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ZERO-HOPF BIFURCATION IN NUCLEAR SPIN GENERATOR SYSTEM*

Renxiang Shi[1,†] and Jiang Yu[2]

Abstract By computing we obtain that $P_1(0,0,1)$ is a zero-Hopf equilibrium point of nuclear spin generator system. We prove that there exist two families of nuclear spin generator system which has the zero-Hopf equilibrium point $P_1(0,0,1)$. Furthermore we prove that a limit cycle bifurcated from $P_1(0,0,1)$ by averaging method of one order and second order respectively.

Keywords Averaging method, nuclear spin generator, limit cycle, zero-Hopf bifurcation.

MSC(2010) 34C23, 34C29.

1. Introduction

In this paper, we study zero-Hopf bifurcation of nuclear spin generator system:

$$\dot{x} = -\beta x + y,\ \dot{y} = -x - \beta y(1 - kz),\ \dot{z} = \beta(\alpha(1 - z) - ky^2) \tag{1.1}$$

where x, y, z are the components of the nuclear magnetization vector in the X, Y, Z direction respectively , $\alpha \in (0, 1]$ for physical consideration $\beta \geq 0, \beta\alpha \geq 0$ are linear damping term, βk is proportional to the amplifier gain in the voltage feedback. It was first introduced by Sherman [12] for describing the nuclear spin generator, he give condition on existence, uniqueness and stability of periodic solution within some parametric region. Besides there have been many works about the dynamics of this system. In Li etc [8], the author discuss the Hopf bifurcation by Liapunov-Schmidt reduction. In Valls [14], the author study the integrability of system (1.1). In [13,15], chaos in nuclear spin generator system is discussed. In [7,10], the authors study the synchronization problem of this system.

Zero-Hopf equilibrium point refer to the equilibrium point of 3-dimensional autonomous differential system which has eigenvalue $\pm i\omega, 0$. Generally zero-Hopf bifurcation refer to 2-parameter unfolding of a 3-dimensional autonomous differential system at zero-Hopf equilibrium point. As the two parameters varies in a small neighborhood of the isolated equilibrium point, the unfolding has different dynamics, where the isolated equilibrium point have the eigenvalues $\pm i\omega, 0$ when the two parameters take zero value. As we know, one hand classical Hopf bifurcation refer

†The corresponding author. Email address:srxahu@aliyun.com(R. Shi)
[1]Department of Mathematics, Nanjing University of Chinese Medicine, 210023 Nanjing, China
[2]Department of Mathematics, Shanghai Jiaotong University, 200240 Shanghai, China
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to the case that eigenvalue can not be zero, so it is not feasible to zero-Hopf equilibrium, on the other hand averaging method is matured tool for finding periodic solution of nonlinear system, for example, Han [6] study the maximum number of periodic solution of piecewise smooth periodic equation by the first order average.

Recently by averaging method, there exist many works about zero-Hopf bifurcation [2–5, 9], several kinds of system are discussed such as Chua's system, Fizhugh system and Lorenz systems. But in my opinion, there is not work about zero-Hopf bifurcation of nuclear spin generator system, so we should analysis the zero-Hopf bifurcation of this system with averaging method.

The averaging method is introduced by Lagrange and Laplace. Faton, Krylov and Bogolinbov develop the method both in practice and theory. For convenience, we should give the description of averaging theory in section 3, which is similar to the appendix in Euzebio etc [4].

By calculating, if $\alpha(\beta^2(k-1)-1) > 0$ hold, nuclear spin generator system (1.1) has the equilibrium point $P_1 = (0, 0, 1)$ and other equilibriums point:
$P_2(\frac{\sqrt{\alpha(\beta^2(k-1)-1)}}{\beta^2 k}, \frac{\sqrt{\alpha(\beta^2(k-1)-1)}}{\beta k}, \frac{\beta^2+1}{\beta^2 k}), P_3(-\frac{\sqrt{\alpha(\beta^2(k-1)-1)}}{\beta^2 k}, -\frac{\sqrt{\alpha(\beta^2(k-1)-1)}}{\beta k}, \frac{\beta^2+1}{\beta^2 k})$
We shall only discuss the zero-Hopf bifurcation about equilibrium $P_1(0, 0, 1)$ since we conclude that the points P_2 and P_3 are not zero-Hopf equilibrium point.

In the following we should give the main results of this paper. First we show that $P_1(0, 0, 1)$ is a zero-Hopf equilibrium point.

