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ON EXACT SOLUTIONS TO EPIDEMIC DYNAMIC MODELS

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Abstract In this study, we address an SIR (susceptible-infected-recovered) model that is given as a system of first order differential equations and propose the SIR model on time scales which unifies and extends continuous and discrete models. More precisely, we derive the exact solution to the SIR model and discuss the asymptotic behavior of the number of susceptibles and infectives. Next, we introduce an SIS (susceptible-infected-susceptible) model on time scales and find the exact solution. We solve the models by using the Bernoulli equation on time scales which provides an alternative method to the existing methods. Having the models on time scales also leads to new discrete models. We illustrate our results with examples where the number of infectives in the population is obtained on different time scales.

Keywords Dynamic equations, time scales, epidemic models, asymptotic behavior.

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1. Introduction

Epidemic models are used for understanding infectious disease dynamics where the population dynamics is divided into compartments. In the susceptible-infected-recovered (SIR) epidemic model, susceptible individuals may become infected, and infected individuals may recover and become immune. No other transitions are considered in this model. The structure of the SIR model dates back to Kermack and McKendrick in 1927 [11] which has provided the basic framework for almost all later epidemic models ever since. In the susceptible-infected-susceptible (SIS) epidemic model, susceptible individuals may become infected, and infected individuals may recover and revert to being susceptible.

The continuous and discrete SIR and SIS models have been investigated in a number of recent works, see [5,10,12]. One of the continuous SIR models is presented in [13] as

$$\begin{cases} S^{'} = -\beta SI - \gamma S + \gamma \\ I^{'} = \beta SI - \gamma I, \end{cases} \tag{1.1}$$

where $S(t)$ and $I(t)$ are the number of susceptibles and the number of infectives at time t, respectively with constant population N and the average number of adequate

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contacts of a person per unit time, i.e, the transmission rate β and the recovery rate γ. The authors eliminate the variable S and obtain the second equation of (1.1) in the form of the Bernoulli equation, and by using a suitable substitution the authors find a solution to (1.1). In the recent articles [3] and [2], the authors call attention to the importance of discrete modeling of HIV-1, Pseudomonas putida bacteria and mammary tumor dynamics, respectively. Especially, comparison of discrete and continuous models of HIV-1 dynamics shows that the data collected in a clinical trial is described by the discrete models better than the continuous model, see [3]. According to our knowledge, the discrete case of system (1.1) has not been studied earlier. Therefore, our purpose is to unify and extend the continuous and the discrete systems on time scales $\mathbb{T}$, nonempty closed subset of real numbers. Motivated by system (1.1) and the Bernoulli equation on time scales, see [4], we propose the SIR model on time scales in the following form

$$\begin{cases} S^{\Delta} = \dfrac{\gamma(t)}{\ominus(-\gamma(t))}\big(\ominus(\beta(t)I)\big)S - \gamma(t)S + \gamma(t) \\ \\ I^{\Delta} = -\dfrac{\gamma(t)}{\ominus(-\gamma(t))}\big(\ominus(\beta(t)I)\big)S - \gamma(t)I, \end{cases} \tag{1.2}$$

where $S, I \in C^1_{rd}([0,\infty)_{\mathbb{T}}, \mathbb{R}^+)$ with the initial conditions $S(0) = S_0 > 0$, $I(0) = I_0 > 0$, and $\beta, \gamma \in C_{rd}([0,\infty)_{\mathbb{T}}, \mathbb{R}^+)$. If $\mathbb{T} = \mathbb{R}$, then $\ominus p = -p$, and system (1.2) with positive β and γ constants turns out to be system (1.1). If $\mathbb{T} = \mathbb{Z}$, then $\ominus p = \frac{-p}{1+p}$ for $p \neq -1$, and system (1.2) is equivalent to the system of first order difference equations as follows

$$\begin{cases} S_{n+1} = \dfrac{1-\gamma_n}{1+\beta_n I_n} S_n + \gamma_n \\ \\ I_{n+1} = \dfrac{\beta_n(1-\gamma_n)}{1+\beta_n I_n} S_n I_n + (1-\gamma_n) I_n. \end{cases} \tag{1.3}$$

In Section 3, we find the exact number of susceptibles and infectives of system (1.2) and discuss their asymptotic behaviors. Furthermore, we illustrate the behavior of infectives of the continuous and the discrete SIR models by examples.

The exact solution of the following SIS model

$$\begin{cases} S' = -\beta SI + \gamma I \\ I' = \beta SI - \gamma I \end{cases} \tag{1.4}$$

with the initial conditions $S(0) = S_0 > 0$, $I(0) = I_0 > 0$ satisfying $S_0 + I_0 = N$, where β and γ are positive constants is studied in [13] while the discrete model of (1.4)

$$\begin{cases} S_{n+1} = S_n\left(1 - \beta I_n\right) + \gamma I_n \\ I_{n+1} = I_n\left(1 - \gamma + \beta S_n\right) \end{cases} \tag{1.5}$$

is studied in [1]. Motivated by system (1.4) and the Bernoulli equation on time scales, see [4], we propose the SIS model on time scales as

$$\begin{cases} S^{\Delta} = \ominus(\beta(t)I)S - \ominus(\beta(t)I)\dfrac{\gamma(t)}{\beta(t)} \\ I^{\Delta} = -\ominus(\beta(t)I)S + \ominus(\beta(t)I)\dfrac{\gamma(t)}{\beta(t)}, \end{cases} \tag{1.6}$$

where $\beta, \gamma \in C_{rd}([0,\infty)_{\mathbb{T}}, \mathbb{R}^+)$, $S, I \in C^1_{rd}([0,\infty)_{\mathbb{T}}, \mathbb{R}^+)$. If $\mathbb{T} = \mathbb{R}$, then system (1.6) with positive constants β and γ is equivalent to system (1.4). However, the discrete model of (1.6) when $\mathbb{T} = \mathbb{Z}$ is

