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ANALYSIS OF AUTONOMOUS LOTKA-VOLTERRA SYSTEMS BY LÉVY NOISE*

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Abstract The present paper deals with the problem of autonomous Lotka-Volterra systems by Lévy noise. The essential mathematical features are analyzed with the help of the existence and uniqueness of the positive solution, the pth moment boundedness, asymptotic pathwise estimation, extinction, asymptotic stability and persistence by Lyapunov analysis methods. An example of three species predator-prey chain model is presented to illustrate the analytical findings.

Keywords Lévy noise, extinction, asymptotic stability, persistence.

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1. Introduction

Recently, stochastic Lotka-Volterra population system driven by Brownian motion has been studied extensively, see [4–13, 15, 17, 18, 20, 21]. A classical stochastic Lotka-Volterra system can be expressed as follows

$$\begin{cases} dx_i(t) = x_i(t)\left[\left(b_i + \sum_{j=1}^{n} a_{ij}x_j(t)\right)dt + \sigma_i dW_i(t)\right], \\ x(0) = x_0, \end{cases} \tag{1.1}$$

where $x_i(t)$ is the density of the ith population, b_i is the intrinsic growth rate of the ith population and the coefficient a_{ij} describes the influence of the jth population upon the ith population. The signs of a_{ij} and $a_{ij}(i \neq j)$ determine the nature of the interaction between the populations i and j. All parameters b_i, a_{ij} for

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$i, j = 1, 2, \cdots, n$ and σ_i are constants, $W_i(t)$ is mutually independent Brownian motion with $W_i(0) = 0$. The authors in [21] investigated n-species model of facultative mutualism in random environments. The environment variability in this study is characterized with both white noise and color noise modeled by Markovian switching. They established new sufficient conditions ensuring that the system model is positive recurrent. They also showed the existence of a unique ergodic stationary distribution. The authors in [11] discussed a randomized n-species Lotka-Volterra competition system. They showed that this system is stable in time average under certain conditions and there is a stationary distribution of this system if extra conditions are satisfied. They obtained the same sufficient condition that all species become extinct. They also yielded the sufficient condition that some species will die out while the others will tend to the equilibrium states. The authors in [18] examined the asymptotic behavior of the stochastic extension of the Lotka-Volterra model. The stochastic version of this process appears to have far more intriguing properties than its deterministic counterpart. It is essentially a continuation of the moment results derived by the authors in [17]. The authors in [13] investigated the dynamical behavior of the non-autonomous stochastic Lotka-Volterra competitive system. They obtained the sufficient conditions for the existence of global positive solutions, stochastic permanence, extinction and global attractivity.

However, the effects due to sudden environmental shocks (earthquakes, hurricanes, epidemics, etc.) have been neglected. These phenomena can not be described by the stochastic system (1.1). To describe these phenomena, introducing a jump process into the underlying population dynamics is important. The authors in [2] proposed a stochastic competitive Lotka-Volterra population model with jumps and they considered the existence and uniqueness, boundedness, tightness, Lyapunov exponents and extinction of positive solutions. The authors in [3] developed a general Lotka-Volterra population model with jumps. They showed that the stochastic differential equation has a unique global positive solution by using the Khasminskii-Mao theorem, and discussed the asymptotic pathwise estimation of such a model by applying an exponential martingale inequality with jumps. The authors in [16] established a new sufficient condition for stochastic permanence which is much weaker than [2]. They proposed sufficient and necessary conditions for persistence in the mean and extinction of each population for three stochastic Lotka-Volterra models of two interacting species perturbed by Lévy noise.

Keeping this in mind, we discuss a modification of the model (1.1) taking into account the effect of jumps:

$$\begin{cases} dx_i(t) = x_i(t-)\left[\left(b_i + \sum_{j=1}^{n} a_{ij}x_j(t-)\right)dt + \sigma_i dW_i(t) + \int_Y H_i(u)\widetilde{N}(dt,du)\right], \\ x(0) = x_0. \end{cases} \tag{1.2}$$

Here $x(t-)$ is the left limit of $x(t), b_i, a_{ij}$ and σ_i are defined as in system (1.1), $W_i(t)$ is mutually independent Brownian motion defined on a complete probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ with a filtration $\{\mathscr{F}_t\}_{t\geq 0}$ satisfying the usual conditions, N is a Poisson counting measure with characteristic measure λ on a measurable subset Y of $[0, \infty)$ with $\lambda(Y) < \infty$, and $\widetilde{N}(dt, du) := N(dt, du) - \lambda(du)dt$. Throughout the paper, we assume that W and N are independent. We also assume:

$(A) - H_0 \leq H_i(u) \leq H,\ i = 1, 2, \cdots, n,$ where $0 < H_0 < 1$ and $H > 0$;

(B)There exists a positive diagonal matrix C such that $-\dfrac{1}{2}(CA+A^TC)$ is positive-definite, where $C=\mathrm{diag}(\mathrm{c}_1,\cdots,\mathrm{c}_\mathrm{n}), \mathrm{c}_\mathrm{i}>0$, i.e., for each $\mathrm{x}=(\mathrm{x}_1,\cdots,\mathrm{x}_\mathrm{n})^\mathrm{T}$ there exists $\lambda>0$ such that $-\dfrac{1}{2}x^T(CA+A^TC)x\geq\lambda\|x\|^2$.

Remark 1.1. From assumption (B), it is easy to see that $\sum\limits_{i,j=1}^{n} c_i a_{ij} x_i x_j \leq -\lambda\|x\|^2$.

This paper is organized as follows. We show that the solution of system (1.2) is global and positive under appropriate conditions (i.e. the population will not explode in a finite time) in Section 2. We obtain the pth moment boundedness of the solution for $p\geq 2$ in Section 3 and asymptotic pathwise estimation of the solution which is much better than [2] in Section 4. We present conditions for all species of system (1.2) to be extinct in Section 5. The asymptotic behavior of this model is analyzed via Lyapunov functions in Section 6. We discuss that this system is persistent in mean in Section 7. We use three species predator-prey chain model as an example to confirm our analytical results in Section 8.

