Journal of Applied Analysis and Computation
Volume 11, Number 1, February 2021, 138–160
Website:http://www.jaac-online.com
DOI:10.11948/20190201

STABILITY ANALYSIS OF A NONLOCAL FRACTIONAL IMPULSIVE COUPLED EVOLUTION DIFFERENTIAL EQUATION*

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Abstract This work is committed to establish the necessary assumptions related with the existence and uniqueness of solutions to a nonlocal coupled impulsive fractional differential equation. We attain our main results by the use of Krasnoselskii's fixed point theorem and Banach contraction principle. Additionally, we create a framework for studying the Hyers–Ulam stability of the considered problem. For the applications of theoretical result, we discuss an example at the end.

Keywords Caputo fractional derivative, Hyers–Ulam stability, impulsive switched coupled system.

MSC(2010) 26A33, 34A08, 34B27.

1. Introduction

In the last few decades, the theory of fractional differential equations (FDEs) has become one of the most attractive research area for finding new results. The reason behind this attractiveness is the fact that it precisely describes a large number of nonlinear phenomena in different branches of science and engineering like, viscoelasticity, control hypothesis, speculation, fluid dynamics, hydrodynamics, aerodynamics, information processing, system networking, picture processing etc. It is also a useful instrument for the depiction of memory and inherited properties of many materials and processes. As a result, FDEs theory gained a significant development in recent years, for details we refer the reader to [1,7–9,11,12,15,19,20,22,25,39–42].

Qualitative analysis of solutions to dynamical systems is a great tool for analyzing its different behaviors. Among these properties, surety of existence and uniqueness of solutions to the given dynamical systems is a challenging task for mathematicians. The aforementioned properties has been explored well for integer order differential equations (DEs). However, for FDEs there are many aspects that

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[*]The authors were supported by the National Natural Science Foundation of China(No. 11371117), the China Postdoctoral Science Foundation (No. 2019M652348), Technology Research Foundation of Chongqing Educational Committee(No. KJQN201900539).

requires further investigations. The literature devoted to the existence and uniqueness of solutions has been marvelously studied by adapting Riemann–Liouville and Caputo factional derivatives, for more details we recommend [2, 16, 26, 44, 45].

In the study of dynamical systems, stability analysis is a basic requirement for the applicability of results. In stability theory, especially Ulam's stability, which was first established by Ulam [30], in 1940 and extended by Hyers [10] to DEs plays a pivot role. Many mathematicians further worked on the Hyers result in different directions, as can be seen in [3, 13, 14, 17, 18, 21, 23, 24, 27–29, 33, 34, 38, 46–52].

In [35], Wang *et al.* studied the existence of solutions to the following system of FDES

$$\begin{cases} {}^cD^q z(t) = \theta(t, z(t)),\ t \in [0, \mathcal{T}],\ \mathcal{T} > 0, \\ z_*(0) + g(z) = z_0, \end{cases}$$

where ${}^cD^q$ is the Caputo fractional derivative of order $q \in (0, 1)$, $\theta : [0, \mathcal{T}] \times \mathbb{R} \to \mathbb{R}$ is continuous and $z_0 \in \mathbb{R}$.

In [43], Zhang investigated the existence and uniqueness of solutions for the model given by

$$\begin{cases} {}^cD^q z(t) = \theta_*(t, z(t)),\ t \in (0, 1),\ 1 < q \le 2, \\ z(0) + z'(0) = 0,\ z(1) + z'(1) = 0, \end{cases}$$

where $\theta_* : [0, 1] \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous.

Many targets have been achieved about stability analysis of integer order DEs, but for FDEs only few monographs are devoted. Recently, Wang *et. al* in [36] studied Ulam's type stability of different kinds for FDEs. In [37], the authors studied the aforementioned stabilities for:

$$\begin{cases} {}^cD^q z(t) = \theta(t, z(t)),\ t \in [a, \infty),\ 0 < q < 1, \\ z_*(a) = 0, \end{cases}$$

and

$$\begin{cases} {}^cD^q z(t) = \theta(t, z(t)),\ t \in [a, \infty),\ 1 < q < 2, \\ z(a) = z(b) = 0, \end{cases}$$

where $\theta : [a, \infty) \times \mathbb{R} \to \mathbb{R}$ is continuous.

Impulsive FDEs play a significant role in the applied models, for details see [4,32]. As pointed out in [4], the theory of initial and boundary value problems (BVPs) for the nonlinear impulsive FDEs is still in the early stage. In [4], the authors studied the following impulsive hybrid BVPs of FDEs:

$$\begin{cases} {}^cD^\delta z(t) + f_1(t, z(t)) = 0,\ t \in \mathsf{J} = [0, 1] - \{t_1, t_2, \dots, t_m\},\ 1 < \delta \le 2, \\ z(0) + z'(0) = 0,\ z(1) + z'(1) = 0,\ k = 1, 2, \dots, m, \\ \Delta z(t_k^-) = I_k(z(t_k^-)),\ \Delta z'(t_k^-) = \tilde{I}(z(t_k^-)),\ t_k \in (0, 1), \end{cases}$$

and

$$\begin{cases} {}^cD^\delta z(t) + f_1(t, z(t)) = 0,\ t \in \mathsf{J} = [0,1] - \{t_1, t_2, \ldots, t_m\},\ 1 < \delta \leq 2, \\ \alpha z(0) + \beta z'(0) = \int_0^1 q_1(s) z(s) ds,\ \alpha z(1) + \beta z'(1) = \int_0^1 q_2(s) z(s) ds, \\ \Delta z(t_k^-) = I_k(z(t_k^-)),\ \Delta z'(t_k^-) = \tilde{I}(z(t_k^-)),\ t_k \in (0,1). \end{cases}$$

Motivated by the above mentioned work, in this article our target is to investigate the existence, uniqueness and Hyers–Ulam stability for the following system of FDEs

$$\begin{cases} {}^cD^\alpha z(t) - \chi_1(t) z(t) = \varphi(t, z(t), \omega(t)),\ t \in \mathsf{J} = [0,1] - \{t_1, t_2, \ldots, t_m\}, 1 < \alpha \leq 2, \\ {}^cD^\alpha \omega(t) - \chi_2(t) \omega(t) = \psi(t, z(t), \omega(t)),\ t \in \mathsf{J} = [0,1] - \{t_1, t_2, \ldots, t_m\}, 1 < \alpha \leq 2, \\ \lambda z(0) + \xi z'(0) = h(z), \quad \lambda z(1) + \xi z'(1) = g(z), \\ \lambda \omega(0) + \xi \omega'(0) = h(\omega),\ \lambda \omega(1) + \xi \omega'(1) = g(\omega), \\ \Delta z(t_k) = I_k(z(t_k)),\ \Delta z'(t_k) = \tilde{I}(z(t_k)), \\ \Delta \omega(t_k) = I_k(\omega(t_k)),\ \Delta \omega'(t_k) = \tilde{I}(\omega(t_k), \quad 0 < t_k < 1, \end{cases} \tag{1.1}$$

where ${}^cD^\alpha$ presents the Caputo derivative of order $\alpha \in (1,2]$ of z and ω with the lower limits t_k, $k = 1, 2, \ldots, m$, $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = 1$, $\mathsf{J} = [0,1] - \{t_1, t_2, \ldots, t_m\}$ and $\chi_1(\cdot), \chi_2(\cdot)$, are linear and bounded operators on $\mathbb{R}$. Furthermore, I_k and $\tilde{I}_k$ are the impulsive operators. The nonlinear functions $\phi : \mathrm{C}(\mathsf{J}, \mathbb{R}) \to \underline{D}(\chi_1(.))$, $\varphi : \mathrm{C}(\mathsf{J}, \mathbb{R}) \to \underline{D}(\chi_2(.))$ are continuous. Moreover, $\Delta z(t_k)|_{t_0 = t_k} = z(t_k^+) - z(t_k^-)$, $\Delta \omega(t_k)|_{t_0 = t_k} = \omega(t_k^+) - \omega(t_k^-)$, $\Delta z'(t_k)|_{t_0 = t_k} = z'(t_k^+) - z'(t_k^-)$ and $\Delta \omega'(t_k)|_{t_0 = t_k} = \omega'(t_k^+) - \omega'(t_k^-)$, where $z(t_k^+)$, $\omega(t_k^+)$, $z'(t_k^+)$, $\omega'(t_k^+)$ are right and $z(t_k^-)$, $\omega(t_k^-)$, $z'(t_k^-)$, $\omega'(t_k^-)$ left limits, respectively.

The manuscript is organized as follows: In Section 2, we give essential definitions, lemmas and theorems. In Section 3, we develop suitable conditions for the existence and uniqueness of solution to (1.1), using Krasnoselskii's fixed point theorem and Banach contraction principle. In Section 4, we built up generalized results according to which problem (1.1) satisfies the conditions of Hyers–Ulam stability. In Section 5, we verify our results by discussing a particular example.

2. Preliminaries

In this part, we assemble some fundamental facts, definitions and lemmas used throughout this article, for detail reader should study [1, 15, 22].

For $t_k \in \mathsf{J}$, such that $0 = t_0 < t_1 < \cdots < t_m < \mathrm{T}$ and $\mathsf{J} = [0,1] - \{t_1, t_2, \ldots, t_m\}$, define the space $\mathrm{PC}(\mathsf{J}, \mathbb{R}) = \{z : \mathsf{J} \to \mathbb{R} : z \in \mathrm{C}(\mathsf{J})\}$, where left limit $z(t_k^-)$ and right limit $z(t_k^+)$ exist and

$$\Delta z(t)|_{t = t_k} = z(t_k^+) - z(t_k^-),\ 1 \leq k \leq m.$$

Similarly, we also define $\mathrm{PC}(\mathsf{J}, \mathbb{R}) = \{\omega : \mathsf{J} \to \mathbb{R} : \omega \in \mathrm{C}(\mathsf{J})$, where left limit $\omega(t_k^-)$ and right limit $\omega(t_k^+)$ exist and

$$\Delta \omega(t)|_{t = t_k} = \omega(t_k^+) - \omega(t_k^-), \quad 1 \leq k \leq m.$$

Bequeathing the norms, $\|z\|_{\mathrm{PC}} = \max_{t\in \mathsf{J}} |z(t)|$ and $\|\omega\|_{\mathrm{PC}} = \max_{t\in \mathsf{J}} |\omega(t)|$ in $\mathrm{PC}(\mathsf{J},\mathbb{R})$, which form a Banach space under these norms, and hence their product $\mathrm{E} = \mathrm{PC}\times\mathrm{PC}$ is again a Banach space with norm defined by $\|(z,\omega)\|_{\mathrm{PC}} = \|z\|_{\mathrm{PC}} + \|\omega\|_{\mathrm{PC}}$.

