutive relation for viscoelastic materials in which the customary time derivatives of integer order are replaced by derivatives of fractional order and homogeneous Bagley-Torvik equation was also obtained

$$Ay''(t) + B_0 D_t^{\frac{3}{2}} y(t) + Cy(t) = 0.$$

In [20], Podlubny studied the initial value problems for the inhomogeneous Bagley-Torvik equation

$$\begin{cases} Ay''(t) + B_0 D_t^{\frac{3}{2}} y(t) + Cy(t) = f(t),\ t > 0, \\ y(0) = 0,\ y'(0) = 0, \end{cases}$$

and the numerical solutions are presented (see [20], page 229). After that, there are many research results on this model, see [1, 5, 24] and the references therein.

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As it well known that fractional differential equations have been studied for a long time. In the recent decades, a lot of research results have been published on the theory of boundary value problems of fractional differential equations, see [1–3, 5, 7–9, 15–18, 23, 26–28]. As far as we know, the resonance problem has to be considered in the theoretical study of vibration equation. In [4, 10, 12, 19, 24], the authors have studied the solvability of the fractional vibration equation under the resonant conditions. In [24], Stanek investigated the nonlocal fractional boundary value problems at resonance

$$\begin{cases} u'' = A^c D^\alpha u + f(t, u, ^c D^\mu u, u'), \\ u'(0) = u'(T), \ \Lambda(u) = 0, \end{cases}$$

where $\alpha \in (1, 2)$, $\mu \in (0, 1)$. The existence of solutions of the problem are given by using the Leray-Schauder degree method. Since Riemann-Stieltjes integral boundary conditions not only contain the classical Riemann integral boundary conditions but also two-point boundary value and multi-point boundary conditions, Riemann-Stieltjes integral boundary value problems have much wider application.

Motivated by the above works, we study the following Riemann-Stieltjes integral boundary value problems of Bagley-Torvik equation with Caputo fractional derivative under the resonant condition

$$\begin{cases} x''(t) + b^c D_{0^+}^\alpha x(t) + ax(t) = f(t, x(t), x'(t)), \ t \in (0, 1), \\ x(0) = 0, \ x(1) = \int_0^1 x(t) \mathrm{d}A(t), \end{cases} \tag{1.1}$$

where $0 < \alpha \leq 1$, $0 \leq a \leq 1$ and $0 \leq b \leq \min\{1, \frac{\Gamma(5-\alpha)}{2((3-\alpha)^2+1-\alpha)}\}$ are real constants, ${}^c D_{0^+}^\alpha$ is the Caputo fractional derivative of order α, $\int_0^1 x(t)\mathrm{d}A(t)$ is the Riemann-Stieltjes integral of x with respect to A and $A(t)$ is a monotone increasing function and not a constant on $t \in [0, 1]$, $f : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous.

By using the Laplace transform, the kernel function is obtained. And then, by using Mawhin's coincidence degree theory, we establish some new results of the solvability for boundary value problem (1.1) under the resonance condition $g_1(1) = \int_0^1 g_1(t)\mathrm{d}A(t)$, where the definition of $g_1(t)$ see (2.2). In order to illustrate the applicability of our main results, two examples are presented.

2. Preliminaries

The definitions of fractional integral, fractional derivative and Laplace transform and the related lemmas can be found in [6, 11, 20].

Definition 2.1 (See [6], P68). Let $\delta,\ \beta,\ z \in \mathbb{C}$ and $\mathrm{Re}(\delta) > 0$. The function $E_{\delta,\beta}(z)$ is defined by

$$E_{\delta,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\delta + \beta)},$$

whenever the series converges is called the two-parameter Mittag-Leffler function with parameters β and δ.

Lemma 2.1 (See [6], P68). *Let $\delta,\ \beta > 0$. The power series $E_{\delta,\beta}(z)$ is convergent for all $z \in \mathbb{C}$. In other words, $E_{\delta,\beta}$ is an entire function.*

Let

$$E^{(k)}_{\delta,\beta}(z) = \frac{\mathrm{d}^k}{\mathrm{d}z^k} E_{\delta,\beta}(z),\ k = 0, 1, 2, \cdots .$$

Lemma 2.2. *Let* $z \in \mathbb{R},\ k = 0, 1, 2, \cdots$. *Then*

$$E^{(k)}_{\delta,\beta}(z) = \sum_{j=0}^{\infty} \frac{z^j \Gamma(k + j + 1)}{\Gamma(j + 1)\Gamma(\delta(k + j) + \beta)}.$$

Proof. By Definition 2.1 and Lemma 2.1, we can get

$$\begin{aligned} E^{(k)}_{\delta,\beta}(z) &= \sum_{j=0}^{\infty} \frac{\mathrm{d}^k}{\mathrm{d}z^k}\Big(\frac{z^j}{\Gamma(j\delta + \beta)}\Big) = \sum_{j=k}^{\infty} \frac{\Gamma(j + 1)z^{j-k}}{\Gamma(j - k + 1)\Gamma(\delta j + \beta)} \\ &= \sum_{j=0}^{\infty} \frac{z^j \Gamma(k + j + 1)}{\Gamma(j + 1)\Gamma(\delta(k + j) + \beta)}. \end{aligned}$$

Thus, the lemma can be obtained. □

Lemma 2.3 (See [11]). *Let* $n - 1 < \delta \le n,\ n \in \mathbb{N}$. *The Laplace transform formula for* ${}^cD^{\delta}_{0+}g(t)$ *is*

$$\mathcal{L}[{}^cD^{\delta}_{0+}g(t)](p) = p^{\delta}\mathcal{L}[g(t)](p) - \sum_{j=0}^{n-1} p^{\delta-1-j} g^{(j)}(0),\ p > 0.$$

Lemma 2.4 (See [20]). *Let* $\delta > 0,\ \beta > 0,\ E_{\delta,\beta}(z)$ *be a two-parameter Mittag-Leffler function. Then*

$$\mathcal{L}[t^{\delta k+\beta-1} E^{(k)}_{\delta,\beta}(\pm a t^{\delta})](p) = \frac{k!p^{\delta-\beta}}{(p^{\delta} \mp a)^{k+1}},\ p > |a|^{\frac{1}{\delta}}.$$

Lemma 2.5. *The function* $E^{(n)}_{2-\alpha,\gamma+\alpha n}(-bt^{2-\alpha})$ *has the following properties, where* $n = 0, 1, 2, \cdots,\ t \in [0, 1]$.

(1) $0 < E^{(n)}_{2-\alpha,\gamma+\alpha n}(-bt^{2-\alpha}) \le \mathrm{e}^b$ *if* $\gamma \ge 1$ *and* $n \ge 0$, *or* $\gamma = 0$ *and* $n \ge 1$;

(2) $E^{(n)}_{2-\alpha,\gamma+\alpha n}(-bt^{2-\alpha})$ *is monotone decreasing with respect to* n *if* $\gamma \ge 2$ *and* $n \ge 0$, $\gamma = 1$ *and* $n \ge 2$, *or* $\gamma = 0$ *and* $n \ge 3$.

