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ON ADMM-BASED METHODS FOR SOLVING THE NEARNESS SYMMETRIC SOLUTION OF THE SYSTEM OF MATRIX EQUATIONS $A_1XB_1 = C_1$ AND $A_2XB_2 = C_2$*

Yu-Ning Wu[1,2,†] and Min-Li Zeng[3]

Abstract Two new equivalent forms of the matrix nearness problem are developed. Some sufficient and necessary conditions for a symmetric matrix X^* being a solution of the considered problem are presented. Based on the new equivalent forms of the above problem and the idea of the alternating direction method with multipliers (ADMM), we establish two new iterative methods to compute its solution, and analyze the global convergence of the proposed algorithms. Numerical results demonstrate the efficiency of our methods. The development here is an extension of the recent work of Peng, Fang, Xiao and Du [*SpringerPlus*, 5:1005, 2016] on the nearness symmetric solution of the matrix equation $AXB = C$.

Keywords Matrix nearness problem, symmetric solution, ADMM, iterative method, matrix equations.

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1. Introduction

Throughout this paper, we will use the following notations. The set of all $m \times n$ real matrices is denoted by $\mathbb{R}^{m\times n}$. $\mathbb{SR}^{n\times n}$ denotes the set of all symmetric matrices in $\mathbb{R}^{n\times n}$. For $A \in \mathbb{R}^{m\times n}$, A^T and $A^\dagger$ will denote the transpose and Moore-Penrose generalized inverse of the matrix A, respectively. The inner product in space $\mathbb{R}^{m\times n}$ is defined by $\langle A, B\rangle = \mathrm{trace}(A^T B)$ for all $A, B \in \mathbb{R}^{m\times n}$, and the induced matrix norm $\|A\| = \sqrt{\langle A, A\rangle}$ is the so-called Frobenius norm. Accordingly, $\mathbb{R}^{m\times n}$ can be seen as a real Hilbert space. For $A = (a_{ij}), B = (b_{ij}) \in \mathbb{R}^{m\times n}$, $A \circ B = (a_{ij}b_{ij})$ denotes the Hadamard product of A and B.

Assume that O is a closed convex subset in a real Hilbert space $\mathbb{H}$ and x is a point in $\mathbb{H}$. It was known that the point in O nearest to x is called the projection of x onto O and represented by $P_O(x)$. More precisely, $P_O(x)$ is the solution of the

†The corresponding author. Email address:wuyuning330@163.com(Y. Wu)
1College of Mathematics and Informatics, Fujian Normal University, Fuzhou 350007, China
2School of Basic Education, Putian University, Putian 351100, China
3School of Mathematics and Finance, Putian University, Putian 351100, China
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problem $\min_{y\in O}\|y-x\|_H$, that is,

$$\|P_O(x)-x\|_H=\min_{y\in O}\|y-x\|_H.$$

Here, $\|\cdot\|_H$ denotes some norm defined in $\mathbb{H}$(see [2,9]). Now, we turn to matrix space $\mathbb{R}^{m\times n}$. The problem of finding a nearness matrix X^* in a constraint matrix set to a given matrix $\bar{X}$ is refered to as the matrix nearness problem. Since the preliminary estimation $\bar{X}$ is frequently obtained from experiments, it may not satisfy the given restrictions. Thus, it is necessary to find a nearness matrix X^* in this constraint matrix set to replace the estimation $\bar{X}$ [13]. In the area of scientific computing and engineering applications, including structure design, finite element model updating and control theory, and so forth, the matrix set is always the (constraint) solution set or the least square (constraint) solution set of some matrix equations [8, 11].

In recent years, there has been a surge of interest in the research on matrix nearness problems. Peng et al. [18] constructed an iterative method for solving symmetric solutions and optimal approximation solution of the system of matrix equations $A_1XB_1=C_1$, $A_2XB_2=C_2$. Cai and Chen [6] presented an iterative algorithm for the least squares bisymmetric solutions of the matrix equations $A_1XB_1=C_1$, $A_2XB_2=C_2$. Chen et al. [7], based on LSQR Algorithm [17], proposed an matrix based iterative method for solving common symmetric solution or common symmetric least-squares solution of the pair of matrix equations $AXB=E$ and $CXD=F$. In addition, they studied the corresponding matrix nearness problems. Based on the alternating projection algorithm [16], Duan and Li [10] proposed a new iterative algorithm to solve the matrix nearness problem associated with the matrix equations $AXB=E$, $CXD=F$. Li et al. [15] applied Dykstra's alternating projection algorithm to compute the optimal approximate symmetric positive semidefinite solution of the matrix equations $AXB=E$, $CXD=F$. Based on the idea of the alternating variable minimization with multiplier (AVMM) method, Peng et al. [20] developed two iterative methods to solve the nearness symmetric solution of the matrix equation $AXB=C$ to a given matrix $\tilde{X}$ in the sense of the Frobenius norm. Ke and Ma [14] applied the unified frame of alternating direction method of multipliers to solve three classes of matrix equations arising in control like $AXB=E$,$AXB+CXD=E$,and $AX^2+BX+C=0$. Peng [19] presented a matrix iterative method to compute the solutions of the matrix equation $AXB=C$, based on LSQR algorithm. Zhang and Nagy [23] presented an alternating direction method of multipliers to solve a linear ill-posed inverse problem $g=\mathbb{K}x+e$, where $\mathbb{K}$ has a Kronecker product structure.

Considering the above introduction, the algorithm of Peng et al. [20] attracted our interest because of the relatively high efficiency in solving the nearness symmetric solution of the matrix equation $AXB=C$. However, when we deal with some more complicated computational problems, we need to solve the symmetric solution of a pair of matrix equations with different high dimension, at this time, the algorithm of Peng et al. can not be directly used to solve the pair of matrix equations. Therefore, we try to extend of the idea of Peng et al., in order to obtain a new efficient algorithm which can be used more widely. In this paper, we consider the following matrix nearness problem

$$\begin{aligned}&\min_{X\in\mathbb{SR}^{n\times n}} \tfrac{1}{2}\|X-\bar{X}\|^2\\ &\text{subject to}\\ &\quad\begin{cases} A_1XB_1=C_1,\\ A_2XB_2=C_2,\end{cases}\end{aligned}\tag{1.1}$$

where $\bar{X}\in\mathbb{R}^{n\times n}$ is a given matrix and $A_1\in\mathbb{R}^{m_1\times n}$, $B_1\in\mathbb{R}^{n\times p_1}$, $C_1\in\mathbb{R}^{m_1\times p_1}$, $A_2\in\mathbb{R}^{m_2\times n}$, $B_2\in\mathbb{R}^{n\times p_2}$ and $C_2\in\mathbb{R}^{m_2\times p_2}$ are known matrices.

