Journal of Applied Analysis and Computation Website:http://www.jaac-online.com
Volume 11, Number 1, February 2021, 271–286 DOI:10.11948/20190361

GENERAL HIGH-ORDER BREATHER SOLUTIONS, LUMP SOLUTIONS AND MIXED SOLUTIONS IN THE (2+1)-DIMENSIONAL BIDIRECTIONAL SAWADA-KOTERA EQUATION*

Jiao-Jiao Dong[1], Biao Li[1,†] and Manwai Yuen[2]

Abstract In this paper, we investigate some interesting solution structures of the (2+1)-dimensional bidirectional Sawada-Kotera (bSK) equation. We obtain general high-order breather solutions by utilizing the Hirota's bilinear method united with the perturbation expansion technique. Taking a long-wave limit of the obtained breather solutions and then making particular parameter constraints, smooth rational solutions are generated, which include high-order lumps and mixed solutions consisting of lumps and stripe. In order to easily explore the dynamical behaviors, some plots are presented to analyse these solutions.

Keywords Hirota's bilinear, (2+1)-dimensional bSK equation, breather solutions, lump solutions, mixed solution.

MSC(2010) 35Q53, 37K40.

1. Introduction

Nonlinear evolution equations (NLEEs) can depict a wide variety of nonlinear phenomena in plasma physics and nonlinear optics, and many others, especially in mathematical physics. It is very important to seek exact solutions of NLEEs to study of many complex mathematical physical phenomena and other nonlinear engineering problems [5, 9, 19, 20, 22, 25]. In order to find the solutions to NLEEs and to examine the physical properties of these solutions, many powerful methods are used to deal with NLEEs, such as Darboux transformation [6, 13], Hirota's bilinear method [7, 14], inverse scattering transform method [1], the Lie group method [4, 8] and so on. Such a variety of the searchs may lead to new developments of analytical solution. We usually obtain several kinds of solutions, such as soliton, peakons,

[†]The corresponding author. Email address:libiao@nbu.edu.cn(B. Li)
[1]School of Mathematics and Statistics, Ningbo University, Ningbo 315211, China
[2]Department of Mathematics and Information Technology,The Education University of Hong Kong, 10 Po Ling Road, Tai Po, New Territories, Hong Kong, China
[*]J. Dong and B. Li were supported by National Natural Science Foundation of China (Nos. 11775121 and 11435005) and K. C. Wong Magna Fund in Ningbo University. M. Yuen was partially supported by the Research Output Prize of Dean's Research Fund 2018-19 (No. ROP-9) from The Education University of Hong Kong.

kinks, breather, lump, rogue wave solutions and many others.

Recently, the study of breather solution and lump solution of physical equations have attracted a growing amount of attention. In 2015, Ma gave a way to get the lump solution of a evolution equation by using the Hirota's bilinear method [16]. By the way and symbolic computations, many kinds of integrable equations obtain their lump solutions, such as the Kadomtsev-Petviashvili equation [16], the BKP equation [27], the (2+1)-dimensional Sawada-Kotera equation [29], reduced generalized KP equation [26], (2+1)-dimensional Boussinesq equation [15], (2+1)-dimensional Burgers equation [26] and so on. Moreover, many scholars have explored the mixed solution of nonlinear evolution equations(NLEEs), which include lump-stripe, lump-kink, lump-soliton, and other mixed solutions [2,10,18,21,24,28,30].

In this paper, we study a (2+1)-dimensional bidirectional Sawada-Kotera (bSK) equation [11]

$$\begin{aligned}&-45u^2u_x-15uu_{xxx}-15u_xu_{xx}-15uu_t-15u_x(\partial_x)^{-1}(u_t)+5(\partial_x)^{-1}u_{tt}\\&-5u_{xxt}-u_{xxxxx}+9u_y=0,\end{aligned} \tag{1.1}$$

where

$$(\partial_x)^{-n}=\left(\frac{d}{dx}\right)^{-n}, \tag{1.2}$$

it was formulated there as a bidirectional generalization of the Sawada-Kotera (SK) equation

$$u_t+45u^2u_x-15u_xu_{xx}-15uu_{xxx}+u_{xxxxx}=0. \tag{1.3}$$

Because of its connection with the SK equation Eq. (1.3), the bSK Eq. (1.1) also belongs to the Kadomtsev-Petviashvili equations of B type (BKP) hierarchy [3]. The approximate and exact solutions of the bSK Eq. (1.1) are discussed in [12,17]. In [17], with the help of the resulting Riccati equation, Ma and Geng derived some exact solutions, Darboux and Bӑklund transformations of the bSK Eq. (1.1). In [12], Lai and Cai calculated Adomian polynomials and obtained several classes of explicit solutions of bSK Eq. (1.1) by means of Maple, which include solitary wave solutions, doubly periodic solutions, two-soliton solutions. However, there are only a few works to study other dynamic behavior of Eq. (1.1).

To author's knowledge, many works have been done for bSK Eq. (1.1), but higher-order breather and lump solutions for bSK Eq. (1.1) have not been investigated before. The main purpose of this paper is to construct general higher-order breather and lump solutions, and to explore their fascinating dynamical behaviors. The structure of the paper is arranged as follows: In Section 2, we construct the bilinear form of the bSK equation. In Section 3, 4, and 5, based on the obtained bilinear form, the general higher-order breathers are derived. By taking a long-wave limit of these obtained breather solution, general higher-order lumps are generated, and dynamics of these solutions are discussed. In Section 6 and 7, mixed solutions consisting of lumps and stripe are yielded. Our results are summarised in the final section.

2. Bilinear formalism to (2+1)-D bSK equation

In this section, let us consider a dependent variable transformation,

$$u=2(\ln f)_{xx}, \tag{2.1}$$

substituting Eq. (2.1) into Eq. (1.1), we obtain the following Hirota's bilinear form [21]

$$\begin{aligned}&(5D_t^2-5D_x^3D_t-D_x^6+9D_xD_y)f\cdot f\\=&10ff_{tt}-10f_t^2-10f_{xxxt}f+30f_xf_{xxt}-30f_{xx}f_{xt}+10f_{xxx}f_t-18f_{xy}f\\&-18f_xf_y-2f_{xxxxxx}f+12f_{xxxxx}f_x-30f_{xx}f_{xxxx}+20f_{xxx}f_{xxx}\\=&0,\end{aligned} \tag{2.2}$$

here f is a real function with respect to variable x, y, and t, and the operates

$$\begin{aligned}&(D_t^mD_x^nD_y^k)f\cdot g\\=&\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\right)^m\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right)^n\left(\frac{\partial}{\partial y}-\frac{\partial}{\partial y'}\right)^k\\&f(x,y,t)\cdot g(x',y',t')\mid x=x',y=y',t=t'.\end{aligned} \tag{2.3}$$