2. Existence and uniqueness of the positive solution

In this section, we show that there is a unique globally positive solution of system (1.2).

Theorem 2.1. *Let assumptions (A) and (B) hold. For any initial value $x_0\in R^n_+$, system (1.2) has a unique positive solution $x(t)$ for $t\geq 0$ almost surely.*

Proof. Since the drift coefficient does not satisfy the linear growth condition, the general theorems of existence and uniqueness can not be implemented for this equation. However, it is locally Lipschitz continuous, so for any given initial condition $x_0\in R^n_+$ there is a unique local solution $x(t)$ for $t\in[0,\tau_e)$, where τ_e is the explosion time.

Now we show that this solution is global, i.e. we show that $\tau_e=\infty$ a.s. Let $m_0\geq 1$ be sufficiently large so that $x_1(0), x_2(0),\cdots,x_n(0)$ all lie within the interval $[\frac{1}{m_0}, m_0]$. For each integer $m\geq m_0$, define the stopping time

$$\tau_m=\inf\Big\{t\in[0,\tau_e):\min\{x_1(t),x_2(t),\cdots,x_n(t)\}\leq\frac{1}{m}\ \text{ or }\ \max\{x_1(t),x_2(t),\cdots,x_n(t)\}\geq m\Big\},$$

where throughout this paper, we set $\inf\emptyset=\infty$(as usual, $\emptyset$ denotes the empty set). Clearly, τ_m is increasing as $m\to\infty$. Set $\tau_\infty=\lim_{m\to\infty}\tau_m$, whence $\tau_\infty\leq\tau_e$ a.s. If we show that $\tau_\infty=\infty$ a.s., then $\tau_e=\infty$ and $x(t)\in R^n_+$ a.s. for all $t\geq 0$. In other words, to complete the proof all we need to show is that $\tau_\infty=\infty$ a.s. If this statement is false, then there is a pair of constants $T>0$ and $\varepsilon\in(0,1)$ such that

$$P\{\tau_\infty\leq T\}>\varepsilon.$$

Hence there is an integer $m_1\geq m_0$ such that

$$P\{\tau_m\leq T\}\geq\varepsilon \quad \text{for all} \quad m\geq m_1.$$

Define a C^2-function $V : R_+^n \to R_+$ by

$$V(x) = \sum_{i=1}^{n} c_i(x_i - 1 - \log x_i),$$

where c_i is defined as in the assumption (B). Applying Ito's formula, we obtain

$$\begin{aligned}
dV(x(t)) =& \{\sum_{i=1}^{n} c_i\left(x_i(t-) - 1\right)\left(b_i + \sum_{j=1}^{n} a_{ij}x_j(t-)\right) + \sum_{i=1}^{n}\frac{1}{2}c_i\sigma_i^2 \\
&- \sum_{i=1}^{n} c_i \int_Y [\log\left(1 + H_i(u)\right) - H_i(u)]\lambda(du)\}dt \\
&+ \sum_{i=1}^{n} c_i(x_i(t-) - 1)\sigma_i dW_i(t) \\
&+ \sum_{i=1}^{n} c_i \int_Y [x_i(t-)H_i(u) - \log(1 + H_i(u))]\widetilde{N}(dt, du) \\
=& LV dt + \sum_{i=1}^{n} c_i(x_i(t-) - 1)\sigma_i dW_i(t) \\
&+ \sum_{i=1}^{n} c_i \int_Y [x_i(t-)H_i(u) - \log(1 + H_i(u))]\widetilde{N}(dt, du)
\end{aligned}$$

where

$$\begin{aligned}
LV =& \sum_{i=1}^{n} c_i(x_i - 1)\left(b_i + \sum_{j=1}^{n} a_{ij}x_j\right) + \sum_{i=1}^{n}\frac{1}{2}c_i\sigma_i^2 \\
&- \sum_{i=1}^{n} c_i \int_Y [\log(1 + H_i(u)) - H_i(u)]\lambda(du).
\end{aligned} \tag{2.1}$$

By Taylor's formula, we deduce that there exist $\theta \in (0, 1)$ such that

$$\log(1 + H_i(u)) = H_i(u) - \frac{H_i^2(u)}{2(1 + \theta H_i(u))^2}. \tag{2.2}$$

Substituting this into (2.1), with assumption (A) and (B), we have

$$\begin{aligned}
LV =& \sum_{i=1}^{n} c_i(x_i - 1)\left(b_i + \sum_{j=1}^{n} a_{ij}x_j\right) + \sum_{i=1}^{n}\frac{1}{2}c_i\sigma_i^2 + \sum_{i=1}^{n} c_i \int_Y \frac{H_i^2(u)}{2(1 + \theta H_i(u))^2}\lambda(du) \\
\le& -\lambda\|x\|^2 + \sum_{i=1}^{n} c_i b_i x_i - \sum_{i,j=1}^{n} c_i a_{ij} x_j - \sum_{i=1}^{n} c_i b_i + \sum_{i=1}^{n}\frac{1}{2}c_i\sigma_i^2 \\
&+ \sum_{i=1}^{n} c_i \frac{(H_0 \vee H)^2}{2(1 - H_0 \wedge H)^2}\lambda(Y) \\
\le& \max\left\{-\lambda\|x\|^2 + \sum_{i=1}^{n} c_i b_i x_i - \sum_{i,j=1}^{n} c_i a_{ij} x_j\right\} + \sum_{i=1}^{n}\frac{1}{2}c_i\sigma_i^2
\end{aligned}$$

$$+\sum_{i=1}^{n} c_i \frac{(H_0 \vee H)^2}{2(1-H_0 \wedge H)^2}\lambda(Y).$$

From a proof similar to that in A. Gray and X. Mao([9, Theorem 3.1]), we obtain the desired assertion. □

3. The pth moment boundedness of the solution

In this section, we show that the pth moment of the solution is bounded.