Proof. (1) Denote

$$\begin{aligned} E^{(n)}_{2-\alpha,\gamma+\alpha n}(-bt^{2-\alpha}) &= \sum_{j=0}^{\infty} \frac{(-1)^j (bt^{2-\alpha})^j \Gamma(k + j + 1)}{\Gamma(j + 1)\Gamma\big((2 - \alpha)(k + j) + (\gamma + \alpha n)\big)} \\ &:= \sum_{j=0}^{\infty} (-1)^j u_j. \end{aligned} \tag{2.1}$$

From (2.1), we have

$$\frac{u_{j+1}}{u_j} = bt^{2-\alpha} \frac{\Gamma((2 - \alpha)j + \gamma)}{\Gamma((2 - \alpha)j + \gamma + 2 - \alpha)} < 1,\ \gamma \ge 1 \text{ and } n = 0,$$

and

$$\frac{u_{j+1}}{u_j} \le \frac{(n + j + 1)\Gamma((2 - \alpha)j + \gamma + 2n)}{((2 - \alpha)j + \gamma + 2n + 1 - \alpha)\Gamma((2 - \alpha)j + \gamma + 2n + 1 - \alpha)} \le 1,$$

$$\gamma \geq 1 \text{ and } n \geq 1.$$

So $\{u_j\}$ is monotone decreasing. In addition, by Lemma 2.1, $u_j \to 0$ as $j \to \infty$. Therefore, $E^{(n)}_{2-\alpha,\gamma+\alpha n}(-bt^{2-\alpha}) > 0$ by Leibniz test for alternating series.

Since $0 < \alpha \leq 1$, $|(-1)^j u_j| \leq \frac{(bt^{2-\alpha})^j}{j!}$ and $\sum\limits_{j=0}^{\infty} \frac{(bt^{2-\alpha})^j}{j!} = \mathrm{e}^{bt^{2-\alpha}}$, then

$$E^{(n)}_{2-\alpha,\gamma+\alpha n}(-bt^{2-\alpha}) \leq \mathrm{e}^{b}.$$

Similarly, (1) holds if $\gamma = 0$ and $n \geq 1$.

(2) Since $\big(((2-\alpha)j+\gamma+2n)^2+(1-\alpha)j+\gamma+n-1\big) > 0$ if $\gamma \geq 2$ and $n \geq 0$, $\gamma = 1$ and $n \geq 2$, or $\gamma = 0$ and $n \geq 3$, then

$$\begin{aligned}
&E^{(n)}_{2-\alpha,\gamma+\alpha n}(-bt^{2-\alpha}) - E^{(n+1)}_{2-\alpha,\gamma+\alpha(n+1)}(-bt^{2-\alpha})\\
=&\sum_{j=0}^{\infty} \frac{(-1)^j (bt^{2-\alpha})^j \Gamma(n+j+1)}{\Gamma(j+1)\Gamma((2-\alpha)j+2+\gamma+2n)} \big(((2-\alpha)j+\gamma+2n)^2 + (1-\alpha)j + \gamma + n - 1\big)\\
:=&\sum_{j=0}^{\infty} (-1)^j v_j
\end{aligned}$$

is alternating series and

$$\begin{aligned}
\frac{v_{j+1}}{v_j} <& \frac{(n+j+1)\Big(\big((2-\alpha)j+\gamma+2-\alpha+2n\big)^2+n+(1-\alpha)j+\gamma-\alpha\Big)\Gamma((2-\alpha)j+2+\gamma+2n)}{\Big(\big((2-\alpha)j+\gamma+2n\big)^2+n+(1-\alpha)j+\gamma-1\Big)\big((2-\alpha)j+\gamma+3-\alpha+2n\big)\Gamma((2-\alpha)j+\gamma+3-\alpha+2n)}\\
\leq& \frac{(n+j+1)(((2-\alpha)j+\gamma+2-\alpha+2n)^2+n+(1-\alpha)j+\gamma-\alpha)}{(((2-\alpha)j+\gamma+2n)^2+n+(1-\alpha)j+\gamma-1)((2-\alpha)j+\gamma+3-\alpha+2n)}\\
\leq& \frac{(n+j+1)(((2-\alpha)j+\gamma+2-\alpha+2n)}{(((2-\alpha)j+\gamma+2n)^2+n+(1-\alpha)j+\gamma-1)}\\
=& \frac{(n+j+1)(((2-\alpha)j+\gamma+2n)+(n+j+1)(2-\alpha)}{((2-\alpha)j+\gamma+2n)(j+n+1)+(2-\alpha)j+\gamma+2n)((1-\alpha)j+\gamma+n-1)+n+(1-\alpha)j+\gamma-1}\\
\leq& 1.
\end{aligned}$$

We have $\{v_j\}$ is monotone decreasing. Besides, by Lemma 2.1, we obtain $v_j \to 0$ as $j \to \infty$. Thus,

$$E^{(n)}_{2-\alpha,\gamma+\alpha n}(-bt^{2-\alpha}) - E^{(n+1)}_{2-\alpha,\gamma+\alpha(n+1)}(-bt^{2-\alpha}) > 0,$$

by Leibniz test for alternating series. Namely, $E^{(n)}_{2-\alpha,\gamma+\alpha n}(-bt^{2-\alpha})$ is monotone decreasing with respect to n. □

Denote

$$g_1(t) = \sum_{n=0}^{\infty} \frac{(-a)^n}{n!} t^{2n+1} E^{(n)}_{2-\alpha,2+\alpha n}(-bt^{2-\alpha}), \tag{2.2}$$

$$g_2(t) = \sum_{n=0}^{\infty} \frac{(-a)^n}{n!} t^{2n+2-\alpha} E^{(n)}_{2-\alpha,3-\alpha+\alpha n}(-bt^{2-\alpha}).$$

Lemma 2.6. *The functions g_1 and g_2, defined above, have the following properties.*

(1) *$g_1(t)$ and $g_2(t)$ are represented by absolutely and uniformly convergent series and $|g_1(t)|$, $|g_2(t)| \leq \mathrm{e}^{a+b}$ on $t \in [0,1]$;*

(2) $g_1'(t)$ *is represented by absolutely and uniformly convergent series on* $[0,1]$,

$$g_1'(t) = \sum_{n=0}^{\infty} \frac{(-at^2)^n}{n!} E^{(n)}_{2-\alpha,1+\alpha n}(-bt^{2-\alpha}),$$

$|g_1'(t)| \leq \mathrm{e}^{a+b}$ *for* $t \in [0,1]$ *and* $g_1'(0) = 1$;

(3) $g_1(t)$, $g_2(t) > 0$ *and* $g_1'(t) > 0$ *for* $t \in (0,1]$;

(4) $g_1''(t) \leq 0$ *for* $t \in [0,1]$.

Proof. (1) By Lemma 2.5, we have

$$\Big|\frac{(-a)^n}{n!} t^{2n+1} E^{(n)}_{2-\alpha,2+\alpha n}(-bt^{2-\alpha})\Big| \leq t\mathrm{e}^b \frac{(at^2)^n}{n!}.$$

Furthermore, $\sum\limits_{n=0}^{\infty} \frac{(at^2)^n}{n!} = \mathrm{e}^{at^2}$, $t \in (-\infty,+\infty)$. So $\sum\limits_{n=0}^{\infty} \frac{(-a)^n}{n!} t^{2n+1} E^{(n)}_{2-\alpha,2+\alpha n}(-bt^{2-\alpha})$ is a absolutely and uniformly convergent series on $[0,1]$, that is, $g_1(t)$ is represented by absolutely and uniformly convergent series on $[0,1]$ and $|g_1(t)| \leq \mathrm{e}^{a+b}$.

Similarly, $g_2(t)$ is also represented by absolutely and uniformly convergent series on $[0,1]$ and $|g_2(t)| \leq \mathrm{e}^{a+b}$.

