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A NOTE ON SOME FIXED POINT THEOREMS ON G-METRIC SPACES*

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Abstract In this paper, we prove some fixed point theorems in the framework of G-metric spaces that cannot be obtained from the existence results in the context of quasi-metric spaces.

Keywords G-metric, quasi-metric, fixed point.

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1. Introduction

As well as we know, functional analysis is made up of three main methods which are variational methods, degree methods and fixed point methods. Fixed point method is a very useful tool on solving the differential equations, integral equations and so on. In 2006 Zead Mustafa introduced the notion of G-metric spaces [13] as the generalization of ordinary metric and analysed the topological structure of the G-metric spaces. From then on, G-metric spaces have been studied and applied to obtain different kinds of fixed point theorems, see [8, 9, 14, 15, 17, 21, 23]. The topic of G-metric is still concerned by many peoples [4, 6, 8, 9, 12, 19, 24, 25]. In 2012, Jleli and Samet [10], Samet and Vetro [20], and An. *et al*. [1] reported that most of fixed point results on G-metric spaces can be derived from the fixed point theorems on the usual metric spaces or quasi-metric spaces. In 2013 Asadi *et al*. [2], Karapınar and Agarwal [11] stated and proved some theorems that cannot be obtained from the existence results on metric spaces or quasi-metric spaces. Very recently Agarwal *et al*. [3] announced that many contractive conditions in G-metric spaces can be expressed in the terms of quasi-metric spaces after a suitable substitution, for example $y = Tx$, even in [2,11]. Inspired by the results of paper [3] we have different opinions with [3] and give some theorems on G-metric spaces that cannot be expressed in the terms of quasi-metric spaces.

In the sequel, let $\mathbb{N}$, $\mathbb{R}$ respectively denote the set of all nonnegative integers and real numbers.

Definition 1.1 ([13]). Let X be a non-empty set, $G : X \times X \times X \longrightarrow \mathbb{R}^+$ be a mapping satisfying the following properties:

(G1) $G(x, y, z) = 0$ if $x = y = z$;

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(G2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$;

(G3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$;

(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ (symmetry in all three variables);

(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z,\ a \in X$ (rectangle inequality).

Then the mapping G is called a generalized metric, or, more specially, a G-metric on X, and the pair (X, G) is called a G-metric space.

It should be noticed that $G(x, x, y)$ may not be equal to $G(x, y, y)$. In the case $G(x, y, y) = G(x, x, y)$, we have the following definition.

Definition 1.2 ([13]). A G-metric space (X, G) is symmetric if $G(x, y, y) = G(x, x, y)$ for all $x, y \in X$.

Example 1.1. Let (X, d) be a metric space, and

(1) $G_1(x, y, z) = \max\{d(x, y), d(x, z), d(y, z)\}$,

(2) $G_2(x, y, z) = d(x, y) + d(x, z) + d(y, z)$,

then (X, G_1), and (X, G_2) are symmetric G-metric spaces.

The following definitions are about convergence and completeness on G-metric spaces.

Definition 1.3 ([13]). Let (X, G) be a G-metric space.

(1) A sequence $\{x_n\}$ in X G-converges to x if and only if $G(x_n, x, x) \to 0$ as $n \to \infty$. That is for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $G(x, x_n, x_n) < \varepsilon$, or $G(x_n, x, x) < \varepsilon$.

(2) Sequence $\{x_n\}$ in X is called a G-Cauchy sequence if for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \varepsilon$ for each $n, m, l \geq n_0$.

(3) The G-metric space (X, G) is said to be G-complete if every Cauchy sequence is G-convergent.

Proposition 1.1 ([13]). *In a G-metric space (X, G)*

(1) the sequence $\{x_n\}$ is G-Cauchy if and only if $\lim_{n,m\to\infty} G(x_n, x_n, x_m) = 0$*;*

(2) $G(x, x, y) \leq 2G(x, y, y)$*;*

(3) $G(x, y, z) = 0 \Rightarrow x = y = z$*.*

Proposition 1.2 ([13]). *Let (X, G) be a G-metric space, then the function $G(x, y, z)$ is jointly continuous in all three of its variables.*

Definition 1.4 ([13]). Let (X, G) be a G-metric space and $T : X \to X$ be a self-map. T is said to be G-continuous if for any G-convergent sequence $\{x_n\}$ to x, then $\{Tx_n\}$ is G-convergent to $T(x)$.