Definition 2.1 ([15]). Let $\delta \in \mathbb{R}^+$, then the arbitrary order integral in the Riemann–Liouville sense for a function $p : \mathsf{J} \to \mathbb{R}$ is given as

$$\mathcal{I}^{\delta} p(t) = \frac{1}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-1} p(s) ds,$$

such that the integral on the right side is pointwise defined on $\mathbb{R}^+$.

Definition 2.2 ([15]). Let p be a given function on close interval $[x_0, \omega_0]$. Then the noninteger order derivative in the Caputo sense of p is stated as

$$\frac{d^{\delta}}{dt^{\delta}} p(t) = \int_{x_0}^{t} \frac{(t-s)^{\eta-\delta-1}}{\Gamma(\eta-\alpha)} \left(\frac{d^{\eta}}{ds^{\eta}} p(s)\right) ds, \ \delta \in (\eta-1, \eta],$$

where $\eta = 1 + [\eta]$. Particularly, if p is defined on the closed interval $[x_0, \omega_0]$ and $\delta \in (0,1]$, then

$$\frac{d^{\delta}}{dt^{\delta}} p(t) = \frac{1}{\Gamma(1-\delta)} \int_{x_0}^{t} \frac{p'(s)}{(t-s)^{\delta}} ds, \text{ where } \varphi'(s) = \frac{d\varphi(s)}{ds}.$$

It is to be noted that the integral on the right hand side is pointwise defined on $\mathbb{R}^+$.

Theorem 2.1 ([6]). *Let $\delta \in [\eta-1, \eta)$. For $p \in \mathrm{C}([x_0, \omega_0])$, the unique solution of $\frac{d^{\alpha}}{dt^{\alpha}} p(t) = 0$ has the following form $p(t) = \sum_{k=0}^{[\delta]} c_k t^k$, where $c_k \in \mathbb{R}$, $k = 1, 2, \ldots, [\delta]$, $1 + [\delta] = \eta$.*

Theorem 2.2 ([6]). *Let $\delta \in [\eta-1, \eta)$. For $p \in \mathrm{C}^n([x_0, \omega_0])$, $\mathcal{I}^{\delta} \frac{d^{\delta}}{dt^{\delta}} p(t) = p(t) + \sum_{k=0}^{[\delta]} a_k t^k$, for some $a_k \in \mathbb{R}$, $k = 1, 2, \ldots, [\delta]$, $1 + [\delta] = \eta$.*

Theorem 2.3 ([5]). *Let $\mathcal{S} \neq \emptyset$ be a closed convex subset of Banach space* E. *Let Υ_1, Υ_2 be two operators such that*

(i). $\Upsilon_1(z,\omega) + \Upsilon_2(\bar{z},\bar{\omega}) \in \mathcal{S}$, where $(z,\omega), (\bar{z},\bar{\omega}) \in \mathcal{S}$;

(ii). Υ_1 is contraction;

(iii). Υ_2 is completely continuous.

Then the operating system $(z,\omega) = \Upsilon_1(z,\omega) + \Upsilon_2(z,\omega)$ has a a solution in $(z,\omega) \in \mathcal{S}$.

Definition 2.3 (Urs [31], Definition 2). Consider a Banach space E such that $\Phi_1, \Phi_2 : \mathrm{E} \to \mathrm{E}$ be two operators. Then the operator system provided by

$$\begin{cases} z(t) = \Phi_1(z,\omega)(t), \\ \omega(t) = \Phi_2(z,\omega)(t), \end{cases} \tag{2.1}$$

is called Hyers–Ulam stable if we can find constants $C_{i=1,2,3,4} > 0$ such that for each $\varrho_{j=1,2} > 0$ and each solution $(\widehat{z}, \widehat{\omega}) \in \mathrm{E}$ of the inequalities given by

$$\begin{cases} \|\widehat{z} - \phi(\widehat{z}, \widehat{\omega})\|_{\mathrm{PC}} \leq \varrho_1, \\ \|\widehat{\omega} - \varphi(\widehat{z}, \widehat{\omega})\|_{\mathrm{PC}} \leq \varrho_2, \end{cases} \tag{2.2}$$

there exists a solution $(\tilde{z}, \tilde{\omega}) \in \mathrm{E}$ of system (2.1) which satisfy

$$\begin{cases} \|\widehat{z} - \tilde{z}\|_{\mathrm{PC}} \leq C_1\varrho_1 + C_2\varrho_2, \\ \|\widehat{\omega} - \tilde{\omega}\|_{\mathrm{PC}} \leq C_3\varrho_1 + C_4\varrho_2. \end{cases} \tag{2.3}$$

Definition 2.4. If the matrix $\mathcal{H}^* \in \mathbb{C}^{m\times m}$ has eigenvalues μ_j, for $j = 1, 2, \dots, m$, then $\rho(\mathcal{H}^*)$ (spectral radius) is defined by

$$\rho(\mathcal{H}^*) = \max\{|\mu_j|, \quad j = 1, 2, \dots, m\}.$$

Furthermore, the matrix $\mathcal{H}^*$ converge to 0 if $\rho(\mathcal{H}^*) < 1$.

Theorem 2.4 ([31]). *Consider a Banach space* E *with* $\Phi_1, \Phi_2 : \mathrm{E} \to \mathrm{E}$ *be two operators such that*

$$\begin{cases} \|\Phi_1(z, \omega) - \Phi_1(\widehat{z}, \widehat{\omega})\|_{\mathrm{PC}} \leq \Lambda_1\|z - \widehat{z}\|_{\mathrm{PC}} + \Lambda_2\|\omega - \widehat{\omega}\|_{\mathrm{PC}}, \\ \|\Phi_2(z, \omega) - \Phi_2(\widehat{z}, \widehat{\omega})\|_{\mathrm{PC}} \leq \Lambda_3\|z - \widehat{z}\|_{\mathrm{PC}} + \Lambda_4\|\omega - \widehat{\omega}\|_{\mathrm{PC}}, \end{cases}$$

and if the matrix

$$\mathcal{H}^* = \begin{pmatrix} \Lambda_1 & \Lambda_2 \\ \Lambda_3 & \Lambda_4 \end{pmatrix}$$

converges to 0*, then the fixed points consequential to the operational system* (2.1) *are Hyers–Ulam stable.*

3. Existence Results

Before coming to the main result, we follow some restrictions.
($\mathbf{H}_1$). The bounded linear operators $\chi_{j,j=1,2} : \underline{D}(\chi_j) \to \mathbb{R}^+$ are closed and for any $t \in \mathsf{J}$, $\sup_{t\in\mathsf{J}}\Big|\chi_1(t)z(t)\Big| \leq M_p$, $\sup_{t\in\mathsf{J}}\Big|\chi_2(t)\omega(t)\Big| \leq M_q$ and

$$\begin{aligned} |\chi_1(t)z(t) - \chi_1(t)\tilde{z}(t)| &\leq M_{p,\tilde{p}}\|z - \tilde{z}\|, \\ |\chi_2(t)\omega(t) - \chi_2(t)\tilde{\omega}(t)| &\leq M_{q,\tilde{q}}\|\omega - \tilde{\omega}\|, \end{aligned}$$

where $M_p, M_q, M_{p,\tilde{p}}$ and $M_{q,\tilde{q}}$ are positive constants.
($\mathbf{H}_2$). The functions $\phi, \varphi : \mathsf{J}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ are continuous such that $\forall\,(z, \omega), (\tilde{z}, \tilde{\omega}) \in \mathrm{E}$ and $t \in \mathsf{J}$, there exist $M'_\phi, M'_\varphi > 0$, satisfying

$$|\phi(t, z(t), \omega(t)) - \phi(t, \tilde{z}(t), \tilde{\omega}(t))| \leq M'_\phi\big(|z(t) - \tilde{z}(t)| + |\omega(t) - \tilde{\omega}(t)|\big) = M'_\phi\|(z - \tilde{z}, \omega - \tilde{\omega})\|_{\mathrm{PC}},$$

$$|\varphi(t,z(t),\omega(t))-\varphi(t,\tilde{z}(t),\tilde{\omega}(t))|\leq M'_\varphi\big(|z(t)-\tilde{z}(t)|+|\omega(t)-\tilde{\omega}(t)|\big)=M'_\varphi\|(z-\tilde{z},\omega-\tilde{\omega})\|_{\mathrm{PC}}.$$

$(\mathbf{H}_3)$. For all $(z,\omega)\in \mathrm{E}$ and $t\in \mathsf{J}$ there exist $M_\phi, M_\varphi>0$, such that

$$\begin{aligned}|\phi(t,z(t),\omega(t))| \leq& M_\phi\{\|z\|_{\mathrm{PC}}+\|\omega\|_{\mathrm{PC}}\}\leq M_\phi\|(z,\omega)\|_{\mathrm{PC}},\\ |\varphi(t,z(t),\omega(t))| \leq& M_\varphi\{\|z\|_{\mathrm{PC}}+\|\omega\|_{\mathrm{PC}}\}\leq M_\varphi\|(z,\omega)\|_{\mathrm{PC}}.\end{aligned}$$