The remainder of this paper is organized as follows: in Section 2, we derive some necessary and sufficient conditions for the matrix X^* being a solution of the matrix nearness problem (1.1). In Section 3, we propose two iterative methods for solving the matrix nearness problem (1.1). Furthermore, we analyze the global convergence properties of the new algorithms. In Section 4, some numerical experiments are provided to show the performance of our methods. Finally, some concluding remarks are given in Section 5.

2. Two equivalent forms of the matrix nearness problem (1.1)

In this section, we will give two equivalent constrained optimization problems of the matrix nearness problem (1.1), and explore the properties of the solutions of the mentioned constrained optimization problems.

Extending the line of the idea of Peng et al. [20], the matrix nearness problem (1.1) is equivalent to the following constrained optimization problem

$$\begin{aligned}&\min_{X\in\mathbb{SR}^{n\times n},Y\in\mathbb{R}^{m_1\times n},Z\in\mathbb{R}^{m_2\times n}} F(X,Y,Z)=\tfrac{1}{2}\|X-\bar{X}\|^2\\ &\text{subject to}\\ &\quad\begin{cases} A_1X-Y=0,\\ YB_1-C_1=0,\\ A_2X-Z=0,\\ ZB_2-C_2=0\end{cases}\end{aligned}\tag{2.1}$$

or

$$\begin{aligned}&\min_{X\in\mathbb{SR}^{n\times n},Y\in\mathbb{R}^{n\times p_1},Z\in\mathbb{R}^{n\times p_2}} F(X,Y,Z)=\tfrac{1}{2}\|X-\bar{X}\|^2\\ &\text{subject to}\\ &\quad\begin{cases} XB_1-Y=0,\\ A_1Y-C_1=0,\\ XB_2-Z=0,\\ A_2Z-C_2=0.\end{cases}\end{aligned}\tag{2.2}$$

Now, we are in position to discuss the properties of the solutions of the constrained optimization problems (2.1) and (2.2).

Theorem 2.1. *Matrix triple $[X^* : Y^* : Z^*]$ is a solution of the constrained optimization problem* (2.1) *if and only if there exists matrices $M^* \in \mathbb{R}^{m_1 \times n}$, $N^* \in \mathbb{R}^{m_1 \times p_1}$, $S^* \in \mathbb{R}^{m_2 \times n}$ and $T^* \in \mathbb{R}^{m_2 \times p_2}$ such that the following equalities* (2.3-2.9) *hold.*

$$(X^* - \bar{X} - A_1^T M^* - A_2^T S^*) + (X^* - \bar{X} - A_1^T M^* - A_2^T S^*)^T = 0, \tag{2.3}$$
$$M^* - N^* B_1^T = 0, \tag{2.4}$$
$$S^* - T^* B_2^T = 0, \tag{2.5}$$
$$A_1 X^* - Y^* = 0, \tag{2.6}$$
$$Y^* B_1 - C_1 = 0, \tag{2.7}$$
$$A_2 X^* - Z^* = 0, \tag{2.8}$$
$$Z^* B_2 - C_2 = 0. \tag{2.9}$$

Proof. The proof is inspired by the proof of Theorem 1 in [20].

On one hand, assume that there exists matrices $M^* \in \mathbb{R}^{m_1 \times n}$, $N^* \in \mathbb{R}^{m_1 \times p_1}$, $S^* \in \mathbb{R}^{m_2 \times n}$ and $T^* \in \mathbb{R}^{m_2 \times p_2}$ such that the equalities (2.3-2.9) hold. Define

$$\bar{F}(X,Y,Z) = F(X,Y,Z) - \langle M^*, A_1 X - Y\rangle - \langle N^*, Y B_1 - C_1\rangle - \langle S^*, A_2 X - Z\rangle - \langle T^*, Z B_2 - C_2\rangle.$$

Then, for any matrices $U \in \mathbb{SR}^{n \times n}$, $V \in \mathbb{R}^{m_1 \times n}$ and $W \in \mathbb{R}^{m_2 \times n}$, it follows from (2.3), (2.4) and (2.5) that

$$\begin{aligned}
&\bar{F}(X^* + U, Y^* + V, Z^* + W) \\
=&\frac{1}{2}\|X^* + U - \bar{X}\|^2 - \langle M^*, A_1(X^* + U) - (Y^* + V)\rangle - \langle N^*, (Y^* + V)B_1 - C_1\rangle \\
&- \langle S^*, A_2(X^* + U) - (Z^* + W)\rangle - \langle T^*, (Z^* + W)B_2 - C_2\rangle \\
=&\bar{F}(X^*, Y^*, Z^*) + \frac{1}{2}\|U\|^2 + \langle X^* - \bar{X} - A_1^T M^* - A_2^T S^*, U\rangle + \langle M^* - N^* B_1^T, V\rangle \\
&+ \langle S^* - T^* B_2^T, W\rangle \\
=&\bar{F}(X^*, Y^*, Z^*) + \frac{1}{2}\|U\|^2 + \frac{1}{2}\langle X^* - \bar{X} - A_1^T M^* - A_2^T S^* \\
&+ (X^* - \bar{X} - A_1^T M^* - A_2^T S^*)^T, U\rangle + \langle M^* - N^* B_1^T, V\rangle + \langle S^* - T^* B_2^T, W\rangle \\
=&\bar{F}(X^*, Y^*, Z^*) + \frac{1}{2}\|U\|^2 \\
\geq&\bar{F}(X^*, Y^*, Z^*),
\end{aligned} \tag{2.10}$$

where the third equality exploits the fact that $\langle A, U\rangle = \frac{1}{2}\langle A + A^T, U\rangle$ since $U \in \mathbb{SR}^{n \times n}$. The above inequality (2.10) implies that the matrix triple $[X^* : Y^* : Z^*]$ is a global minimizer of the matrix function $\bar{F}(X,Y,Z)$. Thus, $\bar{F}(X,Y,Z) \geq \bar{F}(X^*, Y^*, Z^*)$ holds for all $X \in \mathbb{SR}^{n \times n}$, $Y \in \mathbb{R}^{m_1 \times n}$ and $Z \in \mathbb{R}^{m_2 \times n}$, which together with (2.6), (2.7), (2.8) and (2.9) imply that

$$\begin{aligned}
F(X,Y,Z) \geq& F(X^*, Y^*, Z^*) + \langle M^*, A_1 X - Y\rangle - \langle N^*, Y B_1 - C_1\rangle \\
&- \langle S^*, A_2 X - Z\rangle - \langle T^*, Z B_2 - C_2\rangle,
\end{aligned}$$

from which we can conclude that $F(X,Y,Z) \geq F(X^*, Y^*, Z^*)$ holds for all $X \in \mathbb{SR}^{n \times n}$ with $A_1 X - Y = 0$, $Y B_1 - C_1 = 0$, $A_2 X - Z = 0$ and $Z B_2 - C_2 = 0$. That is, the matrix triple $[X^* : Y^* : Z^*]$ is a solution of the constrained optimization problem (2.1).