(2) In view of

$$\begin{aligned}
&\sum_{n=0}^{\infty} \Big(\frac{(-a)^n}{n!} t^{2n+1} \sum_{j=0}^{\infty} \frac{(-bt^{2-\alpha})^j \Gamma(n+j+1)}{\Gamma(j+1)\Gamma((2-\alpha)j+2+2n)}\Big)' \\
&= \sum_{n=0}^{\infty} \frac{(-a)^n}{n!} \Big((2n+1)t^{2n} \sum_{j=0}^{\infty} \frac{(-bt^{2-\alpha})^j \Gamma(n+j+1)}{\Gamma(j+1)\Gamma((2-\alpha)j+2+2n)} \\
&\quad + t^{2n+1} \sum_{j=1}^{\infty} (2-\alpha) \frac{(-b)^j (t^{(2-\alpha)j-1} \Gamma(n+j+1)}{\Gamma(j)\Gamma((2-\alpha)j+2+2n)}\Big) \\
&= \sum_{n=0}^{\infty} \frac{(-at^2)^n \Gamma(n+1)}{n!\Gamma(2n+1)} + \sum_{n=0}^{\infty} \frac{(-at^2)^n}{n!} \sum_{j=1}^{\infty} \frac{(-bt^{2-\alpha})^j \Gamma(n+j+1)}{\Gamma((2-\alpha)j+2+2n)} \Big(\frac{2n+1}{\Gamma(j+1)} + \frac{2-\alpha}{\Gamma(j)}\Big) \\
&= \sum_{n=0}^{\infty} \frac{(-at^2)^n \Gamma(n+1)}{n!\Gamma(2n+1)} + \sum_{n=0}^{\infty} \frac{(-at^2)^n}{n!} \sum_{j=1}^{\infty} \frac{(-bt^{2-\alpha})^j \Gamma(n+j+1)}{\Gamma(j+1)\Gamma((2-\alpha)j+1+2n)} \\
&= \sum_{n=0}^{\infty} \frac{(-at^2)^n}{n!} \sum_{j=0}^{\infty} \frac{(-bt^{2-\alpha})^j \Gamma(n+j+1)}{\Gamma(j+1)\Gamma((2-\alpha)j+1+2n)} \\
&= \sum_{n=0}^{\infty} \frac{(-at^2)^n}{n!} E^{(n)}_{2-\alpha,1+\alpha n}(-bt^{2-\alpha}).
\end{aligned}$$

Similar to the proof of (1), we can get (2) holds.

(3) From Lemma 2.5, we can get $E^{(n)}_{2-\alpha,2+\alpha n}(-bt^{2-\alpha}) > 0$ and $E^{(n)}_{2-\alpha,2+\alpha n}(-bt^{2-\alpha})$ is monotone decreasing with respect to $n \in \mathbb{N}_+$. Then $g_1(t)$ is alternating series.

Furthermore, $\frac{a^n}{n!} t^{2n+1}$ is monotone decreasing with $n \in \mathbb{N}_+$. Thus,

$$\frac{a^n}{n!} t^{2n+1} E^{(n)}_{2-\alpha,2+\alpha n}(-bt^{2-\alpha})$$

is monotone decreasing with $n \in \mathbb{N}_+$ and converges to 0 as $n \to \infty$.

Therefore, according to Leibniz test for alternating series, we have $g_1(t) > 0$ for $t \in (0,1]$.

Similar to the proof above, we can get $g_2(t) > 0$ for $t \in (0,1]$.

By Lemma 2.5, we can get $E^{(n)}_{2-\alpha,1+\alpha n}(-bt^{2-\alpha}) > 0$ and $g_1'(t)$ is alternating series for $t \in (0,1]$.

Similar to the proof above, we can get

$$\sum_{n=2}^{\infty} \frac{(-at^2)^n}{n!} E^{(n)}_{2-\alpha,1+\alpha n}(-bt^{2-\alpha}) > 0.$$

Since

$$\begin{aligned} E_{2-\alpha,1}(-bt^{2-\alpha}) - E^{(1)}_{2-\alpha,1+\alpha}(-bt^{2-\alpha}) &= \sum_{j=0}^{\infty} \frac{(-bt^{2-\alpha})^j \big(((2-\alpha)j+1)^2 + (1-\alpha)j\big)}{\Gamma((2-\alpha)j+3)} \\ &:= \sum_{j=0}^{\infty} (-1)^j w_j. \end{aligned}$$

Then for $j \geq 1$, we can get

$$\begin{aligned} \frac{w_{j+1}}{w_j} &\leq \frac{((2-\alpha)j+3-\alpha)^2 + (1-\alpha)j + 1 - \alpha}{\big(((2-\alpha)j+1)^2 + (1-\alpha)j\big)((2-\alpha)j+4-\alpha)} \\ &\leq \frac{((2-\alpha)j+3-\alpha)((2-\alpha)j+4-\alpha)}{\big(((2-\alpha)j+1)^2 + (1-\alpha)j\big)((2-\alpha)j+4-\alpha)} \\ &< 1. \end{aligned}$$

Thus, $\{w_j\}$ is monotone decreasing. By Lemma 2.1, $w_j \to 0$ as $j \to \infty$. Therefore, according to Leibniz test for alternating series, we have

$$\sum_{j=2}^{\infty} \frac{(-bt^{2-\alpha})^j \big(((2-\alpha)j+1)^2 + (1-\alpha)j\big)}{\Gamma((2-\alpha)j+3)} > 0.$$

On the other hand, since $0 \leq b \leq \min\{1, \frac{\Gamma(5-\alpha)}{2((3-\alpha)^2+1-\alpha)}\}$, we have

$$\begin{aligned} \sum_{j=0}^{1} \frac{(-bt^{2-\alpha})^j \big(((2-\alpha)j+1)^2 + (1-\alpha)j\big)}{\Gamma((2-\alpha)j+3)} &= \frac{1}{2} - b\frac{t^{2-\alpha}((3-\alpha)^2 + 1 - \alpha)}{\Gamma(5-\alpha)} \\ &\geq \frac{1}{2} - \frac{b((3-\alpha)^2 + 1 - \alpha)}{\Gamma(5-\alpha)} \\ &\geq 0. \end{aligned}$$

Thus, $E_{2-\alpha,1}(-bt^{2-\alpha}) - E^{(1)}_{2-\alpha,1+\alpha}(-bt^{2-\alpha}) \geq 0$. Moreover, for $t \in (0,1]$,

$$g_1'(t) \geq E_{2-\alpha,1}(-bt^{2-\alpha}) - E^{(1)}_{2-\alpha,1+\alpha}(-bt^{2-\alpha}) + \sum_{n=2}^{\infty} \frac{(-at^2)^n}{n!} E^{(n)}_{2-\alpha,1+\alpha n}(-bt^{2-\alpha}) > 0.$$