3. First-order breather and lump solutions of (2+1)-D bSK equation

We first begin from the two-soliton solution. We assume f in Eq. (2.1) to have the following formal form:

$$f=1+\epsilon f_1+\epsilon^2f_2, \tag{3.1}$$

with

$$\begin{aligned}f_1&=\exp(\eta_1)+\exp(\eta_2),\\f_2&=\exp(\eta_1+\eta_2+A_{12}),\end{aligned} \tag{3.2}$$

where

$$\eta_s=k_sx+\omega_sy+p_st+\phi_s,\ s=1,2, \tag{3.3}$$

and k_s, ω_s, p_s and ϕ_s are undetermined complex parameters. We substitute f in (3.1) into (2.2) and collect the power orders of ϵ, then, we can obtain the following equations at the ascending power orders of ϵ:

$$\begin{aligned}&\epsilon^0:(5D_t^2-5D_x^3D_t-D_x^6+9D_xD_y)(1\cdot1+1\cdot1)=0,\\&\epsilon^1:(5D_t^2-5D_x^3D_t-D_x^6+9D_xD_y)(1\cdot f_1+f_1\cdot1)=0,\\&\epsilon^2:(5D_t^2-5D_x^3D_t-D_x^6+9D_xD_y)(1\cdot f_2+f_1\cdot f_1+f_2\cdot1)=0.\end{aligned} \tag{3.4}$$

By solving the three equations in (3.4), we can obtain the formula

$$\exp(A_{12})=-\frac{(k_1-k_2)^6+5(w_1-w_2)(k_1-k_2)^3-9(p_1-p_2)(k_1-k_2)-5(w_1-w_2)^2}{(k_1+k_2)^6+5(w_1+w_2)(k_1+k_2)^3-9(p_1+p_2)(k_1+k_2)-5(w_1+w_2)^2}, \tag{3.5}$$

and the dispersion relation

$$9p_sk_s-k_s^6-5k_s^3w_s+5w_s^2=0. \tag{3.6}$$

In order to guarantee the correspond breather solutions to be real functions, there are two constrain conditions for a valid calculation: (1) The wave number k_s must be pure imaginary numbers and must satisfy $k_1=-k_2$; (2) The angular frequency p_s must be real numbers and $w_1 = \overline{w_2}$. Under these constrain conditions, we can take parameter constraints

$$w_1 = \overline{w_2} = w, k_1 = -k_2 = ik, \phi_1 = \phi_2 = \phi_0, \tag{3.7}$$

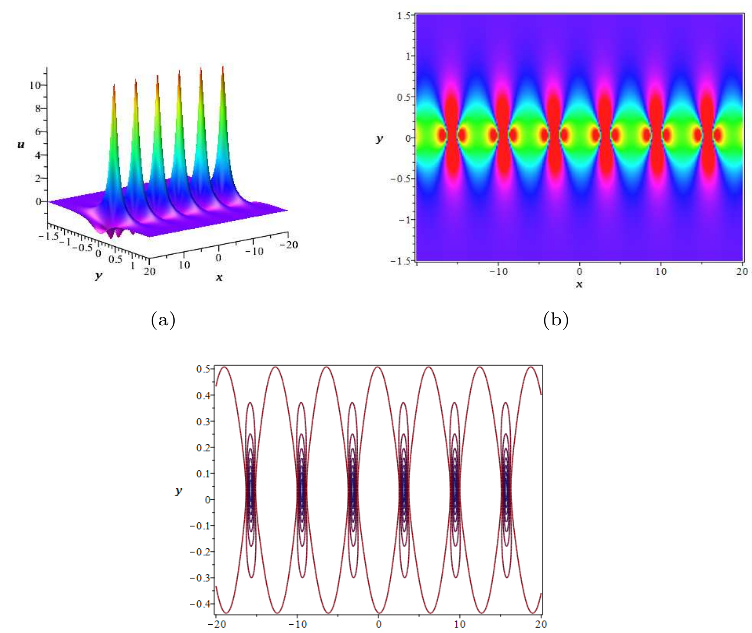
where w, k, ϕ_0 are freely real parameters. Therefore, we can obtain $\eta_1 = \overline{\eta_2}$, and the function f in (3.1) can be rewritten as

$$f = \sqrt{M}\cosh(\Theta) + \cos(kx), \tag{3.8}$$

where

$$\begin{aligned} &\Theta = \omega y + pt + t_0, \\ &t_0 = \sqrt{M}e^{\phi_0}, \\ &M = \frac{(-10w^2 - 2k^6)i}{9k}. \end{aligned} \tag{3.9}$$

The evolution of u in the (x, y)-plane is shown in Fig.1. Because k_s are pure imaginary and real, u is only periodic in x direction and the period is $\frac{2\pi}{k}$.



(a) (b) (c)

Figure 1. (Color online) First-order breather solution u of Eq. (1.1) with the parameters $K_1 = -i, K_2 = i, W_1 = 2 - \frac{3}{2}i, W_2 = 2 + \frac{3}{2}i$; (a)3-D plot, (b)density plot, (c)contour plot.

To generate rational solution, we can take a long-wave limit with the provision

$$w_s = W_s\epsilon, k_s = K_s\epsilon, \epsilon \to 0, e^{\phi_s} = -1, s = 1, 2, \tag{3.10}$$

in (3.8). The expression of f is given as follows

$$f = \theta_1\theta_2 + \theta_0, \tag{3.11}$$

here

$$\begin{aligned}\theta_s &= \frac{9tK_sW_s + 9xK_s^2 - 5yW_s^2}{9K_s}, s = 1, 2.\\ \theta_0 &= -\frac{6K_2^2K_1^2(K_1W_2 + K_2W_1)}{(K_1W_2 - K_2W_1)^2}.\end{aligned} \tag{3.12}$$

To guarantee the corresponding rational solutions to be lump solutions, we take parameters $K_1 = \overline{K_2}, W_1 = \overline{W_2}$ in Eq. (3.12). Then, the final expression of rational solutions u reads

$$u = \frac{2\theta_0\theta_{1x}\theta_{2x} - \theta_1^2\theta_{2x}^2 - \theta_2^2\theta_{1x}^2}{(\theta_1\theta_2 + \theta_0)^2}. \tag{3.13}$$

This lump solution u possesses three critical points

$$\begin{aligned}A_1 &= \left(-\frac{tW_1W_2}{K_1W_2 + K_2W_1}, \frac{9}{5}\frac{K_1K_2t}{K_1W_2 + K_2W_1}\right),\\ A_2 &= \left(\frac{3\sqrt{-2K_1K_2(K_1W_2 + K_2W_1)^3} - tW_1W_2(K_1W_2 - K_2W_1)}{(K_1W_2 + K_2W_1)(K_1W_2 - K_2W_1)}, \frac{9}{5}\frac{K_1K_2t}{K_1W_2 + K_2W_1}\right),\\ A_3 &= \left(\frac{3\sqrt{-2K_1K_2(K_1W_2 + K_2W_1)^3} + tW_1W_2(K_1W_2 - K_2W_1)}{(K_1W_2 + K_2W_1)(K_1W_2 - K_2W_1)}, \frac{9}{5}\frac{K_1K_2t}{K_1W_2 + K_2W_1}\right),\end{aligned} \tag{3.14}$$

which are derived by solving $\frac{\partial u}{\partial x} = 0$ and $\frac{\partial u}{\partial y} = 0$. Because of the analysis of three critical points at second-order derivatives $\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial x \partial y}$, we can obtain u has one local maximum (point A_1), and two minimum points (point A_2, A_3). The patterns of lump solution is shown in Fig.2.