Lemma 1.1. *Let (X, G) be a G-metric space. Let $T : X \to X$ be a G-continuous mapping. If a sequence of T based on $x \in X$ $\{T^n x\}$ is G-convergent to $z \in X$, then z is a fixed point of T.*

Proof. Since T is G-continuous, then $\lim_{n\to\infty} G(T^{n+1}x, Tz, Tz) = 0$. Meanwhile $\lim_{n\to\infty} G(T^{n+1}x, z, z) = 0$, we can get $z = Tz$ from the unique limit of sequence on G-metric spaces. □

In 1931 Wilson [22] introduced the notion of $quasi-$metric spaces. The notion of a (left, right) Cauchy sequence of a $quasi-$metric spaces were raised in Reilly et al.'s work [18]. For more detail, please refer to book [7].

Definition 1.5. Let X be a nonempty set and $q : X \times X \to [0, \infty)$ be a given function which satisfies

(1) $q(x, y) = 0$ if and only if $x = y$;

(2) $q(x, y) \leq q(x, z) + q(z, y)$ for any points $x, y, z \in X$.

Then q is called a quasi-metric and the pair (X, q) is called a quasi-metric space.

The quasi-metric is a generalization of a metric. A metric is a quasi-metric, but a quasi-metric probably is not a metric because $q(x, y) \neq q(y, x)$.

Definition 1.6. Let (X, q) be a quasi-metric space, $\{x_n\}$ be a sequence in X, and $x \in X$. We say that:

(1) $\{x_n\}$ right-converges to x if $\lim_{n\to\infty} q(x_n, x) = 0$;

(2) $\{x_n\}$ left-converges to x if $\lim_{n\to\infty} q(x, x_n) = 0$;

(3) $\{x_n\}$ converges to x if and only if $\{x_n\}$ right-converges and left-converges to x;

(4) $\{x_n\}$ is a right-Cauchy sequence if for all $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $q(x_n, x_m) < \varepsilon$ for all $m > n > n_0$;

(5) $\{x_n\}$ is a left-Cauchy sequence if for all $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $q(x_m, x_n) < \varepsilon$ for all $m > n > n_0$;

(6) $\{x_n\}$ is Cauchy if and only if it is left-Cauchy and right-Cauchy;

(7) (X, q) is complete if every Cauchy sequence in X is convergent.

Proposition 1.3 ([20]). *Let (X, G) be a G-metric space. Let $q : X \times X \to [0, \infty)$ be the function defined by $q(x, y) = G(x, y, y)$ or $q(x, y) = G(x, x, y)$. Then*

(1) (X, q) is a quasi-metric space;

(2) $\{x_n\} \subset X$ is G-convergent to $x \in X$ if and only if $\{x_n\}$ is convergent in (X, q);

(3) $\{x_n\} \subset X$ is G-Cauchy if and only if $\{x_n\}$ is Cauchy in (X, q);

(4) (X, G) is G-complete if and only if (X, q) is complete;

(5) $\{x_n\}$ is right-cauchy if and only if it is left-cauchy in (X,q).

Definition 1.7 ([3]). If function $\varphi : [0, \infty) \to [0, \infty)$ satisfying the following conditions:

(1) φ is monotonous non-decreasing;

(2) $\Sigma_{n=1}^{\infty} \varphi^n(t) < \infty$ for all $t > 0$, where φ^n is the nth iteration of φ.

Then the function is called a (c)-comparison function. If φ is a (c)-comparison function then $\varphi(t) < t$ for all $t > 0$. Indeed, there are many functions satisfying the conditions (1)-(2). For example, $\varphi(t) = kt$ for all $t \geq 0$, where $k \in [0, 1)$, and $\varphi(t) = \frac{t}{1+t}$ for all $t \geq 0$.

2. Main results

In the definition of G-metric space the condition (G3) and (G5) are the basic inequalities of G-metric space. The proposition 1.1 (2) is also a key inequality. It is important to know how to use these inequalities effectively to get some available theorems.