$(\mathbf{H}_4)$. $I_k,\tilde{I}_k:\mathbb{R}\to\mathbb{R}$ are continuous, and there exist constants $l_I, l_{\tilde{I}}>0$ for any $(z,\omega),(\tilde{z},\tilde{\omega})\in \mathrm{E}$ such that

$$\begin{aligned}|I_k(z(t))-I_k(\tilde{z}(t))| \leq& l_I|z-\tilde{z}|,\\ |I_k(\omega(t))-I_k(\tilde{\omega}(t))| \leq& l'_I|\omega-\tilde{\omega}|,\\ |\tilde{I}_k(z(t))-\tilde{I}_k(\tilde{z}(t))| \leq& l_{\tilde{I}}|z-\tilde{z}|,\\ |\tilde{I}_k(\omega(t))-\tilde{I}_k(\tilde{\omega}(t))| \leq& l'_{\tilde{I}}|\omega-\tilde{\omega}|,\end{aligned}$$

where $k=0,1,\dots,m$.
$(\mathbf{H}_5)$. $I_k,\tilde{I}_k:\mathbb{R}\to\mathbb{R}$ are continuous, and there exist constants $M_I, M_{\tilde{I}}>0$ for any $(z,\omega)\in \mathrm{E}$ such that

$$\begin{aligned}|I_k(z(t))| \leq& M_I|z|,\ |\tilde{I}_k(z(t))|\leq M_{\tilde{I}}|z|,\\ |I_k(\omega(t))| \leq& M'_I|\omega|,\ |\tilde{I}_k(\omega(t))|\leq M'_{\tilde{I}}|\omega|,\end{aligned}$$

where $k=0,1,\dots,m$.
$(\mathbf{H}_6)$. $h,g:\mathbb{R}\to\mathbb{R}$ are continuous and $\forall\ (z,\omega),(\tilde{z},\tilde{\omega})\in \mathrm{E}$ there exist, constants $l_h, l_g, l_{gh}, \widehat{l}_{gh}>0$ such that

$$\begin{aligned}|h(z)-h(\tilde{z})| \leq& l_h|z-\tilde{z}|,\ |g(\omega)-g(\tilde{\omega})|\leq l_g|\omega-\tilde{\omega}|,\\ |g(z)-h(z)| \leq& l_{gh}|z|,\ |g(\omega)-h(\omega)|\leq l_{gh}|\omega|.\end{aligned}$$

Theorem 3.1. *Let $\rho_1,\rho_2\in \mathrm{PC}(\mathsf{J},\mathbb{R})$ and $\underline{D}(\chi_1)$, $\underline{D}(\chi_2)$ are bounded linear operators. Then the solution of the coupled system*

$$\begin{cases}{}^cD^\alpha z(t)-\chi_1(t)z(t)=\rho_1(t),\ 1<\alpha\leq 2,\ t\in \mathsf{J},\\ {}^cD^\alpha \omega(t)-\chi_2(t)\omega(t)=\rho_2(t),\ 1<\alpha\leq 2,\ t\in \mathsf{J},\end{cases}$$

supplemented with the boundary conditions in (1.1) *is equivalent to the solution of the following integral equations*

$$\begin{aligned}z(t)=&\frac{1}{\Gamma(\alpha)}\int_{t_m}^t (t-s)^{\alpha-1}\big(\chi_1(s)z(s)+\rho_1(s)\big)ds+(\frac{\xi}{\lambda}-t)\frac{1}{\Gamma(\alpha)}\\ &\times\int_{t_m}^1 (1-s)^{\alpha-1}\big(\chi_1(s)z(s)+\rho_1(s)\big)ds\\ &+(\frac{\xi}{\lambda}-t)\frac{\xi}{\lambda\Gamma(\alpha-1)}\int_{t_m}^1(1-s)^{\alpha-2}\big(\chi_1(s)z(s)+\rho_1(s)\big)ds\\ &+(\frac{\xi}{\lambda}-t)\frac{1}{\Gamma(\alpha)}\sum_{0<t_k<1}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}\big(\chi_1(s)z(s)+\rho_1(s)\big)ds\end{aligned}$$

$$
\begin{aligned}
&+(\frac{\xi}{\lambda}-t)\frac{1}{\Gamma(\alpha-1)}\sum_{0<t_k<1}(\frac{\xi}{\lambda}+1-t_k)\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-2}\big(\chi_1(s)z(s)+\rho_1(s)\big)ds\\
&+\frac{1}{\Gamma(\alpha)}\sum_{0<t_k<1}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}\big(\chi_1(s)z(s)+\rho_1(s)\big)ds\\
&+\frac{1}{\Gamma(\alpha-1)}\sum_{0<t_k<1}(t-t_k)\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-2}\big(\chi_1(s)z(s)+\rho_1(s)\big)ds\\
&+\sum_{k=1}^{m}\{I_kz(t_k)+(t-t_k)\tilde{I}_kz(t_k)\}+\frac{1}{\lambda^2}\{\lambda h(z)+(\lambda t-\xi)(g(z)-h(z))\}\\
&+(\frac{\xi}{\lambda}-t)\sum_{0<t_k<1}(\frac{\xi}{\lambda}+1-t_k)\tilde{I}_k(z(t_k))+(\frac{\xi}{\lambda}-t)\sum_{0<t_k<1}I_kz(t_k), \qquad (3.1)
\end{aligned}
$$

and

$$
\begin{aligned}
\omega(t)=&\frac{1}{\Gamma(\alpha)}\int_{t_m}^{t}(t-s)^{\alpha-1}\big(\chi_2(s)\omega(s)+\rho_2(s)\big)ds+(\frac{\xi}{\lambda}-t)\frac{1}{\Gamma(\alpha)}\\
&\times\int_{t_m}^{1}(1-s)^{\alpha-1}\big(\chi_2(s)\omega(s)+\rho_2(s)\big)ds\\
&+(\frac{\xi}{\lambda}-t)\frac{\xi}{\lambda\Gamma(\alpha-1)}\int_{t_m}^{1}(1-s)^{\alpha-2}\big(\chi_2(s)\omega(s)+\rho_2(s)\big)ds\\
&+(\frac{\xi}{\lambda}-t)\frac{1}{\Gamma(\alpha)}\sum_{0<t_k<1}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}\big(\chi_2(s)\omega(s)+\rho_2(s)\big)ds\\
&+(\frac{\xi}{\lambda}-t)\frac{1}{\Gamma(\alpha-1)}\sum_{0<t_k<1}(\frac{\xi}{\lambda}+1-t_k)\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-2}\big(\chi_2(s)\omega(s)+\rho_2(s)\big)ds\\
&+\frac{1}{\Gamma(\alpha)}\sum_{0<t_k<1}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}\big(\chi_2(s)\omega(s)+\rho_2(s)\big)ds\\
&+\frac{1}{\Gamma(\alpha-1)}\sum_{0<t_k<1}(t-t_k)\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-2}\big(\chi_2(s)\omega(s)+\rho_2(s)\big)ds\\
&+\sum_{k=1}^{m}I_k\omega(t_k)+\sum_{k=1}^{m}(t-t_k)\tilde{I}_k\omega(t_k)+\frac{1}{\lambda^2}\{\lambda h(\omega)+(\lambda t-\xi)(g(\omega)-h(\omega))\}\\
&+(\frac{\xi}{\lambda}-t)\sum_{0<t_k<1}(\frac{\xi}{\lambda}+1-t_k)\tilde{I}_k(\omega(t_k))+(\frac{\xi}{\lambda}-t)\sum_{0<t_k<1}I_k\omega(t_k). \qquad (3.2)
\end{aligned}
$$

Proof. First we consider

$$ {}^cD^{\alpha}z(t)-\chi_1(t)z(t)=\rho_1(t),\quad 1<\alpha\le 2,\ t\in \mathsf{J}. \qquad (3.3)$$

For $t\in[0,t_1]$, the use of $\mathcal{I}^{\alpha}$ on each side of (3.3), gives

$$ z(t)=\frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t-s)^{\alpha-1}(\chi_1(s)z(s)+\rho_1(s))ds-b_0-tb_1. \qquad (3.4)$$

Differentiating (3.4) with respect to t, we obtain

$$ z'(t)=\frac{1}{\Gamma(\alpha-1)}\int_0^{t_1}(t-s)^{\alpha-2}(\chi_1(s)z(s)+\rho_1(s))ds-b_1. $$

Applying the initial conditions, we gain $-b_0 = \frac{\xi}{\lambda} b_1 + \frac{1}{\lambda} h(z)$. Therefore, (3.4) becomes

$$z(t) = \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t-s)^{\alpha-1} (\chi_1(s)z(s) + \rho_1(s))ds + b_1(\frac{\xi}{\lambda} - t) + \frac{1}{\lambda} h(z). \tag{3.5}$$

Now for $t \in (t_1, t_2]$ and using $\mathcal{I}^\alpha$ on (3.3), we get

$$z(t) = \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t} (t-s)^{\alpha-1} (\chi_1(s)z(s) + \rho_1(s))ds - c_0 - (t-t_1)c_1. \tag{3.6}$$

Differentiating (3.6) with respect to t, we have

$$z'(t) = \frac{1}{\Gamma(\alpha-1)} \int_{t_1}^{t} (t-s)^{\alpha-2} (\chi_1(s)z(s) + \rho_1(s))ds - c_1.$$

Using the initial conditions, we get

$$z(t_1^-) = \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1-s)^{\alpha-1} (\chi_1(s)z(s) + \rho_1(s))ds + b_1(\frac{\xi}{\lambda} - t_1) + \frac{1}{\lambda} h(z),$$

and $z(t_1^+) = -c_0$.