Theorem 3.1. *Let assumptions (A) and (B) hold. For any initial value $x_0 \in R_+^n$, there exists a positive constant $M(p)$ such that the solution $x(t)$ of system* (1.2) *has the property*

$$E\left[\left(\sum_{i=1}^{n} c_i x_i\right)^p\right] \le M(p), \quad \textit{for all } t \in [0,\infty), \quad p \ge 2,$$

where c_i is defined as in the assumption (B).

Proof. Define a Lyapunov function for $p \ge 2$

$$V(x) := \left(\sum_{i=1}^{n} c_i x_i\right)^p, \quad x \in R_+^n.$$

Let $H(x,u) = \frac{\sum_{i=1}^{n} c_i x_i H_i(u)}{\sum_{i=1}^{n} c_i x_i}$, we have $-H_0 \le H(x,u) \le H$. Applying Ito's formula, we obtain

$$E(e^t V(x(t))) = V(x(0)) + E\int_0^t e^s[V(x(s)) + LV(x(s),s)]ds,$$

where

$$\begin{aligned} LV =& pV^{\frac{p-1}{p}} \sum_{i=1}^{n} c_i x_i \left(b_i + \sum_{j=1}^{n} a_{ij} x_j\right) + \frac{1}{2}p(p-1)V^{\frac{p-2}{p}}\left(\sum_{i=1}^{n} c_i \sigma_i x_i\right)^2 \\ &+ V\int_Y [(1+H(x,u))^p - 1 - pH(x,u)]\lambda(du). \end{aligned} \tag{3.1}$$

By Taylor's formula, we deduce that there exist $\theta \in (0,1)$ such that

$$(1+H(x,u))^p = 1 + pH(x,u) + \frac{1}{2}p(p-1)(1+\theta H(x,u))^{p-2}H^2(x,u).$$

Substituting this into (3.1) and with assumption (B), we have

$$\begin{aligned} LV \le& pV \max_{1\le i\le n} b_i - p\lambda V^{\frac{p-1}{p}}\|x\|^2 + \frac{1}{2}p(p-1)V \max_{1\le i\le n}\sigma_i^2 \\ &+ V\int_Y \frac{1}{2}p(p-1)(1+\theta H(x,u))^{p-2}H^2(x,u)\lambda(du). \end{aligned}$$

By the inequality $V^{\frac{1}{p}}(x) = \sum_{i=1}^{n} c_i x_i \leq \|c\| \|x\|$ and assumption (A), we obtain

$$\begin{aligned} LV \leq & pV \max_{1\leq i\leq n} b_i - \frac{p\lambda}{\|c\|^2} V^{\frac{p+1}{p}} + \frac{1}{2}p(p-1)V \max_{1\leq i\leq n} \sigma_i^2 + \frac{1}{2}p(p-1)(1+H)^p \lambda(Y) V \\ = & \left[p \max_{1\leq i\leq n} b_i + \frac{1}{2}p(p-1) \max_{1\leq i\leq n} \sigma_i^2 + \frac{1}{2}p(p-1)(1+H)^p \lambda(Y) \right] V - \frac{p\lambda}{\|c\|^2} V^{\frac{p+1}{p}}. \end{aligned} \tag{3.2}$$

Then, we can deduce that there exists a positive constant M such that

$$\begin{aligned} V + LV \leq & \left[1 + p \max_{1\leq i\leq n} b_i + \frac{1}{2}p(p-1) \max_{1\leq i\leq n} \sigma_i^2 + \frac{1}{2}p(p-1)(1+H)^p \lambda(Y) \right] V \\ & - \frac{p\lambda}{\|c\|^2} V^{\frac{p+1}{p}} \\ \leq & M. \end{aligned}$$

Hence

$$EV(x(t)) = e^{-t} V(x(0)) + M(1 - e^{-t}) \leq V(x(0)) + M := M(p).$$

Letting $t \to \infty$, we obtain

$$\limsup_{t\to\infty} EV(x(t)) \leq M(p).$$

Hence the proof of this theorem is completed. □

4. Asymptotic pathwise estimation

In this section, we show pathwise properties of the solutions. Before giving the main theorem, we first give a lemma.

Lemma 4.1 ([1]). *Assume that $g : [0, \infty) \to R$ and $h : [0, \infty) \times Y \to R$ are both predictable adapted processes such that for any $T > 0$,*

$$\int_0^T \mid g(t) \mid^2 dt < \infty \quad a.s. \text{ and} \int_0^T \int_Y \mid h(t,u) \mid^2 \lambda(du) dt < \infty \quad a.s.$$

Then for any constants $\alpha, \beta > 0$,

$$\begin{aligned} P\{ \sup_{0\leq t\leq T} [\int_0^T g(s) dW(s) - \frac{\alpha}{2} \int_0^t \mid g(s) \mid^2 ds + \int_0^t \int_Y h(s,u) \widetilde{N}(ds,du) \\ - \frac{1}{\alpha} \int_0^t \int_Y [e^{\alpha h(s,u)} - 1 - \alpha h(s,u)] \lambda(du) ds] > \beta \} \leq e^{-\alpha\beta}. \end{aligned}$$

Theorem 4.1. *Let assumptions (A) and (B) hold. For any initial value $x_0 \in R_+^n$, the solution of system* (1.2) *has the property*

$$\limsup_{t\to\infty} \frac{\log \|x(t)\|}{\log t} \leq \frac{1}{p}, \quad p \geq 2.$$