Proposition 1.1. *There are two families of nuclear spin generator system, where* $P_1(0, 0, 1)$ *is a zero-Hopf equilibrium point, namely* (1) $\beta \neq 0$, $\alpha = 0, k = 2$ (2) $\beta = 0$.

Remark 1.1. when $\beta = 0$, system (1.1) become

$$\dot{x} = y, \ \dot{y} = -x, \ \dot{z} = 0, \tag{1.2}$$

so system (1.2) has a family no-isolated zero-Hopf equilibrium point $(0, 0, b)(b \in R)$.

In the following theorems, we study the zero-Hopf bifurcation producing periodic orbit from zero-Hopf equilibrium point $(0, 0, 1)$.

Theorem 1.1. *Let* $\beta = \sqrt{1-\omega^2} + \beta_1\varepsilon$, $\alpha = \alpha_1\varepsilon$, $k = 2 + k_1\varepsilon$. *If* $1 - \omega^2 > 0$, $\alpha_1 k_1 > 0$. *The nuclear spin generator system (1.1) have a zero-Hopf bifurcation at equilibrium* $(0, 0, 1)$ *and a limit cycle appears at this equilibrium point* $(0, 0, 1)$ *when* $\varepsilon > 0$ *sufficiently small.*

Theorem 1.2. *Let* $\beta = \beta_1\varepsilon + \beta_2\varepsilon^2$, $\alpha = \alpha_0 + \alpha_1\varepsilon + \alpha_2\varepsilon^2$ *and* $k = k_0 + k_1\varepsilon + k_2\varepsilon^2$, *if* $\alpha_0 = 0, k_0 = 2$, $\alpha_1 k_1 > 0$ *and* $\beta_1 \neq 0$. *The nuclear spin generator system (1.1) have a zero-Hopf bifurcation at equilibrium* $(0, 0, 1)$ *and a limit cycle appear at this equilibrium* $(0, 0, 1)$ *when* $\varepsilon > 0$ *sufficiently small.*

In section 2, Proposition 1.1, Theorems 1.1 and 1.2 are proved respectively. In the appendix we give the averaging theory of first and second order.

2. The proof of the main results

In this section we give the proofs of the results presented in section 1.

Proof of proposition 1.1. The characteristic polynomial of linear part of system (1.1) at the singular point $(0,0,1)$ is

$$p_1 = \lambda^3 + \beta(-k+\alpha+2)\lambda^2 + (\beta^2(-k\alpha-k+2\alpha+1)+1)\lambda + \beta\alpha(\beta^2(-k+1)+1).$$

Imposing that $p_1(\lambda) = \lambda(\lambda^2+\omega^2)$. Hence $\beta(-k+\alpha+2)=0$, $\beta\alpha(\beta^2(-k+1)+1)=0$, $\beta^2(-k\alpha-k+2\alpha+1)+1=\omega^2$,
so we have

(i) $\beta \neq 0, \alpha = 0, k = 2, 1-\beta^2 = \omega^2$;

(ii) $\beta = 0, 1 = \omega^2$;

(iii) $\beta \neq 0, \alpha \neq 0, k = \alpha+2, \beta^2(-k+1)+1 = 0, \beta^2(-k\alpha-k+2\alpha+1)+1 = \omega^2$.

Form (iii), we obtain

$$\beta^2(-k\alpha-k+2\alpha+1)+1 = \omega^2 = -\frac{(k-2)^2}{k-1} < 0,$$

which is a contradiction for $k-1 = \alpha+2 > 0$, so statement (1) and (2) follows. □

Proof of theorem 1.1. Let

$$\beta = \sqrt{1-\omega^2} + \varepsilon\beta_1, \quad \alpha = \varepsilon\alpha_1, \quad k = 2+\varepsilon k_1$$

with $\varepsilon > 0$ sufficiently small, system (1.1) becomes

$$\begin{cases} \dot{x} = -(\sqrt{1-\omega^2}+\varepsilon\beta_1)x + y, \\ \dot{y} = -x - (\sqrt{1-\omega^2}+\varepsilon\beta_1)y + (2+\varepsilon k_1)(\sqrt{1-\omega^2}+\beta_1\varepsilon)zy, \\ \dot{z} = -(2+\varepsilon k_1)(\sqrt{1-\omega^2}+\varepsilon\beta_1)y^2 - \varepsilon\alpha_1(\sqrt{1-\omega^2}+\varepsilon\beta_1)z + \varepsilon\alpha_1(\sqrt{1-\omega^2}+\varepsilon\beta_1). \end{cases} \tag{2.1}$$