$$\begin{cases} S_{n+1} = S_n \left(1 - \dfrac{\beta_n}{1+\beta_n I_n} I_n\right) + \dfrac{\gamma_n}{1+\beta_n I_n} I_n \\ I_{n+1} = I_n \left(1 - \dfrac{\gamma_n}{1+\beta_n I_n} + \dfrac{\beta_n}{1+\beta_n I_n} S_n\right), \end{cases} \tag{1.7}$$

which is not same as (1.5). Observe that continuous systems (1.1) and (1.4) are equivalent if $S + I = 1$. However, this is not true for discrete systems (1.5) and (1.7). Note that a different form of system (1.6) with constant coefficients is studied in [9].

In Section 4, we find the exact number of susceptibles and infectives of (1.6) and demostrate the behavior of the infectives on a quantum calculus with an example.

Now let us present some preliminary concepts regarding the calculus on time scales without proofs to help understanding the key points in our main results. We refer readers to books by Bohner and Peterson [7,8] and manuscripts [4,6].

2. Essentials of Time Scales

There are two important operators in $\mathbb{T}$. The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined as $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$ for $t \in \mathbb{T}$ while the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined as $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$. The graininess function $\mu : \mathbb{T} \to [0,\infty)$ is defined as $\mu(t) := \sigma(t) - t$. If $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called right-dense, and if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is called left-dense. Besides, if $\rho(t) < t$, we say that t is left-scattered. If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^\kappa = \mathbb{T} - \{m\}$. Otherwise, $\mathbb{T}^\kappa = \mathbb{T}$.

Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^\kappa$. Then, the delta (or Hilger) derivative of f, denoted by f^Δ, on $\mathbb{T}^\kappa$ is defined to be the number (provided it exists) such that for given any $\epsilon > 0$, there is a neighborhood $U = (t-\delta, t+\delta)$ for some $\delta > 0$ such that for all $s \in U$

$$|[f^\sigma(t) - f(s)] - f^\Delta(t)[\sigma(t) - s]| \le \epsilon|\sigma(t) - s|,$$

where $f^\sigma(t) = f(\sigma(t))$ for all $t \in \mathbb{T}$, i.e., $f^\sigma = f \circ \sigma$. Here, $[t_0,\infty)_{\mathbb{T}} := [t_0,\infty) \cap \mathbb{T}$.

A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limit exists (finite) at left dense points in $\mathbb{T}$. The set of rd-continuous $f : \mathbb{T} \to \mathbb{R}$ is denoted by $C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R})$. The set of functions $f : \mathbb{T} \to \mathbb{R}$ that are differentiable and whose derivative rd-continuous is denoted by $C^1_{rd} = C^1_{rd}(\mathbb{T}) = C^1_{rd}(\mathbb{T}, \mathbb{R})$. Every rd-continuous function has an antiderivative. In particular, if $t_0 \in \mathbb{T}$, then for $t \in T$

$$F := \int_{t_0}^{t} f(\tau)\Delta\tau$$

is an antiderivative of f. The set of functions $f \in C^1_{rd}(\mathbb{T}, \mathbb{R})$, the so-called simple useful formula

$$f^\sigma(t) = f(t) + \mu(t) f^\Delta(t) \tag{2.1}$$

holds for all $t \in \mathbb{T}^\kappa$. For any left-dense $t_0 \in \mathbb{T}$ and any $\epsilon > 0$, let $L_\epsilon(t_0) = \{t \in \mathbb{T} : 0 < t_0 - t < \epsilon\}$, and $\bar{\mathbb{T}} = \mathbb{T} \cup \{\sup \mathbb{T}\} \cup \{\inf \mathbb{T}\}$. The following theorem is one of several L'Hôpital Rules on time scales.

Theorem 2.1 (Theorem 1.120, [7]). *Assume f and g are differentiable on $\mathbb{T}$ with*

$$\lim_{t \to t_0^-} g(x) = \infty$$

for some left-dense $t_0 \in \bar{\mathbb{T}}$. Suppose there exists $\epsilon > 0$ with $g(t) > 0$ and $g^\Delta(t) > 0$ for all $t \in L_\epsilon(t_0)$. Then,

$$\lim_{t \to t_0^-} \frac{f^\Delta(t)}{g^\Delta(t)} = r \in \bar{\mathbb{R}}$$

implies

$$\lim_{t \to t_0^-} \frac{f(t)}{g(t)} = r.$$

A function $f : \mathbb{T} \to \mathbb{R}$ is called regressive if $1 + \mu(t)f(t) \neq 0$ for all $t \in \mathbb{T}^\kappa$. The set of all regressive and rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R})$. Besides, $f \in \mathcal{R}$ is called positively regressive for all $t \in \mathbb{T}$ if $1 + \mu(t)f(t) > 0$, and is denoted by $\mathcal{R}^+$. Note that $\mathcal{R}(\alpha) = \mathcal{R}$ if $\alpha \in \mathbb{N}$ and $\mathcal{R}(\alpha) = \mathcal{R}^+$ if $\alpha \in \mathbb{R} \setminus \mathbb{N}$. If $p, q \in \mathcal{R}$, then the circle minus subtraction is defined by

$$(\ominus p)(t) := -\frac{p(t)}{1 + \mu(t)q(t)}$$

and

$$(p \ominus q)(t) := \frac{p(t) - q(t)}{1 + \mu(t)q(t)} \tag{2.2}$$

for all $t \in \mathbb{T}^\kappa$, while the circle dot multiplication is defined by

$$(\alpha \odot p)(t) := \alpha p(t) \int_0^1 (1 + \mu(t)p(t)h)^{\alpha - 1} dh$$

to find a simple form of the derivative of p^α on time scales.