On the other hand, if the matrix triple $[X^* : Y^* : Z^*]$ is a solution of the constrained optimization problem (2.1), then the matrix triple $[X^* : Y^* : Z^*]$ definitely satisfies Karush-Kuhn-Tucker (KKT) conditions of the problem (2.1) [4, Chapter 5], which are

$$\frac{\partial L(X,Y,Z,M,N,S,T,H)}{\partial X} = X - \bar{X} - A_1^T M - A_2^T S - H^T + H = 0, \tag{2.11}$$

$$\frac{\partial L(X,Y,Z,M,N,S,T,H)}{\partial Y} = M - NB_1^T = 0, \tag{2.12}$$

$$\frac{\partial L(X,Y,Z,M,N,S,T,H)}{\partial Z} = S - TB_2^T = 0, \tag{2.13}$$

$$A_1X - Y = 0, \tag{2.14}$$

$$YB_1 - C_1 = 0, \tag{2.15}$$

$$A_2X - Z = 0, \tag{2.16}$$

$$ZB_2 - C_2 = 0, \tag{2.17}$$

$$X^T - X = 0, \tag{2.18}$$

where

$$\begin{aligned} & L(X,Y,Z,M,N,S,T,H) \\ =& \frac{1}{2}\left\|X - \bar{X}\right\|^2 - \langle M, A_1X - Y\rangle - \langle N, YB_1 - C_1\rangle - \langle S, A_2X - Z\rangle \\ & - \langle T, ZB_2 - C_2\rangle - \langle H, X^T - X\rangle \end{aligned}$$

is the Lagrange function of the convex optimization problem (2.1) in $\mathbb{R}^{n\times n}\times\mathbb{R}^{m_1\times n}\times\mathbb{R}^{m_2\times n}\times\mathbb{R}^{m_1\times n}\times\mathbb{R}^{m_1\times p_1}\times\mathbb{R}^{m_2\times n}\times\mathbb{R}^{m_2\times p_2}\times\mathbb{R}^{n\times n}$. It follows from (2.11) that

$$(X - \bar{X} - A_1^T M - A_2^T S) + (X - \bar{X} - A_1^T M - A_2^T S)^T = 0. \tag{2.19}$$

Combining (2.12)-(2.19), if the matrix triple $[X^* : Y^* : Z^*]$ is a solution of the constrained optimization problem (2.1), then there exist matrices $M^* \in \mathbb{R}^{m_1\times n}$, $N^* \in \mathbb{R}^{m_1\times p_1}$, $S^* \in \mathbb{R}^{m_2\times n}$ and $T^* \in \mathbb{R}^{m_2\times p_2}$ such that (2.3)-(2.9) hold. □

Similarly, we can prove the following Theorem 2.2.

Theorem 2.2. *Matrix triple $[X^* : Y^* : Z^*]$ is a solution of the constrained optimization problem* (2.2) *if and only if there exists matrices $M^* \in \mathbb{R}^{n\times p_1}$, $N^* \in \mathbb{R}^{m_1\times p_1}$, $S^* \in \mathbb{R}^{n\times p_2}$ and $T^* \in \mathbb{R}^{m_2\times p_2}$ such that the following equalities* (2.20-2.26) *hold.*

$$(X^* - \bar{X} - M^*B_1^T - S^*B_2^T) + (X^* - \bar{X} - M^*B_1^T - S^*B_2^T)^T = 0, \tag{2.20}$$

$$M^* - A_1^T N^* = 0, \tag{2.21}$$

$$S^* - A_2^T T^* = 0, \tag{2.22}$$

$$X^*B_1 - Y^* = 0, \tag{2.23}$$

$$A_1Y^* - C_1 = 0, \tag{2.24}$$

$$X^*B_2 - Z^* = 0, \tag{2.25}$$

$$A_2Z^* - C_2 = 0. \tag{2.26}$$

3. Iterative methods to solve the matrix nearness problem (1.1)

In this section, based on the idea of AVMM method [1], we develop iterative methods to compute the solutions of the equivalent constrained optimization problems (2.1) and (2.2), and hence to compute the solution of the matrix nearness problem (1.1). It should be mentioned here that the AVMM method is the so-called alternating direction method with multipliers (ADMM) in the field of optimization [12].

For the constrained optimization (2.1), its augmented Lagrangian in $\mathbb{SR}^{n\times n} \times \mathbb{R}^{m_1\times n} \times \mathbb{R}^{m_2\times n} \times \mathbb{R}^{m_1\times n} \times \mathbb{R}^{m_1\times p_1} \times \mathbb{R}^{m_2\times n} \times \mathbb{R}^{m_2\times p_2}$ is

$$\begin{aligned}\mathcal{L}_{\alpha,\beta,\gamma,\delta}(X,Y,Z,M,N,S,T) =& \frac{1}{2}\|X-\bar{X}\|^2 - \langle M, A_1X-Y\rangle - \langle N, YB_1-C_1\rangle \\ &- \langle S, A_2X-Z\rangle - \langle T, ZB_2-C_2\rangle + \frac{\alpha}{2}\|A_1X-Y\|^2 \\ &+ \frac{\beta}{2}\|YB_1-C_1\|^2 + \frac{\gamma}{2}\|A_2X-Z\|^2 + \frac{\delta}{2}\|ZB_2-C_2\|^2,\end{aligned} \tag{3.1}$$

where $\alpha,\beta,\gamma,\delta > 0$ are penalty parameters. Based on the idea of ADMM, at each iteration step, we first alternatively minimizes the augmented Lagrangian function $\mathcal{L}_{\alpha,\beta,\gamma,\delta}(X,Y,Z,M,N,S,T)$ defined as in (3.1) with respect to the variables $X \in \mathbb{SR}^{n\times n}, Y \in \mathbb{R}^{m_1\times n}, Z \in \mathbb{R}^{m_2\times n}$, and then update the Lagrange multipliers M,N,S,T according to the steepest ascent principle [5]. More precisely, we propose the ADMM-based iterative method for solving the constrained optimization (2.1) as the Algorithm 1.