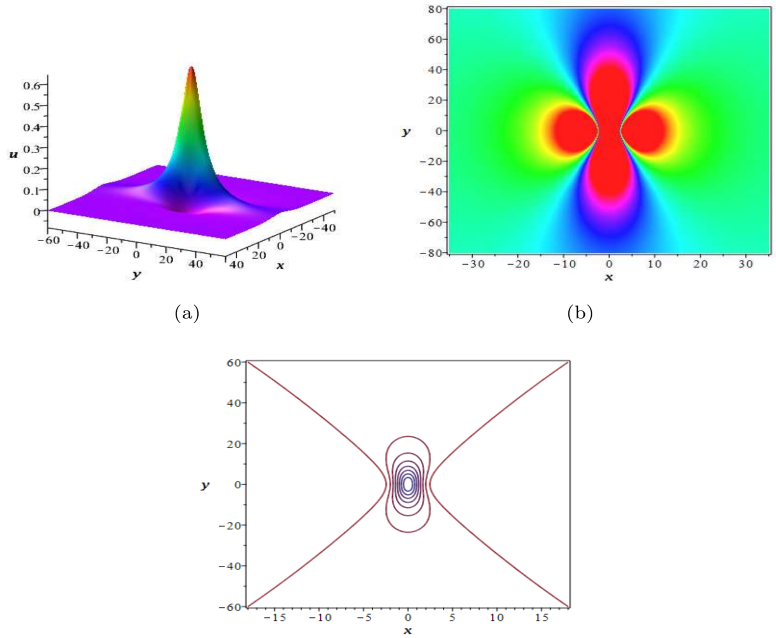
4. Second-order breather and lump solutions of (2+1)-D bSK equation

We can derive the second-order breather solutions by similar procedures to the first-order breather, and assume that f has the following expansions in terms of ϵ:

$$f = 1 + \epsilon f_1 + \epsilon^2 f_2 + \epsilon^3 f_3 + \epsilon^4 f_4, \tag{4.1}$$

then, we substitute (4.1) into bilinear equation (2.2) and collect the coefficients of ϵ; Equations of different orders of ϵ would be yielded. However, it is troublesome to solve these equations, but worthy to get the second-order breather solutions. We present the corresponding results as follow:

$$\begin{aligned}f_1 &= \sum_j e^{\eta_j}, f_2 = \sum_{j<s} e^{\eta_j+\eta_s+A_{js}},\\ f_3 &= \sum_{j<s<l} e^{\eta_j+\eta_s+\eta_l+A_{jsl}}, f_4 = e^{\eta_1+\eta_2+\eta_3+\eta_4+A},\end{aligned} \tag{4.2}$$



(a) (b)

(c)

Figure 2. (Color online) First-order lump solution u of equation (1.1) with the parameters $K_1 = 1-i, K_2 = 1+i, W_1 = 1, W_2 = 1$: ($a$) 3-D plot; ($b$) density plot; ($c$) contour plot.

where

$$\begin{aligned}
&\eta_j = k_j x + w_j y + p_j t + \phi_j,\\
&e^{A_{js}} = -\frac{(k_j-k_s)^6 + 5(w_j-w_s)(k_j-k_s)^3 - 9(p_j-p_s)(k_j-k_s) - 5(w_j-w_s)^2}{(k_j+k_s)^6 + 5(w_j+w_s)(k_j+k_s)^3 - 9(p_j+p_s)(k_j+k_s) - 5(w_j+w_s)^2}\\
&e^{A_{jst}} = e^{A_{js}} e^{A_{jl}} e^{A_{sl}},\\
&e^{A} = \prod_{j<s} e^{A_{js}},
\end{aligned} \tag{4.3}$$

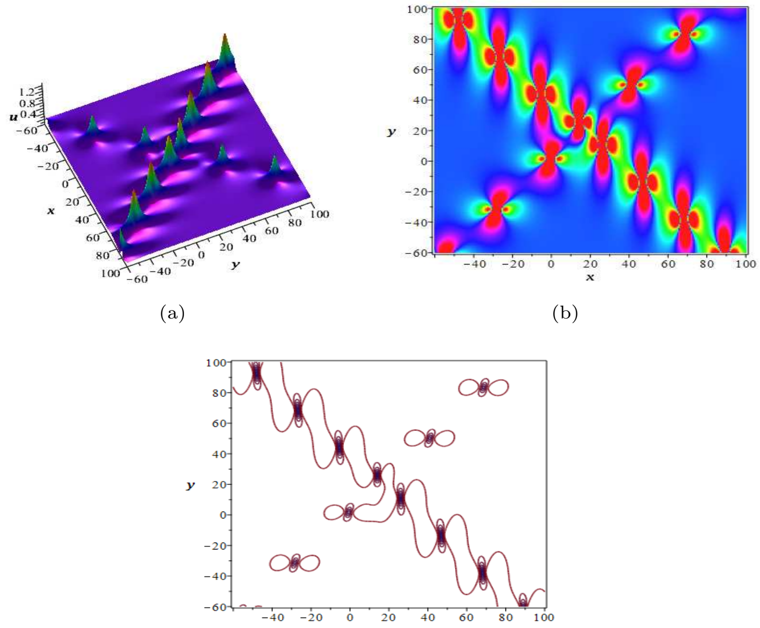
where and the dispersion relation

$$9p_j k_j - k_j^6 - 5k_j^3 w_j + 5w_j^2 = 0, \tag{4.4}$$

here $j = 1,2,3,4$; $s = 1,2,3,4$; $l = 3,4$. The second-order breather solutions also have two constrain conditions: (1) The wave number k_s must be pure imaginary numbers and satisfy $k_1 = -k_2$; (2) The angular frequency p_s must be real number and $w_1 = \overline{w_2}$. Under these constrain conditions, we can obtain $\eta_1 = \overline{\eta_2}$, $\eta_3 = \overline{\eta_4}$. The picture of second-order breather solution is shown in Fig.3:

We carry out similar procedures obtained the first-order lump solutions to generate second-order lump solutions, and take the long wave limit $(p, k \to 0)$ with the provision

$$e^{\phi_0} = -1, \tag{4.5}$$



(a) (b)

(c)

Figure 3. (Color online) Second-order breather solution u of equation (1.1) with the parameters $K_1 = \frac{1}{7} + \frac{i}{9}, K_2 = \frac{1}{7} - \frac{i}{9}, W_1 = \frac{2}{9}, W_2 = \frac{2}{9}, K_3 = -\frac{1}{8} + \frac{i}{4}, K_4 = -\frac{1}{8} - \frac{i}{9}, W_3 = \frac{1}{10} - \frac{2}{9}i, W_4 = \frac{1}{10} + \frac{2}{9}i, \phi_1 = 0, \phi_2 = 0, \phi_3 = 4, \phi_4 = 4$: ($a$) the 3-D plot; ($b$) density plot; ($c$) contour plot.

in (4.1). Then the expansion of f is expressed as follow

$$f = \prod_j \theta_j + \sum a_{j<s} \prod_{k \neq j,s} \theta_k + \sum_{j<s} \prod a_{js}, \tag{4.6}$$

where

$$\begin{aligned} \theta_j &= \frac{9tK_jW_j + 9xK_j^2 - 5yW_j^2}{9K_j}, \\ \theta_0 &= -\frac{6K_s^2K_j^2(K_jW_s + K_sW_j)}{(K_jW_s - K_sW_j)^2}, \end{aligned} \tag{4.7}$$

here $j = 1,2,3,4,\ s = 1,2,3,4$, and $j < s$, K_j, W_j are complex parameters. In order to ensure the corresponding rational solutions to be lump solutions, the constrain conditions are $K_1 = \overline{K_2}, K_3 = \overline{K_4}, W_1 = \overline{W_2}, W_3 = \overline{W_4}$. This solution with parameters

$$\begin{aligned} &K_1 = 2 - \frac{3}{2}i, K_2 = 2 + \frac{3}{2}i, K_3 = 2 - \frac{3}{2}i, K_4 = 2 + \frac{3}{2}i, \\ &W_1 = W_2 = 2, W_3 = W_4 = 3, \end{aligned} \tag{4.8}$$

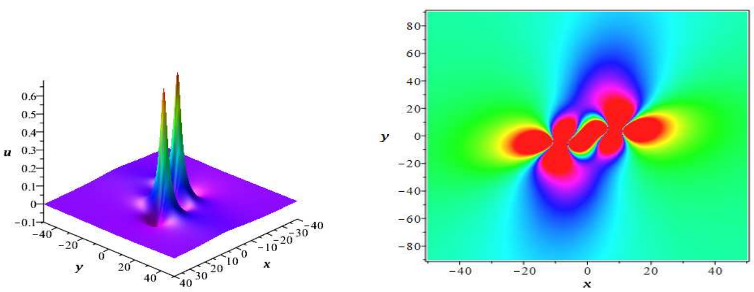
is shown as the following form

$$u = 2\ln(f_2)_{xx}, \tag{4.9}$$

with

$$\begin{aligned} f_2 =& 36t^4 + 120t^3x - 64t^3y + \frac{709}{4}t^2x^2 - \frac{400}{3}t^2xy + \frac{11344}{225}t^2y^2 + 125tx^3 \\ & - \frac{1036}{9}tx^2y + \frac{8288}{135}txy^2 - \frac{512}{27}ty^3 + \frac{625}{16}x^4 - \frac{455}{18}x^3y + \frac{6781}{2025}x^2y^2 \\ & - \frac{2912}{405}xy^3 + \frac{256}{81}y^4 + \frac{6184085555}{2090916}t^2 - \frac{26602957225}{9409122}xy - \frac{29340955220}{4704561}ty \\ & - \frac{31443971875}{8363664}x^2 - \frac{1765346875}{1045458}tx + \frac{424945749199}{211705245}y^2 + \frac{349834765625}{1568187}. \end{aligned} \tag{4.10}$$

The second-order lump solution is shown in Fig.4.



(a) (b)

Figure 4. (Color online) Second-order lump solution u of equation (1.1) with the parameters $K_1 = 2 - \frac{3}{2}i, K_2 = 2 + \frac{3}{2}i, K_3 = 2 - \frac{3}{2}i, K_4 = 2 + \frac{3}{2}i, W_1 = 2, W_2 = 2, W_3 = 3, W_4 = 3$: (a) 3-D plot; (b) density plot.

5. Higher-order breather and lump solutions of (2+1)-D bSK equation

Higher-order breather and lump solutions are generalized in this section. Firstly, we assume f has the folowing higher-order expansions in terms of ϵ:

$$f = 1 + \epsilon f_1 + \epsilon^2 f_2 + \epsilon^3 f_3 + \cdots \epsilon^n f_n \cdots. \tag{5.1}$$

Then, we substitute f into the bilinear equation (2.2) and collect the coefficients of ϵ; $2n+1$ equations of different orders of ϵ would be yielded. We can solve these equations and present the corresponding results as follow:

$$f = \sum_{\mu=0,1} e^{\sum_{j<s}^{(N)} \mu_j \mu_s A_{js} + \sum_{j<s}^{N} \mu_j \eta_j}, \tag{5.2}$$

where

$$\begin{aligned} \eta_j &= k_j x + w_j y + p_j t + \phi_j, \\ e^{A_{js}} &= -\frac{(k_j - k_s)^6 + 5(w_j - w_s)(k_j - k_s)^3 - 9(p_j - p_s)(k_j - k_s) - 5(w_j - w_s)^2}{(k_j + k_s)^6 + 5(w_j + w_s)(k_j + k_s)^3 - 9(p_j + p_s)(k_j + k_s) - 5(w_j + w_s)^2}, \end{aligned} \tag{5.3}$$

and

$$9p_jk_j - k_j^6 - 5k_j^3w_j + 5w_j^2 = 0, \tag{5.4}$$

here k_j, p_j, w_j, ϕ_j are arbitrary complex constants. The notation $\sum_{\mu=0}$ indicates over all possible combinations of $\mu_1 = 0,1, \mu_2 = 0,1, \cdots, \mu_N = 0,1$; the $\sum_{\mu=0}^{N}$ is over all possible combinations of the N elements with the special condition $j < s$.