Theorem 2.2. *Suppose $p \in \mathcal{R}$ and fix $t_0 \in \mathbb{T}$. Then the initial value problem*

$$y^\Delta = p(t)y, \quad y(t_0) = 1$$

has a unique solution $e_p(\cdot, t_0)$, so called the exponential function on time scales.

Let $\alpha \in \mathcal{R}$ be constant and $t_0 \in \mathbb{T}$. If $\mathbb{T} = \mathbb{R}$, then

$$e_\alpha(t, t_0) = e^{\alpha(t - t_0)}. \tag{2.3}$$

For the discrete time scales, if $\mathbb{T} = \mathbb{Z}$,

$$e_\alpha(t, t_0) = (1 + \alpha)^{t - t_0} \tag{2.4}$$

and if $\mathbb{T} = q^{\mathbb{N}_0} = \{q^n : n \in \mathbb{N}\}$, where $q > 1$ and $q \in \mathbb{R}$, i.e., the quantum calculus

$$e_\alpha(t, t_0) = \prod_{s \in [t_0, t)_{q^{\mathbb{N}_0}}} [1 + (q - 1)\alpha s], \quad t > t_0. \tag{2.5}$$

We use the following properties of exponential functions on time scales in our proofs, see Theorems 2.36, 2.39 and 2.44 in [7].

Theorem 2.3. *If $p, q \in \mathcal{R}$ and $t_0, t, s \in \mathbb{T}$, then*

(i) $e_0(t,s) = 1$ *and* $e_p(t,t) = 1$;

(ii) $e_p(t,s) = \dfrac{1}{e_p(s,t)} = e_{\ominus p}(s,t)$;

(iii) $e_p(t,s)e_p(s,r) = e_p(t,r)$;

(iv) $\displaystyle\int_{t_0}^{t} p(\tau)e_p(s,\sigma(\tau))\Delta\tau = e_p(s,t_0) - e_p(s,t)$;

(v) If $p \in \mathcal{R}^+$ *on* $\mathbb{T}^\kappa$, *then* $e_p(t,t_0) > 0$ *for all* $t \in \mathbb{T}$.

One of the Variation of Constants Formulas in [7, Theorem 2.77] is stated as follows.

Theorem 2.4. *Suppose $p \in \mathcal{R}$ and $f \in C_{rd}$. Then the unique solution of the initial value problem*

$$y^\Delta = p(t)y + f(t), \quad y(t_0) = y_0$$

is given by

$$y(t) = e_p(t,t_0)y_0 + \int_{t_0}^{t} e_p(t,\sigma(\tau))f(\tau)\Delta\tau,$$

where $t_0 \in \mathbb{T}$ and $y_0 \in \mathbb{R}$.

As we mention in the introduction, our main results are based on solutions of the Bernoulli equation on time scales of the form

$$x^\Delta = \left[p(t) \ominus \left(\frac{1}{\alpha} \odot (f(t)x^\alpha)\right)\right] x, \tag{2.6}$$

where $\alpha \in \mathbb{R} \setminus 0$, and the proof of the existence of solutions of (2.6) can be found in [4, Theorem 6.1].

Theorem 2.5. *Suppose $\alpha \in \mathbb{R} \setminus 0$, $p \in \mathcal{R}(\alpha)$ and $f \in C_{rd}$. If*

$$\frac{1}{x_0^\alpha} + \int_{t_0}^{t} e_p^\alpha(\tau,t_0)f(\tau)\Delta\tau > 0$$

for all $t \in \mathbb{T}$, then

$$x(t) = \frac{e_p(t,t_0)}{\left[\frac{1}{x_0^\alpha} + \int_{t_0}^{t} e_p^\alpha(\tau,t_0)f(\tau)\Delta\tau\right]^{1/\alpha}}$$

solves the Bernoulli equation (2.6) with $x(t_0) = x_0$.

Note that in the case of $\alpha = 1$ in (2.6), we have

$$(1 \odot fx)(t) := f(t)x \int_0^1 (1 + \mu(t)f(t)xh)^0 dh = f(t)x. \tag{2.7}$$

Hence, the Bernoulli equation (2.6) is equivalent to

$$x^\Delta = \left[\frac{p(t) - f(t)x}{1 + \mu(t)f(t)x}\right] x, \tag{2.8}$$

where we use (2.2) and (2.7), and the solution of (2.8) with $x(t_0) = x_0$ is

$$x(t) = \frac{e_p(t,t_0)}{\frac{1}{x_0} + \int_{t_0}^t e_p(\tau,t_0)f(\tau)\Delta\tau}$$

by Theorem 2.5.

The following inequalities, see [6, Lemma 2] and [6, Remark 2], are necessary to show the asymptotic behavior of solutions of system (1.2). For nonnegative f if $-f \in \mathcal{R}^+$, then

$$1 - \int_s^t f(u)\Delta u \leq e_{-f}(t,s) \leq \exp\left\{-\int_s^t f(u)\Delta u\right\} \tag{2.9}$$

and if f is rd-continuous, then

$$1 + \int_s^t f(u)\Delta u \leq e_f(t,s) \leq \exp\left\{\int_s^t f(u)\Delta u\right\} \tag{2.10}$$

for all $t \geq s$.

3. An SIR Model on Time Scales

In this section, we find the exact solution to SIR model (1.2) with the initial conditions (S_0, I_0). Then, we discuss the asymptotic behavior of the solutions and illustrate the behavior of infectives on continuous and discrete time scales.