(4) By Lemma 2.6 (2), we can show that

$$\begin{aligned}
g_1''(t)=&\sum_{j=1}^{\infty}\frac{(-b)^j t^{(2-\alpha)j-1}}{\Gamma((2-\alpha)j)}+\sum_{n=1}^{\infty}\frac{(-a)^n}{n!}\Big(2nt^{2n-1}\sum_{j=0}^{\infty}\frac{(-bt^{2-\alpha})^j\Gamma(n+j+1)}{\Gamma(j+1)\Gamma((2-\alpha)j+1+2n)}\\
&+(2-\alpha)t^{2n}\sum_{j=1}^{\infty}\frac{(-b)^j t^{(2-\alpha)j-1}\Gamma(j+n+1)}{\Gamma(j(2-\alpha)+1+2n)\Gamma(j)}\Big)\\
=&\sum_{j=1}^{\infty}\frac{(-b)^j t^{(2-\alpha)j-1}}{\Gamma((2-\alpha)j)}+\sum_{n=1}^{\infty}\frac{(-a)^n}{n!}t^{2n-1}\Big(2n\frac{\Gamma(n+1)}{\Gamma(1+2n)}\\
&+\sum_{j=1}^{\infty}\frac{(-bt^{2-\alpha})^j\Gamma(n+j+1)}{\Gamma(j+1)\Gamma((2-\alpha)j+1+2n)}\big(2n+(2-\alpha)j\big)\Big)\\
=&-\sum_{j=1}^{\infty}\frac{(-1)^{j-1}b^j t^{(2-\alpha)j-1}}{\Gamma((2-\alpha)j)}-\sum_{n=1}^{\infty}\frac{(-1)^{n-1}a^n}{n!}t^{2n-1}\sum_{j=0}^{\infty}\frac{(-bt^{2-\alpha})^j\Gamma(n+j+1)}{\Gamma(j+1)\Gamma((2-\alpha)j+2n)}\\
=&-\sum_{j=1}^{\infty}\frac{(-1)^{j-1}b^j t^{(2-\alpha)j-1}}{\Gamma((2-\alpha)j)}-\sum_{n=1}^{\infty}\frac{(-1)^{n-1}a^n}{n!}t^{2n-1}E^{(n)}_{2-\alpha,\alpha n}(-bt^{2-\alpha}). \qquad (2.3)
\end{aligned}$$

Because $0<\alpha\leq 1$ and $0\leq b\leq \min\{1,\ \frac{\Gamma(5-\alpha)}{2((3-\alpha)^2+1-\alpha)}\}$, according to Leibniz test for alternating series, we have

$$\sum_{j=1}^{\infty}\frac{(-1)^{j-1}b^j t^{(2-\alpha)j-1}}{\Gamma((2-\alpha)j)}\geq 0.$$

On the other hand, by Lemma 2.5, similar to the proof of (3), we can show that

$$\sum_{n=3}^{\infty}\frac{(-1)^{n-1}a^n}{n!}t^{2n-1}E^{(n)}_{2-\alpha,\alpha n}(-bt^{2-\alpha})\geq 0.$$

Let

$$\begin{aligned}
E^{(1)}_{2-\alpha,\alpha}(-bt^{2-\alpha})-E^{(2)}_{2-\alpha,2\alpha}(-bt^{2-\alpha})&=\sum_{j=0}^{\infty}\frac{(-bt^{2-\alpha})^j(j+1)\big(((2-\alpha)j+2)^2+(1-\alpha)j\big)}{\Gamma((2-\alpha)j+4)}\\
&:=\sum_{j=0}^{\infty}(-1)^j z_j.
\end{aligned}$$

If $j\geq 1$, we have

$$\begin{aligned}
\frac{z_{j+1}}{z_j}\leq&\frac{(j+2)\big(((2-\alpha)j+4-\alpha)^2+(1-\alpha)j+1-\alpha\big)\Gamma((2-\alpha)j+4)}{(j+1)\big(((2-\alpha)j+2)^2+(1-\alpha)j\big)((2-\alpha)j+5-\alpha)\Gamma((2-\alpha)j+5-\alpha)}\\
\leq&\frac{(j+2)\big(((2-\alpha)j+4-\alpha)^2+(1-\alpha)j+1-\alpha\big)}{(j+1)\big(((2-\alpha)j+2)^2+(1-\alpha)j\big)((2-\alpha)j+5-\alpha)}\\
\leq&\frac{(j+2)((2-\alpha)j+2-\alpha)+2(j+2)}{(j+1)\big(((2-\alpha)j+1+1)^2+(1-\alpha)j\big)}\\
=&\frac{(j+1)((2-\alpha)j+2-\alpha)+(2-\alpha)j+4-\alpha+2(j+1)}{(j+1)((2-\alpha)j+2)(2-\alpha)j+(j+1)((2-\alpha)j+2)+(j+1)((2-\alpha)j+2)+(j+1)(1-\alpha)j}\\
\leq&1.
\end{aligned}$$

Thus, $\{z_j\}$ is monotone decreasing. By Lemma 2.1, we have $z_j \to 0$ as $j \to \infty$. Therefore, according to Leibniz test for alternating series,

$$\sum_{j=2}^{\infty} \frac{(-bt^{2-\alpha})^j (j+1)\big(((2-\alpha)j+2)^2+(1-\alpha)j\big)}{\Gamma((2-\alpha)j+4)} \geq 0.$$

In addition,

$$\begin{aligned}&\sum_{j=0}^{1} \frac{(-bt^{2-\alpha})^j (j+1)\big(((2-\alpha)j+2)^2+(1-\alpha)j\big)}{\Gamma((2-\alpha)j+4)}\\ =&\frac{2}{3}-2b\frac{t^{2-\alpha}((4-\alpha)^2+1-\alpha)}{\Gamma(6-\alpha)}\\ \geq&\frac{2}{3}-\frac{(4-\alpha)^2+1-\alpha}{\Gamma(6-\alpha)}\cdot\frac{\Gamma(5-\alpha)}{(3-\alpha)^2+1-\alpha}\\ =&\frac{2}{3}-\frac{(4-\alpha)^2+1-\alpha}{(5-\alpha)\big((3-\alpha)^2+1-\alpha\big)}\\ >&0.\end{aligned}$$

Hence,

$$E^{(1)}_{2-\alpha,\alpha}(-bt^{2-\alpha}) - E^{(2)}_{2-\alpha,2\alpha}(-bt^{2-\alpha}) \geq 0.$$

We can obtain

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{(-1)^{n-1}a^n}{n!}t^{2n-1}E^{(n)}_{2-\alpha,\alpha n}(-bt^{2-\alpha})\\ \geq&at\big(E^{(1)}_{2-\alpha,\alpha}(-bt^{2-\alpha})-E^{(2)}_{2-\alpha,2\alpha}(-bt^{2-\alpha})\big)+\sum_{n=3}^{\infty}\frac{(-1)^{n-1}a^n t^{2n-1}}{n!}E^{(n)}_{2-\alpha,\alpha n}(-bt^{2-\alpha})\\ \geq&0.\end{aligned}$$

Therefore, by (2.3), $g_1''(t) \leq 0$ for $t \in [0,1]$. □

Let

$$G(t,s)=\begin{cases}\frac{g_1(t)g_1(1-s)}{g_1(1)}-g_1(t-s), & 0\leq s\leq t\leq 1,\\ \frac{g_1(t)g_1(1-s)}{g_1(1)}, & 0\leq t< s\leq 1.\end{cases} \tag{2.4}$$

Lemma 2.7. *The function $G(t,s)$ is continuous on $(t,s) \in [0,1]\times[0,1]$ and $G(t,s)>0$ in $(t,s)\in(0,1)\times(0,1)$.*

Proof. By Lemma 2.6, $G(t,s)$ is continuous for $(t,s) \in [0,1]\times[0,1]$ and if $0 < t < s < 1$, we can also have $G(t,s) > 0$.

If $0 < s \leq t \leq 1$, it follows

$$\frac{\partial}{\partial t}\Big(\frac{g_1(t)}{g_1(t-s)}\Big)=\frac{g_1'(t)g_1(t-s)-g_1(t)g_1'(t-s)}{g_1^2(t-s)}<0$$

from Lemma 2.6, which implies $\frac{g_1(t)}{g_1(t-s)}$ is monotone decreasing with respect to t.

So, $\frac{g_1(t)}{g_1(t-s)} > \frac{g_1(1)}{g_1(1-s)}$ and $G(t,s) > 0$.