The n-th order breather solutions can be generated from $2n$-soliton solutions and parameters must be taken conjugations in (5.2)

$$N = 2n, k_{n+j}^* = k_j, p_{n+j}^* = p_j, w_{n+j}^* = w_j, \phi_{n+j}^* = \phi_j, \tag{5.5}$$

and these parameters also satisfy the same restrictions as first-order breather solution, then n-breather solutions would be obtained.

We apply a long-wave limit to generate rational solutions, and take parameters in (5.2)

$$k_s = K_s\epsilon, p_s = P_s\epsilon, w_s = W_s\epsilon, \tag{5.6}$$

with the provision

$$e^{\phi_s} = -1, s = 1,2,\cdots N, \tag{5.7}$$

then taking the limit as $\epsilon \to 0$, the function f given by (5.2) is changed into pure rational function. General higher-order rational solutions of (2+1)-dimensional bSK equation can be presented as follow:

$$u = 2(\ln f_N)_{xx}, \tag{5.8}$$

where

$$\begin{aligned} f_N =& \prod_{j=1}^{N}\theta_j + \frac{1}{2}\sum_{j,k}^{N}\alpha_{jk}\prod_{l\neq j,k}^{N}\theta_l + \cdots \\ &+ \frac{1}{M!2^M}\sum_{j,k,\cdots,m,n}^{N}\overbrace{\alpha_{jk}\alpha_{sl}\cdots\alpha_{mn}}^{M}\prod_{P\neq j,k,\cdots m,n}^{N}\theta_p + \cdots, \end{aligned} \tag{5.9}$$

with

$$\begin{aligned} \theta_s &= \frac{9K_sW_st + 9K_s^2x - 5W_s^2y}{9K_s}, s = 1,2, \\ \alpha_{js} &= -\frac{6K_2^2K_1^2(K_1W_2 + K_2W_1)}{(K_1W_2 - K_2W_1)^2}, \end{aligned} \tag{5.10}$$

parameters j, k are positive integers and not large than N, but p_S, K_sW_s are complex constants.

In order to obtain third-order lump solution, we can consider the case of $N = 6$ in (5.9). We take special parameters:

$$\begin{aligned} &K_1 = (1+2i), K_2 = (1-2i), K_3 = (2 - \frac{3i}{2}), \\ &K_4 = (2 + \frac{3i}{2}), K_5 = (-\frac{2}{3} + 2i), K_6 = (-\frac{2}{3} - 2i), \\ &W_1 = W_2 = 2, W_3 = W_4 = \frac{5}{3}, \\ &W_5 = W_6 = -1, \phi_i = i\pi, i = 1,2,\cdots 6, \end{aligned} \tag{5.11}$$

then we can derive the third-order lump solution by valid calculation.

This solution u in the (x, y)-plane is illustrated in Fig.5. From the picture, it is seen that the maximum amplitudes of third-order lump solution u is up to 1.6 when these third-order lumps mix with each other (see (b) of Fig.5). At $t = -14, t = 14$, third-order lumps respectively separate, and the maximum amplitudes are below 1.6.

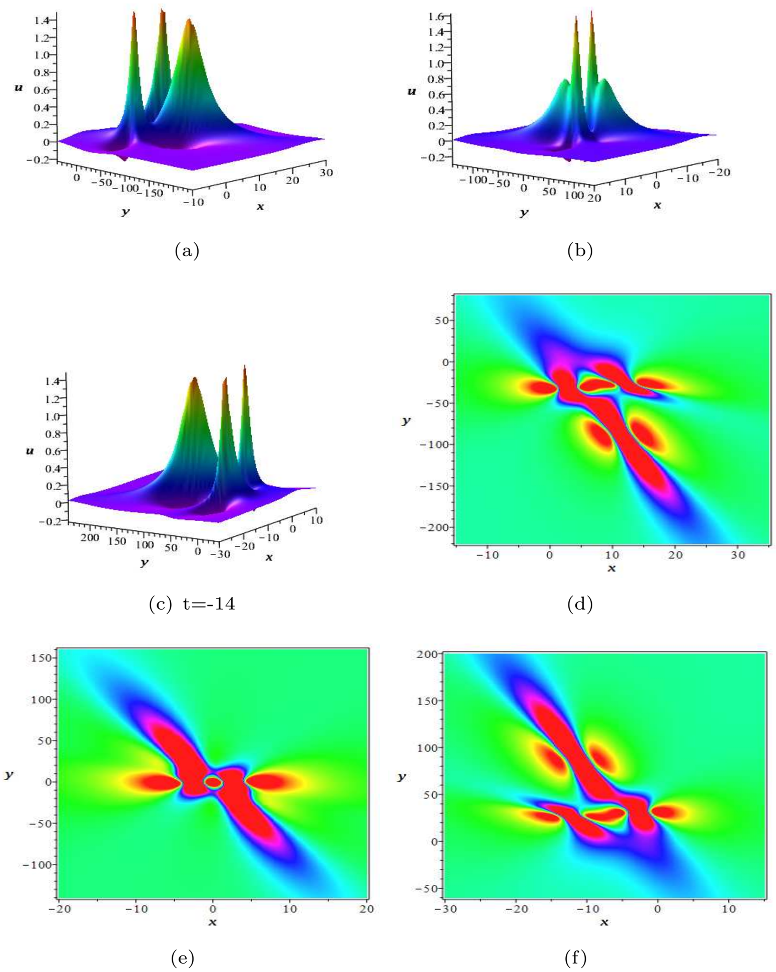


Figure 5. (Color online) Time evolution of three-order lump solution u of equation (1.1): $(a)(b)(c)$: 3-D plot; (d):density plot with $t = -14$, (e):density plot with $t = 0$, (f):density plot with $t = 14$.

6. Mixed solutions between first-order lump and a stripe of (2+1)-D bSK equation

The mixed solution between first-order lump and a stripe of the (2+1)-dimensional bidirectional Sawada-Kotera (bSK) equation will be studied in this section. For the purpose of obtaining the mixed solution between lump solution and a stripe, we first substitute $N=3$, $k_1=K_1\epsilon$, $k_2=K_2\epsilon$, $w_1=W_1\epsilon$, $w_2=W_2\epsilon$, $\phi_1=i\pi$, $\phi_2=i\pi$ into (5.9), then we derive f^*, expand the f^* in terms of ϵ at $\epsilon=0$ and take the limit while $\epsilon\to 0$. Therefore we can obtain another new f, the expression of f is:

$$f=\theta_1\theta_2+a_{12}+e^{\eta_3}(\theta_1\theta_2+a_{12}+\theta_1 a_{23}+\theta_2 a_{13}+a_{13}a_{23}), \tag{6.1}$$

here

$$\begin{aligned}&\theta_s=\frac{9K_sW_jt+9K_s^2x-5W_s^2y}{9K_s}, s=1,2,3,\\ &9p_sk_s-k_s^6-5k_s^3w_s+5w_s^2=0,\end{aligned} \tag{6.2}$$

and

$$a_{js}=\begin{cases}\frac{-6{K_s}^2{K_j}^2(K_jW_s+K_sW_j)}{(K_jW_s-K_sW_j)^2}, s<3\\ \frac{-6{k_s}^2{K_j}^2\left({k_s}^3K_j+K_jw_s+W_jk_s\right)}{{K_j}^2{k_s}^6+2\,{K_j}^2{k_s}^3w_s+{k_s}^4W_jK_j+{K_j}^2{w_s}^2-2\,W_jw_sk_sK_j+{W_j}^2{k_s}^2}, s=3.\end{cases} \tag{6.3}$$