Algorithm 1

Step 0: Input the matrices $A_1, A_2, B_1, B_2, C_1, C_2, \bar{X}$.
Step 1: Choose the initial matrices $Y_0, Z_0, M_0, N_0, S_0, T_0$ and the parameters $\alpha,\beta,\gamma,\delta > 0$. Set $k = 0$.
Step 2: Exit if a stopping criterion has been met.
Step 3: Compute

$$X_{k+1} = \arg\min_{X\in\mathbb{SR}^{n\times n}} \mathcal{L}_{\alpha,\beta,\gamma,\delta}(X, Y_k, Z_k, M_k, N_k, S_k, T_k), \tag{3.2}$$

$$Y_{k+1} = \arg\min_{Y\in\mathbb{R}^{m_1\times n}} \mathcal{L}_{\alpha,\beta,\gamma,\delta}(X_{k+1}, Y, Z_k, M_k, N_k, S_k, T_k), \tag{3.3}$$

$$Z_{k+1} = \arg\min_{Z\in\mathbb{R}^{m_2\times n}} \mathcal{L}_{\alpha,\beta,\gamma,\delta}(X_{k+1}, Y_{k+1}, Z, M_k, N_k, S_k, T_k), \tag{3.4}$$

$$M_{k+1} = M_k - \alpha(A_1X_{k+1} - Y_{k+1}), \tag{3.5}$$

$$N_{k+1} = N_k - \beta(Y_{k+1}B_1 - C_1), \tag{3.6}$$

$$S_{k+1} = S_k - \gamma(A_2X_{k+1} - Z_{k+1}), \tag{3.7}$$

$$T_{k+1} = T_k - \delta(Z_{k+1}B_2 - C_2). \tag{3.8}$$

Step 4: Set $k = k + 1$ and go to Step 2.

For the constrained optimization (2.2), its augmented Lagrangian in $\mathbb{SR}^{n\times n} \times$

$\mathbb{R}^{n\times p_1} \times \mathbb{R}^{n\times p_2} \times \mathbb{R}^{n\times p_1} \times \mathbb{R}^{m_1\times p_1} \times \mathbb{R}^{n\times p_2} \times \mathbb{R}^{m_2\times p_2}$ is

$$\begin{aligned}\bar{\mathcal{L}}_{\alpha,\beta,\gamma,\delta}(X,Y,Z,M,N,S,T) =& \frac{1}{2}\|X-\bar{X}\|^2 - \langle M, XB_1 - Y\rangle - \langle N, A_1Y - C_1\rangle \\ & - \langle S, XB_2 - Z\rangle - \langle T, A_2Z - C_2\rangle + \frac{\alpha}{2}\|XB_1 - Y\|^2 \\ & + \frac{\beta}{2}\|A_1Y - C_1\|^2 + \frac{\gamma}{2}\|XB_2 - Z\|^2 + \frac{\delta}{2}\|A_2Z - C_2\|^2.\end{aligned} \tag{3.9}$$

Analogously, according to (3.9), we propose the ADMM-based iterative method for solving the constrained optimization (2.2) as the Algorithm 2.

Algorithm 2

Step 0: Input the matrices $A_1, A_2, B_1, B_2, C_1, C_2, \bar{X}$.
Step 1: Choose the initial matrices $Y_0, Z_0, M_0, N_0, S_0, T_0$ and the parameters $\alpha, \beta, \gamma, \delta > 0$. Set $k = 0$.
Step 2: Exit if a stopping criterion has been met.
Step 3: Compute

$$X_{k+1} = \arg\min_{X\in\mathbb{SR}^{n\times n}} \bar{\mathcal{L}}_{\alpha,\beta,\gamma,\delta}(X, Y_k, Z_k, M_k, N_k, S_k, T_k), \tag{3.10}$$

$$Y_{k+1} = \arg\min_{Y\in\mathbb{R}^{n\times p_1}} \bar{\mathcal{L}}_{\alpha,\beta,\gamma,\delta}(X_{k+1}, Y, Z_k, M_k, N_k, S_k, T_k), \tag{3.11}$$

$$Z_{k+1} = \arg\min_{Z\in\mathbb{R}^{n\times p_2}} \bar{\mathcal{L}}_{\alpha,\beta,\gamma,\delta}(X_{k+1}, Y_{k+1}, Z, M_k, N_k, S_k, T_k), \tag{3.12}$$

$$M_{k+1} = M_k - \alpha(X_{k+1}B_1 - Y_{k+1}), \tag{3.13}$$

$$N_{k+1} = N_k - \beta(A_1Y_{k+1} - C_1), \tag{3.14}$$

$$S_{k+1} = S_k - \gamma(X_{k+1}B_2 - Z_{k+1}), \tag{3.15}$$

$$T_{k+1} = T_k - \delta(A_2Z_{k+1} - C_2). \tag{3.16}$$

Step 4: Set $k = k + 1$ and go to Step 2.

For these two algorithms, the iteration kernel involves computing X_{k+1}, Y_{k+1} and Z_{k+1}. In what follows, we will probe how to compute X_{k+1}, Y_{k+1} and Z_{k+1}.

Firstly, we devote ourselves to compute X_{k+1}. By straightforward calculations, X_{k+1} in (3.2) can be reformulated as

$$\begin{aligned} X_{k+1} =& \arg\min_{X\in\mathbb{SR}^{n\times n}} \frac{1}{2}\|X-\bar{X}\|^2 - \langle M_k, A_1X - Y_k\rangle + \frac{\alpha}{2}\|A_1X - Y_k\|^2 \\ & - \langle S_k, A_2X - Z_k\rangle + \frac{\gamma}{2}\|A_2X - Z_k\|^2 \\ =& \arg\min_{X\in\mathbb{SR}^{n\times n}} \frac{1}{2}\|X-\bar{X}\|^2 + \frac{\alpha}{2}\left\|A_1X - \left(Y_k + \frac{1}{\alpha}M_k\right)\right\|^2 \\ & + \frac{\gamma}{2}\left\|A_2X - \left(Z_k + \frac{1}{\gamma}S_k\right)\right\|^2 \\ =& \arg\min_{X\in\mathbb{SR}^{n\times n}} \frac{1}{2}\left\|\begin{bmatrix}\sqrt{\alpha}A_1 \\ \sqrt{\gamma}A_2 \\ I_n\end{bmatrix} X - \begin{bmatrix}\sqrt{\alpha}Y_k + M_k/\sqrt{\alpha} \\ \sqrt{\gamma}Z_k + S_k/\sqrt{\gamma} \\ \bar{X}\end{bmatrix}\right\|^2 \end{aligned}$$

$$= \arg \min_{X \in \mathbb{SR}^{n\times n}} \frac{1}{2}\|AX - B\|^2, \tag{3.17}$$

where $A = \begin{bmatrix} \sqrt{\alpha}A_1 \\ \sqrt{\gamma}A_2 \\ I_n \end{bmatrix} \in \mathbb{R}^{(m_1+m_2+n)\times n}$, $B = \begin{bmatrix} \sqrt{\alpha}Y_k + M_k/\sqrt{\alpha} \\ \sqrt{\gamma}Z_k + S_k/\sqrt{\gamma} \\ \bar{X} \end{bmatrix} \in \mathbb{R}^{(m_1+m_2+n)\times n}$.