Because

$$\begin{aligned}\Delta(z(t_1)) &= I_1(z(t_1)) = z(t_1^+) - z(t_1^-) \\ &= -c_0 - \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1-s)^{\alpha-1} (\chi_1(s)z(s) + \rho_1(s))ds - b_1(\frac{\xi}{\lambda} - t_1) - \frac{1}{\lambda} h(z).\end{aligned}$$

Which gives

$$-c_0 = I_1(z(t_1)) + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1-s)^{\alpha-1} (\chi_1(s)z(s) + \rho_1(s))ds + b_1(\frac{\xi}{\lambda} - t_1) + \frac{1}{\lambda} h(z). \tag{3.7}$$

Similarly using $\Delta(z'(t_1)) = \tilde{I}_1(z(t_1)) = z'(t_1^+) - z'(t_1^-)$, gives

$$-c_1 = \tilde{I}_1(z(t_1)) + \frac{1}{\Gamma(\alpha-1)} \int_0^{t_1} (t_1-s)^{\alpha-2} (\chi_1(s)z(s) + \rho_1(s))ds - b_1. \tag{3.8}$$

Putting (3.7) and (3.8) in (3.6), we get

$$\begin{aligned}z(t) =& \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t} (t-s)^{\alpha-1} (\chi_1(s)z(s) + \rho_1(s))ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1-s)^{\alpha-1} (\chi_1(s)z(s) + \rho_1(s))ds \\ &+ \frac{t-t_1}{\Gamma(\alpha-1)} \int_0^{t_1} (t_1-s)^{\alpha-2} (\chi_1(s)z(s) \\ &+ \rho_1(s))ds + I_1(p(t_1)) + (t-t_1)\tilde{I}_1(p(t_1)) + b_1(\frac{\xi}{\lambda} - t) + \frac{1}{\lambda} h(z).\end{aligned}$$

In identical way for any $t \in (t_k, 1)$, we gain

$$z(t) = \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} (\chi_1(s)z(s)$$

$$
\begin{aligned}
&+\rho_1(s))ds+\frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}(\chi_1(s)z(s)\\
&+\rho_1(s))ds+\sum_{k=1}^{m}\frac{t-t_k}{\Gamma(\alpha-1)}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-2}(\chi_1(s)z(s)+\rho_1(s))ds\\
&+\sum_{k=1}^{m}I_k(z(t_k))+\sum_{k=1}^{m}(t-t_k)\tilde{I}_k(z(t_k))+b_1(\frac{\xi}{\lambda}-t)+\frac{1}{\lambda}h(z).
\end{aligned}
$$

By differentiating with respect to t, we get

$$
\begin{aligned}
z'(t)=&\frac{1}{\Gamma(\alpha-1)}\int_{t_k}^{t}(t-s)^{\alpha-2}(\chi_1(s)z(s)+\rho_1(s))ds\\
&+\frac{1}{\Gamma(\alpha-1)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-2}(\chi_1(s)z(s)+\rho_1(s))ds+\sum_{k=1}^{m}\tilde{I}_k(z(t_k))-b_1.
\end{aligned}
$$

Utilizing $\lambda z(1)+\xi z'(1)=g(z)$, we obtain

$$
\begin{aligned}
b_1=&\frac{1}{\Gamma(\alpha)}\int_{t_m}^{1}(t-s)^{\alpha-1}(\chi_1(s)z(s)+\rho_1(s))ds\\
&+\frac{\xi}{\lambda\Gamma(\alpha-1)}\int_{t_m}^{1}(1-s)^{\alpha-1}(\chi_1(s)z(s)+\rho_1(s))ds\\
&+\frac{1}{\Gamma(\alpha)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}(\chi_1(s)z(s)+\rho_1(s))ds\\
&+\sum_{k=1}^{m}I_k(z(t_k))+\sum_{k=1}^{m}(t-t_k)\tilde{I}(z(t_k))\\
&+\frac{1}{\Gamma(\alpha-1)}\sum_{k=1}^{m}\Big(\frac{\xi}{\lambda}+1-t_k\Big)\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-2}(\chi_1(s)z(s)\\
&+\rho_1(s))ds+\frac{1}{\lambda}\{h(z)-g(z)\}.
\end{aligned}
$$

Substituting the value of b_1 in (3.5), we obtain (3.1). On the same process, we can obtain (3.2). The proof is complete. □

($\mathbf{H}_7$). Suppose that $\xi^*=\max\{\xi_1^*,\xi_2^*\}<1$.
Choose a closed ball

$$
\mathcal{B}_r=\{(z,\omega)\in \mathrm{E},\|(z,\omega)\|\leq r,\|z\|\leq\frac{r}{2},\|\omega\|\leq\frac{r}{2}\}\subset \mathrm{E}
$$

where

$$
r\geq\frac{M^*C_1^*+M^{**}C_1^{**}}{1-(C_m+M_\phi C_3^*+M_\varphi C_3^{**})}.
$$

Define the operators $\mathbb{F}=(\mathbb{F}_1,\mathbb{F}_2)$, $\mathbb{G}=(\mathbb{G}_1,\mathbb{G}_2)$ on $\mathcal{B}_r$ by

$$
\begin{cases}
\mathbb{F}_1(z(t)) = & \frac{1}{\Gamma(\alpha)} \int_{t_m}^{t} (t-s)^{\alpha-1} \chi_1(s) z(s) ds \\
& + (\frac{\xi}{\lambda} - t) \Bigg[\frac{1}{\Gamma(\alpha)} \int_{t_m}^{1} (1-s)^{\alpha-1} \chi_1(s) z(s) ds \\
& + \frac{\xi}{\lambda \Gamma(\alpha-1)} \int_{t_m}^{1} (1-s)^{\alpha-2} \chi_1(s) z(s) ds \\
& + \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<1} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} \chi_1(s) z(s) ds \\
& + \frac{1}{\Gamma(\alpha-1)} \sum_{0<t_k<1} (\frac{\xi}{\lambda} + 1 - t_k) \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-2} \chi_1(s) z(s) ds \\
& + \sum_{0<t_k<1} (\frac{\xi}{\lambda} + 1 - t_k) \tilde{I}_k(z(t_k)) + \sum_{0<t_k<1} I_k(z(t_k)) \Bigg] \\
& + \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<1} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} \chi_1(s) z(s) ds \\
& + \frac{1}{\Gamma(\alpha-1)} \sum_{0<t_k<1} (t - t_k) \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-2} \chi_1(s) z(s) ds \\
& + \sum_{k=1}^{m} \{ I_k z(t_k) + (t - t_k) \tilde{I}_k z(t_k) \} + \frac{1}{\lambda^2} \{ \lambda h(z) + (\lambda t - \xi)(g(z) - h(z)) \}, \\
\mathbb{F}_2(\omega(t)) = & \frac{1}{\Gamma(\alpha)} \int_{t_m}^{t} (t-s)^{\alpha-1} \chi_2(s) \omega(s) ds \\
& + (\frac{\xi}{\lambda} - t) \Bigg[\frac{1}{\Gamma(\alpha)} \int_{t_m}^{1} (1-s)^{\alpha-1} \chi_2(s) \omega(s) ds \\
& + \frac{\xi}{\lambda \Gamma(\alpha-1)} \int_{t_m}^{1} (1-s)^{\alpha-2} \chi_2(s) \omega(s) ds \\
& + \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<1} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} \chi_2(s) \omega(s) ds \\
& + \frac{1}{\Gamma(\alpha-1)} \sum_{0<t_k<1} (\frac{\xi}{\lambda} + 1 - t_k) \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-2} \chi_2(s) \omega(s) ds \\
& + \sum_{0<t_k<1} (\frac{\xi}{\lambda} + 1 - t_k) \tilde{I}_k(\omega(t_k)) + \sum_{0<t_k<1} I_k(\omega(t_k)) \Bigg] \\
& + \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<1} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} \chi_2(s) \omega(s) ds \\
& + \frac{1}{\Gamma(\alpha-1)} \sum_{0<t_k<1} (t - t_k) \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-2} \chi_2(s) \omega(s) ds \\
& + \sum_{k=1}^{m} I_k \omega(t_k) + \sum_{k=1}^{m} (t - t_k) \tilde{I}_k \omega(t_k) \\
& + \frac{1}{\lambda^2} \{ \lambda h(\omega) + (\lambda t - \xi)(g(\omega) - h(\omega)) \},
\end{cases}
\tag{3.9}
$$

and

$$\begin{cases}\mathbb{G}_1(z(t),\omega(t)) =\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_{t_m}^{t}(t-s)^{\alpha-1}\phi(s,z(s),\omega(s))ds\\ \qquad +(\dfrac{\xi}{\lambda}-t)\Bigg[\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_{t_m}^{1}(1-s)^{\alpha-1}\phi(s,z(s),\omega(s))ds\\ \qquad +\dfrac{\xi}{\lambda\Gamma(\alpha-1)}\displaystyle\int_{t_m}^{1}(1-s)^{\alpha-2}\phi(s,z(s),\omega(s))ds\\ \qquad +\dfrac{1}{\Gamma(\alpha)}\displaystyle\sum_{0<t_k<1}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}\phi(s,z(s),\omega(s))ds\\ \qquad +\dfrac{1}{\Gamma(\alpha-1)}\displaystyle\sum_{0<t_k<1}(\dfrac{\xi}{\lambda}+1-t_k)\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-2}\phi(s,z(s),\omega(s))ds\Bigg]\\ \qquad +\dfrac{1}{\Gamma(\alpha)}\displaystyle\sum_{0<t_k<1}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}\phi(s,z(s),\omega(s))ds\\ \qquad +\dfrac{1}{\Gamma(\alpha-1)}\displaystyle\sum_{0<t_k<1}(t-t_k)\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-2}\phi(s,z(s),\omega(s))ds,\\ \mathbb{G}_2(z(t),\omega(t)) =\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_{t_m}^{t}(t-s)^{\alpha-1}\varphi(s,z(s),\omega(s))ds\\ \qquad +(\dfrac{\xi}{\lambda}-t)\Bigg[\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_{t_m}^{1}(1-s)^{\alpha-1}\varphi(s,z(s),\omega(s))ds\\ \qquad +\dfrac{\xi}{\lambda\Gamma(\alpha-1)}\displaystyle\int_{t_m}^{1}(1-s)^{\alpha-2}\varphi(s,z(s),\omega(s))ds\\ \qquad +\dfrac{1}{\Gamma(\alpha)}\displaystyle\sum_{0<t_k<1}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}\varphi(s,z(s),\omega(s))ds\\ \qquad +\dfrac{1}{\Gamma(\alpha-1)}\displaystyle\sum_{0<t_k<1}(\dfrac{\xi}{\lambda}+1-t_k)\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-2}\varphi(s,z(s),\omega(s))ds\Bigg]\\ \qquad +\dfrac{1}{\Gamma(\alpha)}\displaystyle\sum_{0<t_k<1}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}\varphi(s,z(s),\omega(s))ds\\ \qquad +\dfrac{1}{\Gamma(\alpha-1)}\displaystyle\sum_{0<t_k<1}(t-t_k)\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-2}\varphi(s,z(s),\omega(s))ds.\end{cases} \tag{3.10}$$