Definition 2.1. Let (X, G) be a G-metric space. A mapping $T : X \to X$ is said to be a $G\varphi-$contraction if there exists a (c)-comparison function φ such that for all $x, y \in X$:

$$G(Tx, Ty, T^2y) \leq \varphi(G(x, y, Ty)). \tag{2.1}$$

Definition 2.2. Let (X, G) be a G-metric space. A mapping $T : X \to X$ is said to be a weak $G\varphi-$contraction if there exists a (c)-comparison function φ such that for all $x \in X$:

$$G(Tx, T^2x, T^3x) \leq \varphi(G(x, Tx, T^2x)). \tag{2.2}$$

We denote by $\Omega(X, G\varphi)$ the collection of all $G\varphi-$contraction mappings and by $\Omega(X, \mathcal{W}G\varphi)$ the collection of all weak $G\varphi-$contraction mappings on a G-metric space (X, G). Obiviously

$$\Omega(X, G\varphi) \subseteq \Omega(X, \mathcal{W}G\varphi).$$

Theorem 2.1. *Let (X, G) be a complete G-metric space and $T : X \to X$ be a G-continuous mapping. If T is a weak $G\varphi-$contraction mapping, then T has a fixed point.*

Proof. Let $x_0 \in X$. We define an iterative sequence $\{x_n\}$ as follows:

$$x_{n+1} = Tx_n = T^{n+1}x_0$$

for all $n \in \mathbb{N}$. If $x_{n_0+1} = x_{n_0}$ for some $n_0 \in \mathbb{N}$, then x_{n_0} is a fixed point of T. Throughout the proof, we assume that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$. Consequently, we have $G(x_{n+1}, x_n, x_n) > 0$ for every $n \in \mathbb{N}$.

From (2.2), with $x = x_{n-1}$, and φ is monotonous non-decreasing, we have

$$G(x_n, x_{n+1}, x_{n+2}) \leq \varphi(G(x_{n-1}, x_n, x_{n+1})) \leq \cdots \leq \varphi^n(G(x_0, x_1, x_2))$$

for all $n \in \mathbb{N}$. Since

$$G(x_n, x_n, x_{n+1}) \leq G(x_n, x_{n+1}, x_{n+2})$$

with $x_{n+1} \neq x_{n+2}$, so

$$G(x_n, x_n, x_{n+1}) \leq \varphi^n(G(x_0, x_1, x_2)).$$

For all $n, m \in \mathbb{N}$, $n < m$ we have

$$\begin{aligned} G(x_n, x_n, x_m) &\leq G(x_n, x_n, x_{n+1}) + G(x_{n+1}, x_{n+1}, x_{n+2}) + \cdots + G(x_{m-1}, x_{m-1}, x_m) \\ &\leq (\varphi^n + \varphi^{n+1} + \cdots + \varphi^{m-1})(G(x_0, x_1, x_2)) \\ &= \sum_{k=n}^{m-1} \varphi^k(G(x_0, x_1, x_2)). \end{aligned} \tag{2.3}$$

Since $\sum_{n=1}^{\infty} \varphi^n(t) < \infty$ for all $t > 0$, so $\lim G(x_n, x_n, x_m) = 0$, as $n, m \to \infty$. Thus, $\{x_n\}$ is a G-Cauchy sequence. Due to the completeness of (X, G), there exists $z \in X$ such that $\{x_n\}$ is G-convergent to z. Since T is G-continuous, z is the fixed point of T. □

Corollary 2.1. *Let (X, G) be a complete G-metric space and $T : X \to X$ be a G-continuous mapping. If T satisfy the following condition for all $x \in X$:*

$$G(Tx, T^2x, T^3x) \leq kG(x, Tx, T^2x) \tag{2.4}$$

where $0 \leq k < 1$, then T has a fixed point.

Proof. $\varphi(t) = kt$ is a (c)-comparison function for all $t \geq 0$, where $k \in [0, 1)$. □

Remark 2.1. Corollary 2.1 cannot be expressed in quasi-metric spaces because $G(Tx, T^2x, T^3x)$ cannot be expressed in the style of $G(z, y, y)$ or $G(z, z, y)$. The following corollaries cann't be expressed in quasi-metric spaces too.