Theorem 3.1. *Let $\beta, \gamma \in C_{rd}([0,\infty)_{\mathbb{T}}, \mathbb{R}^+)$ and $-\gamma \in \mathcal{R}^+$. Then the unique solution (S, I) of SIR model (1.2) with the initial conditions (S_0, I_0) is given by*

$$\begin{cases} S = e_{-\gamma}(t,0)\left(D_0 - 1\right) + 1 - \dfrac{e_p(t,0)}{\dfrac{1}{I_0} + \displaystyle\int_0^t \beta(\tau)e_p(\tau,0)\Delta\tau} \\ I = \dfrac{e_p(t,0)}{\dfrac{1}{I_0} + \displaystyle\int_0^t \beta(\tau)e_p(\tau,0)\Delta\tau}, \end{cases} \tag{3.1}$$

where $S, I \in C^1_{rd}([0,\infty)_{\mathbb{T}}, \mathbb{R}^+)$, $D = S + I$ with $D(0) = D_0$, and

$$p(t) = \beta(t)D(t)(1 - \mu(t)\gamma(t)) - \gamma(t) \quad \text{for} \quad t \in [0,\infty)_{\mathbb{T}}. \tag{3.2}$$

Proof. Suppose $\beta, \gamma \in C_{rd}([0,\infty)_{\mathbb{T}}, \mathbb{R}^+)$ and $-\gamma \in \mathcal{R}^+$. First of all, from the assumption $-\gamma \in \mathcal{R}^+$, $1 + \mu(t)p(t) = 1 + \mu(t)\big[\beta(t)D(t)\big(1 - \mu(t)\gamma(t)\big) - \gamma(t)\big] > 1 - \mu(t)\gamma(t) > 0$, that is $p \in \mathcal{R}^+$. Note that $S^\Delta + I^\Delta = -\gamma(t)(S + I) + \gamma(t)$, that is

$$D^\Delta = -\gamma(t)D + \gamma(t), \quad t \in [0,\infty)_{\mathbb{T}}. \tag{3.3}$$

Since $-\gamma \in \mathcal{R}^+$, from Theorem 2.4, the solution to (3.3) with $D(0) = D_0$ is

$$D(t) = e_{-\gamma}(t,0)D_0 + \int_0^t e_{-\gamma}(t,\sigma(\tau))\gamma(\tau)\Delta\tau$$

$$
\begin{aligned}
&= e_{-\gamma}(t,0)D_0 - e_{-\gamma}(t,0)\int_0^t \big(-\gamma(\tau)\big)e_{-\gamma}(0,\sigma(\tau))\Delta\tau \\
&= e_{-\gamma}(t,0)D_0 - e_{-\gamma}(t,0)\big[e_{-\gamma}(0,0) - e_{-\gamma}(0,t)\big] \\
&= e_{-\gamma}(t,0)D_0 - e_{-\gamma}(t,0) + 1 \\
&= e_{-\gamma}(t,0)\big(D_0 - 1\big) + 1
\end{aligned}
\tag{3.4}
$$

for $t \in [0,\infty)_{\mathbb{T}}$, where we use Theorem 2.3 (iv). SIR model (1.2) on time scales can be rewritten as

$$
\begin{cases}
S^{\Delta} = -\dfrac{\beta(t)(1-\mu(t)\gamma(t))}{1+\mu(t)\beta(t)I}SI - \gamma(t)S + \gamma(t) \\[2ex]
I^{\Delta} = \dfrac{\beta(t)(1-\mu(t)\gamma(t))}{1+\mu(t)\beta(t)I}SI - \gamma(t)I.
\end{cases}
\tag{3.5}
$$

Note that the positivity of β and I implies that $1+\mu(t)\beta(t)I \neq 0$ for $t \in [0,\infty)_{\mathbb{T}}$. By plugging $S = D - I$ into the second equation of (3.5), we have

$$
\begin{aligned}
I^{\Delta} &= \frac{\beta(t)(1-\mu(t)\gamma(t))}{1+\mu(t)\beta(t)I} I\big[D(t) - I\big] - \gamma(t)I \\
&= \frac{\beta(t)(1-\mu(t)\gamma(t))\big[D(t)-I\big]I - \big[1+\mu(t)\beta(t)I\big]\gamma(t)I}{1+\mu(t)\beta(t)I} \\
&= \frac{\big[\beta(t)D(t) - \beta(t)D(t)\mu(t)\gamma(t) - \gamma(t) - \beta(t)I\big]I}{1+\mu(t)\beta(t)I} \\
&= \left[\frac{p(t)-\beta(t)I}{1+\mu(t)\beta(t)I}\right] I,
\end{aligned}
\tag{3.6}
$$

where p is defined as in (3.2). Note that (3.6) is a Bernoulli equation in the form of (2.8). Therefore, by Theorem 2.5 when $\alpha = 1$, we obtain I as in (3.1). This implies that $S = D - I$ is obtained as in (3.1). Therefore, the proof is completed. □

We now consider system (1.2) with positive β and γ constants for the following examples.

Example 3.1. Let $\mathbb{T} = [0,\infty)$ and $D = 1$ in system (1.2). Then, since $\mu = 0$, we have $p = \beta - \gamma$ from (3.2). From Theorem 3.1, the number of infectives to the continuous SIR model with initial conditions S_0 and I_0 is given by

$$
I(t) = \frac{1}{\dfrac{1}{I_0} + \beta t}, \quad t \in [0,\infty) \tag{3.7}
$$

if $p = 0$, that is $\beta = \gamma$. Moreover, if $p \neq 0$ then

$$
\begin{aligned}
I(t) &= \frac{e^{\int_0^t (\beta-\gamma)du}}{\frac{1}{I_0} + \frac{\beta}{\beta-\gamma}e^{(\beta-\gamma)t} - \frac{\beta}{\beta-\gamma}} \\
&= \frac{e^{(\beta-\gamma)t}}{\frac{1}{I_0} + \frac{\beta}{\beta-\gamma}\big[e^{(\beta-\gamma)t} - 1\big]}, \quad t \in [0,\infty),
\end{aligned}
\tag{3.8}
$$

where we use (2.3), and so $S = 1 - I$.