Hence, $G(t,s) > 0$ for $(t,s) \in (0,1)\times(0,1)$. □

By Lemma 2.7, we can obtain the following Lemma 2.8 holds.

Lemma 2.8. *Let the function G be defined by* (2.4)*, then*

$$\int_0^1 \int_0^1 G(t,s)\mathrm{d}s\mathrm{d}A(t) \neq 0. \tag{2.5}$$

Lemma 2.9 (See [4, 21])**.** *Let X and Y be Banach spaces and let $\Omega \subset X$ be a bounded open symmetric set with $0 \in \Omega$. Let $L : \mathrm{Dom}L \subset X \to Y$ be a Fredholm operator of index zero with $\mathrm{Dom}L \cap \overline{\Omega} \neq \emptyset$ and $N : X \to Y$ be an L-compact operator on $\overline{\Omega}$. Assume that*

$$Lx - Nx \neq -\lambda(Lx + N(-x))$$

for all $x \in \mathrm{Dom}L \cap \partial\Omega$ and all $\lambda \in (0,1]$, where $\partial\Omega$ is the boundary of Ω with respect to X. Then the equation $Lx = Nx$ has at least one solution on $\mathrm{Dom}L \cap \overline{\Omega}$.

3. The existence of the solutions

Throughout this paper, we always suppose that the following resonance condition is satisfied

$$g_1(1) = \int_0^1 g_1(t)\mathrm{d}A(t). \tag{3.1}$$

Lemma 3.1. *For $y \in C[0,1]$, the equation*

$$x''(t) + b\,{}^cD^{\alpha}_{0+}x(t) + ax(t) = y(t) \tag{3.2}$$

has general solution

$$x(t) = x'(0)g_1(t) + x(0)(g_1'(t) + bg_2(t)) + \int_0^t g_1(t-s)y(s)\mathrm{d}s. \tag{3.3}$$

Proof. By Lemma 2.3, we have

$$\mathcal{L}[x''(t)](p) = p^2\mathcal{L}[x(t)](p) - x(0)p - x'(0),$$

and

$$\mathcal{L}[{}^cD^{\alpha}_{0+}x(t)](p) = p^{\alpha}\mathcal{L}[x(t)](p) - x(0)p^{\alpha-1}.$$

Apply Laplace transform to both sides of the equation (3.2), we can easily obtain

$$\mathcal{L}[x(t)](p) = \frac{x(0)p + x'(0) + bx(0)p^{\alpha-1} + \mathcal{L}[y(t)](p)}{p^2 + a + bp^{\alpha}}. \tag{3.4}$$

If $0 \leq \frac{a}{p^2+bp^{\alpha}} < 1$, we have

$$\begin{aligned}\frac{1}{p^2 + a + bp^{\alpha}} &= \frac{p^{-\alpha}}{p^{2-\alpha} + b} \cdot \frac{1}{1 + \frac{ap^{-\alpha}}{p^{2-\alpha}+b}} = \frac{p^{-\alpha}}{p^{2-\alpha} + b}\sum_{n=0}^{\infty}\left(\frac{-ap^{-\alpha}}{p^{2-\alpha}+b}\right)^n \\ &= \sum_{n=0}^{\infty}\frac{(-a)^n p^{-\alpha(n+1)}}{(p^{2-\alpha}+b)^{n+1}}.\end{aligned}$$

By virtue of Lemma 2.4, we can show

$$\frac{1}{p^2+a+bp^\alpha}=\sum_{n=0}^{\infty}\frac{(-a)^n p^{-\alpha(n+1)}}{(p^{2-\alpha}+b)^{n+1}}=\mathcal{L}[g_1(t)](p),\ p>a^{\frac{1}{2-\alpha}},$$

$$\frac{p}{p^2+a+bp^\alpha}=\sum_{n=0}^{\infty}\frac{(-a)^n p^{1-\alpha(n+1)}}{(p^{2-\alpha}+b)^{n+1}}=\mathcal{L}[g_1'(t)](p),\ 0\le\frac{a}{p^2+bp^\alpha}<1 \text{ and } p>a^{\frac{1}{2-\alpha}},$$

$$\frac{p^{\alpha-1}}{p^2+a+bp^\alpha}=\sum_{n=0}^{\infty}\frac{(-a)^n p^{-\alpha n-1}}{(p^{2-\alpha}+b)^{n+1}}=\mathcal{L}[g_2(t)](p),\ 0\le\frac{a}{p^2+bp^\alpha}<1 \text{ and } p>a^{\frac{1}{2-\alpha}}.$$

So (3.4) is equivalent to

$$\mathcal{L}[x(t)](p)=x'(0)\mathcal{L}[g_1(t)](p)+\mathcal{L}[y(t)](p)\mathcal{L}[g_1(t)](p)+x(0)\mathcal{L}[g_1'(t)](p)+bx(0)\mathcal{L}[g_2(t)](p). \tag{3.5}$$

On the other hand,

$$\mathcal{L}[y(t)](p)\mathcal{L}[g_1(t)](p)=\mathcal{L}[\int_0^t g_1(t-s)y(s)\mathrm{d}s](p),$$

then we can get the inverse Laplace transform for (3.5) is (3.3). □

By Lemma 3.1, we can obtain the following Lemma 3.2 holds.

Lemma 3.2. *Boundary value problem* (1.1) *is equivalent to the following problem*

$$\begin{cases} x(t)=x'(0)g_1(t)+\int_0^t g_1(t-s)f(s,x(s),x'(s))\mathrm{d}s,\ t\in(0,1),\\ x(1)=\int_0^1 x(t)\mathrm{d}A(t). \end{cases} \tag{3.6}$$

Let $X=C^1[0,1]$, with the norm $\|x\|=\max\{\|x\|_\infty,\ \|x'\|_\infty\}$ where $\|x\|_\infty=\max_{t\in[0,1]}|x(t)|$. Obviously, $(X,\|\cdot\|)$ is a Banach space.

We denote

(H1) There exist nonnegative functions $\phi,\ \varphi,\ \psi\in C[0,1]$ such that

$$|f(t,x,y)|\le\phi(t)+\varphi(t)|x|+\psi(t)|y|,\ t\in[0,1],\ \text{and } (x,y)\in\mathbb{R}\times\mathbb{R}.$$

(H2) There exists a constant $M_0>0$ such that if $|y|>M_0$, then $f(t,x,y)-\lambda f(t,-x,-y)>0$ or $f(t,x,y)-\lambda f(t,-x,-y)<0$, where $\lambda\in(0,1]$, $t\in[0,1]$ and $(x,y)\in\mathbb{R}\times\mathbb{R}$.

Define the operators

$$L:\mathrm{Dom}L\subset X\to X,\ Lx=x''+b{}^cD_{0+}^\alpha x+ax, \tag{3.7}$$

$$N:\ X\to X,\ Nx(t)=f(t,x(t),x'(t)),\ t\in[0,1], \tag{3.8}$$

where $\mathrm{Dom}L=\{x\in X\cap C^2[0,1]:x(0)=0,\ x(1)=\int_0^1 x(t)\mathrm{d}A(t)\}$.

Theorem 3.1. *Suppose (H1) and (H2) hold. If* $\int_0^1(\varphi(s)+\psi(s))\mathrm{d}s+\frac{b}{\Gamma(3-\alpha)}+a<1$, *then boundary value problem* (1.1) *has at least one solution.*

Proof. Step 1: L is a Fredholm operator of index zero.