Corollary 2.2. *Let (X, G) be a complete G-metric space and $T : X \to X$ be a G-continuous mapping. If T satisfy the following condition for all $x, y \in X$:*

$$G(Tx, Ty, T^2y) \leq aG(x, y, Ty) \tag{2.5}$$

where $0 \leq a < 1$ then T has a fixed point.

Proof. Let $y = Tx$, we get $G(Tx, T^2x, T^3x) \leq aG(x, Tx, T^2x)$. Then the condition of Corollary 2.1 is satisfied, so we can get this conclusion. □

Remark 2.2. The condition of Corollary 2.2 is stronger than Corollary 2.1. Because for every $x, y \in X$ the $G(Tx, Ty, T^2y) \leq aG(x, y, Ty)$ is satisfied, we can choose the special point $y = Tx$. Then using the Corollary 2.1 we know T has a fixed point.

Remark 2.3. In Corollary 2.2 the fixed point is unique. We suppose $Tu = u$ and $Tv = v$. Let $x = u$, $y = v$ in (2.5), then

$$\begin{aligned} G(Tu, Tv, T^2v) \leq aG(u, v, Tv) &\Rightarrow G(u, v, v) \leq aG(u, v, v) \\ &\Rightarrow (1 - a)G(u, v, v) \leq 0 \\ &\Rightarrow G(u, v, v) = 0 \\ &\Rightarrow u = v. \end{aligned} \tag{2.6}$$

If the contractive condition contains two variables x and y, then the uniqueness of fixed point can be deduced. The T need not to be continuous to guarantee that the limit of $\{x_n\}$ is the fixed point of T. Using the triangle inequality and contraction condition we get

$$\begin{aligned} G(Tu, u, u) \leq& G(Tu, x_{n+1}, x_{n+1}) + G(x_{n+1}, u, u) \\ \leq& G(Tu, x_{n+1}, x_{n+2}) + G(x_{n+2}, x_{n+2}, x_{n+1}) + G(x_{n+1}, u, u) \\ \leq& aG(u, x_n, x_{n+1}) + G(x_{n+2}, x_{n+2}, x_{n+1}) + G(x_{n+1}, u, u). \end{aligned} \tag{2.7}$$

Let $n \to \infty$, we get $G(Tu, u, u) = 0$, so $Tu = u$.

We can get many other corollaries by replacing Tx with y or T^2x with y in the condition of Corollary 2.1.

Corollary 2.3. *Let (X, G) be a complete G-metric space. Let $T : X \to X$ be a G-continuous mapping satisfying the following condition for all $x, y \in X$:*

$$G(Tx, T^2x, Ty) \leq aG(x, Tx, y) \tag{2.8}$$

or

$$G(Tx, Ty, T^3x) \leq aG(x, y, T^2x) \tag{2.9}$$

or

$$G(Tx, T^2x, T^2y) \leq aG(x, Tx, Ty) \tag{2.10}$$

where $0 \leq a < 1$. Then T has a fixed point.

Corollary 2.4. *Let (X, G) be a complete G-metric space. Let $T : X \to X$ be a G-continuous mapping satisfying the following condition for all $x, y, z \in X$:*

$$G(Tx, T^2x, Ty) + G(Tx, T^2x, Tz) \leq aG(x, Tx, y) + bG(x, Tx, z) \tag{2.11}$$

where $0 \leq a + b < 2$. Then T has a fixed point.

Proof. Let $y = z = T^2x$, we get $G(Tx, T^2x, T^3x) \leq \frac{a+b}{2} G(x, Tx, T^2x)$. Then the condition of Corollary 2.1 is satisfied, so we can get this conclusion. □

The following corollary is from [2], but we add G-continuity to T to let this theorem be the corollary of Theorem 2.1.

Corollary 2.5. *Let (X, G) be a complete G-metric space. Let $T : X \to X$ be a G-continuous mapping satisfying the following condition for all $x, y, z \in X$:*

$$G(Tx, Ty, T^2y) \leq aG(x, Tx, T^2x) + bG(y, Ty, T^2y) + cG(x, Tx, Ty) + dG(y, Ty, T^3x) \tag{2.12}$$

where a, b, c, d are non-negative and $a + b + c + d < 1$. Then T has a fixed point.