Similarly, X_{k+1} as defined in (3.10) can be expressed as

$$X_{k+1} = \arg \min_{X \in \mathbb{SR}^{n\times n}} \frac{1}{2}\|XA - B\|^2, \tag{3.18}$$

where $A = [I_n, \sqrt{\alpha}B_1, \sqrt{\gamma}B_2] \in \mathbb{R}^{n\times(n+p_1+p_2)}$, $B = [\bar{X}, \sqrt{\alpha}Y_k + M_k/\sqrt{\alpha}, \sqrt{\gamma}Z_k + S_k/\sqrt{\gamma}] \in \mathbb{R}^{n\times(n+p_1+p_2)}$.

In order to solve the problems (3.17) and (3.18), we need the following Lemma 3.1.

Lemma 3.1 ([22]). *Given matrix $B \in \mathbb{R}^{n\times n}$ and $\Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \cdots, \sigma_n)$ with $\sigma_i > 0$ $(i = 1, \cdots, n)$, then the problem $\|X\Sigma - B\|^2 = \min$ has a unique least squares symmetric solution in $\mathbb{SR}^{n\times n}$ with the following expression*

$$\hat{X} = \Phi \circ (B\Sigma + \Sigma B^T),$$

where $\Phi_{ij} = \frac{1}{\sigma_i^2+\sigma_j^2}$ $(1 \le i, j \le n)$, $\Phi = (\Phi_{ij}) \in \mathbb{R}^{n\times n}$.

It follows from the fact that the matrix A in (3.17) is full column rank that the singular value decomposition (SVD) of the matrix A is

$$A = U \begin{bmatrix} \Sigma \\ 0 \end{bmatrix} V^T,$$

where $\Sigma = \mathrm{diag}(\sigma_1, \cdots, \sigma_n)$ with $\sigma_i > 0, i = 1, \cdots, n$, and $U = [U_1, U_2] \in \mathbb{R}^{(m_1+m_2+n)\times(m_1+m_2+n)}$, $V \in \mathbb{R}^{n\times n}$ are orthogonal matrices, $U_1 \in \mathbb{R}^{(m_1+m_2+n)\times n}$. Therefore, formula (3.17) can be expressed as

$$\begin{aligned} X_{k+1} &= \arg \min_{X \in \mathbb{SR}^{n\times n}} \frac{1}{2}\left\| U \begin{bmatrix} \Sigma \\ 0 \end{bmatrix} V^T X - B \right\|^2 \\ &= \arg \min_{X \in \mathbb{SR}^{n\times n}} \frac{1}{2}\left\| \begin{bmatrix} \Sigma \\ 0 \end{bmatrix} V^T X V - U^T B V \right\|^2 \\ &= \arg \min_{X \in \mathbb{SR}^{n\times n}} \frac{1}{2}\left\| \begin{bmatrix} \Sigma \\ 0 \end{bmatrix} V^T X V - \begin{bmatrix} U_1^T \\ U_2^T \end{bmatrix} B V \right\|^2 \\ &= \arg \min_{X \in \mathbb{SR}^{n\times n}} \frac{1}{2}\|\Sigma V^T X V - U_1^T B V\|^2 \\ &= \arg \min_{\tilde{X} \in \mathbb{SR}^{n\times n}} \frac{1}{2}\|\tilde{X}\Sigma - \tilde{B}^T\|^2, \end{aligned} \tag{3.19}$$

where $\tilde{X} = V^T X V$ and $\tilde{B} = U_1^T B V$. According to (3.19) and Lemma 3.1, the solution of the problem (3.2) can be expressed as

$$X_{k+1} = V(\Phi \circ (\Sigma\tilde{B} + \tilde{B}^T\Sigma))V^T.$$

For the optimization problem (3.18), the matrix A is full row rank, and the SVD of A is

$$A = P[\Sigma, 0]Q^T,$$

where $\Sigma = \mathrm{diag}(\sigma_1, \cdots, \sigma_n)$ with $\sigma_i > 0, i = 1, \cdots, n$, and $P \in \mathbb{R}^{n\times n}$, $Q = [Q_1, Q_2] \in \mathbb{R}^{(n+p_1+p_2)\times(n+p_1+p_2)}$ are orthogonal matrices, $Q_1 \in \mathbb{R}^{(n+p_1+p_2)\times n}$. Then similar to the above discussion, X_{k+1} in (3.18) can be expressed as

$$X_{k+1} = P(\Phi \circ (\tilde{B}\Sigma + \Sigma\tilde{B}^T))P^T,$$

where $\tilde{B} = P^T B Q_1$.

Then, we focus to compute Y_{k+1}. By straightforward calculations, we can rewrite Y_{k+1} in (3.3) as

$$\begin{aligned} Y_{k+1} &= \arg\min_{Y\in\mathbb{R}^{m_1\times n}} \frac{1}{2}\left\| Y[\sqrt{\alpha}I_n, \sqrt{\beta}B_1] - [\sqrt{\alpha}A_1X_{k+1} - M_k/\sqrt{\alpha}, \sqrt{\beta}C_1 + N_k/\sqrt{\beta}]\right\|^2 \\ &= [\sqrt{\alpha}A_1X_{k+1} - M_k/\sqrt{\alpha}, \sqrt{\beta}C_1 + N_k/\sqrt{\beta}][\sqrt{\alpha}I_n, \sqrt{\beta}B_1]^\dagger, \end{aligned}$$

and Y_{k+1} in (3.11) as

$$\begin{aligned} Y_{k+1} &= \arg\min_{Y\in\mathbb{R}^{n\times p_1}} \frac{1}{2}\left\| \begin{bmatrix} \sqrt{\alpha}I_n \\ \sqrt{\beta}A_1 \end{bmatrix} Y - \begin{bmatrix} \sqrt{\alpha}X_{k+1}B_1 - M_k/\sqrt{\alpha} \\ \sqrt{\beta}C_1 + N_k/\sqrt{\beta} \end{bmatrix} \right\|^2 \\ &= \begin{bmatrix} \sqrt{\alpha}I_n \\ \sqrt{\beta}A_1 \end{bmatrix}^\dagger \begin{bmatrix} \sqrt{\alpha}X_{k+1}B_1 - M_k/\sqrt{\alpha} \\ \sqrt{\beta}C_1 + N_k/\sqrt{\beta} \end{bmatrix}. \end{aligned}$$