Example 3.2. Let $\mathbb{T} = \mathbb{Z}_0^+$ and $D = 1$ in system (1.2). Then, since $\mu = 1$ and $-\gamma \in \mathcal{R}^+$, we have $p = \beta - \beta\gamma - \gamma$ from (3.2) and so $1 + \mu p = 1 - \gamma + \beta(1-\gamma) = (1-\gamma)(1+\beta) > 0$, i.e., $p \in \mathcal{R}^+$. Theorem 3.1 states that the number of infectives to discrete SIR model (1.3) with initial conditions S_0 and I_0 is given by

$$I_n = \frac{1}{\dfrac{1}{I_0} + \beta n}, \quad n \in \mathbb{Z}_0^+ \tag{3.9}$$

if $p = 0$. Moreover, if $p \neq 0$, then

$$\begin{aligned} I_n &= \frac{(1+p)^n}{\dfrac{1}{I_0} + \beta \displaystyle\sum_{k=0}^{n-1} (1+p)^k} \\ &= \frac{(1+p)^n}{\dfrac{1}{I_0} + \beta \left[\dfrac{(1+p)^n - 1}{p} \right]} \\ &= \frac{(1+p)^n p I_0}{p + \beta I_0 \left[(1+p)^n - 1 \right]}, \quad n \in \mathbb{Z}_0^+ \end{aligned} \tag{3.10}$$

following from (2.4) and so $S = 1 - I$.

Remark 3.1. Since $\gamma > 0$ and $-\gamma \in \mathcal{R}^+$, from Theorem 2.3 (v) and (2.9) we have

$$0 < e_{-\gamma}(t, 0) \leq e^{-\int_0^t \gamma \Delta u} = e^{-\gamma t}, \quad t \in [0, \infty)_{\mathbb{T}}.$$

This implies that $e_{-\gamma}(t, 0) \to 0$ as $t \to \infty$. Therefore, $D(t) \to 1$ as $t \to \infty$ by (3.4). Note that $D^{\Delta}(t) = -\gamma e_{-\gamma}(t, 0)\big(D_0 - 1\big)$ for all $t \in [0, \infty)_{\mathbb{T}}$. Hence, $D^{\Delta}(t) > 0$ if $0 < D_0 < 1$ and $D^{\Delta}(t) < 0$ if $D_0 > 1$ for $t \in [0, \infty)_{\mathbb{T}}$.

The results in Remark 3.1 are important to analyze the asymptotic behavior of infectives and susceptibles to system (1.2) with positive constants β and γ in the following theorem.

Theorem 3.2. *Consider system* (1.2) *with positive constants β and γ. Let $-\gamma \in \mathcal{R}^+$ and p be as in* (3.2).

(i) If $p(t) = 0$ on $[0, \infty)_{\mathbb{T}}$, then all solutions (S, I) of system (1.2) *with $D_0 = S_0 + I_0$ converge to* $(1, 0)$.

(ii) If $p(t) < 0$ on $[0, \infty)_{\mathbb{T}}$ and $\gamma > k\beta$ for some $k > 0$, then all solutions (S, I) of system (1.2) *with $D_0 = S_0 + I_0$ converge to* $(1, 0)$.

(iii) If $p(t) > 0$ for $t \in [0, \infty)_{\mathbb{T}}$ with the constant graininess μ, then all solutions (S, I) of system (1.2) *with $D_0 = S_0 + I_0$ converge to* $(\gamma\mu + \frac{\gamma}{\beta}, 1 - \gamma\mu - \frac{\gamma}{\beta})$.

Proof. Assume that β and γ are positive constants, $-\gamma \in \mathcal{R}^+$, and p is as in (3.2) for $t \in [0, \infty)_{\mathbb{T}}$. In the proof of Theorem 3.1, we show that $p \in \mathcal{R}^+$.

(i) Let $p(t) = 0$ on $[0, \infty)_{\mathbb{T}}$. Then, $e_p(t, 0) \equiv 1$ and by Theorem 3.1, the number of infectives is (3.7) and so $I(t) \to 0$ as $t \to \infty$. Since $D(t) \to 1$ as $t \to \infty$ by Remark 3.1 and $D = S + I$, then $S(t) \to 1$ as $t \to \infty$.

(ii) Suppose $p(t) < 0$ on $[0,\infty)_{\mathbb{T}}$ and $\gamma > k\beta$ for some $k > 0$. Since $p \in \mathcal{R}^+$, then $e_p(t,0) > 0$ for all $t \in [0,\infty)_{\mathbb{T}}$ by Theorem 2.3 (v). Therefore, we have

$$0 < I(t) \leq e_p(t,0)I_0, \quad t \in [0,\infty)_{\mathbb{T}}, \tag{3.11}$$

where we use Theorem 3.1. If $0 < D_0 \leq 1$, then $D(t) \leq 1$ for $t \in [0,\infty)_{\mathbb{T}}$ by Remark 3.1. Therefore, $p(t) \leq \beta D - \gamma \leq \beta - \gamma < 0$ for $t \in [0,\infty)_{\mathbb{T}}$. Now let $-f = p < 0$, then $f > 0$ and $-f \in \mathcal{R}^+$. We can apply (2.9) for nonnegative f as follows

$$0 < e_p(t,0) \leq e^{\int_0^t p(u)\Delta u} \leq e^{\int_0^t (\beta-\gamma)\Delta u} = e^{(\beta-\gamma)t} \to 0 \quad \text{as} \quad t \to \infty.$$