Proof. Applying Ito's formula, we have

$$\begin{aligned}
d\log V(x(t)) =& p\{\frac{\sum_{i=1}^n c_i x_i(t-)\left(b_i+\sum_{j=1}^n a_{ij}x_j(t-)\right)}{\sum_{i=1}^n c_i x_i(t-)} - \frac{1}{2}\left(\frac{\sum_{i=1}^n c_i\sigma_i x_i(t-)}{\sum_{i=1}^n c_i x_i(t-)}\right)^2 \\
&+ \int_Y [\log(1+H_i(u)) - H_i(u)]\lambda(du)\}dt + p\frac{\sum_{i=1}^n c_i\sigma_i x_i(t-)}{\sum_{i=1}^n c_i x_i(t-)}dW_i(t) \\
&+ \int_Y \log(1+H(x,u))^p \widetilde{N}(dt,du) \\
=& \{\frac{LV}{V} - \frac{1}{2}p^2 V^{\frac{2(p-1)}{p}}\frac{(\sum_{i=1}^n c_i\sigma_i x_i(t-))^2}{V^2} + \int_Y [\log(1+H(x,u))^p \\
&- (1+H(x,u))^p + 1]\lambda(du)\}dt + p\frac{\sum_{i=1}^n c_i\sigma_i x_i(t-)}{\sum_{i=1}^n c_i x_i(t-)}dW_i(t) \\
&+ \int_Y \log(1+H(x,u))^p \widetilde{N}(dt,du),
\end{aligned}$$

where V, LV and $H(x,u)$ are defined as in the proof of Theorem 3.1. Then, by Ito's formula,

$$\begin{aligned}
& e^t \log V(x(t)) \\
=& \log V(x(0)) + \int_0^t e^s\{\log V(x(s)) + \frac{LV}{V} - \frac{1}{2}p^2\frac{(\sum_{i=1}^n c_i\sigma_i x_i(s))^2}{(\sum_{i=1}^n c_i x_i(s))^2} \\
&+ \int_Y [\log(1+H(x,u))^p - (1+H(x,u))^p + 1]\lambda(du)\}ds \\
&+ \int_0^t e^s p\frac{\sum_{i=1}^n c_i\sigma_i x_i(s)}{\sum_{i=1}^n c_i x_i(s)}dW_i(s) + \int_0^t\int_Y e^s \log(1+H(x,u))^p\widetilde{N}(ds,du).
\end{aligned} \tag{4.1}$$

Let

$$Z(x(s)) = p\frac{\sum_{i=1}^n c_i\sigma_i x_i(s)}{\sum_{i=1}^n c_i x_i(s)},\ Q(x,u) = (1+H(x,u))^p.$$

In the light of Lemma 4.1, for any $\alpha, \beta, T > 0$,

$$\begin{aligned}
P\{\sup_{0\le t\le T}[&\int_0^t e^s Z(x(s))dW_i(s) - \frac{\alpha}{2}\int_0^t e^{2s}Z^2(x(s))ds + \int_0^t\int_Y e^s\log Q(x,u)\widetilde{N}(ds,du) \\
&- \frac{1}{\alpha}\int_0^t\int_Y [e^{\alpha e^s\log Q(x,u)} - 1 - \alpha e^s\log Q(x,u)]\lambda(du)ds] > \beta\} \le e^{-\alpha\beta}.
\end{aligned}$$

Choose $T = k\delta$, $\alpha = \varepsilon e^{-k\delta}$ and $\beta = \frac{(1+\delta)e^{k\delta}\log k\delta}{\varepsilon}$, where $K \in N$, $0<\delta<1$ and $0<\varepsilon<1$ in the above equation. Since $\sum_{k=1}^\infty \frac{1}{(k\delta)^{1+\delta}} < \infty$, we can deduce from the Borel-Cantelli lemma that there exists an $\Omega_i \subseteq \Omega$ with $P(\Omega_i) = 1$ such that for any $\varepsilon \in \Omega_i$, an integer $k_i = k_i(\omega,\varepsilon)$ can be found such that

$$\begin{aligned}
&\int_0^t e^s Z(x(s))dW_i(s) + \int_0^t\int_Y \log Q(x,u)\widetilde{N}(dt,du) \\
\le& \frac{(1+\delta)e^{k\delta}\log k\delta}{\varepsilon} + \frac{\varepsilon e^{-k\delta}}{2}\int_0^t e^{2s}Z^2(x(s))ds
\end{aligned}$$

$$+\frac{1}{\varepsilon e^{-k\delta}}\int_0^t\int_Y[e^{\varepsilon e^{s-k\delta}\log Q(x,u)}-1-\varepsilon e^{s-k\delta}\log Q(x,u)]\lambda(du)ds$$

whenever $k\geq k_i,\ 0\leq t\leq k\delta$. Substituting this into (4.1), we have

$$\begin{aligned}
&\log V(x(t))\\
\leq & e^{-t}\log V(x(0))+\frac{(1+\delta)e^{k\delta-t}\log k\delta}{\varepsilon}+\int_0^t e^{s-t}[\log V(x(s))+\frac{LV(x(s))}{V(x(s))}\\
&-\frac{1-\varepsilon}{2}Z^2(x(s))]ds+\int_0^t e^{s-t}\int_Y[\log Q(x(s),u)-Q(x(s),u)+1]\lambda(du)ds\\
&+\frac{1}{\varepsilon e^{t-k\delta}}\int_0^t\int_Y\{[Q(x(s),u)]^{\varepsilon e^{s-k\delta}}-1-\varepsilon e^{s-k\delta}\log Q(x(s),u)\}\lambda(du)ds\\
= & e^{-t}\log V(x(0))+\frac{(1+\delta)e^{k\delta-t}\log k\delta}{\varepsilon}\\
&+\int_0^t e^{s-t}[\log V(x(s))+\frac{LV(x(s))}{V(x(s))}-\frac{1-\varepsilon}{2}Z^2(x(s))]ds\\
&+\frac{1}{\varepsilon e^{t-k\delta}}\int_0^t\int_Y\{\varepsilon e^{s-k\delta}(1-Q(x(s),u))+[Q(x(s),u)]^{\varepsilon e^{s-k\delta}}-1\}\lambda(du)ds.
\end{aligned}$$