Finally, we change our attention to compute Z_{k+1}. By simple calculations, we can rewrite Z_{k+1} in (3.4) as

$$\begin{aligned} Z_{k+1} &= \arg\min_{Z\in\mathbb{R}^{m_2\times n}} \frac{1}{2}\left\| Z[\sqrt{\gamma}I_n, \sqrt{\delta}B_2] - [\sqrt{\gamma}A_2X_{k+1} - S_k/\sqrt{\gamma}, \sqrt{\delta}C_2 + T_k/\sqrt{\delta}]\right\|^2 \\ &= [\sqrt{\gamma}A_2X_{k+1} - S_k/\sqrt{\gamma}, \sqrt{\delta}C_2 + T_k/\sqrt{\delta}][\sqrt{\gamma}I_n, \sqrt{\delta}B_2]^\dagger, \end{aligned}$$

and Z_{k+1} in (3.12) as

$$\begin{aligned} Z_{k+1} &= \arg\min_{Z\in\mathbb{R}^{n\times p_2}} \frac{1}{2}\left\| \begin{bmatrix} \sqrt{\gamma}I_n \\ \sqrt{\delta}A_2 \end{bmatrix} Z - \begin{bmatrix} \sqrt{\gamma}X_{k+1}B_2 - S_k/\sqrt{\gamma} \\ \sqrt{\delta}C_2 + T_k/\sqrt{\delta} \end{bmatrix} \right\|^2 \\ &= \begin{bmatrix} \sqrt{\gamma}I_n \\ \sqrt{\delta}A_2 \end{bmatrix}^\dagger \begin{bmatrix} \sqrt{\gamma}X_{k+1}B_2 - S_k/\sqrt{\gamma} \\ \sqrt{\delta}C_2 + T_k/\sqrt{\delta} \end{bmatrix}. \end{aligned}$$

In the following, we will probe into the global convergence of Algorithm 1 and 2.

Theorem 3.1. *Let $(X^*, Y^*, Z^*, M^*, N^*, S^*, T^*)$ be a KKT point for the constrained optimization problem* (2.1)*, that is, the matrices X^*, Y^*, Z^*, M^*, N^*, S^* and T^* satisfy conditions* (2.3-2.9)*. Let matrix sequences $\{X_k\}, \{Y_k\}, \{Z_k\}, \{M_k\}, \{N_k\}, \{S_k\}, \{T_k\}$ be generated by Algorithm* 1*. Define*

$$P_{k+1} = A_1X_{k+1} - Y_{k+1}, \quad Q_{k+1} = Y_{k+1}B_1 - C_1, \tag{3.20}$$

$$U_{k+1} = A_2X_{k+1} - Z_{k+1}, \quad V_{k+1} = Z_{k+1}B_2 - C_2, \tag{3.21}$$

$$\begin{aligned}\Psi_k = \alpha\|Y_k - Y^*\|^2 + \frac{1}{\alpha}\|M_k - M^*\|^2 + \frac{1}{\beta}\|N_k - N^*\|^2 \\ + \gamma\|Z_k - Z^*\|^2 + \frac{1}{\gamma}\|S_k - S^*\|^2 + \frac{1}{\delta}\|T_k - T^*\|^2,\end{aligned} \tag{3.22}$$

then, we have

$$\Psi_{k+1} \le \Psi_k - \beta\|Q_{k+1}\|^2 - \delta\|V_{k+1}\|^2 - \alpha\|P_{k+1} + Y_{k+1} - Y_k\|^2 - \gamma\|U_{k+1} + Z_{k+1} - Z_k\|^2. \tag{3.23}$$

Proof. The proof is motivated by the proof of Theorem 3 in [20].

Since $(X^*, Y^*, Z^*, M^*, N^*, S^*, T^*)$ is a KKT point of the constrained optimization problem (2.1), it is a saddle point of

$$\begin{aligned}\mathcal{L}(X, Y, Z, M, N, S, T) =& \frac{1}{2}\|X - \bar{X}\|^2 - \langle M, A_1X - Y\rangle - \langle N, YB_1 - C_1\rangle \\ &- \langle S, A_2X - Z\rangle - \langle T, ZB_2 - C_2\rangle,\end{aligned}$$

the Lagrange function of the constrained optimization problem (2.1) in $\mathbb{SR}^{n\times n} \times \mathbb{R}^{m_1\times n} \times \mathbb{R}^{m_2\times n} \times \mathbb{R}^{m_1\times n} \times \mathbb{R}^{m_1\times p_1} \times \mathbb{R}^{m_2\times n} \times \mathbb{R}^{m_2\times p_2}$. It follows from the saddle point theorem [3] that

$$\begin{aligned}\mathcal{L}(X^*, Y^*, Z^*, M, N, S, T) &\le \mathcal{L}(X^*, Y^*, Z^*, M^*, N^*, S^*, T^*) \\ &\le \mathcal{L}(X, Y, Z, M^*, N^*, S^*, T^*)\end{aligned}$$

for all X, Y, Z, M, N, S and T. Thus, we have

$$\mathcal{L}(X^*, Y^*, Z^*, M^*, N^*, S^*, T^*) \le \mathcal{L}(X_{k+1}, Y_{k+1}, Z_{k+1}, M^*, N^*, S^*, T^*). \tag{3.24}$$

It follows from (2.6), (2.7), (2.8), (2.9), (3.20), (3.21) and (3.24) that

$$\begin{aligned}&\frac{1}{2}\|X^* - \bar{X}\|^2 - \frac{1}{2}\|X_{k+1} - \bar{X}\|^2 \\ &\le - \langle M^*, P_{k+1}\rangle - \langle N^*, Q_{k+1}\rangle - \langle S^*, U_{k+1}\rangle - \langle T^*, V_{k+1}\rangle.\end{aligned} \tag{3.25}$$

On the other hand, it follows from the first-order optimality condition of the optimization problem (3.2) and the iterative relations (3.5) and (3.7) that

$$\begin{aligned}0 =&[X_{k+1} - \bar{X} - A_1^TM_k + \alpha A_1^T(A_1X_{k+1} - Y_k) - A_2^TS_k + \gamma A_2^T(A_2X_{k+1} - Z_k)] \\ &+ [X_{k+1} - \bar{X} - A_1^TM_k + \alpha A_1^T(A_1X_{k+1} - Y_k) - A_2^TS_k + \gamma A_2^T(A_2X_{k+1} - Z_k)]^T \\ =&[X_{k+1} - \bar{X} - A_1^TM_{k+1} - \alpha A_1^T(Y_k - Y_{k+1}) - A_2^TS_{k+1} - \gamma A_2^T(Z_k - Z_{k+1})] \\ &+ [X_{k+1} - \bar{X} - A_1^TM_{k+1} - \alpha A_1^T(Y_k - Y_{k+1}) - A_2^TS_{k+1} - \gamma A_2^T(Z_k - Z_{k+1})]^T.\end{aligned} \tag{3.26}$$

From (3.26), we can conclude that

$$X_{k+1}=\arg\min_{X\in\mathbb{SR}^{n\times n}}\frac{1}{2}\|X-\bar{X}\|^2-\langle M_{k+1}-\alpha Y_{k+1}+\alpha Y_k, A_1X\rangle-\langle S_{k+1}-\gamma Z_{k+1}+\gamma Z_k, A_2X\rangle,$$

from which we have

$$\begin{aligned}&\frac{1}{2}\|X_{k+1}-\bar{X}\|^2-\langle M_{k+1}-\alpha Y_{k+1}+\alpha Y_k, A_1X_{k+1}\rangle-\langle S_{k+1}-\gamma Z_{k+1}+\gamma Z_k, A_2X_{k+1}\rangle\\ \le&\frac{1}{2}\|X^*-\bar{X}\|^2-\langle M_{k+1}-\alpha Y_{k+1}+\alpha Y_k, A_1X^*\rangle-\langle S_{k+1}-\gamma Z_{k+1}+\gamma Z_k, A_2X^*\rangle.\end{aligned} \tag{3.27}$$