It is easy to see that

$$\mathrm{Ker}L=\{x\in X:\ x=cg_1(t),\ c\in\mathbb{R}\}.$$

For $y \in \mathrm{Im}L$, there exists $x \in \mathrm{Dom}L$ such that $x(0) = 0$ and $Lx = y$. By Lemma 3.1, we have

$$x(t) = x'(0)g_1(t) + \int_0^t g_1(t-s)y(s)\mathrm{d}s,\ t \in [0,1],$$

and $x(1) = \int_0^1 x(t)\mathrm{d}A(t)$. Since $g_1(1) = \int_0^1 g_1(t)\mathrm{d}A(t)$, then

$$\int_0^1 \frac{g_1(t)}{g_1(1)} \int_0^1 g_1(1-s)y(s)\mathrm{d}s\mathrm{d}A(t) = \int_0^1 \int_0^t g_1(t-s)y(s)\mathrm{d}s\mathrm{d}A(t).$$

Thus, $\int_0^1 \int_0^1 G(t,s)y(s)\mathrm{d}s\mathrm{d}A(t) = 0$. We can show

$$\mathrm{Im}L \subseteq \{y \in X : \int_0^1 \int_0^1 G(t,s)y(s)\mathrm{d}s\mathrm{d}A(t) = 0\}.$$

If $y \in \{y \in X : \int_0^1 \int_0^1 G(t,s)y(s)\mathrm{d}s\mathrm{d}A(t) = 0\}$, let

$$x(t) = \int_0^t g_1(t-s)y(s)\mathrm{d}s, \tag{3.9}$$

then $x \in X \cap C^2[0,1]$ and $x(0) = 0$. Since $g_1(1) = \int_0^1 g_1(t)\mathrm{d}A(t)$, we have

$$\begin{aligned} x(1) &= \int_0^1 g_1(1-s)y(s)\mathrm{d}s = \int_0^1 g_1(1-s)y(s)\mathrm{d}s \int_0^1 \frac{g_1(t)}{g_1(1)}\mathrm{d}A(t) \\ &= \int_0^1 \int_0^1 \frac{g_1(t)g_1(1-s)}{g_1(1)} y(s)\mathrm{d}s\mathrm{d}A(t). \end{aligned}$$

On the other hand,

$$\int_0^1 x(t)\mathrm{d}A(t) = \int_0^1 \int_0^t g_1(t-s)y(s)\mathrm{d}s\mathrm{d}A(t).$$

Because $\int_0^1 \int_0^1 G(t,s)y(s)\mathrm{d}s\mathrm{d}A(t) = 0$, we can show

$$\begin{aligned} 0 &= \int_0^1 \int_0^1 G(t,s)y(s)\mathrm{d}s\mathrm{d}A(t) \\ &= \int_0^1 \Big(\int_0^t \Big(\frac{g_1(t)g_1(1-s)}{g_1(1)} - g_1(t-s) \Big) y(s)\mathrm{d}s + \int_t^1 \frac{g_1(t)g_1(1-s)}{g_1(1)} y(s)\mathrm{d}s \Big) \mathrm{d}A(t) \\ &= \int_0^1 \int_0^1 \frac{g_1(t)g_1(1-s)}{g_1(1)} y(s)\mathrm{d}s\mathrm{d}A(t) - \int_0^1 \int_0^t g_1(t-s)y(s)\mathrm{d}s\mathrm{d}A(t) \\ &= x(1) - \int_0^1 x(t)\mathrm{d}A(t). \end{aligned}$$

Thus,

$$x \in \mathrm{Dom}L,\ Lx = y. \tag{3.10}$$

Hence,

$$\mathrm{Im}L \supseteq \{y \in X : \int_0^1 \int_0^1 G(t,s)y(s)\mathrm{d}s\mathrm{d}A(t) = 0\}.$$

Therefore,

$$\mathrm{Im}L = \{y \in X : \int_0^1 \int_0^1 G(t,s)y(s)\mathrm{d}s\mathrm{d}A(t) = 0\}.$$

Define linear continuous projector operators $P : X \to X$ and $Q : X \to X$ by

$$Px(t) = x'(0)g_1(t), \quad Qx(t) = \frac{\int_0^1 \int_0^1 G(t,s)x(s)\mathrm{d}s\mathrm{d}A(t)}{\int_0^1 \int_0^1 G(t,s)\mathrm{d}s\mathrm{d}A(t)}.$$

We can easily obtain that $\mathrm{Im}P = \mathrm{Ker}L$, $X = \mathrm{Ker}P \oplus \mathrm{Ker}L$, $\mathrm{Im}L = \mathrm{Ker}Q$, $X = \mathrm{Im}L \oplus \mathrm{Im}Q$. So

$$1 = \dim \mathrm{Ker}L = \dim \mathrm{Im}Q = \mathrm{codim}\mathrm{Im}L,$$

that is, L is a Fredholm operator with index zero.

Step 2: The operator N is L-compact on any bounded open set $\Omega \subset X$.

Let $K_P : \mathrm{Im}L \to X$,

$$K_P y(t) = \int_0^t g_1(t-s)y(s)\mathrm{d}s.$$

In views of (3.9) and (3.10), we can get $K_P : \mathrm{Im}L \to \mathrm{Dom}L \cap \mathrm{Ker}P$ and for $y \in \mathrm{Im}L$,

$$LK_P y(t) = L(\int_0^t g_1(t-s)y(s)\mathrm{d}s) = y(t).$$

For $x \in \mathrm{Dom}L \cap \mathrm{Ker}P$, $K_P Lx(t) := K_P y(t) = x(t)$.

Then K_P is the inverse mapping of $L|_{\mathrm{Dom}L\cap\mathrm{Ker}P}$. Hence,

$$QNx(t) = \frac{1}{\int_0^1 \int_0^1 G(t,s)\mathrm{d}s\mathrm{d}A(t)} \int_0^1 \int_0^1 G(t,s)f(s,x(s),x'(s))\mathrm{d}s\mathrm{d}A(t),$$

and

$$\begin{aligned} K_P(I-Q)Nx(t) = &\int_0^t g_1(t-s)f(s,x(s),x'(s))\mathrm{d}s \\ &- \frac{\int_0^1 \int_0^1 G(\tau,s)f(s,x(s),x'(s))\mathrm{d}s\mathrm{d}A(\tau)}{\int_0^1 \int_0^1 G(\tau,s)\mathrm{d}s\mathrm{d}A(\tau)} \int_0^t g_1(t-s)\mathrm{d}s. \end{aligned}$$

Since f is measurable, G and g_1 are continuous, we can easily get that $QN : \overline{\Omega} \to X$ and $K_P(I-Q)N : \overline{\Omega} \to X$ are continuous and compact operators, that is, the operator N is L-compact on any bounded open set $\Omega \subset X$.

Step 3: Boundary value problem (1.1) has at least one solution.