Next, from the inequality $x^r-1\leq r(x-1),\ x>0,\ 0<r<1$, for any $\omega\in\Omega_i$ and $0<\varepsilon<1,\ 0\leq t\leq k\delta$ with $k\geq k_i$,

$$\begin{aligned}
&\frac{1}{\varepsilon e^{t-k\delta}}\int_0^t\int_Y\{\varepsilon e^{s-k\delta}(1-Q(x(s),u))+[Q(x(s),u)]^{\varepsilon e^{s-k\delta}}-1\}\lambda(du)ds\\
\leq&\frac{1}{\varepsilon e^{t-k\delta}}\int_0^t\int_Y\{\varepsilon e^{s-k\delta}(1-Q(x(s),u))+\varepsilon e^{s-k\delta}[Q(x(s),u)-1]\}\lambda(du)ds=0.
\end{aligned}$$

It then follows from (3.2) that

$$\begin{aligned}
&\int_0^t e^{s-t}[\log V(x(s))+\frac{LV(x(s))}{V(x(s))}-\frac{1-\varepsilon}{2}Z^2(x(s))]ds\\
\leq&\int_0^t e^{s-t}[\log V(x(s))+p\max_{1\leq i\leq n}b_i+\frac{p(p-1)}{2}\max_{1\leq i\leq n}\sigma_i^2+\frac{p(p-1)}{2}(1+H)^p\lambda(Y)\\
&-\frac{p\lambda}{\|c\|^2}V^{\frac{1}{p}}]ds\\
\leq&\int_0^t e^{s-t}[p\max_{1\leq i\leq n}b_i+\frac{p(p-1)}{2}\max_{1\leq i\leq n}\sigma_i^2+\frac{p(p-1)}{2}(1+H)^p\lambda(Y)-p-p\log\frac{\lambda}{\|c\|^2}]ds\\
\leq&p\max_{1\leq i\leq n}b_i+\frac{p(p-1)}{2}\max_{1\leq i\leq n}\sigma_i^2+\frac{p(p-1)}{2}(1+H)^p\lambda(Y)-p-p\log\frac{\lambda}{\|c\|^2}:=K_1(p)
\end{aligned}$$

where

$$\log V-\frac{p\lambda}{\|c\|^2}V^{\frac{1}{p}}=p[\log(\frac{\lambda}{\|c\|^2}V^{\frac{1}{p}})-\frac{\lambda}{\|c\|^2}V^{\frac{1}{p}}]-p\log\frac{\lambda}{\|c\|^2}\leq-p-p\log\frac{\lambda}{\|c\|^2}$$

according to the inequality $x-1-\log x\geq 0,\forall x>0$. Thus, for $\omega\in\Omega_i$ and $(k-1)\delta\leq t\leq k\delta$ with $k\geq k_i$, we have

$$\frac{\log V(x(t))}{\log t}\leq\frac{e^{-t}\log V(0)}{\log(k-1)\delta}+\frac{(1+\delta)e^{\delta}\log k\delta}{\varepsilon\log(k-1)\delta}+\frac{K_1(p)}{\log(k-1)\delta}.$$

Supposing that $k \to \infty$, we have

$$\limsup_{t\to\infty} \frac{\log V(x(t))}{\log t} \le \frac{(1+\delta)e^{\delta}}{\varepsilon}.$$

Let $\delta \downarrow 0$ and $\varepsilon \to 1$, we have

$$\limsup_{t\to\infty} \frac{\log V(x(t))}{\log t} \le 1$$

and

$$\limsup_{t\to\infty} \frac{\log \|x(t)\|}{\log t} \le \frac{1}{p}.$$

Hence the proof of this theorem is completed. □

5. Extinction

In this section, we present conditions for all species of system (1.2) to be extinct.

Lemma 5.1 ([14]). *Let $M(t), t \ge 0$, be a local martingale vanishing at time 0 and define*

$$\rho_M(t) := \int_0^t \frac{d\langle M\rangle(s)}{(1+s)^2}, t \ge 0,$$

where $\langle M\rangle(t) := \langle M, M\rangle(t)$is Meyers angle bracket process. Then

$$\lim_{t\to\infty} \frac{M(t)}{t} = 0 \quad a.s. \quad \text{provided that } \lim_{t\to\infty} \rho_M(t) < \infty \quad a.s.$$

Theorem 5.1. *Let assumptions (A) and (B) hold. For any initial value $x_0 \in R^n_+$, if*

$$\max_{1\le i\le n} b_i < \frac{1}{2}\min_{1\le i\le n} \sigma_i^2 + \frac{(H_0 \wedge H)^2}{2(1+H_0 \vee H)^2}\lambda(Y), \tag{5.1}$$

the solution of system (1.2) has the property

$$\limsup_{t\to\infty} \frac{\log \sum_{i=1}^n c_i x_i(t)}{t} \le \max_{1\le i\le n} b_i - \frac{1}{2}\min_{1\le i\le n} \sigma_i^2 - \frac{(H_0 \wedge H)^2}{2(1+H_0 \vee H)^2}\lambda(Y) < 0 \quad a.s.$$

where c_i is defined as in the assumption (B).

Proof. By Ito's formula, we have

$$\begin{aligned} d\log \sum_{i=1}^n c_i x_i(t) =& \Big\{ \frac{\sum_{i=1}^n c_i x_i(t-)\left(b_i + \sum_{j=1}^n a_{ij} x_j(t-)\right)}{\sum_{i=1}^n c_i x_i(t-)} - \frac{1}{2}\left(\frac{\sum_{i=1}^n c_i \sigma_i x_i(t-)}{\sum_{i=1}^n c_i x_i(t-)}\right)^2 \\ &+ \int_Y [\log(1+H(x,u)) - H(x,u)]\lambda(du)\Big\} dt \\ &+ \frac{\sum_{i=1}^n c_i \sigma_i x_i(t-)}{\sum_{i=1}^n c_i x_i(t-)} dW_i(t) + \int_Y \log(1+H(x,u))\widetilde{N}(dt,du). \end{aligned}$$