In addition, from the first-order optimality condition of the problem (3.3) and the iteration relations (3.5) and (3.6), we have

$$0=M_k-N_kB_1^T-\alpha(A_1X_{k+1}-Y_{k+1})+\beta(Y_{k+1}B_1-C_1)B_1^T=M_{k+1}-N_{k+1}B_1^T, \tag{3.28}$$

which implies that

$$Y_{k+1}=\arg\min_{Y\in\mathbb{R}^{m_1\times n}}\langle M_{k+1}-N_{k+1}B_1^T, Y\rangle. \tag{3.29}$$

From (3.29), we have

$$\langle M_{k+1}-N_{k+1}B_1^T, Y_{k+1}\rangle\le\langle M_{k+1}-N_{k+1}B_1^T, Y^*\rangle. \tag{3.30}$$

Similarly, from the first-order optimality condition of the problem (3.4) and the iteration relations (3.7) and (3.8), we have

$$0=S_k-T_kB_2^T-\gamma(A_2X_{k+1}-Z_{k+1})+\delta(Z_{k+1}B_2-C_2)B_2^T=S_{k+1}-T_{k+1}B_2^T, \tag{3.31}$$

which implies that

$$Z_{k+1}=\arg\min_{Z\in\mathbb{R}^{m_2\times n}}\langle S_{k+1}-T_{k+1}B_2^T, Z\rangle. \tag{3.32}$$

By (3.32), we obtain

$$\langle S_{k+1}-T_{k+1}B_2^T, Z_{k+1}\rangle\le\langle S_{k+1}-T_{k+1}B_2^T, Z^*\rangle. \tag{3.33}$$

Adding the inequalities (3.27), (3.30) and (3.33), and using (2.6), (2.7), (2.8) and (2.9), we can conclude that

$$\begin{aligned}&\frac{1}{2}\|X_{k+1}-\bar{X}\|^2-\frac{1}{2}\|X^*-\bar{X}\|^2\\ \le&\langle M_{k+1}, P_{k+1}\rangle+\langle N_{k+1}, Q_{k+1}\rangle-\alpha\langle Y_{k+1}-Y_k, P_{k+1}+Y_{k+1}-Y^*\rangle\\ &+\langle S_{k+1}, U_{k+1}\rangle+\langle T_{k+1}, V_{k+1}\rangle-\gamma\langle Z_{k+1}-Z_k, S_{k+1}+Z_{k+1}-Z^*\rangle.\end{aligned} \tag{3.34}$$

Combining the inequalities (3.25) and (3.34), we can derive that

$$\begin{aligned}&\langle M^*-M_{k+1}, P_{k+1}\rangle+\langle N^*-N_{k+1}, Q_{k+1}\rangle+\langle S^*-S_{k+1}, U_{k+1}\rangle+\langle T^*-T_{k+1}, V_{k+1}\rangle\\ &+\alpha\langle Y_{k+1}-Y_k, P_{k+1}+Y_{k+1}-Y^*\rangle+\gamma\langle Z_{k+1}-Z_k, S_{k+1}+Z_{k+1}-Z^*\rangle\le 0.\end{aligned} \tag{3.35}$$

Noting that

$$\begin{aligned}
&2\langle M^* - M_{k+1}, P_{k+1}\rangle + 2\langle N^* - N_{k+1}, Q_{k+1}\rangle \\
=&\frac{1}{\alpha}\left(\|M_{k+1} - M^*\|^2 - \|M_k - M^*\|^2\right) + \alpha\|P_{k+1}\|^2 \\
&+ \frac{1}{\beta}\left(\|N_{k+1} - N^*\|^2 - \|N_k - N^*\|^2\right) + \beta\|Q_{k+1}\|^2,
\end{aligned} \tag{3.36}$$

$$\begin{aligned}
&2\langle S^* - S_{k+1}, U_{k+1}\rangle + 2\langle T^* - T_{k+1}, V_{k+1}\rangle \\
=&\frac{1}{\gamma}\left(\|S_{k+1} - S^*\|^2 - \|S_k - S^*\|^2\right) + \gamma\|U_{k+1}\|^2 \\
&+ \frac{1}{\delta}\left(\|T_{k+1} - T^*\|^2 - \|T_k - T^*\|^2\right) + \delta\|V_{k+1}\|^2,
\end{aligned} \tag{3.37}$$

$$\begin{aligned}
&\alpha\|P_{k+1}\|^2 + 2\alpha\langle Y_{k+1} - Y_k, P_{k+1} + Y_{k+1} - Y^*\rangle \\
=&\alpha\|P_{k+1} + Y_{k+1} - Y_k\|^2 + \alpha\left(\|Y_{k+1} - Y^*\|^2 - \|Y_k - Y^*\|^2\right)
\end{aligned} \tag{3.38}$$

and

$$\begin{aligned}
&\gamma\|U_{k+1}\|^2 + 2\alpha\langle Z_{k+1} - Z_k, U_{k+1} + Z_{k+1} - Z^*\rangle \\
=&\gamma\|U_{k+1} + Z_{k+1} - Z_k\|^2 + \gamma\left(\|Z_{k+1} - Z^*\|^2 - \|Z_k - Z^*\|^2\right).
\end{aligned} \tag{3.39}$$

Combing the inequality (3.35) with the equalities (3.36), (3.37), (3.38), (3.39) and the definition of Ψ_k in (3.22), we have

$$\Psi_{k+1} \leq \Psi_k - \beta\|Q_{k+1}\|^2 - \delta\|V_{k+1}\|^2 - \alpha\|P_{k+1} + Y_{k+1} - Y_k\|^2 - \gamma\|U_{k+1} + Z_{k+1} - Z_k\|^2,$$