Theorem 3.2. *Let the assumptions,* (H_1) *to* (H_7) *are satisfied. Then the problem* (1.1) *has at least one solution.*

Proof. For any $(z,\omega)\in\mathcal{B}_r$, we have

$$\begin{aligned}\|\mathbb{F}(z,\omega)+\mathbb{G}(z,\omega)\|_{\mathrm{PC}} &\leq\|\mathbb{F}(z,\omega)\|_{\mathrm{PC}}+\|\mathbb{G}(z,\omega)\|_{\mathrm{PC}}\\ &\leq\|\mathbb{F}_1(z)\|_{\mathrm{PC}}+\|\mathbb{F}_2(\omega)\|_{\mathrm{PC}}\\ &\quad+\|\mathbb{G}_1(z,\omega)\|_{\mathrm{PC}}+\|\mathbb{G}_2(z,\omega)\|_{\mathrm{PC}}.\end{aligned} \tag{3.11}$$

From (3.9), we get

$$
\begin{aligned}
\big|\mathbb{F}_1 z(t)\big| \leq & \bigg|\frac{1}{\Gamma(\alpha)}\int_{t_m}^{t}(t-s)^{\alpha-1}\chi_1(s)z(s)ds \\
& + (\frac{\xi}{\lambda}-t)\bigg[\frac{1}{\Gamma(\alpha)}\int_{t_m}^{1}(1-s)^{\alpha-1}\chi_1(s)z(s)ds \\
& + \frac{\xi}{\lambda\Gamma(\alpha-1)}\int_{t_m}^{1}(1-s)^{\alpha-2}\chi_1(s)z(s)ds \\
& + \frac{1}{\Gamma(\alpha)}\sum_{0<t_k<1}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}\chi_1(s)z(s)ds \\
& + \frac{1}{\Gamma(\alpha-1)}\sum_{0<t_k<1}(\frac{\xi}{\lambda}+1-t_k)\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-2}\chi_1(s)z(s)ds \\
& + \sum_{0<t_k<1}(\frac{\xi}{\lambda}+1-t_k)\tilde{I}_k(z(t_k)) + \sum_{0<t_k<1} I_k(z(t_k))\bigg]\bigg| \\
& + \bigg|\frac{1}{\Gamma(\alpha)}\sum_{0<t_k<1}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}\chi_1(s)z(s)ds \\
& + \frac{1}{\Gamma(\alpha-1)}\sum_{0<t_k<1}(t-t_k)\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-2}\chi_1(s)z(s)ds \\
& + \sum_{k=1}^{m}\{I_k z(t_k) + (t-t_k)(\tilde{I}_k z(t_k))\}\bigg| \\
& + \bigg|\frac{1}{\lambda^2}\{\lambda h(z) + (\lambda t-\xi)(g(z)-h(z))\}\bigg|.
\end{aligned}
$$

Which gives

$$
\begin{aligned}
\big\|\mathbb{F}_1 z(t)\big\| \leq & \frac{M_p}{\Gamma(\alpha+1)} + \frac{\xi+\lambda}{\lambda}\bigg[\frac{M_p}{\Gamma(\alpha+1)} + \frac{\xi M_p}{\lambda\Gamma(\alpha)} \\
& + \frac{mM_p}{\Gamma(\alpha+1)} + mM_I + m\frac{(\xi+2\lambda)}{\lambda}\bigg(\frac{M_p}{\Gamma(\alpha)} + M_{\tilde{I}}\bigg)\bigg] \\
& + m\bigg(\frac{M_p}{\Gamma(\alpha+1)} + M_{\tilde{I}}\bigg) + m\bigg(\frac{M_p}{\Gamma(\alpha)} + M_I\bigg) \\
& + \frac{1}{\lambda^2}\bigg(\lambda l_h\|z\|_{\mathrm{PC}} + (\xi+\lambda)l_{gh}\|z\|_{\mathrm{PC}}\bigg).
\end{aligned}
$$

Here $M^* = \max\{M_p, M_I, M_{\tilde{I}}\}$,

$$
C_1^* = (m+1)\frac{2\lambda+\xi}{\lambda\Gamma(\alpha+1)} + \frac{(m+1)\xi^2 + (3m+1)\xi\lambda + 3m\lambda^2}{\lambda^2\Gamma(\alpha)} + \frac{m\xi^2 + 4m\xi\lambda + 5m\lambda^2}{\lambda^2}
$$

and

$$
C_2^* = \frac{1}{\lambda^2}\bigg(\lambda l_h + (\xi+\lambda)l_{gh}\bigg).
$$

Alternatively, we can write

$$\|\mathbb{F}_1 z\|_{\mathrm{PC}} \leq M^* C_1^* + \frac{1}{2} C_2^*. \tag{3.12}$$

Following the same procedure, we get

$$\|\mathbb{F}_2 \omega\|_{\mathrm{PC}} \leq M^{**} C_1^{**} + \frac{1}{2} C_2^{**}. \tag{3.13}$$

Furthermore,

$$\begin{aligned}
\Big|\mathbb{G}_1(z,\omega)(t)\Big| \leq& \Big|\frac{1}{\Gamma(\alpha)} \int_{t_m}^{t} (t-s)^{\alpha-1} \phi(s, z(s), \omega(s)) ds \\
&+ \big(\frac{\xi}{\lambda}+1\big)\Bigg[\frac{1}{\Gamma(\alpha)} \int_{t_m}^{1} (1-s)^{\alpha-1} \phi(s, z(s), \omega(s)) ds \\
&+ \frac{\xi}{\lambda\Gamma(\alpha-1)} \int_{t_m}^{1} (1-s)^{\alpha-2} \phi(s, z(s), \omega(s)) ds \\
&+ \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<1} \int_{t_{k-1}}^{t_k} (t_k-s)^{\alpha-1} \phi(s, z(s), \omega(s)) ds \\
&+ \frac{1}{\Gamma(\alpha-1)} \sum_{0<t_k<1} (\frac{\xi}{\lambda}+2) \int_{t_{k-1}}^{t_k} (t_k-s)^{\alpha-2} \phi(s, z(s), \omega(s)) ds\Bigg]\Bigg| \\
&+ \Big|\frac{1}{\Gamma(\alpha)} \sum_{0<t_k<1} \int_{t_{k-1}}^{t_k} (t_k-s)^{\alpha-1} \phi(s, z(s), \omega(s)) ds \\
&+ \frac{1}{\Gamma(\alpha-1)} \sum_{0<t_k<1} (t-t_k) \int_{t_{k-1}}^{t_k} (t_k-s)^{\alpha-2} \phi(s, z(s), \omega(s)) ds\Big| \\
\leq& \Bigg[\frac{M_\phi}{\Gamma(\alpha+1)} + \frac{\xi+\lambda}{\lambda}\Bigg(\frac{M_\phi}{\Gamma(\alpha+1)} + \frac{\xi M_\phi}{\lambda\Gamma(\alpha)} + \frac{m M_\phi}{\Gamma(\alpha+1)} \qquad (3.14) \\
&+ \frac{m M_\phi(\xi+2\lambda)}{\lambda\Gamma(\alpha)}\Bigg) + \frac{m M_\phi}{\Gamma(\alpha+1)} + \frac{m M_\phi}{\Gamma(\alpha)}\Bigg] \|(z,\omega)\|_{\mathrm{PC}} \\
\leq& M_\phi C_3^* r, \qquad (3.15)
\end{aligned}$$

where

$$C_3^* = \frac{1}{\Gamma(\alpha+1)} + \frac{\xi+\lambda}{\lambda}\left[\frac{1}{\Gamma(\alpha+1)} + \frac{\xi}{\lambda\Gamma(\alpha)} + \frac{m}{\Gamma(\alpha+1)} + \frac{m(\xi+2\lambda)}{\lambda\Gamma(\alpha)}\right] + \frac{m}{\Gamma(\alpha+1)} + \frac{m}{\Gamma(\alpha)}.$$

Similarly

$$\|\mathbb{G}_2(z,\omega)(t)\|_{\mathrm{PC}} \leq M_\varphi C_3^{**} \|(z,\omega)\|_{\mathrm{PC}}. \tag{3.16}$$

Thus, by using (3.12), (3.13), (3.14) and (3.16) in (3.11), we get

$$\begin{aligned}
\|\mathbb{F}(z,\omega) + \mathbb{G}(z,\omega)\|_{\mathrm{PC}} \leq& M^* C_1^* + M^{**} C_1^{**} + (C_m + M_\phi C_3^* + M_\varphi C_3^{**}) r \\
\leq& r,
\end{aligned}$$

where $C_m = \max\{C_2^*, C_2^{**}\}$. Therefore, $\mathbb{F}(z,\omega) + \mathbb{G}(z,\omega) \in \mathcal{B}_r$.