Proof. Let $y = Tx$, we get $G(Tx, T^2x, T^3x) \leq \frac{a+c}{1-b-d} G(x, Tx, T^2x)$. Then the condition of Corollary 2.1 is satisfied, so we can get this conclusion. □

As we all know, the fixed point theorem always be proofed by an iteration of mapping on one point. Although the G-metric contains three variables but we always take the place of deferent variable by the same form during the proof. So we can reduce the variables to only one variable and add the continuity of the mapping to guarantee the existence of fixed point. Inspired by [9] (Theorem 2.6, Theorem 2.7, Theorem 2.8 in [9]), we get the following corollaries.

Corollary 2.6. *Let (X, G) be a complete G-metric space. Let $T : X \to X$ be a G-continuous mapping satisfying the following condition for all $x, y, z \in X$:*

$$G(Tx, Ty, Tz) \leq \alpha\left(\frac{\min\{G(y, Ty, Tz), G(y, z, Tz)\}[1 + G(x, Tx, Ty)]}{1 + G(x, y, z)}\right) + \beta G(x, y, z),$$

where α, β are nonnegative reals, satisfying $\alpha + \beta < 1$. Then T has a fixed point in X.

Proof. Let $y = Tx$,$z = T^2x$ we get $G(Tx, T^2x, T^3x) \leq \frac{\beta}{1-\alpha} G(x, Tx, T^2x)$. Then the condition of Corollary 2.1 is satisfied, so we can get this conclusion. □

Corollary 2.7. *Let (X, G) be a complete G-metric space. Let $T : X \to X$ be a G-continuous mapping satisfying the following condition for all $x, y, z \in X$:*

$$\begin{aligned} G(Tx, Ty, Tz) \leq & a_1\left(\frac{G(y, Ty, Tz)[1 + G(x, Tx, Ty)]}{1 + G(x, y, z)}\right) \\ & + a_2\left(\frac{G(y, z, Tz)[1 + G(x, Tx, Ty)]}{1 + G(x, y, z)}\right) + a_3 G(x, y, z), \end{aligned} \tag{2.13}$$

where a_1, a_2, a_3 are nonnegative reals, satisfying $a_1 + a_2 + a_3 < 1$. Then T has a fixed point in X.

Proof. Let $y = Tx, z = T^2x$, we get $G(Tx, T^2x, T^3x) \leq \frac{a_3}{1-a_1-a_2} G(x, Tx, T^2x)$. Then the condition of Corollary 2.1 is satisfied, so we can get this conclusion. □

Corollary 2.8. *Let (X, G) be a complete G-metric space. Let $T : X \to X$ be a G-continuous mapping satisfying the following condition for all $x, y, z \in X$:*

$$\begin{aligned} G(Tx, Ty, Tz) \leq & a_1 G(x, y, z) + a_2[G(x, Tx, T^2x) + G(y, Ty, T^2y)] \\ & + a_3[G(T^2x, Ty, z) + 2G(x, y, z) + 2G(Tx, Ty, Tz)] \\ & + a_4 min\{G(y, Ty, Tz), G(y, z, Tz)\} \frac{[1 + G(x, Tx, T^2x)]}{1 + G(x, y, z)} \\ & + a_5 G(T^2x, Ty, z)[1 + G(x, Ty, z) + G(x, y, Tz)][1 + G(x, y, z)]^{-1} \\ & + a_6 G(x, y, z)[1 + G(x, Tx, T^2x) + G(T^2x, Ty, z)][1 + G(x, y, z)]^{-1} \\ & + a_7 G(T^2x, Ty, z), \end{aligned} \tag{2.14}$$

where a_i are nonnegative reals, $i = 1, 2, 3, 4, 5, 6, 7$, satisfying $a_1 + 2a_2 + 4a_3 + a_4 + a_6 < 1$. Then T has a fixed point in X.