First we translate equilibrium $(0,0,1)$ to origin by $x = X, y = Y, z = Z+1$. Then do the rescaling of variables $X = \varepsilon U$, $Y = \varepsilon V$, $Z = \varepsilon W$, system (2.1) becomes

$$\begin{cases} \dot{U} &= -\sqrt{1-\omega^2}U + V + \varepsilon F_{11}(U,V,W), \\ \dot{V} &= -U + \sqrt{1-\omega^2}V + \varepsilon F_{21}(U,V,W) + \varepsilon^2 F_{22}(U,V,W) + \varepsilon^3 F_{23}(U,V,W), \\ \dot{W} &= \varepsilon F_{31}(U,V,W) + \varepsilon^2 F_{32}(U,V,W) + \varepsilon^3 F_{33}(U,V,W), \end{cases} \tag{2.2}$$

where

$F_{11} = -\beta_1 U$,

$F_{21} = V(\beta_1 + \sqrt{1-\omega^2}(2W+k_1))$, $F_{22} = V(W(k_1\sqrt{1-\omega^2}+2\beta_1)+\beta_1 k_1)$, $F_{23} = V\beta_1 k_1 W$,

$F_{31} = -\sqrt{1-\omega^2}(2V^2+\alpha_1 W)$, $F_{32} = -(V^2(2\beta_1+k_1\sqrt{1-\omega^2})+\beta_1\alpha_1 W)$, $F_{33} = -\beta_1 k_1 V^2$,

when $\varepsilon = 0$, we shall write the linear part at the origin of (2.2) into real Jordan normal form

$$\begin{pmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

For doing that we do the linear change of variables

$$(U,V,W)^\top = P(u,v,w)^\top, \tag{2.3}$$

where

$$P = \begin{pmatrix} -\omega & \sqrt{1-\omega^2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

system (2.2) becomes

$$\begin{cases} \dot{u} &= -v\omega + \varepsilon f_{11}(u,v,w) + O(\varepsilon^2), \\ \dot{v} &= u\omega + \varepsilon f_{21}(u,v,w) + O(\varepsilon^2), \\ \dot{w} &= \varepsilon f_{31}(u,v,w) + O(\varepsilon^2), \end{cases} \tag{2.4}$$

where

$$\begin{aligned} f_{11}(u,v,w) &= -\beta_1 u + v(\frac{1}{\omega}(2\beta_1\sqrt{1-\omega^2} + k_1 + 2w) - \omega(k_1+2w)), \\ f_{21}(u,v,w) &= v\beta_1 + v\sqrt{1-\omega^2}(k_1+2w), \\ f_{31}(u,v,w) &= -(\alpha_1 w + 2v^2)\sqrt{1-\omega^2}. \end{aligned}$$

Writing system (2.4) in cylindrical coordinates (θ, r, w) by $u = r\cos\theta, v = r\sin\theta$ and $w = w$, then we take θ as independent variable:

$$\begin{cases} \frac{dr}{d\theta} &= \varepsilon R_{11}(\theta,r,w) + O(\varepsilon^2), \\ \frac{dw}{d\theta} &= \varepsilon R_{12}(\theta,r,w) + O(\varepsilon^2), \end{cases} \tag{2.5}$$

where

$$\begin{aligned} R_{11}(\theta,r,w) = \frac{1}{\omega}(&-r\beta_1(\sin^2\theta - \cos^2\theta) + r\sqrt{1-\omega^2}\sin^2\theta(k_1+2w) \\ &+ r\sin\theta\cos\theta(\frac{1}{\omega}(k_1+2w+2\beta\sqrt{1-\omega^2}) - \omega(k_1+2w))), \\ R_{12}(\theta,r,w) = -&\frac{\sqrt{1-\omega^2}}{\omega}(w\alpha_1 + 2r^2\sin^2\theta). \end{aligned}$$