It follows from assumption (B), (2.2) and the inequality $\sum_{i=1}^n c_i x_i \le \|c\| \|x\|$ that

$$
\begin{aligned}
& d \log \sum_{i=1}^n c_i x_i(t) \\
\le & \{\max_{1\le i\le n} b_i - \frac{\lambda}{\|c\|^2} \sum_{i=1}^n c_i x_i(t-) - \frac{1}{2} \min_{1\le i\le n} \sigma_i^2 - \frac{(H_0 \wedge H)^2}{2(1+H_0 \vee H)^2} \lambda(Y)\} dt \\
& + \frac{\sum_{i=1}^n c_i \sigma_i x_i(t-)}{\sum_{i=1}^n c_i x_i(t-)} dW_i(t) + \int_Y \log(1+H(x,u)) \widetilde{N}(dt,du)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\log \sum_{i=1}^n c_i x_i(t)}{t} \le & \frac{\log \sum_{i=1}^n c_i x_i(0)}{t} + \max_{1\le i\le n} b_i - \frac{\lambda}{\|c\|^2} \sum_{i=1}^n c_i x_i - \frac{1}{2} \min_{1\le i\le n} \sigma_i^2 \\
& - \frac{(H_0 \wedge H)^2}{2(1+H_0 \vee H)^2} \lambda(Y) + \frac{1}{t} \int_0^t \frac{\sum_{i=1}^n c_i \sigma_i x_i(s)}{\sum_{i=1}^n c_i x_i(s)} dW_i(s) \\
& + \frac{1}{t} \int_0^t \int_Y \log(1+H(x,u)) \widetilde{N}(ds,du).
\end{aligned}
$$

Let $M_1(t) = \int_0^t \frac{\sum_{i=1}^n c_i \sigma_i x_i(s)}{\sum_{i=1}^n c_i x_i(s)} dW_i(s)$, which is a real-valued continuous local martingale, $M_1(0) = 0$ and

$$
\limsup_{t\to\infty} \frac{\langle M_1, M_1 \rangle_t}{t} = \limsup_{t\to\infty} \frac{\int_0^t (\frac{\sum_{i=1}^n c_i \sigma_i x_i(s)}{\sum_{i=1}^n c_i x_i(s)})^2 ds}{t} \le \max_{1\le i\le n} \sigma_i^2 .
$$

Then by the strong law of large numbers, we have

$$
\lim_{t\to\infty} \frac{M_1(t)}{t} = 0 \quad \text{a.s.}
$$

Let $M_2(t) = \int_0^t \int_Y \log(1+H(x,u)) \widetilde{N}(ds,du)$, which is a local martingale, $M_2(0) = 0$ and

$$
\langle M_2 \rangle(t) := \langle M_2, M_2 \rangle_t = \int_0^t \int_Y (\log(1+H(x,u)))^2 \lambda(du) ds.
$$

Note

$$
\begin{aligned}
\rho_{M_2}(t) := \int_0^t \frac{d\langle M_2 \rangle(s)}{(1+s)^2} &= \int_0^t \frac{\int_Y (\log(1+H(x,u)))^2 \lambda(du)}{(1+s)^2} ds \\
&\le \int_0^t \frac{[\log^2(1+H) \vee \log^2(1-H_0)] \lambda(Y)}{(1+s)^2} ds < \infty.
\end{aligned}
$$

Then by Lemma 5.1, we have

$$
\lim_{t\to\infty} \frac{M_2(t)}{t} = 0 \quad \text{a.s.}
$$

which implies that

$$
\limsup_{t\to\infty} \frac{\log \sum\limits_{i=1}^n c_i x_i(t)}{t} \le \max_{1\le i\le n} b_i - \frac{1}{2} \min_{1\le i\le n} \sigma_i^2 - \frac{(H_0 \wedge H)^2}{2(1+H_0 \vee H)^2} \lambda(Y) \quad \text{a.s.}
$$

From the assumption (5.1), we get

$$\limsup_{t\to\infty} \frac{\log \sum_{i=1}^{n} c_i x_i(t)}{t} < 0 \quad \text{a.s.}$$

Namely,

$$\lim_{t\to\infty} \sum_{i=1}^{n} c_i x_i(t) = 0 \quad \text{a.s.}$$

Hence the proof of this theorem is completed. □

6. Asymptotic behavior around the equilibrium x^* of system (1.2)

In this section, we show that the solution of system (1.2) is going around x^* under some conditions.

Let $B = (b_1, \cdots, b_n)^T$. Assume that there exists $x^* \in R_+^n$ such that $B + Ax^* = 0$. Then system (1.2) can be written as

$$dx_i(t) = x_i(t-)\left[\sum_{j=1}^{n} a_{ij}(x_j(t-) - x_j^*)dt + \sigma_i dW_i(t) + \int_Y H_i(u)\widetilde{N}(dt, du)\right].$$

Theorem 6.1. *Let assumptions (A) and (B) hold. For any initial value $x_0 \in R_+^n$, the solution of system (1.2) has the property*

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t \|x(s) - x^*\|^2 ds \le K(\sigma, H_0, H) \quad a.s.$$

where $K(\sigma, H_0, H) = \frac{1}{2\lambda}\sum_{i=1}^{n} c_i x_i^*[\sigma_i^2 + \frac{(H_0\vee H)^2}{(1-H_0\wedge H)^2}\lambda(Y)]$, λ and c_i are defined as in the assumption (B).

