Journal of Applied Analysis and Computation
Volume 11, Number 1, February 2021, 287–308
Website:http://www.jaac-online.com
DOI:10.11948/20190405

A DELAYED DISCRETE MULTI-GROUP NONLINEAR EPIDEMIC MODEL WITH VACCINATION AND LATENCY*

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Abstract A delayed discrete multi-group nonlinear epidemic model with vaccination and latency is derived by the application of a nonstandard finite difference scheme. It is proved that the extinction and persistence of the disease are determined by a threshold parameter in term of the basic reproduction number R. More precisely, when $R \leq 1$ the disease goes to extinction with the globally asymptotically stable disease-free equilibrium, while the disease is persistent with the globally asymptotically stable endemic equilibrium when $R > 1$.

Keywords Discrete epidemic model, multi-group, delay, vaccination, latency.

MSC(2010) 34K20, 34K25, 92D30.

1. Introduction

Mumps is an acute viral disease with high infectiousness that is transmitted via respiratory droplets. Over the last decade, childhood vaccination has resulted in a significant reduction in mumps incidence. However, the resurgences of mumps outbreaks are being frequently reported worldwide, including many high-income countries with high vaccine coverage such as USA [4,8], Australia [20], Scotland [21]. During multiple outbreaks, most cases diagnosed occurred adolescents and young adults among highly vaccinated populations [4,8,20,21], which supports that waning of vaccine-induced immunity has protracted mumps outbreak [20]. On the other hand, approximately 1/3 of mumps infections present only with nonspecific respiratory symptoms, namely, asymptomatic/inapparent infections, which are still contagious [1, 15]. Consequently, both the patients and the asymptomatic individuals are the infection sources. The incubation period after Mumps virus (MuV) infection varies from 12-25 days [4].

In mathematical epidemiology, the epidemic compartmental models can provide a better understanding of the disease transmission dynamics. In order to capture the mechanism of the recent global resurgence of mumps, we developed a multi-group nonlinear epidemic model with infinite distributed delays of vaccination and latency, asymptomatic infection (see [9]). In particular, when the probability distributions

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*The authors were supported by National Natural Science Foundation of China (Nos. 11871201; 11961023) and Natural Science Foundation of Hubei Province (No. 2019CFB241).

take the form of the step-functions, the model reduces to the following counterpart version of delay differential equations (DDEs)

$$\begin{cases} S_i' = q_i\Lambda_i - (\mu_i+\nu_i)S_i - \sum_{j=1}^{m} S_iG_{ij} + (p_i\Lambda_i + \nu_i S_i(t-\tau_i))e^{-\mu_i\tau_i}, \\ I_i' = \rho_i \sum_{j=1}^{m} S_i(t-\sigma_i)G_{ij}(t-\sigma_i)e^{-\mu_i\sigma_i} - (\mu_i+\delta_i+\gamma_i)I_i, \\ A_i' = \varrho_i \sum_{j=1}^{m} S_i(t-\sigma_i)G_{ij}(t-\sigma_i)e^{-\mu_i\sigma_i} - (\mu_i+\lambda_i+r_i)A_i, \\ G_{ij} := g_{ij}(I_j) + \mathcal{G}_{ij}(A_j), \quad \rho_i+\varrho_i = 1, \quad i \in \mathbb{N}^+ = \{1,2,\dots m\}, \end{cases} \tag{1.1}$$

where S_i, I_i, A_i respectively represent the numbers of susceptible, symptomatic and asymptomatic individuals at time t in the ith group. And the biological meanings of the parameters are described as follows. For certain a group $i \in \mathbb{N}^+$, Λ_i denotes the rate at which new recruits enter into group i; p_i, q_i are the fractions of vaccinated, unvaccinated new recruits in group i, respectively; the vaccination rate for susceptible individuals is denoted by ν_i; $\kappa_j \in [0,1]$ measures the infectivity factor for A_j class; the natural death rate of each class, the death rates due to symptomatic, asymptomatic infections are respectively denoted by μ_i, δ_i and λ_i; γ_i and r_i are the recovery rates of I_i and A_j classes. And β_{ij} measure the transmission rate between S_i and I_j. Meanwhile, the infectious force functions $g_{ij}(I_j)$ and $\mathcal{G}_{ij}(A_j)$ satisfy

$$\beta_{ij} = g_{ij}'(0), \quad \kappa_j\beta_{ij} = \mathcal{G}_{ij}'(0), \quad \mathcal{G}_{ij}(A_j) = \kappa_j g_{ij}(A_j), \quad \text{for} \quad \kappa_j > 0, \tag{1.2}$$

where the functions $g_{ij}(x)(i,j = 1,2,\dots,m)$ satisfy the following assumption (**A**):

(A) For $x \geq 0$, $g_{ij}(x) \geq 0$ with "=" holding iff $x = 0$, and $g_{ij}'(x) \geq 0$, $g_{ij}''(x) \leq 0$.