Next, for any $t \in \mathsf{J}$, (z,ω) and $(\bar{z},\bar{\omega}) \in \mathcal{B}_r$, we have

$$\begin{aligned}
&\|\mathbb{F}(z,\omega)-\mathbb{F}(\bar{z},\bar{\omega})\|_{\mathrm{PC}}\\
\leq&\|\mathbb{F}_1(z)-\mathbb{F}_1(\bar{z})\|_{\mathrm{PC}}+\|\mathbb{F}_2(\omega)-\mathbb{F}_2(\bar{\omega})\|_{\mathrm{PC}}\\
\leq&\Bigg\{\frac{M_{p,\tilde{p}}}{\Gamma(\alpha+1)}+\frac{\xi+\lambda}{\lambda}\Bigg[\frac{M_{p,\tilde{p}}}{\Gamma(\alpha+1)}+\frac{\xi M_{p,\tilde{p}}}{\lambda\Gamma(\alpha)}+\frac{mM_{p,\tilde{p}}}{\Gamma(\alpha+1)}+ml_I\\
&+m\frac{(\xi+2\lambda)}{\lambda}\left(\frac{M_{p,\tilde{p}}}{\Gamma(\alpha)}+l_{\tilde{I}}\right)\Bigg]\\
&+m\left(\frac{M_{p,\tilde{p}}}{\Gamma(\alpha+1)}+l_{\tilde{I}}\right)+m\left(\frac{M_{p,\tilde{p}}}{\Gamma(\alpha)}+l_I\right)+\frac{1}{\lambda^2}\Big(\lambda l_h+(\xi+\lambda)l_{gh}\Big)\Bigg\}\|z-\bar{z}\|_{\mathrm{PC}}\\
&+\Bigg\{\frac{M_{q,\tilde{q}}}{\Gamma(\alpha+1)}+\frac{\xi+\lambda}{\lambda}\Bigg[\frac{M_{q,\tilde{q}}}{\Gamma(\alpha+1)}+\frac{\xi M_{q,\tilde{q}}}{\lambda\Gamma(\alpha)}+\frac{mM_{q,\tilde{q}}}{\Gamma(\alpha+1)}+ml_I\\
&+m\frac{(\xi+2\lambda)}{\lambda}\left(\frac{M_{q,\tilde{q}}}{\Gamma(\alpha)}+l_{\tilde{I}}\right)\Bigg]\\
&+m\left(\frac{M_{q,\tilde{q}}}{\Gamma(\alpha+1)}+l_{\tilde{I}}\right)+m\left(\frac{M_{q,\tilde{q}}}{\Gamma(\alpha)}+l_I\right)+\frac{1}{\lambda^2}\Big(\lambda l_h+(\xi+\lambda)l_{gh}\Big)\Bigg\}\|\omega-\bar{\omega}\|_{\mathrm{PC}}\\
\leq&\xi^*\|(z,\omega)-(\bar{z},\bar{\omega})\|_{\mathrm{PC}}.
\end{aligned}$$

This yields

$$\|\mathbb{F}(z,\omega)-\mathbb{F}(\bar{z},\bar{\omega})\|_{\mathrm{PC}}\leq\xi^*\|(z,\omega)-(\bar{z},\bar{\omega})\|_{\mathrm{PC}},\ 0<\xi^*<1.$$

The last inequality shows that $\mathbb{F}$ is contractive. Here $\xi^*=\max\{\xi_1^*,\xi_2^*\}$, while

$$\begin{aligned}
\xi_1^*=&\frac{M_{p,\tilde{p}}}{\Gamma(\alpha+1)}+\frac{\xi+\lambda}{\lambda}\Bigg[\frac{M_{p,\tilde{p}}}{\Gamma(\alpha+1)}+\frac{\xi M_{p,\tilde{p}}}{\lambda\Gamma(\alpha)}\\
&+\frac{mM_{p,\tilde{p}}}{\Gamma(\alpha+1)}+ml_I+m\frac{(\xi+2\lambda)}{\lambda}\left(\frac{M_{p,\tilde{p}}}{\Gamma(\alpha)}+l_{\tilde{I}}\right)\Bigg]\\
&+m\left(\frac{M_{p,\tilde{p}}}{\Gamma(\alpha+1)}+l_{\tilde{I}}\right)+m\left(\frac{M_{p,\tilde{p}}}{\Gamma(\alpha)}+l_I\right)+\frac{1}{\lambda^2}\Big(\lambda l_h+(\xi+\lambda)l_{gh}\Big)
\end{aligned}$$

and

$$\begin{aligned}
\xi_2^*=&\frac{M_{q,\tilde{q}}}{\Gamma(\alpha+1)}+\frac{\xi+\lambda}{\lambda}\Bigg[\frac{M_{q,\tilde{q}}}{\Gamma(\alpha+1)}+\frac{\xi M_{q,\tilde{q}}}{\lambda\Gamma(\alpha)}+\frac{mM_{q,\tilde{q}}}{\Gamma(\alpha+1)}+ml_I\\
&+m\frac{(\xi+2\lambda)}{\lambda}\left(\frac{M_{q,\tilde{q}}}{\Gamma(\alpha)}+l_{\tilde{I}}\right)\Bigg]\\
&+m\left(\frac{M_{q,\tilde{q}}}{\Gamma(\alpha+1)}+l_{\tilde{I}}\right)+m\left(\frac{M_{q,\tilde{q}}}{\Gamma(\alpha)}+l_I\right)+\frac{1}{\lambda^2}\Big(\lambda l_h+(\xi+\lambda)l_{gh}\Big).
\end{aligned}$$

For the continuity and compactness of the operator $\mathbb{G}$, we design a sequence $T_n=(z_n,\omega_n)$ in $\mathcal{B}_r$ such that $(z_n,\omega_n)\to(z,\omega)$ for $n\to\infty$ in $\mathcal{B}_r$. Thus, we have

$$\|\mathbb{G}(z_n,\omega_n)-\mathbb{G}(z,\omega)\|_{\mathrm{PC}}\leq\Bigg[\frac{M'_\phi}{\Gamma(\alpha+1)}+\frac{\xi+\lambda}{\lambda}\Bigg\{\frac{M'_\phi}{\Gamma(\alpha+1)}$$

$$+\frac{\xi M'_\phi}{\lambda\Gamma(\alpha)}+\frac{mM'_\phi}{\Gamma(\alpha+1)}+\frac{mM'_\phi(\xi+2\lambda)}{\lambda\Gamma(\alpha)}\Bigg\}$$
$$+\frac{mM'_\phi}{\Gamma(\alpha+1)}+\frac{mM'_\phi}{\Gamma(\alpha)}+\frac{M'_\varphi}{\Gamma(\alpha+1)}+\frac{\xi+\lambda}{\lambda}$$
$$\times\left\{\frac{M'_\varphi}{\Gamma(\alpha+1)}+\frac{\xi M'_\varphi}{\lambda\Gamma(\alpha)}+\frac{mM'_\varphi}{\Gamma(\alpha+1)}+\frac{mM'_\varphi(\xi+2\lambda)}{\lambda\Gamma(\alpha)}\right\}$$
$$+\frac{mM'_\varphi}{\Gamma(\alpha+1)}+\frac{mM'_\varphi}{\Gamma(\alpha)}\Bigg]\|(z_n,\omega_n)-(z,\omega)\|_{\mathrm{PC}}.$$

Which implies $\|\mathbb{G}(p_n,q_n)-\mathbb{G}(z,\omega)\|_{\mathrm{PC}}\to 0$ as $n\to\infty$, therefore $\mathbb{G}$ is continuous.

Next we confirm that the operator $\mathbb{G}$ is uniformly bounded on $\mathcal{B}_r$. From (3.14) and (3.16), we have

$$\begin{aligned}\|\mathbb{G}(z,\omega)(t)\|_{\mathrm{PC}}&\le\|\mathbb{G}_1(z,\omega)(t)\|_{\mathrm{PC}}+\|\mathbb{G}_2(z,\omega)(t)\|_{\mathrm{PC}}\\&\le\{M_\phi C_3^*+M_\varphi C_3^{**}\}\|(z,\omega)\|_{\mathrm{PC}}\le r.\end{aligned}$$

Thus, $\mathbb{G}$ is uniformly bounded on $\mathcal{B}_r$.

For equi–continuity, take $\tau_1,\tau_2\in\mathsf{J}$ with $\tau_1<\tau_2$ and for any $(z,\omega)\in\mathcal{B}_r\subset\mathrm{E}$, where $\mathbb{G}$ is bounded on $\mathcal{B}_r$, we have

$$\begin{aligned}&\big|\mathbb{G}_1(z,\omega)(\tau_1)-\mathbb{G}_1(z,\omega)(\tau_2)\big|\\
\le&\frac{1}{\Gamma(\alpha)}\int_{\tau_1}^{\tau_2}(\tau_2-s)^{\alpha-1}M_\phi\|(z,\omega)\|ds\\
&+\frac{1}{\Gamma(\alpha)}\int_0^{\tau_1}(\tau_2-s)^{\alpha-1}-(\tau_1-s)^{\alpha-1}M_\phi\|(z,\omega)\|ds\\
&+\frac{\tau_2-\tau_1}{\Gamma(\alpha)}\int_{t_m}^1(1-s)^{\alpha-1}M_\phi\|(z,\omega)\|ds+\frac{\tau_2-\tau_1}{\Gamma(\alpha-1)}\frac{\xi}{\lambda}\int_{t_m}^1(1-s)^{\alpha-2}M_\phi\|(z,\omega)\|ds\\
&+\frac{\tau_2-\tau_1}{\Gamma(\alpha)}\sum_{0<t_k<1}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}M_\phi\|(z,\omega)\|ds+\frac{\tau_2-\tau_1}{\Gamma(\alpha-1)}\sum_{0<t_k<1}(\frac{\xi}{\lambda}+1-t_k)\\
&\times\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-2}M_\phi\|(z,\omega)\|ds\\
&+\frac{1}{\Gamma(\alpha-1)}\sum_{0<t_k<1}(\tau_2-\tau_1)\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-2}\|(z,\omega)|ds\\
\le&M_\phi\Bigg[\frac{(\tau_2-\tau_1)^\alpha}{\Gamma(\alpha+1)}+\frac{(\tau_2^\alpha-\tau_1^\alpha-(\tau_2-\tau_1)^\alpha)}{\Gamma(\alpha+1)}\\
&+(\tau_2-\tau_1)\Bigg\{\int_{t_m}^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{\xi}{\lambda}\int_{t_m}^1\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\\
&+\sum_{0<t_k<1}\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)}+\sum_{0<t_k}(\frac{\xi}{\lambda}+1)\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha-1)}\Bigg\}\\
&+\sum_{0<t_k<1}(\tau_2-\tau_1)\frac{1}{\Gamma(\alpha-1)}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-2}ds\Bigg]\|(z,\omega)\|.\end{aligned}$$