Proof. Let $y = Tx, z = T^2x$, we get $G(Tx, T^2x, T^3x) \leq \frac{a_1+a_2+2a_3+a_6}{1-a_2-2a_3-a_4} G(x, Tx, T^2x)$. Then the condition of Corollary 2.1 is satisfied, so we can get this conclusion. □

Using the same skill we can get many other corollaries. We omit the corollaries of following theorems.

Theorem 2.2. *Let (X, G) be a complete G-metric space. Let $T : X \to X$ be a G-continuous and onto mapping satisfying the following condition for all $x \in X$:*

$$G(Tx, T^2x, T^3x) \geq aG(x, Tx, T^2x) \tag{2.15}$$

where $a > 1$. Then T has a fixed point.

Proof. Let $x_0 \in X$. Since T is onto, there exists $x_1 \in X$ such that $x_0 = Tx_1$. By continuing this process, we get iterative sequence $\{x_n\}$ as follows:

$$x_n = Tx_{n+1}$$

for all $n \in \mathbb{N}$. If $x_{n_0+1} = x_{n_0}$ for some $n_0 \in \mathbb{N}$, then x_{n_0+1} is a fixed point of T. Throughout the proof, we assume that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$. From (2.15), with $x = x_{n+1}$, we have

$$G(x_n, x_{n-1}, x_{n-2}) \geq aG(x_{n+1}, x_n, x_{n-1}),$$

and so

$$G(x_{n+1}, x_n, x_{n-1}) \leq hG(x_n, x_{n-1}, x_{n-2}) \leq h^{n-1}G(x_2, x_1, x_0),$$

for all $n \in \mathbb{N}$, where $h = \frac{1}{a} < 1$. We can readily show that $\{x_n\}$ is a G-Cauchy sequence. Since (X, G) is a G-complete space, then exists $w \in X$ such that $\{x_n\}$ is G-convergent to w. Since T is G-continuous, w is the fixed point of T. □

The following corollary is from [2], but we add G-continuity to T to let this theorem be the corollary of Theorem 2.2.

Corollary 2.9. *Let (X, G) be a complete G-metric space. Let $T : X \to X$ be a G-continuous and onto mapping satisfying the following condition for all $x \in X$:*

$$G(Tx, Ty, T^2y) \geq aG(x, Tx, T^2x) \tag{2.16}$$

where $a > 1$. Then T has a fixed point.

Theorem 2.3. *Let (X, G) be a complete G-metric space. Let $T : X \to X$ be a G-continuous mapping satisfying the following condition for all $x \in X$:*

$$G(Tx, T^2x, T^2x) \leq aG(x, Tx, T^2x), \tag{2.17}$$

or

$$G(Tx, Tx, T^2x) \leq aG(x, Tx, T^2x), \tag{2.18}$$

where $0 \leq a < \frac{1}{3}$. Then T has a fixed point.

Proof. Let $x = x_n$ in (2.17), we get

$$\begin{aligned} G(x_{n+1}, x_{n+2}, x_{n+2}) &\leq aG(x_n, x_{n+1}, x_{n+2}) \\ &\leq a(G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+1}, x_{n+2})) \\ &\leq aG(x_n, x_{n+1}, x_{n+1}) + 2aG(x_{n+1}, x_{n+2}, x_{n+2}) \end{aligned} \tag{2.19}$$

for all $n \in N$. Then

$$G(x_{n+1}, x_{n+2}, x_{n+2}) \leq \frac{a}{1-2a}G(x_n, x_{n+1}, x_{n+1}) \leq (\frac{a}{1-2a})^{n+1}G(x_0, x_1, x_1).$$

The $\{x_n\}$ is obviously a G-Cauchy sequence. Due to the completeness of (X, G), there exists $z \in X$ such that $\{x_n\}$ is G-convergent to z. Since T is G-continuous, z is the fixed point of T. Similarly, the other conclusion can be deduced. □

Theorem 2.4. *Let (X, G) be a complete G-metric space. Let $T : X \to X$ be a G-continuous mapping satisfying the following condition for all $x \in X$:*

$$G(Tx, T^2x, T^3x) \leq \frac{G(x, Tx, T^2x) + G(Tx, T^2x, T^3x)}{G(x, Tx, T^2x) + G(Tx, T^2x, T^3x) + a}G(x, Tx, T^2x) \tag{2.20}$$

where $0 < a$. Then T has a fixed point.