Let $x \in \mathrm{Dom}L$ and satisfy

$$Lx - Nx = -\lambda(Lx + N(-x)),\ \lambda \in (0,1].$$

We have

$$Lx = \frac{1}{1+\lambda}(Nx - \lambda N(-x)).$$

Hence, $\frac{1}{1+\lambda}(Nx - \lambda N(-x)) \in \mathrm{Im}L$ and

$$\int_0^1 \int_0^1 G(t,s)\frac{1}{1+\lambda}(Nx - \lambda N(-x))\mathrm{d}s\mathrm{d}A(t)$$

$$=\frac{1}{1+\lambda}\int_0^1\int_0^1 G(t,s)\big(f(s,x(s),x'(s))-\lambda f(s,-x(s),-x'(s))\big)\mathrm{d}s\mathrm{d}A(t)$$
$$=0.$$

By (H2) and Lemma 2.7, there exists $t_0\in[0,1]$ such that $|x'(t_0)|\le M_0$.
By (H1) and $x(0)=0$, we have

$$\begin{aligned}\int_0^1|f(s,x(s),x'(s))|\mathrm{d}s\le&\int_0^1\phi(s)\mathrm{d}s+\int_0^1\varphi(s)|x(s)|\mathrm{d}s+\int_0^1\psi(s)|x'(s)|\mathrm{d}s\\ \le&\int_0^1\phi(s)\mathrm{d}s+\int_0^1\varphi(s)|x(s)-x(0)|\mathrm{d}s+\int_0^1\psi(s)|x'(s)|\mathrm{d}s\\ \le&\int_0^1\phi(s)\mathrm{d}s+\|x'\|_\infty\Big(\int_0^1\big(\varphi(s)+\psi(s)\big)\mathrm{d}s\Big),\end{aligned}\tag{3.11}$$

$$|x(t)|=|x(0)+\int_0^t x'(s)\mathrm{d}s|=|\int_0^t x'(s)\mathrm{d}s|\le\int_0^t|x'(s)|\mathrm{d}s\le\|x'\|_\infty$$

and

$$\|x\|_\infty\le\|x'\|_\infty.\tag{3.12}$$

We can also get

$$\begin{aligned}\Big|\int_{t_0}^t {}^cD_{0+}^\alpha x(s)\mathrm{d}s\Big|=&\frac{1}{\Gamma(1-\alpha)}\Big|\int_{t_0}^t\int_0^s(s-r)^{-\alpha}x'(s)\mathrm{d}r\mathrm{d}s\Big|\\ \le&\frac{\|x'\|_\infty}{\Gamma(1-\alpha)}\int_0^1\int_0^s(s-r)^{-\alpha}\mathrm{d}r\mathrm{d}s\\ =&\frac{\|x'\|_\infty}{\Gamma(3-\alpha)},\end{aligned}\tag{3.13}$$

and

$$\int_{t_0}^t x''(s)\mathrm{d}s=x'(t)-x'(t_0).\tag{3.14}$$

Then, from t_0 to t, integrate both sides of

$$x''(t)+b{}^cD_{0+}^\alpha x(t)+ax(t)=f(t,x(t),x'(t)),$$

it follows

$$\begin{aligned}|x'(t)|\le&|x'(t_0)|+\int_0^1|f(s,x(s),x'(s))|\mathrm{d}s+b\Big|\int_{t_0}^t {}^cD_{0+}^\alpha x(s)\mathrm{d}s\Big|+a\int_0^1|x(s)|\mathrm{d}s\\ \le&M_0+\int_0^1\phi(s)\mathrm{d}s+\Big(\int_0^1\big(\varphi(s)+\psi(s)\big)\mathrm{d}s+\frac{b}{\Gamma(3-\alpha)}+a\Big)\|x'\|_\infty\end{aligned}$$

from (3.11), (3.12), (3.13) and (3.14). Thus,

$$\|x'\|_\infty\le\frac{M_0+\int_0^1\phi(s)\mathrm{d}s}{1-\Big(\int_0^1\big(\varphi(s)+\psi(s)\big)\mathrm{d}s+\frac{b}{\Gamma(3-\alpha)}+a\Big)}.$$

Hence, there exists a constant $B > 0$ such that $\|x\| \le B$. Thus, there exists a bounded open set $\Omega = \{x \in X : \|x\| < B + 1\} \subset X$ such that

$$Lx - Nx \neq -\lambda(Lx + N(-x)),\ x \in \partial\Omega,\ \lambda \in (0,1].$$

Then, Lemma 2.9 holds. Therefore, boundary value problem (1.1) has at least one solution. □

Remark 3.1. The result of Theorem 3.1 is obtained under the parameters a and b satisfy the conditions $0 \le a \le 1$ and $0 \le b \le \min\{1, \frac{\Gamma(5-\alpha)}{2((3-\alpha)^2+1-\alpha)}\}$ which proves that $G(t,s) > 0$. In fact, if only $a \in \mathbb{R}$ and $|b| \le 1$ are required, then $g_1(t)$ is represented by absolutely and uniformly convergent series on $[0,1]$ and $g_1 \in C[0,1]$, but $G(t,s)$ may be sign-changing in $(t,s) \in (0,1) \times (0,1)$. In this case, if we assume that $\int_0^1 \int_0^1 G(t,s)\mathrm{d}s\mathrm{d}A(t) \neq 0$ and $\int_0^1 (\varphi(s) + \psi(s))\mathrm{d}s + \frac{|b|}{\Gamma(3-\alpha)} + |a| < 1$, then Theorem 3.1 is also valid.

4. Illustration

In order to illustrate the applicability of our main results, we give out the following examples.

Example 4.1. We consider the boundary value problem

$$\begin{cases} x''(t) + 0.1662\,{}^cD_{0^+}^{0.5}x(t) + 0.5x(t) = \frac{t^3}{2}\sin x(t) + \frac{1}{26}x'(t) + t,\ t \in (0,1), \\ x(0) = 0,\ x(1) = \int_0^1 (1.1785801t^2 + 1)x(t)\mathrm{d}t, \end{cases} \tag{4.1}$$

where $\alpha = 0.5$, $a = 0.5$, $b = 0.1662$, $A(t) = 0.595267t^3 + t + 1$ and $f(t,x,y) = \frac{t^3}{2}\sin x + \frac{1}{26}y + t$.

Through calculation, we have $\int_0^1 g_1(t)\mathrm{d}A(t) = 0.872904 = g_1(1)$, $b = 0.1662 < \frac{\Gamma(5-0.5)}{2((3-0.5)^2+1-0.5)} \approx 0.86161$.

Let $\phi(t) = 2t,\ \varphi(t) = t^3,\ \psi(t) = \frac{1}{13}$, then

$$|f(t,x,y)| \le \phi(t) + \varphi(t)|x| + \psi(t)|y|,\ \text{for } t \in [0,1],\ (x,y) \in \mathbb{R} \times \mathbb{R},$$

and $\int_0^1 (\varphi(s) + \psi(s))\mathrm{d}s + \frac{b}{\Gamma(3-\alpha)} + a \approx 0.951947 < 1$.

If $y > 60$, then for any $\lambda \in [0,1]$,

$$f(t,x,y) - \lambda f(t,-x,-y) = \frac{(1+\lambda)t^3}{2}\sin x + \frac{1+\lambda}{26}y + (1-\lambda)t > 0,$$

and if $y < -60$, then for any $\lambda \in [0,1]$,

$$f(t,x,y) - \lambda f(t,-x,-y) = \frac{(1+\lambda)t^3}{2}\sin x + \frac{1+\lambda}{26}y + (1-\lambda)t < 0.$$

It follows from Theorem 3.1 that boundary value problem (4.1) has at least one solution.