Hence, $e_p(t,0) \to 0$ as $t \to \infty$. This concludes that $I(t) \to 0$ because of (3.11) and so $S(t) \to 1$ as $t \to \infty$. If $D_0 > 1$, then there exist $\eta > 1$ and $t_0 \in [0,\infty)_{\mathbb{T}}$ such that $D(t) \leq \eta$ for $t \in [t_0,\infty)_{\mathbb{T}}$ by Remark 3.1. Besides, $p(t) \leq \beta D - \gamma$re the number of infective $\leq \beta\eta - \gamma < 0$ by assumption. Again, by letting $-f = p < 0$ such that $-f \in \mathcal{R}^+$, we can apply (2.9) for nonnegative f and obtain

$$0 < e_p(t,0) \leq e^{\int_0^t p(u)\Delta u} \leq e^{\int_0^t (\beta\eta-\gamma)\Delta u} = e^{(\beta\eta-\gamma)t} \to 0 \quad \text{as} \quad t \to \infty.$$

Therefore, $e_p(t,0) \to 0$ as $t \to \infty$. This implies that $I(t) \to 0$ as $t \to \infty$ because of (3.11) and so $S(t) \to 1$ as $t \to \infty$.

(iii) Let $p(t) > 0$ for $t \in [0,\infty)_{\mathbb{T}}$ with the constant graininess μ. Then, $e_p(t,0) \to \infty$ as $t \to \infty$ and we have

$$e_p(t,0) \geq 1 + \int_0^t p(u)\Delta u, \quad t \in [0,\infty)_{\mathbb{T}}$$

by (2.10). Integrating the above inequality from 0 to ∞ gives $\int_0^\infty e_p(u,0)\Delta u = \infty$. For the limit of I, we apply L'Hôpital Rule. Let $g(t) = \frac{1}{I_0} + \beta\int_0^t e_p(\tau,0)\Delta\tau > 0$ and $f(t) = e_p(t,0) > 0$ on $[0,\infty)_{\mathbb{T}}$ in Theorem 2.1. Hence, $\lim_{t\to\infty} g(t) = \infty$, $g^\Delta = \beta e_p > 0$ by Theorem 2.3 (v), and $f^\Delta = pe_p$. Therefore,

$$\lim_{t\to\infty} \frac{f^\Delta(t)}{g^\Delta(t)} = \lim_{t\to\infty} \frac{p(t)e_p(t,0)}{\beta e_p(t,0)} = \lim_{t\to\infty} \frac{p(t)}{\beta}.$$

Since $D(t) \to 1$ as $t \to \infty$, $\lim_{t\to\infty} p(t) = \beta(1-\gamma\mu) - \gamma$. This implies that

$$\lim_{t\to\infty} I(t) = \lim_{t\to\infty} \frac{f(t)}{g(t)} = \lim_{t\to\infty} \frac{p(t)}{\beta} = 1 - \gamma\mu - \frac{\gamma}{\beta},$$

and so $S(t) \to \gamma\mu + \frac{\gamma}{\beta}$ as $t \to \infty$.

□

The following examples illustrate Theorem 3.2, where the number of infectives is obtained for the continuous and discrete SIR models.

Example 3.3. Consider SIR model (1.2) with $(S_0, I_0) = (0.8, 0.2)$ on $[0,\infty)$. In Example 3.1, we obtain the number of infectives from (3.7) and (3.8) for all $t \in [0,\infty)$. If $\beta = \gamma = 0.5$, then $p = 0$. Hence, $I \to 0$ as $t \to \infty$. If $\beta = 0.3$ and $\gamma = 0.4$ are chosen, then $p = -0.1 < 0$ and in this case, $I(t) \to 0$ as $t \to \infty$. On the other hand, choosing $\beta = 0.4$ and $\gamma = 0.3$ yields $p = 0.1 > 0$. Hence, $I(t) \to \frac{1}{4}$ as $t \to \infty$. Figure 1 shows the number of infectives for all $t \in [0, 50]$ based on the sign of p.

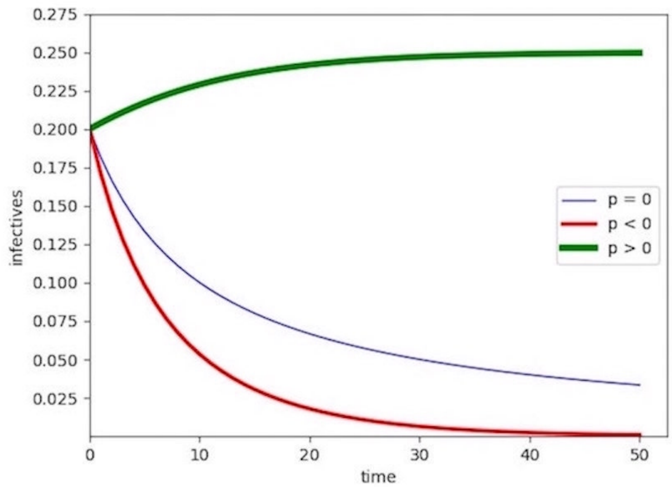

Figure 1. Number of infectives on $[0, 50]$.

Example 3.4. Now consider SIR model (1.2) with $(S_0, I_0) = (0.8, 0.2)$ on $[0, \infty)_{\mathbb{Z}}$. In Example 3.2, we compute the number of infectives for all $n \in [0, \infty)_{\mathbb{Z}}$ from (3.9) and (3.10). If $\beta = 0.25$ and $\gamma = 0.2$, then $p = 0$ and so $I_n \to 0$ as $n \to \infty$. Letting $\beta = 0.1$ and $\gamma = 0.4$ yields $p = -0.34 < 0$. Hence, $I_n \to 0$ as $n \to \infty$. Now let $\beta = 1.5$ and $\gamma = 0.5$. Then, $p = 0.25 > 0$ and $I_n \to \frac{1}{6}$ as $n \to \infty$. Number of infectives on $[0, 25]_{\mathbb{Z}}$ for these cases are demostrated in Figure 2.