Proof. Let $x_0 \in X$. We define an iterative sequence $\{x_n\}$ as follows:

$$x_{n+1} = Tx_n = T^{n+1}x_0$$

for all $n \in \mathbb{N}$. If some $x_{n_0+1} = x_{n_0}$ for some $n \in \mathbb{N}$, then x_{n_0} is a fixed point of T. Throughout the proof, we assume that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$. Consequently, we have $G(x_n, x_n, x_{n+1}) > 0$ for every $n \in \mathbb{N}$.

From(2.20), with $x = x_n$, let $d_n = G(x_n, x_{n+1}, x_{n+2})$ we have

$$d_{n+1} \leq \frac{d_n + d_{n+1}}{d_n + d_{n+1} + a} d_n.$$

Let $a_n = \frac{d_n + d_{n+1}}{d_n + d_{n+1} + a}$, then we get

$$d_{n+1} \leq a_n d_n \leq a_n a_{n-1} d_{n-1} \leq \cdots \leq a_n a_{n-1} \cdots a_0 d_0.$$

Since $0 < G(x_n, x_n, x_{n+1}) \leq G(x_n, x_{n+1}, x_{n+2}) = d_n$, we get $0 < a_n < 1$. And $d_{n+1} \leq a_n d_n < d_n$, so $\{d_n\}$ is a strictly decreasing sequence. $\{a_n\}$ is also a strictly decreasing sequence, because

$$\begin{aligned} d_n < d_{n-1} &\Rightarrow d_n + d_{n+1} < d_{n-1} + d_n \\ &\Rightarrow 1 + \frac{a}{d_{n-1} + d_n} < 1 + \frac{a}{d_n + d_{n+1}} \\ &\Rightarrow \frac{d_{n-1} + d_n + a}{d_{n-1} + d_n} < \frac{d_n + d_{n+1} + a}{d_n + d_{n+1}} \\ &\Rightarrow \frac{1}{a_{n-1}} < \frac{1}{a_n} \\ &\Rightarrow a_n < a_{n-1} \end{aligned} \tag{2.21}$$

for all $n \in \mathbb{N}$. Then

$$d_{n+1} < a_0^{n+1} d_0.$$

Since

$$G(x_n, x_n, x_{n+1}) \leq G(x_n, x_{n+1}, x_{n+2}) = d_n < a_0^n d_0.$$

The $\{x_n\}$ is obviously a G-Cauchy sequence. Due to the completeness of (X, G), there exists $z \in X$ such that $\{x_n\}$ is G-convergent to z. Since T is G-continuous, z is the fixed point of T. □

Remark 2.4. We can change the condition of Theorem 2.4 to

$$G(Tx, T^2x, T^3x) \leq \frac{\Sigma_{i=0}^k G(T^i x, T^{i+1}x, T^{i+2}x)}{\Sigma_{i=0}^k G(T^i x, T^{i+1}x, T^{i+2}x) + a} G(x, Tx, T^2x),$$

where $k \in \mathbb{N}$, and the conclusion is still correct.

Example 2.1. $G(x, y, z) = \max\{|x - y|, |x - z|, |y - z|\}$, $X = (0, 2)$,

$$Tx = \begin{cases} 1, & \text{if } 0 < x < 1, \\ \frac{x}{2} + \frac{1}{2}, & \text{if } 1 \leq x < 2. \end{cases}$$

Let $\varphi(t) = \frac{t}{2}$ for all $t \geq 0$. Case 1, when $0 < x < 1$, $G(Tx, T^2x, T^3x) = 0 \leq \frac{1}{2} G(x, Tx, T^2x) = \frac{1-x}{2}$. Case 2, when $1 \leq x < 2$, we have $\frac{x}{2} + \frac{1}{2} \geq 1$. $G(Tx, T^2x, T^3x) = \frac{3}{8}(x - 1)$, $G(x, Tx, T^2x) = \frac{3}{4}(x - 1)$. From above, we get $G(Tx, T^2x, T^3x) \leq \frac{1}{2} G(x, Tx, T^2x)$. Then all conditions of Theorem 2.1 are satisfied. T has a fixed point 1.