Example 4.2 We consider the boundary value problem

$$\begin{cases} x''(t) + 0.25\,{}^cD_{0^+}^{0.5}x(t) + 0.5x(t) = 0.05\,\mathrm{e}^t \sin \mathrm{e}^{-2t}x(t) + 0.4\,\mathrm{e}^{-t}\arctan x'(t),\ t \in (0,1), \\ x(0) = 0,\ x(1) = \frac{g_1(1)}{g_1(\frac{1}{2})}x(\frac{1}{2}), \end{cases} \tag{4.2}$$

where $\alpha = 0.5,\ a = 0.5,\ b = 0.25,\ f(t,x,y) = 0.05\,\mathrm{e}^t \sin \mathrm{e}^{-2t} x + 0.4\,\mathrm{e}^{-t} \arctan y$ and

$$A(t) = \begin{cases} 0, & 0 \le t < \frac{1}{2}, \\ \frac{g_1(1)}{g_1(\frac{1}{2})}, & \frac{1}{2} \le t < 1 \end{cases} \quad = \begin{cases} 0, & 0 \le t < \frac{1}{2}, \\ 1.78428, & \frac{1}{2} \le t < 1. \end{cases}$$

Through calculation, we have

$$\int_0^1 g_1(t)\mathrm{d}A(t) = \frac{g_1(1)}{g_1(\frac{1}{2})} g_1(\frac{1}{2}) = g_1(1),$$

$b = 0.25 < \frac{\Gamma(5-0.5)}{2((3-0.5)^2+1-0.5)} \approx 0.86161.$

Let $\phi(t) = 0,\ \varphi(t) = 0.05\,\mathrm{e}^{-t},\ \psi(t) = 0.4\,\mathrm{e}^{-t}$, then

$$|f(t,x,y)| \le \phi(t) + \varphi(t)|x| + \psi(t)|y|,\ \text{for } t \in [0,1],\ (x,y) \in \mathbb{R} \times \mathbb{R},$$

and $\int_0^1 (\varphi(s) + \psi(s))\mathrm{d}s + \frac{b}{\Gamma(3-\alpha)} + a \approx 0.972517 < 1.$

If $y > 100$, then for any $\lambda \in [0,1]$,

$$f(t,x,y) - \lambda f(t,-x,-y) = (1+\lambda)(0.05\,\mathrm{e}^t \sin \mathrm{e}^{-2t} x + 0.4\,\mathrm{e}^{-t} \arctan y) > 0,$$

and if $y < -100$, then for any $\lambda \in [0,1]$,

$$f(t,x,y) - \lambda f(t,-x,-y) = (1+\lambda)(0.05\,\mathrm{e}^t \sin \mathrm{e}^{-2t} x + 0.4\,\mathrm{e}^{-t} \arctan y) < 0.$$

It follows from Theorem 3.1 that boundary value problem (4.2) has at least one solution.

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References

[1] O. Abu Arqub and B. Maayah, *Solutions of Bagley-Torvik and Painlev, equations of fractional order using iterative reproducing kernel algorithm with error estimates*, Neural Comput. Appl., 2018, 29, 1465–1479.

[2] Z. Bai, Z. Du and S. Zhang, *Iterative method for a class of fourth-order p-Laplacian beam equation*, J. Appl Anal. Comput., 2019, 9, 1443–1453.

[3] Z. Bai, X. Dong and C. Yin, *Existence results for impulsive nonlinear fractional differential equation with mixed boundary conditions*, Bound. Value Probl., 2016, 2016(63). DOI: 10.1186/s13661–016–0573–z.

[4] M. Benchohra, S. Bouriah and J.R. Graef, *Nonlinear implicit differential equations of fractional order at resonance*, Electron. J. Differ. Eq., 2016, 2016, 1–10.

[5] J. Cermak and T. Kisela, *Exact and discretized stability of the Bagley-Torvik equation*, J. Comput. Appl. Math., 2014, 269, 53–67.

[6] K. Diethelm, *The analysis of fractional differential equations*, in: Lectures Notes in Mathematics, Springer-Verlag, Berlin, 2010.

[7] Z. Han, H. Lu and C. Zhang, *Positive solutions for eigenvalue problems of fractional differential equation with generalized p-Laplacian*, Appl. Math. Comput., 2015, 257, 526–536.

[8] M. Jia, L. Li, X. Liu, et al, *A class of nonlocal problems of fractional differential equations with composition of derivative and parameters*, Adv. Differ. Equ., 2019,2019(280). DOI: 10.1186/s13662–019–2181–6.

[9] M. Jia and X. Liu, *Multiplicity of solutions for integral boundary value problems of fractional differential equations with upper and lower solutions*, Appl. Math. Comput., 2014, 232, 313–323.

[10] W. Jiang, *The existence of solutions to boundary value problems of fractional differential equations at resonance*, Nonlinear Anal., 2011, 74, 1987–1994.

[11] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier, Amsterdam, 2006.

[12] N. Kosmatov and W. Jiang, *Resonant functional problems of fractional order*, Chaos Solitons Fract., 2016, 91, 573–579.

[13] M. Lázaro and J.L. Pérez-Aparicio, *Dynamic analysis of frame structures with free viscoelastic layers: New closed-form solutions of eigenvalues and a viscous approach*, Eng. Struct., 2013, 54, 69–81.

[14] R. Lewandowski and Z. Pawlak, *Dynamic analysis of frames with viscoelastic dampers modelled by rheological models with fractional derivatives*, J. Sound Vib., 2011, 330, 923–936.

[15] P. Li and M. Feng, *Denumerably many positive solutions for a n-dimensional higher-order singular fractional differential system,* Adv. Difer. Equ., 2018, 2018(145). DOI: 10.1186/s13662–018–1602–2.

[16] X. Liu and M. Jia, *The method of lower and upper solutions for the general boundary value problems of fractional differential equations with p-Laplacian*, Adv. Difer. Equ., 2018, 2018(28). DOI: 10.1186/s13662–017–1446–1.

[17] X. Liu and M. Jia, *Solvability and numerical simulations for BVPs of fractional coupled systems involving left and right fractional derivatives*, Appl. Math. Comput., 2019, 353, 230–242.

[18] X. Liu, M. Jia and W. Ge, *The method of lower and upper solutions for mixed fractional four-point boundary value problem with p-Laplacian operator*, Appl. Math. Lett., 2017, 65, 56–62.

[19] P. D. Phung and H. B. Minh, *Existence of solutions to fractional boundary value problems at resonance in Hilbert spaces*, Bound. Value Probl., 2017, 2017(105). DOI: 10.1186/s13661-017-0836-3.

[20] I. Podlubny, *Fractional differential equations*, mathematics in science and engineering, Academic Press, New York, 1999.

[21] D. O' Regan, Y. Chao and Y. Chen, *Topological degree theory and application*, Taylor and Francis Group, Boca Raton, 2006.

[22] T. Sandev, R. Metzler and Z. Tomovski, *Correlation functions for the fractional generalized Langevin equation in the presence of internal and external noise*, J. Math. Phys., 2014, 55(023301).

[23] X. Su, M. Jia and X. Fu, *On positive solutions of eigenvalue problems for a class of p-Laplacian fractional differential equations*, J. Appl. Anal. Comput., 2018, 8, 152–171.

[24] S. Stanek, *Boundary value problems for Bagley-Torvik fractional differential equations at resonance*, Miskolc Math. Notes, 2018, 19, 611–622.

[25] P. J. Torvik and R. L. Bagley, *On the appearance of the fractional derivative in the behavior of real materials*, J. Appl. Mech., 1984, 51, 294–298.

[26] L. Yang, *Application of Avery-Peterson fixed point theorem to nonlinear boundary value problem of fractional differential equation with the Caputo's derivative*, Commun. Nonlinear Sci. Numer. Simulat., 2012, 17, 4576–4584.

[27] X. Zhao, Y. Liu and H. Pang, *Iterative positive solutions to a coupled fractional differential system with the multistrip and multipoint mixed boundary conditions*, Adv. Difer. Equ., 2019, 2019(389). DOI: 10.1186/s13662–019–2259–1.

[28] Y. Zou and G. He, *The Existence of solutions to integral boundary value problems of fractional differential equations at resonance*, J. Funct. Spaces, 2017, 2017(2785937).