Proof. Define a C^2-function $V : R_+^n \to R_+$ by

$$V(x) = \sum_{i=1}^{n} c_i(x_i - x_i^* - x_i^* \log \frac{x_i}{x_i^*}),$$

where c_i is defined as in the assumption (B). Applying Ito's formula, we obtain

$$\begin{aligned} dV(x(t)) ={}& \{\sum_{i,j=1}^{n} c_i a_{ij}(x_i(t-) - x_i^*)(x_j(t-) - x_j^*) + \sum_{i=1}^{n} \frac{1}{2} c_i x_i^* \sigma_i^2 \\ & - \sum_{i=1}^{n} c_i x_i^* \int_Y [\log(1 + H_i(u)) - H_i(u)]\lambda(du)\}dt \\ & + \sum_{i=1}^{n} c_i \sigma_i (x_i(t-) - x_i^*) dW_i(t) \end{aligned}$$

$$+\sum_{i=1}^{n} c_i \int_Y [x_i(t-)H_i(u) - x_i^* \log(1+H_i(u))]\widetilde{N}(dt,du).$$

By assumption (B) and (2.2), one can show that

$$\begin{aligned} dV(x(t)) \leq & \{-\lambda\|x(t-)-x^*\|^2 + \frac{1}{2}\sum_{i=1}^{n} c_i x_i^* [\sigma_i^2 + \frac{(H_0 \vee H)^2}{(1-H_0\wedge H)^2}\lambda(Y)]\}dt \\ & + \sum_{i=1}^{n} c_i \sigma_i (x_i(t-)-x_i^*)dW_i(t) \\ & + \sum_{i=1}^{n} c_i \int_Y [x_i(t-)H_i(u) - x_i^* \log(1+H_i(u))]\widetilde{N}(dt,du) \end{aligned}$$

and

$$\begin{aligned} \frac{V(x(t))}{t} \leq & \frac{V(x(0))}{t} - \lambda \frac{\int_0^t \|x(s)-x^*\|^2 ds}{t} + \frac{1}{2}\sum_{i=1}^{n} c_i x_i^* [\sigma_i^2 + \frac{(H_0 \vee H)^2}{(1-H_0\wedge H)^2}\lambda(Y)] \\ & + \sum_{i=1}^{n} c_i [\frac{1}{t}\int_0^t \sigma_i (x_i(s)-x_i^*)dW_i(s) + \frac{1}{t}\int_0^t \int_Y x_i(s)H_i(u)\widetilde{N}(ds,du) \\ & - \frac{1}{t}\int_0^t \int_Y x_i^* \log(1+H_i(u))\widetilde{N}(ds,du)]. \end{aligned} \tag{6.1}$$

Let $M_3(t) = \int_0^t \sigma_i(x_i(s)-x_i^*)dW_i(s)$, which is a real-valued continuous local martingale, $M_3(0)=0$.

In view of Theorem 4.1, we see that

$$\limsup_{t\to\infty} \frac{\log x_i(t)}{\log t} \leq \limsup_{t\to\infty} \frac{\log\|x(t)\|}{\log t} \leq \frac{1}{p}, \quad p \geq 2.$$

For arbitrary small $0 < \varepsilon < \frac{1}{2} - \frac{1}{p}$, there exist a constant $T = T(\omega)$ and a set Ω_ε such that $P(\Omega_\varepsilon) \geq 1-\varepsilon$ and for $t \geq T$, $\omega \in \Omega_\varepsilon$, $x_i(t) \leq t^{\frac{1}{p}+\varepsilon}$. Then,

$$\begin{aligned} \limsup_{t\to\infty} \frac{\langle M_3, M_3\rangle_t}{t} = & \limsup_{t\to\infty} \frac{\int_0^t \sigma_i^2 (x_i(s)-x_i^*)^2 ds}{t} \\ \leq & \limsup_{t\to\infty} \frac{\int_0^t 2\sigma_i^2 (x_i^2(s)+x_i^{*2}) ds}{t} < \infty. \end{aligned}$$

Let $\varepsilon \to 0$, by the strong law of large numbers, we have

$$\lim_{t\to\infty} \frac{M_3(t)}{t} = 0 \ \text{ a.s.}$$

Let $M_4(t) = \int_0^t \int_Y x_i(s)H_i(u)\widetilde{N}(ds,du)$, which is a local martingale, $M_4(0)=0$ and

$$\langle M_4\rangle(t) := \langle M_4, M_4\rangle_t = \int_0^t \int_Y (x_i(s)H_i(u))^2 \lambda(du)ds.$$

Note

$$\rho_{M_4}(t) := \int_0^t \frac{d\langle M_4\rangle(s)}{(1+s)^2} = \int_0^t \frac{\int_Y (x_i(s)H_i(u))^2\lambda(du)}{(1+s)^2} ds$$

$$\leq \lambda(Y)[H_0^2 \vee H^2] \int_0^t \frac{x_i^2(s)}{(1+s)^2} ds < \infty.$$

Then by Lemma 5.1, we have

$$\lim_{t\to\infty} \frac{M_4(t)}{t} = 0 \quad \text{a.s.}$$

Let $M_5(t) = \int_0^t \int_Y x_i^* \log(1 + H_i(u))\widetilde{N}(ds, du)$, which is a local martingale, $M_5(0) = 0$. In the same way,

$$\lim_{t\to\infty} \frac{M_5(t)}{t} = 0 \quad \text{a.s.}$$

It then follows from (6.1) that

$$\limsup_{t\to\infty} \frac{1}{t} \int_0^t \|x(s) - x^*\|^2 ds \leq K(\sigma, H_0, H) \quad \text{a.s.}$$

Hence the proof of this theorem is completed. □

7. Persistence

In this section, we show that this system is persistent in mean.

Theorem 7.1. *Let assumptions (A) and (B) hold. If $x_i^* > K^{\frac{1}{2}}(\sigma, H_0, H)$, then the solution $x(t)$ of system (1.2) with any initial value $x_0 \in R_+^n$ has the following property*

$$\liminf_{t\to\infty} \frac{1}{t} \int_0^t x_i(s) ds \geq x_i^* - K^{\frac{1}{2}}(\sigma, H_0, H) > 0 \quad a.s.$$

where $K(\sigma, H_0, H)$ is defined in Theorem 6.1, so system (1.2) is persistent in mean.