Example 2.2. Let $G(x, y, z) = \max\{|x - y|, |x - z|, |y - z|\}$, $X = [0, 1)$, and $T : X \to X$

$$Tx = \frac{x}{1 + x}.$$

Let $\varphi(t) = \frac{t}{1+t}$ for $t > 0$. Then we have

$$\begin{aligned} G(Tx, T^2x, T^3x) &= \left| \frac{x}{1+x} - \frac{x}{1+3x} \right| \\ &\leq \frac{2x^2}{(1+x)(1+3x)} \\ &\leq \frac{2x^2}{1+2x+2x^2} \\ &\leq \frac{\frac{2x^2}{1+2x}}{1+\frac{2x^2}{1+2x}} \\ &\leq \varphi(G(x, Tx, T^2x)). \end{aligned}$$

The conditions of Theorem 2.1 are satisfied. Thus T has a fixed point 0. Obviously, the Banach contractive mapping $G(Tx, Ty, Tz) \leq kG(x, y, z)$ in G-metric space does not work, where $k \in [0, 1)$.

3. Application to existence of solutions of integral equations

In this section, we present an example where Corollary 2.1 can be applied to show the existence of solutions for some integral equations. Consider the integral equation

$$u(t) = \int_0^\beta \mathcal{G}(t, s) f(s, u(s)) ds, \quad \text{for all} \quad t \in [0, \beta], \tag{3.1}$$

where $\beta > 0$, $f : [0, \beta] \times \mathbb{R} \to \mathbb{R}$ and $\mathcal{G} : [0, \beta] \times [0, \beta] \to \mathbb{R}$ are continuous functions. Let $X = C([0, \beta])$ be the set of real continuous functions on $[0, \beta]$. We endow X with the G-metric mapping

$$G(u, v, w) = \max_{x \in [0,\beta]} (|u(x) - v(x)| + |u(x) - w(x)| + |v(x) - w(x)|).$$

Consider the self-mapping $T : X \to X$ defined by

$$Tu(x) = \int_0^\beta \mathcal{G}(x, s) f(s, u(s)) ds.$$

Clearly, u^* is a solution of (3.1) if and only if u^* is a fixed point of T.

Suppose the following conditions are satisfied:

(A) $|f(s, x) - f(s, y)| + |f(s, x) - f(s, z)| + |f(s, z) - f(s, y)| \leq |x - y| + |x - z| + |z - y|$ for all $s \in [0, \beta]$ and $x, y, z \in \mathbb{R}$;

(B) $\max_{x \in [0,\beta]} \int_0^\beta |\mathcal{G}(x, s)| ds = r < 1$.

Therefore, we deduce

$$\begin{aligned} & G(Tu, T^2u, T^3u) \\ &= \max_{x \in [0,\beta]} |Tu(x) - T^2u(x)| + \max_{x \in [0,\beta]} |Tu(x) - T^3u(x)| + \max_{x \in [0,\beta]} |T^2u(x) - T^3u(x)| \end{aligned}$$

$$
\begin{aligned}
&= \max_{x\in[0,\beta]} |\int_0^\beta \mathcal{G}(x,s)(f(s,u(s)) - f(s,Tu(s)))ds| \\
&\quad + \max_{x\in[0,\beta]} |\int_0^\beta \mathcal{G}(x,s)(f(s,u(s)) - f(s,T^2u(s)))ds| \\
&\quad + \max_{x\in[0,\beta]} |\int_0^\beta \mathcal{G}(x,s)(f(s,Tu(s)) - f(s,T^2u(s)))ds| \\
&\leq \max_{x\in[0,\beta]} \int_0^\beta |\mathcal{G}(x,s)|(|u(s) - Tu(s)| + |u(s) - T^2u(s)| + |Tu(s) - T^2u(s)|)ds \\
&\leq G(u,Tu,T^2u) \max_{x\in[0,\beta]} \int_0^\beta |\mathcal{G}(x,s)|ds \\
&\leq rG(u,Tu,T^2u). \qquad (3.2)
\end{aligned}
$$

The conditions of Corollary 2.1 are satisfied, so T has a fixed point in X.

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