Denote $t=\theta$, $\vec{x}=(r,w)\in(0,+\infty)\times R$, $T=2\pi$ and $\vec{R}_1(\theta,r,w)=(R_{11}(\theta,r,w), R_{12}(\theta,r,w))$. Then apply the averaging Theorem 3.1 shown in Appendix, we obtain the averaging function of first order

$$\vec{R}_{10}(r,w) = (R_{101}(r,w), R_{102}(r,w)),$$

where

$$R_{101}(r,w) = \frac{1}{2\pi}\int_0^{2\pi} R_{11}(\theta,r,w)d\theta = \frac{r\sqrt{1-\omega^2}(k_1+2w)}{2\omega},$$

$$R_{102}(r,w)=\frac{1}{2\pi}\int_0^{2\pi}R_{12}(\theta,r,w)d\theta=-\frac{\sqrt{1-\omega^2}(r^2+w\alpha_1)}{\omega}.$$

Then system $R_{101}(r,w)=R_{102}(r,w)=0$ have an solution (r^*,w^*) with $r^*>0$, where

$$(r^*,w^*)=\left(\sqrt{\frac{k_1\alpha_1}{2}},-\frac{k_1}{2}\right),$$

the solution exist if $\alpha_1 k_1>0$. It is easy to verify that

$$A:=\frac{\partial(R_{101},R_{102})}{\partial(r,w)}|_{(r,w)=(r^*,w^*)}=\begin{pmatrix}0 & \frac{\sqrt{2\alpha_1k_1(1-\omega^2)}}{2\omega}\\ -\frac{\sqrt{2\alpha_1k_1(1-\omega^2)}}{\omega} & -\frac{\alpha_1\sqrt{1-\omega^2}}{\omega}\end{pmatrix},$$

which implies the Jacobian (3.2) take the value $\det(A)=\frac{(1-\omega^2)k_1\alpha_1}{\omega^2}\neq 0$.

The rest proof of the theorem 1.1 follows immediately from Theorem 3.1, if we show that periodic solution corresponding to equilibrium (r^*,w^*) provides a periodic orbit bifurcated from the origin of system (2.2) at $\varepsilon>0$ sufficiently small.

Theorem 3.1 guarantees for $\varepsilon\neq 0$ sufficiently small, the existence of a periodic orbit $(r(\theta,\varepsilon),w(\theta,\varepsilon))$ corresponding to the point (r^*,w^*) for system (2.5), such that $(r(0,\varepsilon),w(0,\varepsilon))\to(r^*,w^*)$ when $\varepsilon\to 0$. So system (2.4) has a periodic solution $(u(\theta,\varepsilon)=r(\theta,\varepsilon)\cos\theta,v(\theta,\varepsilon)=r(\theta,\varepsilon)\sin\theta,w(\theta,\varepsilon))$ when ε sufficiently small. Consequently, system (2.2) has the period solution $(U(\theta),V(\theta),W(\theta))$ obtained through the change of variables (2.3). Finally, for $\varepsilon\neq 0$ sufficiently small, system (2.1) has a period solution $(x(\theta,\varepsilon),y(\theta,\varepsilon),z(\theta,\varepsilon)=\varepsilon U(\theta),\varepsilon V(\theta),\varepsilon W(\theta)+1)$ which goes to the $(0,0,1)$ when $\varepsilon\to 0$. Thus it is a periodic solution starting at the zero-Hopf equilibrium point $(0,0,1)$ when $\varepsilon=0$ sufficiently small. So we complete the proof of Theorem 1.1. □

Proof of theorem 1.2. Let

$$\beta=\beta_1\varepsilon+\beta_2\varepsilon^2,\ \alpha=\alpha_0+\alpha_1\varepsilon+\alpha_2\varepsilon^2,\ k=k_0+k_1\varepsilon+k_2\varepsilon^2,$$

with $\varepsilon>0$ sufficiently small, system (1.1) becomes

$$\begin{cases}\dot{x}=-(\beta_1\varepsilon+\beta_2\varepsilon^2)x+y,\\ \dot{y}=-x-(\beta_1\varepsilon+\beta_2\varepsilon^2)y+(k_0+k_1\varepsilon+k_2\varepsilon^2)(\beta_1\varepsilon+\beta_2\varepsilon^2)zy,\\ \dot{z}=(\alpha_0+\alpha_1\varepsilon+\alpha_2)(\beta_1\varepsilon+\beta_2\varepsilon^2)-(\alpha_0+\alpha_1\varepsilon+\alpha_2\varepsilon^2)(\beta_1\varepsilon+\beta_2\varepsilon^2)z\\ \qquad -(k_0+k_1\varepsilon+k_2\varepsilon^2)(\beta_1\varepsilon+\beta_2\varepsilon^2)y^2.\end{cases}\tag{2.6}$$