In order to get the collision phenomenon, parameters must satisfy $K_1=\overline{K_2}$, $W_1=\overline{W_2}$, and k_3, w_3, ϕ_3 are real parameters. To illustrate the particular phenomena between first-order lump and a stripe, we select the following parameters

$$\begin{aligned}&K_1=3+3i, K_2=3-3i, W_1=5, W_2=5,\\ &k_3=1, w_3=2, \phi_3=0.\end{aligned} \tag{6.4}$$

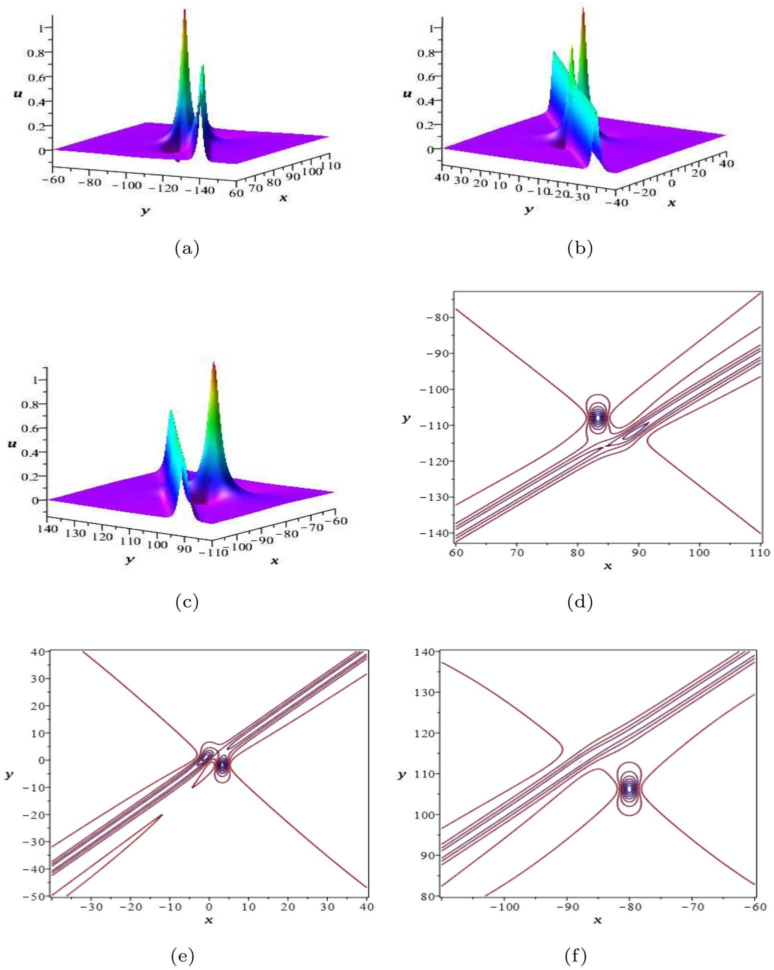
Then take the above parameters into f, we obtain the expression of f as follow:

$$\begin{aligned}f=&(25\,t^2+18\,x^2+\frac{15625\,y^2}{1458}+\frac{(3081264300\,x-2377518750\,y-14313477600)\,t}{102708810}\\ &-\frac{1678968\,x}{14089}+\frac{522500\,y}{14089}+\frac{20719476}{70445})\mathrm{e}^{2\,t+x-y}+25\,t^2+18\,x^2+\frac{15625\,y^2}{1458}\\ &+\frac{(3081264300\,x-2377518750\,y)\,t}{102708810}+\frac{324}{5}.\end{aligned} \tag{6.5}$$

The evolution of u is shown in the Fig.6. For $t<-100$, it is seen that first-order lump and a stripe move on the constant background respectively. When $t\to 0$, first-order lump and a stripe begin to collision and consistent. As time goes, first-order lump and a stripe become to separate and completely separate.

7. Mixed solutions between second-order lump and a stripe of (2+1)-D bSK equation

The mixed solution between second-order lump and a stripe of the (2+1)-dimensional bidirectional Sawada-Kotera (bSK) equation will be studied in this section. The method and process of deriving the mixed solution are the same as first-order lump and a stripe. We first take $k_1=K_1\epsilon$, $k_2=K_2\epsilon$, $w_1=W_1\epsilon$, $w_2=W_2\epsilon$, $\phi_1=i\pi$, $\phi_2=i\pi$ into (5.9), then expand the expression at $\epsilon=0$ and take the limit. We can



(a) (b) (c) (d) (e) (f)

Figure 6. (Color online) Profiles of mixed solution between first-order lump and a stripe with the parameters $K_1 = 3+3i, K_1 = 3-3i, W_1 = 5, W_2 = 5$ at times (a) $t = -100$; (b) $t = 0$; (c) $t = 100$; (d) contour plot at $t = -100$; (e) contour plot at $t = 0$; (f) contour plot at $t = 100$.