This implies that $\|\mathbb{G}_1(z,\omega)(\tau_1) - \mathbb{G}_1(z,\omega)(\tau_2)\|_{\mathrm{PC}} \to 0$ as $\tau_1 \to \tau_2$. Similarly, we can show that, $\|\mathbb{G}_2(z,\omega)(\tau_1) - \mathbb{G}_2(z,\omega)(\tau_2)\|_{\mathrm{PC}} \to 0$ as $\tau_1 \to \tau_2$, Thats why, $\|\mathbb{G}(z,\omega)(\tau_1) - \mathbb{G}(z,\omega)(\tau_2)\|_{\mathrm{PC}} \to 0$ as $\tau_1 \to \tau_2$. Therefore, $\mathbb{G}$ is relatively compact on $\mathcal{B}_r$. By Arzelä–Ascolli theorem, $\mathbb{G}$ is completely continuous and compact operator, so (1.1) has at least one solution, thanks to Theorem 2.3. □

Theorem 3.3. *Let the hypothesis, $(\mathcal{H}_1)$ to $(\mathcal{H}_7)$ be true with $\Lambda < 1$. Then (1.1) has a unique solution.*

Proof. Define an operator $\Phi = (\Phi_1, \Phi_2) : \mathrm{E} \to \mathrm{E}$, such that

$$\Phi(z,\omega)(t) = (\Phi_1(z,\omega), \Phi_2(z,\omega))(t),$$

where

$$\begin{aligned}
\Phi_1(z,\omega)(t) =& \frac{1}{\Gamma(\alpha)} \int_{t_m}^{t} (t-s)^{\alpha-1}(\chi_1(s)z(s) + \phi(s,z(s),\omega(s)))ds \\
&+ (\frac{\xi}{\lambda} - t)\Bigg[\frac{1}{\Gamma(\alpha)} \int_{t_m}^{1} (1-s)^{\alpha-1}(\chi_1(s)z(s) + \phi(s,z(s),\omega(s)))ds \\
&+ \frac{\xi}{\lambda\Gamma(\alpha-1)} \int_{t_m}^{1} (1-s)^{\alpha-2}(\chi_1(s)z(s) + \phi(s,z(s),\omega(s)))ds \\
&+ \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<1} \int_{t_{k-1}}^{t_k} (t_k-s)^{\alpha-1}(\chi_1(s)z(s) + \phi(s,z(s),\omega(s)))ds \\
&+ \frac{1}{\Gamma(\alpha-1)} \sum_{0<t_k<1} (\frac{\xi}{\lambda} + 1 - t_k) \int_{t_{k-1}}^{t_k} (t_k-s)^{\alpha-2}(\chi_1(s)z(s) \\
&+ \phi(s,z(s),\omega(s)))ds + \sum_{0<t_k<1} (\frac{\xi}{\lambda} + 1 - t_k)\tilde{I}_k(z(t_k)) + \sum_{0<t_k<1} I_k(z(t_k))\Bigg] \\
&+ \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<1} \int_{t_{k-1}}^{t_k} (t_k-s)^{\alpha-1}(\chi_1(s)z(s) + \phi(s,z(s),\omega(s)))ds \\
&+ \frac{1}{\Gamma(\alpha-1)} \sum_{0<t_k<1} (t-t_k) \int_{t_{k-1}}^{t_k} (t_k-s)^{\alpha-2}(\chi_1(s)z(s) + \phi(s,z(s),\omega(s)))ds \\
&+ \sum_{k=1}^{m} \{I_k z(t_k) + (t-t_k)\tilde{I}_k z(t_k)\} + \frac{1}{\lambda^2}\{\lambda h(z) + (\lambda t - \xi)(g(z) - h(z))\}
\end{aligned}$$

and

$$\begin{aligned}
\Phi_2(z,\omega)(t) =& \frac{1}{\Gamma(\alpha)} \int_{t_m}^{t} (t-s)^{\alpha-1}(\chi_2(s)\omega(s) + \varphi(s,z(s),\omega(s)))ds \\
&+ (\frac{\xi}{\lambda} - t)\Bigg[\frac{1}{\Gamma(\alpha)} \int_{t_m}^{1} (1-s)^{\alpha-1}(\chi_2(s)\omega(s) + \varphi(s,z(s),\omega(s)))ds \\
&+ \frac{\xi}{\lambda\Gamma(\alpha-1)} \int_{t_m}^{1} (1-s)^{\alpha-2}(\chi_2(s)\omega(s) + \varphi(s,z(s),\omega(s)))ds \\
&+ \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<1} \int_{t_{k-1}}^{t_k} (t_k-s)^{\alpha-1}(\chi_2(s)\omega(s) + \varphi(s,z(s),\omega(s)))ds
\end{aligned}$$

$$+\frac{1}{\Gamma(\alpha-1)}\sum_{0<t_k<1}(\frac{\xi}{\lambda}+1-t_k)\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-2}(\chi_2(s)\omega(s)$$
$$+\varphi(s,z(s),\omega(s)))ds+\sum_{0<t_k<1}(\frac{\xi}{\lambda}+1-t_k)\tilde{I}_k(\omega(t_k))+\sum_{0<t_k<1}I_k(\omega(t_k))\Bigg]$$
$$+\frac{1}{\Gamma(\alpha)}\sum_{0<t_k<1}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}(\chi_2(s)\omega(s)+\varphi(s,z(s),\omega(s)))ds$$
$$+\frac{1}{\Gamma(\alpha-1)}\sum_{0<t_k<1}(t-t_k)\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-2}(\chi_2(s)\omega(s)+\phi(s,z(s),\omega(s)))ds$$
$$+\sum_{k=1}^{m}\{I_k\omega(t_k)+(t-t_k)\tilde{I}_k\omega(t_k)\}+\frac{1}{\lambda^2}\{\lambda h(\omega)+(\lambda t-\xi)(g(\omega)-h(\omega))\}.$$

In view of Theorem 3.2, we have

$$\begin{aligned}
&\|\Phi_1(z,\omega)-\Phi_1(\bar{z},\bar{\omega})\|_{\mathrm{PC}}\\
\leq&\frac{(M_{p,\tilde{p}}+M'_\phi)(\|z-\bar{z}\|_{\mathrm{PC}})}{\Gamma(\alpha+1)}+\frac{M'_\phi\|\omega-\bar{\omega}\|_{\mathrm{PC}}}{\Gamma(\alpha+1)}\\
&+\frac{(\xi+\lambda)}{\lambda}\Bigg[\frac{(M_{p,\tilde{p}}+M'_\phi)(\|z-\bar{z}\|_{\mathrm{PC}})}{\Gamma(\alpha+1)}+\frac{M'_\phi\|\omega-\bar{\omega}\|_{\mathrm{PC}}}{\Gamma(\alpha+1)}\\
&+\frac{\xi}{\lambda}\Bigg\{\frac{(M_{p,\tilde{p}}+M'_\phi)(\|z-\bar{z}\|_{\mathrm{PC}})}{\Gamma(\alpha)}+\frac{M'_\phi\|\omega-\bar{\omega}\|_{\mathrm{PC}}}{\Gamma(\alpha)}+ml_I\|z-\bar{z}\|_{\mathrm{PC}}\Bigg\}\\
&+m\frac{(M_{p,\tilde{p}}+M'_\phi)}{\Gamma(\alpha+1)}\|z-\bar{z}\|_{\mathrm{PC}}+m\frac{M'_\phi}{\Gamma(\alpha+1)}\|\omega-\bar{\omega}\|_{\mathrm{PC}}\\
&+m\frac{(\xi+2\lambda)}{\lambda}\times\Bigg\{\frac{M_{p,\tilde{p}}+M'_\phi}{\Gamma(\alpha)}\|z-\bar{z}\|_{\mathrm{PC}}+\frac{M'_\phi}{\Gamma(\alpha)}\|\omega-\bar{\omega}\|_{\mathrm{PC}}+ml_{\tilde{I}}\|z-\bar{z}\|_{\mathrm{PC}}\Bigg\}\Bigg]\\
&+m\Bigg\{\frac{M_{p,\tilde{p}}+M'_\phi}{\Gamma(\alpha+1)}+l_I\Bigg\}\|z-\bar{z}\|_{\mathrm{PC}}+m\frac{M'_\phi}{\Gamma(\alpha+1)}\|\omega-\bar{\omega}\|_{\mathrm{PC}}\\
&+m\Bigg\{\frac{M_{p,\tilde{p}}+M'_\phi}{\Gamma(\alpha)}+l_{\tilde{I}}\Bigg\}\|z-\bar{z}\|_{\mathrm{PC}}+m\frac{M'_\phi}{\Gamma(\alpha)}\|\omega-\bar{\omega}\|_{\mathrm{PC}}\\
&+\frac{1}{\lambda^2}\Bigg((\xi+\lambda)l_g+(\xi+2\lambda)l_h\Bigg)\|z-\bar{z}\|_{\mathrm{PC}}\\
\leq&\Bigg[\frac{(M_{p,\tilde{p}}+M'_\phi)(1+m)(2\lambda+\xi)}{\lambda\Gamma(\alpha+1)}+\frac{M_{p,\tilde{p}}+M'_\phi}{\Gamma(\alpha)}\Bigg\{\frac{(1+m)\xi^2+\xi\lambda(1+3m)+3m\lambda^2}{\lambda^2}\Bigg\}\\
&+\frac{m(M_{p,\tilde{p}}+M'_\phi)}{\lambda^2}\Bigg(\xi^2+4\xi\lambda+5\lambda^2\Bigg)\\
&+m\Bigg\{\frac{\xi+2\lambda}{\lambda}l_I+\frac{\xi^2+3\xi\lambda+3\lambda^2}{\lambda^2}l_{\tilde{I}}\Bigg\}+\Bigg\{\frac{(\xi+2\lambda)l_h+(\xi+\lambda)l_g}{\lambda^2}\Bigg\}\Bigg]\|z-\bar{z}\|_{\mathrm{PC}}\\
&+\Bigg[\frac{M'_\phi(1+m)(2\lambda+\xi)}{\lambda\Gamma(\alpha+1)}+\frac{M'_\phi}{\Gamma(\alpha)}\Bigg(\frac{(1+m)\xi^2+\xi\lambda(1+3m)+3m\lambda^2}{\lambda^2}\Bigg)
\end{aligned}$$

$$+\frac{mM'_{\phi}(\xi^2+4\xi\lambda+5\lambda^2)}{\lambda^2}\Bigg]\|\omega-\bar{\omega}\|_{\mathrm{PC}}$$
$$\leq \Lambda^{*}\|(z,\omega)-(\bar{z},\bar{\omega})\|_{\mathrm{PC}},$$