Proof. By the result of Theorem 6.1 and the Holder inequality, one can derive that

$$\limsup_{t\to\infty} (\frac{1}{t} \int_0^t \|x(s) - x^*\| ds)^2 \leq \limsup_{t\to\infty} \frac{1}{t} \int_0^t \|x(s) - x^*\|^2 ds \leq K(\sigma, H_0, H) \quad \text{a.s.}$$

i.e.,

$$\limsup_{t\to\infty} \frac{1}{t} \int_0^t \|x(s) - x^*\| ds \leq K^{\frac{1}{2}}(\sigma, H_0, H) \quad \text{a.s.} \tag{7.1}$$

It then follows from (7.1) and the inequality $x_i - x_i^* \leq \|x - x^*\|$ that

$$\limsup_{t\to\infty} \frac{1}{t} \int_0^t x_i(s) ds \leq x_i^* + \limsup_{t\to\infty} \frac{1}{t} \int_0^t \|x(s) - x^*\| ds \leq x_i^* + K^{\frac{1}{2}}(\sigma, H_0, H) \quad \text{a.s.}$$

which implies

$$\liminf_{t\to\infty} \frac{1}{t} \int_0^t (x_i(s) - x_i^*) ds \geq - \limsup_{t\to\infty} \frac{1}{t} \int_0^t \|x(s) - x^*\| ds \geq -K^{\frac{1}{2}}(\sigma, H_0, H) \quad \text{a.s.}$$

By condition $x_i^* > K^{\frac{1}{2}}(\sigma, H_0, H)$, it is easy to see that

$$\liminf_{t\to\infty} \frac{1}{t} \int_0^t x_i(s) ds \geq x_i^* - K^{\frac{1}{2}}(\sigma, H_0, H) > 0 \quad \text{a.s.}$$

Hence the proof of this theorem is completed. □

8. Example

Consider the following three species predator-prey chain model

$$\begin{cases} dx_1(t) = x_1(t)[(b_1 - a_{11}x_1(t) - a_{12}x_2(t))dt + \sigma_1 dW_1(t) + \int_Y H_1(u)\widetilde{N}(dt,du)], \\ dx_2(t) = x_2(t)[(-b_2 + a_{21}x_1(t) - a_{22}x_2(t) - a_{23}x_3(t))dt + \sigma_2 dW_2(t) \\ \qquad + \int_Y H_2(u)\widetilde{N}(dt,du)], \\ dx_3(t) = x_3(t)[(-b_3 + a_{32}x_2(t) - a_{33}x_3(t))dt + \sigma_3 dW_3(t) + \int_Y H_3(u)\widetilde{N}(dt,du)], \end{cases} \tag{8.1}$$

where $x_i(t)(i=1,2,3)$ denotes the population densities of the species at time t. The parameters $b_1, b_2, b_3, a_{ii}(i=1,2,3)$ are positive constants that stand for intrinsic growth rate, predator death rate of the second species, predator death rate of the third species, coefficient of internal competition, respectively. a_{21}, a_{32} represent saturated rate of the second and third predator and a_{12}, a_{23} represent the decrement rate of the predator to prey. $W_i(t)(i=1,2,3)$ are independent white noises with $W_i(0)=0$, $\delta_i^2 > 0(i=1,2,3)$ representing the intensities of the noise. $H_i(u)(i=1,2,3)$ are bounded functions and $-H_0 \le H_i(u) \le H$ $(0 < H_0 < 1,\ H > 0)$.

The matrix A of the model is

$$A = \begin{pmatrix} -a_{11} & -a_{12} & 0 \\ a_{21} & -a_{22} & -a_{23} \\ 0 & a_{32} & -a_{33} \end{pmatrix}.$$

Choose $C = \mathrm{diag}(1, \frac{\mathrm{a}_{12}}{\mathrm{a}_{21}}, \frac{\mathrm{a}_{12}\mathrm{a}_{23}}{\mathrm{a}_{21}\mathrm{a}_{32}})$, then

$$-\frac{1}{2}(CA + A^T C) = \mathrm{diag}(\mathrm{a}_{11}, \frac{\mathrm{a}_{12}\mathrm{a}_{22}}{\mathrm{a}_{21}}, \frac{\mathrm{a}_{12}\mathrm{a}_{23}\mathrm{a}_{33}}{\mathrm{a}_{21}\mathrm{a}_{32}})$$

is positive-definite.

If $b_1 < \frac{1}{2}\min_{1\le i\le n}\sigma_i^2 + \frac{(H_0\wedge H)^2}{2(1+H_0\vee H)^2}\lambda(Y)$, Theorem 5.1 tells that the solution of system (1.2) is extinctive with probability one.

It is well known that if $b_1 - \frac{a_{11}}{a_{21}}b_2 - \frac{a_{11}a_{22}+a_{12}a_{21}}{a_{21}a_{32}}b_3 > 0$, then the corresponding deterministic model has a positive equilibrium $x^* = (x_1^*, x_2^*, x_3^*)$. Note that system (8.1) does not have a positive equilibrium, thus the solution of system (8.1) will not tend to a fixed positive point. However, Theorem 6.1 shows that the difference between the solution of system (8.1) and $x^* = (x_1^*, x_2^*, x_3^*)$ in time average is only related with the intensity of the white noise and Lévy noise. The weaker the white noise and Lévy noise are, the smaller the difference is. If $x_i^* > K^{\frac{1}{2}}(\sigma, H_0, H)$, where $K(\sigma, H_0, H)$ is defined in Theorem 6.1, system (8.1) is persistent in mean by Theorem 7.1.

9. Conclusion

Our aim in this paper is to discuss autonomous Lotka-Volterra systems by Lévy noise. We analyzed the existence and uniqueness of its global positive solution, discussed the boundedness of the pth moment for $p \geq 2$ and obtained the pathwise estimation which is better than that of [2] and [16]. We gave sufficient and necessary conditions for extinction and persistence in the mean. We also discussed the asymptotic stability of the positive solution of this model, and sufficient conditions for this are established.

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