obtain

$$
\begin{aligned}
f =& \theta_1\theta_2\theta_3\theta_4 + a_{12}\theta_3\theta_4 + a_{13}\theta_2\theta_4 + a_{14}\theta_2\theta_3 + a_{23}\theta_1\theta_4 + a_{24}\theta_1\theta_3 + a_{34}\theta_1\theta_2 \\
& + a_{12}a_{34} + a_{13}a_{24} + a_{14}a_{23} + e^{\eta_5}(\theta_1\theta_2\theta_3\theta_4 + a_{45}\theta_1\theta_2\theta_3 + a_{35}\theta_1\theta_2\theta_4 \\
& + a_{25}\theta_1\theta_3\theta_4 + a_{15}\theta_2\theta_3\theta_4 + (a_{35}a_{45} + a_{34})\theta_1\theta_2 + (a_{25}a_{45} + a_{24})\theta_1\theta_3 \\
& + (a_{25}a_{35} + a_{23})\theta_1\theta_4 + (a_{15}a_{45} + a_{14})\theta_2\theta_3 + (a_{15}a_{35} + a_{13})\theta_2\theta_4 \\
& + (a_{15}a_{25} + a_{12})\theta_3\theta_4 + a_{12}a_{34} + a_{24}a_{15}a_{35} \\
& + (a_{25}a_{35}a_{45} + a_{23}a_{45} + a_{35}a_{24} + a_{25}a_{34})\theta_1 \\
& + (a_{15}a_{35}a_{45} + a_{14}a_{35} + a_{13}a_{45} + a_{15}a_{34})\theta_2 \\
& + (a_{15}a_{25}a_{45} + a_{21}a_{45} + a_{24}a_{15} + a_{14}a_{25})\theta_3 \\
& + (a_{15}a_{35}a_{25} + a_{23}a_{15} + a_{21}a_{35} + a_{25}a_{31})\theta_4 \\
& + a_{13}a_{24} + a_{14}a_{23} + a_{12}a_{35}a_{45} + a_{13}a_{25}a_{45} \\
& + a_{14}a_{25}a_{35} + a_{34}a_{15}a_{25} + a_{23}a_{15}a_{45} + a_{15}a_{25}a_{35}a_{45}),
\end{aligned}
\tag{7.1}
$$

here

$$\begin{aligned}\theta_s &= \frac{9K_sW_jt + 9K_s^2x - 5W_s^2y}{9K_s}, s = 1, 2, 3, 4 \\ 9p_sk_s &- k_s^6 - 5k_s^3w_s + 5w_s^2 = 0,\end{aligned} \tag{7.2}$$

and

$$a_{js} = \begin{cases} -6\,\frac{{K_s}^2{K_j}^2(K_jW_s+K_sW_j)}{(K_jW_s-K_sW_j)^2}, s < 5, \\ -6\,\frac{{k_s}^2{K_j}^2\left({k_s}^3K_j+K_jw_s+W_jk_s\right)}{{K_j}^2{k_s}^6+2\,{K_j}^2{k_s}^3w_s+{k_s}^4W_jK_j+{K_j}^2{w_s}^2-2\,W_jw_sk_sK_j+{W_j}^2{k_s}^2}, s = 5. \end{cases} \tag{7.3}$$

In order to get the mixed solutions between second-order lump and a stripe of bSK equation, there are restrictions for a valid calculation: $K_1 = \overline{K_2}$, $K_3 = \overline{K_4}$, $W_1 = \overline{W}_2$, $W_3 = \overline{W}_4$, and k_5, w_5, ϕ_5 are real parameters. We take the following parameters

$$\begin{aligned} &K_1 = \frac{2}{3}, K_2 = \frac{2}{3}, K_3 = -\frac{3}{4} + \frac{i}{3}, K_4 = -\frac{3}{4} - \frac{i}{3}, \\ &W_1 = \frac{3}{4} - i, W_2 = \frac{3}{4} + i, W_3 = -\frac{3}{4} - \frac{i}{4}, W_4 = -\frac{3}{4} + \frac{i}{4}, \\ &k_5 = 1, w_5 = \frac{1}{2}, \phi_5 = 0. \end{aligned} \tag{7.4}$$

Then we substitute above parameters into (6.4). We can derive the mixed solution by valid calculation.

The evolution of u is shown in Fig.7. From the plots, we can see that second-order lump and a stripe wave move on the constant background respectively at $t = -15$. Then the mixed phenomena happen near $t = 0$, second-order lump are in a collision with a stripe. After collision, second-order lump and a stripe become to separate.

8. Conclusion

In this paper, we investigated a (2+1)-dimensional bSK equation. Based on the bilinear form, we derive general high-order breathers by the Hirota's bilinear method. Taking a long-wave limit of the obtained breather solutions, several smooth rational solutions of the (2+1)-dimensional bSK equation are generated, which include n-th lump solution and the mixed solutions comprising lump solution and stripe. First-order breather and second-order breather solution are shown in Figs.1, 3. And their corresponding lump solutions are shown in Figs.2, 4. The third-order lump solution is shown in Fig.5. The mixed solution composed of lumps and a stripe are presented. The corresponding dynamical behaviors are analyzed in Figs.6, 7. Finally, it is expected that these results can also be applied to illustrate the dynamical behavior of nonlinear wave fields.

Acknowledgement

The authors are very grateful to the Editor and Reviewers for their valuable suggestions and useful comments.

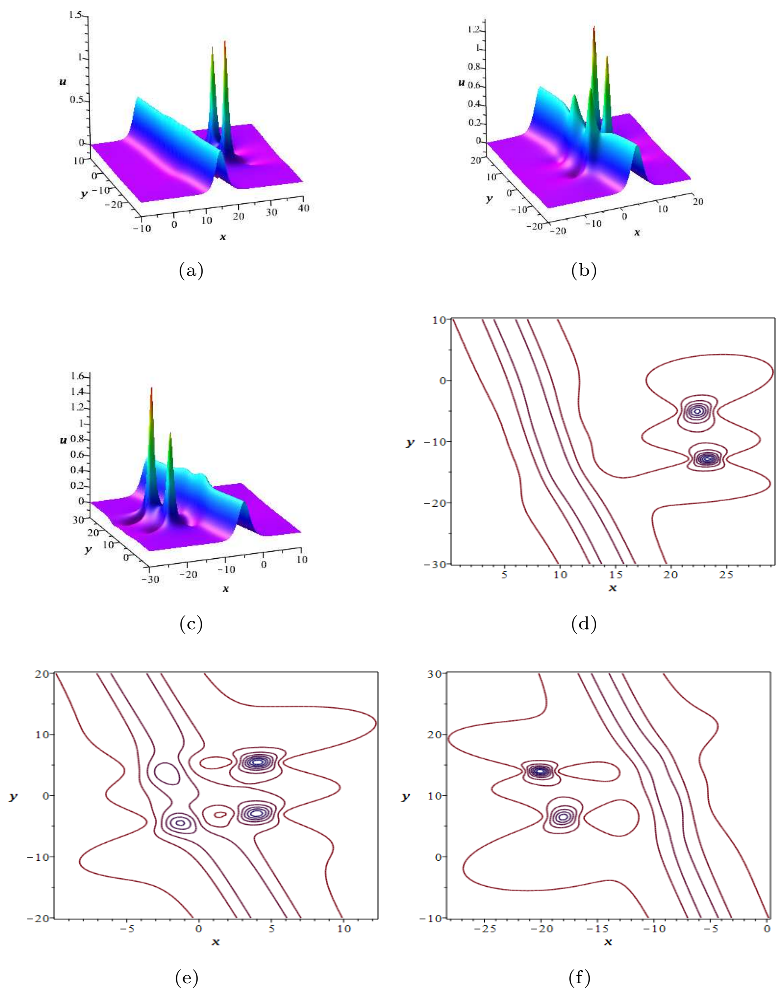


Figure 7. (Color online) Profiles of mixed solution between second-order lump and a stripe with the parameters $K_1 = \frac{2}{3}, K_2 = \frac{2}{3}, K_3 = -\frac{3}{4} + \frac{i}{3}, K_4 = -\frac{3}{4} - \frac{i}{3}, k_5 = 1, W_1 = \frac{4}{3} - i, W_2 = \frac{4}{3} + i, W_3 = -\frac{4}{3} - \frac{i}{4}, W_4 = -\frac{4}{3} + \frac{i}{4}, w_5 = \frac{1}{2}$; (d) contour plot at $t = -15$; (e) contour plot at $t = 0$; (f) contour plot at $t = 15$.