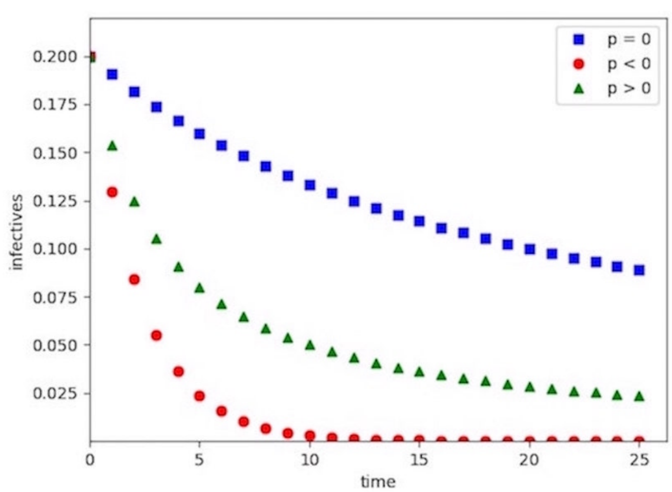

Figure 2. Number of infectives on $[0, 25]_{\mathbb{Z}}$.

4. An SIS Model on Time Scales

We now find the exact solution to SIS model (1.6) with the initial conditions (S_0, I_0) where the population size is constant. An example on quantum calculus is presented at the end of this section.

Theorem 4.1. *Let $\beta, \gamma \in C_{rd}([0,\infty)_{\mathbb{T}}, \mathbb{R}^+)$ and $q(t) = \beta(t)N - \gamma(t) \in \mathcal{R}^+$. Then the unique solution (S, I) of SIS model (1.6) is given by*

$$\begin{cases} S(t) = N - \dfrac{e_q(t,0)}{\dfrac{1}{I_0} + \displaystyle\int_0^t \beta(\tau) e_q(\tau, 0)\Delta\tau} \\ I(t) = \dfrac{e_q(t,0)}{\dfrac{1}{I_0} + \displaystyle\int_0^t \beta(\tau) e_q(\tau, 0)\Delta\tau} \end{cases} \tag{4.1}$$

with the initial conditions $S(0) = S_0$ and $I(0) = I_0$, where $S, I \in C^1_{rd}([0,\infty)_{\mathbb{T}}, \mathbb{R}^+)$ and $N = S_0 + I_0$.

Proof. Suppose $\beta, \gamma \in C_{rd}([0,\infty)_{\mathbb{T}}, \mathbb{R}^+)$ and $q(t) = \beta(t)N - \gamma(t) \in \mathcal{R}^+$. First, adding dynamic equations of system (1.6) yields $S^\Delta + I^\Delta = 0$. This implies that the total population size $N = S + I$ is constant in time and hence $N = S_0 + I_0$. System (1.6) can be rewritten as

$$\begin{cases} S^\Delta = -\dfrac{\beta(t)}{1+\mu(t)\beta(t)I} SI + \dfrac{\gamma(t)}{1+\mu(t)\beta(t)I} I \\ I^\Delta = \dfrac{\beta(t)}{1+\mu(t)\beta(t)I} SI - \dfrac{\gamma(t)}{1+\mu(t)\beta(t)I} I. \end{cases} \tag{4.2}$$

Note that the positivity of β and I implies that $1 + \mu(t)\beta(t)I \neq 0$ for $t \in [0,\infty)_{\mathbb{T}}$. By plugging $S = N - I$ into the second equation of (4.2), we have

$$\begin{aligned} I^\Delta =& \frac{\beta(t)}{1+\mu(t)\beta(t)I}\big(N - I\big)I - \frac{\gamma(t)}{1+\mu(t)\beta(t)I} I \\ =& \frac{\big[\beta(t)N - \gamma(t) - \beta(t)I\big]I}{1+\mu(t)\beta(t)I} \\ =& \left[\frac{q(t) - \beta(t)I}{1+\mu(t)\beta(t)I}\right] I, \end{aligned}$$

where $q(t) = \beta(t)N - \gamma(t) \in \mathcal{R}^+$ for all $t \in [0,\infty)_{\mathbb{T}}$. Therefore, we obtain I as in (4.1) by Theorem 2.5 when $\alpha = 1$. The number of susceptibles can be found by $S = N - I$ as in (4.1). This completes the proof. $\square$

Remark 4.1. In the proof of Theorem 4.1, it is mentioned that SIS model (1.6) can be rewritten as (4.2). Furthermore, if $\beta, \gamma \in C_{rd}([0,\infty)_{\mathbb{T}}, \mathbb{R}^+)$, then from the first equation of (4.2), one can obtain

$$S^\Delta\big(1 + \mu(t)\beta(t)I\big) = -\,\beta(t)SI + \gamma(t)I, \quad t \in [0,\infty)_{\mathbb{T}}$$

and from (2.1)

$$S^\Delta + \big(S^\sigma - S\big)\beta(t)I = -\beta(t)SI + \gamma(t)I, \quad t \in [0,\infty)_{\mathbb{T}}.$$

This implies that

$$S^{\Delta} = -\beta(t)S^{\sigma}I + \gamma(t)I, \quad t \in [0,\infty)_{\mathbb{T}}. \tag{4.3}$$