which means that the inequality (3.23) holds. The proof is completed. □

Theorem 3.1 implies that the sequence $\{\Psi_k\}$ is a nonnegative monotone decreasing with low bounded. Hence, the limit of the sequence $\{\Psi_k\}$ exists which means that the limit of the sequences $\{Y_k\}$, $\{Z_k\}$, $\{M_k\}$, $\{N_k\}$, $\{S_k\}$ and $\{T_k\}$ exist, and $Q_{k+1} \to 0, V_{k+1} \to 0$, $P_{k+1} + Y_{k+1} - Y_k \to 0$, $U_{k+1} + Z_{k+1} - Z_k \to 0$ as $k \to \infty$. Furthermore, from $P_{k+1} + Y_{k+1} - Y_k = AX_{k+1} - Y_k \to 0$, we conclude that the limit of the sequence $\{X_k\}$ exists. Assume that $X_k \to X^*$, $Y_k \to Y^*$, $Z_k \to Z^*$, $M_k \to M^*$, $N_k \to N^*$, $S_k \to S^*$ and $T_k \to T^*$ as $k \to \infty$, then (3.26), (3.28), (3.31) are hold by taking limit respectively. These imply that X^*, M^*, N^*, T^*, S^* satisfy (2.3), (2.4) and (2.5). In addition, X^*, Y^* and Z^* satisfy conditions (2.6)-(2.9) since $P_{k+1} + Y_{k+1} - Y_k \to 0$, $Q_{k+1} \to 0$, $U_{k+1} + Z_{k+1} - Z_k \to 0$ and $V_{k+1} \to 0$ as $k \to \infty$. Noting that $X_k \in \mathbb{SR}^{n\times n}$, so is X^*. In conclusion, it follows from Theorem 2.1 that the matrix triple $[X^*, Y^*, Z^*]$ is a solution of the problem (2.1), and so is a solution of the matrix nearness problem (1.1). In addition, noting that the subjective function of (2.1) is strictly convex and the corresponding constrained set is closed and convex, the matrix triple $[X^*, Y^*, Z^*]$ is the unique solution of (2.1). Hence the sequence generated by Algorithm 1 converges to the unique solution of the matrix nearness problem (1.1). These results can be described as the following Theorem 3.2.

Theorem 3.2. *Assume that $\{X_k\}$ is a sequence generated by Algorithm 1 with any initial matrices Y_0, Z_0, M_0, N_0, S_0 and T_0 and parameters $\alpha, \beta, \gamma, \delta > 0$, then the sequence $\{X_k\}$ converges to the unique solution of the matrix nearness problem* (1.1).

For Algorithm 2, we have similar results and we omit the detail here in order to save space.

4. Numerical experiments

In this section, we conduct some numerical experiments to illustrate the feasibility and effectiveness of the ADMM-based methods for solving the matrix nearness problem (1.1) in the sense of iteration numbers(denoted by 'IT') and the iteration CPU time (denoted by 'CPU'). In our numerical experiments, we compare Algorithm 1 and 2 with the method proposed in Peng et al. [18], denoted by Algorithm PENG_CG. Our computations are all implemented in MATLAB 6.5.1 with a machine precision 2.22×10^{-16} on a personal computer with 2.70GHz central processing unit (Intel(R) Core(TM) i7-7500U), 8GB memory and Windows 10 operating system. The stopping criterion for the tested methods is

$$\sqrt{\|A_1X_kB_1 - C_1\|^2 + \|A_2X_kB_2 - C_2\|^2} \leq \varepsilon,$$

or the maximum iterations numbers $k_{max} = 15000$ is exceeded. Here, ε is a given tolerance and $\varepsilon = 10^{-8}$ in this paper.

We set parameters $\alpha = \beta = \gamma = \delta = 10$ and the initial matrices $Y_0, Z_0, M_0, N_0, S_0, T_0$ are chosen as zeros matrices in the Algorithm 1 and 2. $\hat{X}_1$ in Algorithm PENG_CG is chosen as zero matrix. $\bar{X}$ in the Algorithm 1 and 2 is chosen as random symmetric matrix, which is the same with X_0 in Algorithm PENG_CG.

For the three methods, the matrices $A_1, A_2, B_1, B_2, \bar{X}, C_1, C_2$ are given as follows (in MATLAB style): $A_1 = randn(m_1, n), A_2 = randn(m_2, n), B_1 = randn(n, p_1), B_2 = randn(n, p_2), \bar{X} = randn(n, n), C_1 = A_1X_{00}B_1, C_2 = A_2X_{00}B_2$ with $X_{00} = W + W^T$ and $W = randn(n, n)$. Here the intention of choosing matrices C_1, C_2 in this way is to ensure that the matrix nearness problem (1.1) is solvable. As shown in Table 1, we exhibit the average iteration CPU time(in seconds) and the average iteration numbers based on 12 tests. We repeated 10 times for each test with the same randomly generated matrices $A_1, A_2, B_1, B_2, C_1, C_2$ according to each problem size.

After analysing the results exhibited in Table 1 and many other unreported tests for the problem (1.1), we get the following findings: When $m_1, m_2, p_1, p_2 \gg n$, both iteration CPU time and iteration numbers of Algorithm 1 and 2 are less than those of Algorithm PENG_CG. That is to say, Algorithm 1 and 2 are more effective than Algorithm PENG_CG. But when the values of m_1, m_2, p_1, p_2, n are close to each other, Algorithm PENG_CG is obviously more effective than Algorithm 1 and 2. When $m_1 > p_1, m_2 > p_2$ and $m_1, m_2 \gg n$, Algorithm 2 performs most effectively, and Algorithm 1 gets the most excellent performance when $m_1 < p_1, m_2 < p_2$ and $p_1, p_2 \gg n$.

5. Conclusions

In this paper, by proposing two new equivalent forms of the matrix nearness problem (1.1), we developed two ADMM-based iterative methods to solve it. The global convergence of the proposed methods are studied. Numerical results demonstrate that, in most situations, our methods are more effective than the method proposed

Table 1. Average iteration CPU time and average iteration numbers for three methods

m_1, n, p_1, m_2, n, p_2	Algorithm 1 IT(CPU)	Algorithm 2 IT(CPU)	PENG_CG IT(CPU)
60,40,60,40,40,40	95(0.2816)	93(0.2920)	**87**(**0.0962**)
80,80,80,80,80,80	180(1.9082)	177(1.9588)	**135**(**0.4246**)
100,60,400,80,60,400	**12**(**0.2119**)	31(0.8040)	53(1.1200)
100,80,500,100,80,600	**12**(**0.4280**)	40(2.2092)	64(2.8196)
200,80,600,100,80,600	**12**(**0.6145**)	34(2.5548)	48(3.5943)
200,200,200,200,200,200	169(22.7652)	171(24.4079)	**153**(**7.2288**)
400,80,100,400,80,100	39(1.5937)	**13**(**0.3860**)	65(0.9228)
400,80,400,500,80,600	**13**(**1.6541**)	13(1.7225)	32(5.3922)
500,80,100,500,80,100	38(1.9574)	**12**(**0.4155**)	64(1.1279)
500,80,500,500,80,500	13(1.7635)	**13**(**1.7453**)	31(5.4235)
600,80,600,500,80,500	**12**(**2.0026**)	12(2.0268)	30(6.9631)
600,100,200,500,100,200	17(1.9751)	**12**(**1.0496**)	49(2.7036)

in Peng et al. [18]. In addition, the new methods can abstractly be further extended to solve the nearness symmetric solution associated with matrix equations $(A_1XB_1, \cdots, A_kXB_k) = (C_1, \cdots, C_k)$ with $k \geq 3$ [21].

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