References

[1] M. J. Ablowitz, M. A. Ablowitz and P. A. Clarkson, *Solitons, nonlinear evolution equations and inverse scattering*, Cambridge university press, 1991.

[2] M. Chen, X. Li, Y. Wang and B. Li, *A Pair of Resonance Stripe Solitons and Lump Solutions to a Reduced (3 + 1)-dimensional Nonlinear Evolution Equation*, Commun. Theor. Phys., 2017, 67, 595.

[3] J. M. Dye and A. Parker, *A bidirectional Kaup-Kupershmidt equation and directionally dependent solitons*, J. Math. Phys., 2002, 43(10), 4921–4949.

[4] E. Date, M. Jimbo, M. Kashiwara and T. Miwa, *Transformation groups for*

soliton equations-Euclidean Lie algebras and reduction of the KP hierarchy, Publ. Res. Inst. Math. Sci., 1982, 18, 1077–1110.

[5] L. Feng and T. Zhang, *Breather wave,rogue wave and solitary wave solutions of a coupled nonlinear Schrödinger equation*, Appl. Math. Lett., 2018, 78, 133–140.

[6] C. Gu, H. Hu and Z. Zhou, *Darboux transformation in soliton theory and its geometric applications*, Shanghai. Sci. Tech. Publ. Shanghai, 1999, 1, 999.

[7] R. Hirota, *The Direct Method in Soliton Theory*, Cambridge University Press, New York(NY), 2004.

[8] M. Jimbo and T. Miwa, *Solitons and infinite dimensional Lie algebras*, Publ. Res. Inst. Math. Sci., 1983, 19, 943–1001.

[9] P. G. Kevrekidis and D. J. Frantzeskakis, *Solitons in coupled nonlinear models: a survey of recent developments*, Rev. Phys., 2016, 1, 140–153.

[10] Y. Liu, B. Li and H. An, *General high-order breathers, lumps in the (*$2+1$*)-dimensional Boussinesq equation*, Nonlinear Dyn., 2018, 92, 2061–2076.

[11] J. Lv and S. D. Bilige, *Lump solutions of a (*$2+1$*)-dimensional bSK equation*, Nonlinear Dyn., 2017, 90(3), 2119–2124.

[12] X. Lai and X. Cai, *Adomian decomposition method for approximating the solutions of the bidirectional Sawada-Kotera Equation*, Z. Naturf., 2010, 65a, 658–664.

[13] V. B. Matveev and M. A. Salle, *Darboux transformations and solitons*, Springer, Berlin, 1991.

[14] Y. Matsuno, *Bilinear Transformation Method*, Academic, New York(NY), 1984.

[15] H. Ma and A. Deng, *Lump solution of (*$2+1$*)-dimensional Boussinesq equation*, Commun. Theor. Phys., 2016, 65, 546.

[16] W. Ma, *Lump solutions to the Kadomtsev-Petviashvili equation*, Phys. Lett. A, 2015, 379, 1975–1978.

[17] Y. Ma and X. Geng, *Darboux and Bäcklund transformations of the bidirectional Sawada-Kotera equation*, Appl. Math. Comput., 2012, 218(12), 6963–6965.

[18] W. Ma, X. Yong and H. Q, *Diversity of interaction solutions to the (*$2+1$*)-dimensional Ito equation*, Comput. Math. Appl., 2018, 75, 289–295.

[19] W. Peng, S. Tian and T. Zhang, *Breather waves, high-order rogue waves and their dynamics in the coupled nonlinear Schrödinger equations with alternate signs of nonlinearities*, EPL, 2019. DOI: 10.1209/0295-5075/127/50005.

[20] W. Peng and S. Tian, *Dynamics of breather waves and higher-order rogue waves in a coupled nonlinear Schrödinger equation*, EPL (Europhysics Letters), 2018. DOI: 10.1209/0295-5075/123/50005.

[21] H. Sun and A. Chen, *Lump and lump-kink solutions of the (*$3+1$*)-dimensional Jimbo-Miwa and two extended Jimbo-Miwa equations*, Appl. Math. Lett., 2017, 68, 55–61.

[22] A. M. Wazwaz, *The generalized Kaup-Boussinesq equation: multiple soliton solutions*, Waves. Random. Complex. Media., 2015, 25, 473–481.

[23] H. Wang, *Lump and interaction solutions to the $(2+1)$-dimensional Burgers equation*, Appl. Math. Lett., 2018, 85, 27–34.

[24] Y. Wang, M. Chen, X. Li and B. Li, *Some interaction solutions of a reduced generalised $(3+1)$-dimensional shallow water wave equation for lump solutions and a pair of resonance solitons*, Z. Naturf. A, 2017, 72, 419–424.

[25] X. Yan, S. Tian, M. Dong, and T. Zhang, *Rogue Waves and Their Dynamics on Bright-Dark Soliton Background of the Coupled Higher Order Nonlinear Schrödinger Equation*, J. Phys. Soc. Japan, 2019. DOI: 10.7566/JPSJ.88.074004.

[26] J. Yu and Y. Sun, *Study of lump solutions to dimensionally reduced generalized KP equations*, Nonlinear Dyn., 2017, 87, 2755–2763.

[27] J. Yang and W. Ma, *Lump solutions to the BKP equation by symbolic computation*, Int. J. Mod. Phys. B, 2016, 30, 1640028.

[28] J. Yang, W. Ma and Z. Qin, *Lump and lump-soliton solutions to the $(2+1)$-dimensional Ito equation*, Anal. Math. Phys., 2018, 8, 427–436.

[29] H. Zhang and W. Ma, *Lump solutions to the $(2+1)$-dimensional Sawada-Kotera equation*, Nonlinear Dyn., 2017, 87, 2305–2310.

[30] X. Zhang and Y. Chen, *Rogue wave and a pair of resonance stripe solitons to a reduced $(3+1)$-dimensional Jimbo-Miwa equation*, Commun. Nonlinear. Sci. Numer. Simul., 2017, 52, 24–31.