The linear part of system (2.6) has real Jordan normal form when $\varepsilon=0$, so there is no need for rescaling this system. But for making the computing more easy, we translating $(0,0,1)$ to origin by $x=X,y=Y,z=Z+1$ and doing the rescaling

variables $(X, Y, Z) = (\varepsilon U, \varepsilon V, \varepsilon W)$, we obtain

$$\begin{cases} \dot{U} &= V + \varepsilon G_{11}(U,V,W) + \varepsilon^2 G_{12}(U,V,W), \\ \dot{V} &= -U + \varepsilon G_{21}(U,V,W) + \varepsilon^2 G_{22}(U,V,W) + \varepsilon^3 G_{23}(U,V,W) \\ &\quad + \varepsilon^4 G_{24}(U,V,W) + \varepsilon^5 G_{25}(U,V,W) \\ \dot{W} &= \varepsilon G_{31}(U,V,W) + \varepsilon^2 G_{32}(U,V,W) + \varepsilon^3 G_{33}(U,V,W) + \varepsilon^4 G_{34}(U,V,W) \\ &\quad + \varepsilon^5 G_{35}(U,V,w), \end{cases} \tag{2.7}$$

where

$$\begin{aligned} &G_{11}(U,V,W) = -\beta_1 U, G_{12}(U,V,W) = -\beta_2 U, \\ &G_{21}(U,V,W) = \beta_1 V(k_0 - 1), \\ &G_{22}(U,V,W) = V(\beta_1(k_1 + k_0 W) + \beta_2(k_0 - 1)), \\ &G_{23}(U,V,W) = V(\beta_2(k_0 W + k_1) + \beta_1(k_1 W + k_2)), \\ &G_{24}(U,V,W) = V(k_1\beta_2 W + k_2(\beta_1 W + \beta_2)), \\ &G_{25}(U,V,W) = k_2\beta_2 V W, \\ &G_{31}(U,V,W) = -\beta_1\alpha_0 W, \\ &G_{32}(U,V,W) = -\beta_2\alpha_0 W - \beta_1(\alpha_1 W + k_0 V^2), \\ &G_{33}(U,V,W) = -(\beta_2(\alpha_1 W + k_0 V^2) + \beta_1(k_2 V^2 + \alpha_2 W)), \\ &G_{34}(U,V,W) = -(\beta_1 k_2 V^2 + \beta_2(-\alpha_2 W + k_1 V^2)), \\ &G_{35}(U,V,W) = -\beta_2 k_2 V^2, \end{aligned}$$

when $\varepsilon = 0$, the linear part at origin of (2.7) has real Jordan normal form

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We write system (2.7) into cylindrical coordinate (θ, r, w) by $U = r\cos\theta, V = r\sin\theta, W = w$, and then take θ as new independent variable:

$$\begin{cases} \frac{dr}{d\theta} &= \varepsilon R_{11}(\theta,r,w) + \varepsilon^2 R_{21}(\theta,r,w) + O(\varepsilon^3), \\ \frac{dw}{d\theta} &= \varepsilon R_{12}(\theta,r,w) + \varepsilon^2 R_{22}(\theta,r,w) + O(\varepsilon^3), \end{cases} \tag{2.8}$$

where

$$\begin{aligned} &R_{11}(r,w) = \beta_1 r(1 - k_0 + k_0\cos^2\theta), \quad R_{12}(r,w) = \alpha_0\beta_1 w, \\ &R_{21}(r,w) = r(\beta_1^2 k_0 \sin\theta\cos\theta(1 - k_0\sin^2\theta) - \sin^2\theta(k_1\beta_1 + k_0\beta_2 + \beta_1 k_0 w - \beta_2) + \beta_2\cos^2\theta), \\ &R_{22}(r,w) = \beta_1(\alpha_1 w + \beta_1\alpha_0 k_0 w\sin\theta\cos\theta + k_0 r^2\sin^2\theta) + \alpha_0\beta_2 w. \end{aligned}$$

From Theorem 3.1, we have the averaging function

$$(R_{101}(r,w), R_{102}(r,w)) = (r\beta_1(1 - \frac{k_0}{2}), \alpha_0\beta_1 w).$$

If $k_0 \neq 2$, from the equation $(R_{101}(r,w), R_{102}(r,w)) = 0$, we have $r = 0$. It is not proper because r must be positive. In order to apply the averaging of second order, we need $R_{101} \equiv 0$ and $R_{102} \equiv 0$. So we take $k_0 = 2$ and $\alpha_0 = 0$. From Theorem 3.1 in Appendix, we have

$$R_{201}(r,w) = -\frac{1}{4}\beta_1 r(2k_1 + 4w),$$
$$R_{202}(r,w) = \beta_1(r^2 + \alpha_1 w).$$