Now from the second equation of (4.2), we get

$$\begin{aligned}
I^{\Delta} =& \beta(t)\left[1 - \frac{\mu(t)\beta(t)I}{1+\mu(t)\beta(t)I}\right]SI - \gamma(t)\left[\frac{\mu(t)\beta(t)I}{1+\mu(t)\beta(t)I} - 1\right]I \\
=& \beta(t)SI - \mu(t)\beta(t)I\left[\frac{\beta(t)SI}{1+\mu(t)\beta(t)I}\right] - \mu(t)\beta(t)I\left[\frac{\gamma(t)I}{1+\mu(t)\beta(t)I}\right] - \gamma(t)I \\
=& \beta(t)SI + \mu(t)\beta(t)I\left[-\frac{\beta(t)}{1+\mu(t)\beta(t)I}SI + \frac{\gamma(t)I}{1+\mu(t)\beta(t)I}\right] - \gamma(t)I \\
=& \beta(t)SI + \mu(t)\beta(t)S^{\Delta}I - \gamma(t)I \\
=& \beta(t)\big(S + \mu(t)S^{\Delta}\big)I - \gamma(t)I \\
=& \beta(t)S^{\sigma}I - \gamma(t)I, \quad t \in [0,\infty)_{\mathbb{T}},
\end{aligned} \tag{4.4}$$

where we use (2.1) in the last step. Note that when β and γ are positive constants, (4.3) and (4.4) give SIS model (3.1) in [9].

Remark 4.2. Let β and γ be positive constants and $\mathscr{R}_0 = \frac{\beta N}{\gamma}$ be the reproduction number. If $q = 0$, i.e., $\mathscr{R}_0 = 1$, then Theorem 4.1 states that the number of susceptibles is $S = N - I$, where I is given as in (3.7). If $q \neq 0$, i.e., $\mathscr{R}_0 \neq 1$, then the number of infectives is

$$I(t) = \frac{qI_0 e_q(t,0)}{q - \beta I_0 + \beta I_0 e_q(t,0)}. \tag{4.5}$$

Remark 4.3. Consider SIS model (1.6) when β and γ are positive constants. If $q = 0$, i.e., $\mathscr{R}_0 = 1$, then $I(t) \to 0$ and $S(t) \to N$ as $t \to \infty$ from (3.7). Hence, the disease dies out. The asymptotic behavior of infectives is discussed in [9, Theorem 3.2] when $q \neq 0$, i.e., $\mathscr{R}_0 < 1$ and $\mathscr{R}_0 > 1$.

Example 4.1. Consider SIS model (1.6) on $[0,\infty)_{2^{\mathbb{N}_0}}$ with $N = 1$, and positive constants β, γ. Let $s = 2^n$ and $t = 2^k$, $n, k \in \mathbb{N}$ and $q = \beta - \gamma \in \mathcal{R}^+$. From Remark 4.2, the unique solution to the discrete SIS model with initial conditions S_0 and I_0 is given by

$$I(t) = \frac{1}{\dfrac{1}{I_0} + \beta t} \tag{4.6}$$

if $q = 0$. Moreover, if $q \neq 0$, then $e_q(t,0) = \displaystyle\prod_{s \in [0,t)_{2^{\mathbb{N}_0}}} (1 + qs) = \prod_{n=0}^{k-1}(1 + q2^n)$ by (2.5). Hence, (4.5) implies that the number of infectives can be found as

$$I(t) = \frac{qI_0 \displaystyle\prod_{n=0}^{k-1}(1+q2^n)}{q - \beta I_0 + \beta I_0 \displaystyle\prod_{n=0}^{k-1}(1+q2^n)}, \tag{4.7}$$

and $S(t) = 1 - I(t)$ for $t \in [0,\infty)_{\mathbb{T}}$.

Example 4.2. Consider SIS model (1.6) with $(S_0, I_0) = (0.6, 0.4)$ on $[0, \infty)_{2^{\mathbb{N}_0}}$. If $\beta = \gamma = 0.3$, then $q = 0$ and $I(t) \to 0$ as $t \to \infty$. Choosing $\beta = 0.0008$ and $\gamma = 0.0016$ yields $q = -0.0008 < 0$ and $I(t) \to 0$ as $t \to \infty$. If $\beta = 0.5$ and $\gamma = 0.4$ are chosen, then $q = 0.1 > 0$. Hence, $I(t) \to \frac{q}{\beta} = 0.2$ as $t \to \infty$. Figure 3 illustrates the behavior of infectives on $[0, 1024)_{2^{\mathbb{N}_0}}$ based on the sign of q. Here, the number of infectives is computed from (4.6) and (4.7) in Example 4.1.

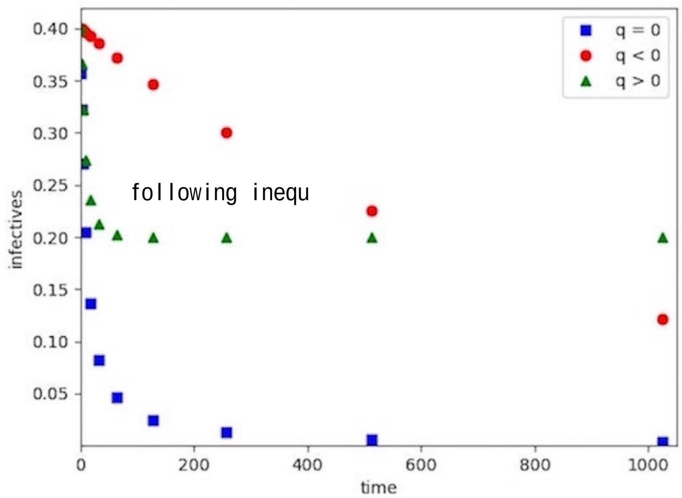

Figure 3. Number of infectives on $[0, 1024)_{2^{\mathbb{N}_0}}$.

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