where $\Lambda^{*}=\max\{\Lambda_1,\Lambda_2\}$ with

$$\begin{aligned}\Lambda_1=&\frac{(M_{p,\tilde{p}}+M'_{\phi})(1+m)(2\lambda+\xi)}{\lambda\Gamma(\alpha+1)}\\&+\frac{M_{p,\tilde{p}}+M'_{\phi}}{\Gamma(\alpha)}\times\left[\frac{(1+m)\xi^2+\xi\lambda(1+3m)+3m\lambda^2}{\lambda^2}\right]\\&+\frac{m(M_{p,\tilde{p}}+M'_{\phi})}{\lambda^2}\left(\xi^2+4\xi\lambda+5\lambda^2\right)\\&+m\left[\frac{\xi+2\lambda}{\lambda}l_I+\frac{\xi^2+3\xi\lambda+3\lambda^2}{\lambda^2}l_{\tilde{I}}\right]+\left[\frac{(\xi+2\lambda)l_h+(\xi+\lambda)l_g}{\lambda^2}\right]\end{aligned}$$

and

$$\begin{aligned}\Lambda_2=&\frac{M'_{\phi}(1+m)(2\lambda+\xi)}{\lambda\Gamma(\alpha+1)}+\frac{M'_{\phi}}{\Gamma(\alpha)}\left(\frac{(1+m)\xi^2+\xi\lambda(1+3m)+3m\lambda^2}{\lambda^2}\right)\\&+\frac{mM'_{\phi}(\xi^2+4\xi\lambda+5\lambda^2)}{\lambda^2}.\end{aligned}$$

Following the same steps, we have

$$\|\Phi_2(z,\omega)-\Phi_2(\bar{z},\bar{\omega})\|_{\mathrm{PC}}\leq\Lambda^{**}\|(z,\omega)-(\bar{z},\bar{\omega})\|_{\mathrm{PC}},$$

where $\Lambda^{**}=\max\{\Lambda_3,\Lambda_4\}$ with

$$\begin{aligned}\Lambda_3=&\frac{(M_{q,\tilde{q}}+M'_{\varphi})(1+m)(2\lambda+\xi)}{\lambda\Gamma(\alpha+1)}+\frac{M_{q,\tilde{q}}+M'_{\varphi}}{\Gamma(\alpha)}\left[\frac{(1+m)\xi^2+\xi\lambda(1+3m)+3m\lambda^2}{\lambda^2}\right]\\&+\frac{m(M_{q,\tilde{q}}+M'_{\varphi})}{\lambda^2}\left(\xi^2+4\xi\lambda+5\lambda^2\right)\\&+m\left[\frac{\xi+2\lambda}{\lambda}l_I+\frac{\xi^2+3\xi\lambda+3\lambda^2}{\lambda^2}l_{\tilde{I}}\right]+\left[\frac{(\xi+2\lambda)l_h+(\xi+\lambda)l_g}{\lambda^2}\right]\end{aligned}$$

and

$$\begin{aligned}\Lambda_4=&\frac{M'_{\varphi}(1+m)(2\lambda+\xi)}{\lambda\Gamma(\alpha+1)}+\frac{M'_{\varphi}}{\Gamma(\alpha)}\left(\frac{(1+m)\xi^2+\xi\lambda(1+3m)+3m\lambda^2}{\lambda^2}\right)\\&+\frac{mM'_{\varphi}(\xi^2+4\xi\lambda+5\lambda^2)}{\lambda^2}.\end{aligned}$$

Hence,

$$\|\Phi(z,\omega)-\Phi(\bar{z},\bar{\omega})\|_{\mathrm{PC}}\leq\Lambda\|(z,\omega)-(\bar{z},\bar{\omega})\|_{\mathrm{PC}},$$

where $\Lambda=\max\{\Lambda^{*},\Lambda^{**}\}<1$. This implies that the operator Φ is contraction. Therefore by Banach contraction mapping theorem, Φ has unique fixed point, which is the unique solution of (1.1). □

4. Hyers–Ulam Stability

This section is devoted to the investigate the Hyers–Ulam stability for the solution of (1.1).

Theorem 4.1. *Suppose that the hypothesis* (H_1) *to* (H_7) *and* $\Lambda < 1$ *hold along with the condition that the matrix* $\mathcal{H}^*$ *is converging to* 0. *Then the solutions of* (1.1) *are Hyers–Ulam stable.*

Proof. In view of Theorem 3.3, we have

$$\begin{cases} \|\Phi_1(z,\omega) - \Phi_1(\bar{z},\bar{\omega})\|_{\mathrm{PC}} \le \Lambda_1\|z-\bar{z}\|_{\mathrm{PC}} + \Lambda_2\|\omega-\bar{\omega}\|_{\mathrm{PC}}, \\ \|\Phi_2(z,\omega) - \Phi_2(\bar{z},\bar{\omega})\|_{\mathrm{PC}} \le \Lambda_3\|z-\bar{z}\|_{\mathrm{PC}} + \Lambda_4\|\omega-\bar{\omega}\|_{\mathrm{PC}}. \end{cases} \tag{4.1}$$

From (4.1), we obtain the following inequality

$$\|\Phi(z,\omega) - \Phi(\bar{z},\bar{\omega})\|_{\mathrm{PC}} \le \begin{pmatrix} \Lambda_1 & \Lambda_2 \\ \Lambda_3 & \Lambda_4 \end{pmatrix} \begin{pmatrix} \|z-\bar{z}\|_{\mathrm{PC}} \\ \|\omega-\bar{\omega}\|_{\mathrm{PC}} \end{pmatrix}, \tag{4.2}$$

where

$$\mathcal{H}^* = \begin{pmatrix} \Lambda_1 & \Lambda_2 \\ \Lambda_3 & \Lambda_4 \end{pmatrix}.$$

Since $\mathcal{H}^*$ converges to 0, thus (1.1) is Hyers–Ulam stable. □

5. Example

For supporting our theoretical results, we discuss a particular example.

Example 5.1. We take the given system of fractional order impulsive differential equations as

$$\begin{cases} {}^cD^{\frac{3}{2}}z(t) - \frac{e^{-t}}{30}z(t) = \frac{t+\sin(|z(t)|)+\cos(|\omega(t)|)}{150}, \quad t \ne \frac{1}{2}, \\ {}^cD^{\frac{3}{2}}\omega(t) - \frac{t^2}{30}\omega(t) = \frac{\sin(|\omega(t)|)+\cos(|z(t)|)}{100+t^3}, \quad t \ne \frac{1}{2}, \\ \lambda z(0)+\xi z'(0) = \sum_{k=1}^{10}\hbar_k|p(\zeta_k)|,\ \lambda z(1)+\xi z'(1) = \sum_{k=1}^{10}\frac{1}{\wp_k}|z(\zeta_k)|,\ 0<\eta_k,\zeta_k<1,\ \hbar_k>0, \\ \lambda\omega(0)+\xi\omega'(0) = \sum_{k=1}^{10}\hbar_k|q(\eta_k)|, \lambda\omega(1)+\xi\omega'(1) = \sum_{k=1}^{10}\frac{1}{\wp_k}|\omega(\eta_k)|,\ 0<\eta_k,\zeta_k<1, \wp_k>0, \\ \Delta z(\frac{1}{2}) = I_k(z(\frac{1}{2})) = \frac{|z(\frac{1}{2})|}{75+|z(\frac{1}{2})|},\ \Delta z'(\frac{1}{2}) = \tilde{I}(z(\frac{1}{2}) = \frac{|z(\frac{1}{2})|}{25+|z(\frac{1}{2})|}, \\ \Delta\omega(\frac{1}{2}) = I_k(\omega(\frac{1}{2})) = \frac{|\omega(\frac{1}{2})|}{75+|\omega(\frac{1}{2})|},\ \Delta\omega'(\frac{1}{2}) = \tilde{I}(\omega(\frac{1}{2}) = \frac{|\omega(\frac{1}{2})|}{25+|\omega(\frac{1}{2})|}, \end{cases}$$

where $\sum_{k=1}^{10}\hbar_k < \frac{1}{25}$, $\sum_{k=1}^{10}\frac{1}{\wp_k} < \frac{1}{75}$, $\lambda = 1$ and $\xi = -1.86 + 1.0011\iota$.

From the given system, we have

$$t_1 \ne \frac{1}{2}, \chi_1(t) = \frac{e^{-t}}{30}, \chi_2(t) = \frac{t^2}{30}, h(z) = \sum_{k=1}^{10}\hbar_k|z(\zeta_k)|, h(\omega) = \sum_{k=1}^{10}\hbar_k|q(\eta_k)|,$$

$$g(z)=\sum_{k=1}^{10}\frac{1}{\wp_k}|p(\zeta_k)| \quad \text{and} \quad g(\omega)=\sum_{k=1}^{10}\frac{1}{\wp_k}|\omega(\zeta_k)|.$$

After finding Λ_1, Λ_2, Λ_3 and Λ_4, we have

$$\mathcal{H}^*=\begin{pmatrix}-0.0303-0.0599\iota & 0.0311+0.0101\iota\\ 0.2001-0.4905\iota & -0.8162-0.0052\iota\end{pmatrix}.$$

By calculations, we get that, if ω_1 and ω_2 are the eigenvalues corresponding to $\mathcal{H}^*$ satisfying the quadratic equation $\omega^2+\omega(0.8465+0.0651\iota)-(0.0649-0.1996\iota)=0$, then the system (5.1) is Hyers–Ulam stable.

6. Conclusion

In this manuscript, we exercise the Arzelä–Ascoli theorem, Banach contraction principle and Krasnoselskii's fixed point theorem to attain the necessary criteria for the existence as well as uniqueness of the solution to considered switched coupled impulsive FDEs system given in (1.1). Similarly under particular assumptions and conditions, we have established the Hyers–Ulam stability result of the solution of the considered problem (1.1). From the obtained results, we conclude that such a method is very powerful, effectual and suitable for the solutions of nonlinear FDEs.

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