The equation $R_{201}(r,w) = R_{202}(r,w) = 0$ has a solution $(\sqrt{\frac{k_1\alpha_1}{2}}, -\frac{k_1}{2})$ when $k_1\alpha_1 > 0$. In this situations the Jacobian is

$$\begin{pmatrix} 0 & -\frac{\sqrt{2}}{2}\beta_1\sqrt{k_1\alpha_1} \\ 2\beta_1\sqrt{k_1\alpha_1} & \beta_1\alpha_1 \end{pmatrix}.$$

Then its Jacobian(3.2) takes the value $\beta_1^2\alpha_1 k_1 \neq 0$. The rest of the proof of theorem 1.2 is similar to Theorem 1.1. □

Remark 2.1. We give the description about stability of bifurcated periodic solution in Appendix.

3. Appendix

In this appendix, we give the averaging theory of first and second order, which is also used for finding period solutions in nonlinear differential equation. In Sanders etc [11], we could see more about averaging theory, besides we could finding proof of averaging theory in Buica etc [1].

Theorem 3.1. *Consider the differential system*

$$\dot{x}(t) = \varepsilon R_1(t,x) + \varepsilon^2 R_2(t,x) + \varepsilon^3 R_3(t,x,\varepsilon) \tag{3.1}$$

where $R_1, R_2 : R \times D \to R^n$, $R_3 : R \times D \times (-\varepsilon_{R_1}, \varepsilon_{R_1}) \to R^n$ are continuous function, T-period in the first variable. D is an open subset of R^n, we assume the hypotheses hold.

(i) $R_1(t,.), R_2(t,.) \in C^1(D)$ for all $t \in R$, R_1, R_2, R_3, D_xR_1 and D_xR_2 are locally Lipschitz with respect to x. R_3 is differentiable with respect to ε, we define $R_{10}, R_{20} : D \to R^n$ as

$$R_{10}(z) = \frac{1}{T}\int_0^T R_1(s,z)ds$$
$$R_{20}(z) = \frac{1}{T}\int_0^T [D_z R_1(s,z)\int_0^s R_1(t,z)dt + R_2(s,z)]ds$$

(ii) For $V \subset D$ an open and bounded set and for each $\varepsilon \in (-\varepsilon_{R_1}, \varepsilon_{R_1})\backslash(0)$, there exist $p \in V$ such that $R_{10}(p) + \varepsilon R_{20}(p) = 0$ and

$$det(\frac{\partial(R_{10} + \varepsilon R_{20})}{\partial z})|_{z=p} \neq 0. \tag{3.2}$$

Then for $|\varepsilon| > 0$ sufficiently small, there exist a $T-$period solution $\varphi(.,\varepsilon)$ of system (3.1) such that $\varphi(0,\varepsilon) \to p$ when $\varepsilon \to 0$.

If the function R_{10} and R_{20} is not identically zero , then the zeros of $R_{10}+\varepsilon R_{20}$ are mainly the zeros of R_{10} for ε sufficiently small. In this case, the theorem is called averaging theory of first order.

If the function R_{10} is identically zero and R_{20} is not identically zero, then the zeros of $R_{10}+\varepsilon R_{20}$ are the zeros of R_{20}. The theorem is called the averaging theory of second order.

In averaging theory of first order, for the averaged differential equation in D:

$$\dot{y} = \varepsilon R_{10}(y), y(0) = x_0, \tag{3.3}$$

where $R_{10}(y) = \frac{1}{T}\int_0^T R_1(t,y)dt$. The stability of periodic solution $\varphi(t,\varepsilon)$ could be given by the stability of equilibrium point p of averaged system (3.3). That is the equilibrium point p have the stability behavior of Poincaré map associated to periodic solution $\varphi(t,\varepsilon)$.

In averaging theory of second order, where $R_{10} \equiv 0$ and R_{20} is no-identically zero, the stability of periodic solution $\varphi(t,\varepsilon)$ is given by the stability of equilibrium point p of averaging system

$$\dot{y} = \varepsilon^2 R_{20}(y), y(0) = x_0,$$

where the equilibrium point p associated with the Poincaré map about periodic solution $\varphi(t,\varepsilon)$.

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