It is well known that the epidemic data are usually discrete in practice since they are collected daily, monthly or yearly, etc. Moreover, for the purpose of performing numerical simulations, it is necessary to discretize the continuous epidemic models of differential equations, and a variety of standard finite discretization (SFD) schemes such as Euler, Runge-Kutta methods have been utilized to obtain the approximate solutions of the models in [6, 17]. However, these SFD schemes may give rise to negative solutions or some spurious dynamical behaviours (e.g., converging to false equilibria or periodic cycles), even lead to numerical instabilities (NI), seeing [6,17]. Instead, Mickens [11–14] developed a robust nonstandard finite difference (NSFD) scheme that performs well in preserving as many dynamical properties of the original continuous models (i.e., ODEs, DDEs, PDEs) according to recent exploration of the scholars (e.g., [2,3,7,16,18,22–25]). As a result of the mentioned themes, we apply

NSFD scheme [11–14] to model (1.1) and obtain the discrete version as follows

$$\begin{cases} \dfrac{S_{i_{n+1}}-S_{i_n}}{\Delta t}=q_i\Lambda_i-(\mu_i+\nu_i)S_{i_{n+1}}-\sum\limits_{j=1}^{m}S_{i_{n+1}}G_{ij}^{n}+(p_i\Lambda_i+\nu_iS_{i_{n-l_i+1}})e^{-\mu_i\tau_i},\\ \dfrac{I_{i_{n+1}}-I_{i_n}}{\Delta t}=\rho_i\sum\limits_{j=1}^{m}S_{i_{n-\ell_i+1}}G_{ij}^{n-\ell_i}e^{-\mu_i\sigma_i}-(\mu_i+\delta_i+\gamma_i)I_{i_{n+1}},\\ \dfrac{A_{i_{n+1}}-A_{i_n}}{\Delta t}=\varrho_i\sum\limits_{j=1}^{m}S_{i_{n-\ell_i+1}}G_{ij}^{n-\ell_i}e^{-\mu_i\sigma_i}-(\mu_i+\lambda_i+r_i)A_{i_{n+1}},\\ G_{ij}^{n}:=g_{ij}(I_{j_n})+\mathcal{G}_{ij}(A_{j_n}),\quad G_{ij}^{n-\ell_i}:=g_{ij}(I_{j_{n-\ell_i}})+\mathcal{G}_{ij}(A_{j_{n-\ell_i}}),\\ \rho_i+\varrho_i=1,\quad i\in\mathbb{N}^+=\{1,2,\dots m\},\quad n\in\mathbb{N}, \end{cases} \tag{1.3}$$

where $\mathbb{N}$ denotes nonnegative integer, $\Delta t>0$ is the time step size, and the parameters remain the same as in model (1.1). It is natural to assume that there exist integers l_i, ℓ_i such that $\tau_i=l_i\Delta t$, $\sigma_i=\ell_i\Delta t$, $i\in\mathbb{N}^+$. The initial condition of model (1.3) reads

$$\begin{aligned} &S_{i_s}=\phi_{1i}(s),\quad I_{i_s}=\phi_{2i}(s),\quad A_{i_s}=\phi_{3i}(s),\quad \phi_{bi}(s)\ge 0,\quad \phi_{bi}(0)>0,\\ &s=-d,-d+1,\dots,0,\quad d=\max\{l_i,\ell_i:\ i\in\mathbb{N}^+\},\quad b=1,2,3. \end{aligned} \tag{1.4}$$

This article as a continuation of the work of our differential model (1.1) (see [9]). We further consider the difference model (1.3) and are aimed at verifying whether this method can effectively guarantee the dynamics of the corresponding continuous model (1.1) or not. The organization of this paper is as follows. Some preliminaries are presented in Section 2. Sections 3 and 4 focus on exploring the global stability of the disease-free equilibrium and the endemic equilibrium of model (1.3), respectively. Several examples are demonstrated to confirm our analysis results in Section 5. We close with a brief discussion section.

2. Preliminaries

Assign $\eta_i=\mu_i+\delta_i+\gamma_i$, $\ h_i=\mu_i+\lambda_i+r_i$. Rewriting model (1.3) give us

$$\begin{cases} S_{i_{n+1}}=\dfrac{S_{i_n}+\Delta t(q_i+p_ie^{-\mu_i\tau_i})\Lambda_i+\Delta t\nu_iS_{i_{n-l_i+1}}e^{-\mu_i\tau_i}}{1+\Delta t(\mu_i+\nu_i)+\Delta t\sum\limits_{j=1}^{m}G_{ij}^{n}},\\ I_{i_{n+1}}=\dfrac{I_{i_n}+\Delta t\rho_i\sum\limits_{j=1}^{m}S_{i_{n-\ell_i+1}}G_{ij}^{n-\ell_i}e^{-\mu_i\sigma_i}}{1+\Delta t\eta_i},\\ A_{i_{n+1}}=\dfrac{A_{i_n}+\Delta t\varrho_i\sum\limits_{j=1}^{m}S_{i_{n-\ell_i+1}}G_{ij}^{n-\ell_i}e^{-\mu_i\sigma_i}}{1+\Delta th_i},\\ \rho_i+\varrho_i=1, i\in\mathbb{N}^+=\{1,2,\dots m\},\quad n\in\mathbb{N}. \end{cases} \tag{2.1}$$

By condition (**A**), we obtain

$$g'_{ij}(I_{j_n})I_{j_n}\le g_{ij}(I_{j_n})\le g'_{ij}(0)I_{j_n}. \tag{2.2}$$

Let $\Psi(x) = x - 1 - \ln x$, similar to Proposition $A.1$ in Ref. [19], one can prove the following Lemma 2.1.

Lemma 2.1. *The condition* (**A**) *implies that for all and* I_{j_n}, $I_j^* > 0$,

$$\Gamma(I_{j_n}) := \Psi\Big(\frac{g_{ij}(I_{j_n})}{g_{ij}(I_j^*)}\Big) - \Psi\Big(\frac{I_{j_n}}{I_j^*}\Big) \leq 0. \tag{2.3}$$

Next, we consider the positivity and boundedness of any solution of model (1.3).

Lemma 2.2. *Let* $(S_{i_n}, I_{i_n}, A_{i_n})$ *be a solution of model* (1.3) *subject to initial condition* (1.4), *then the solution remains nonnegative and bounded for all* $n \in \mathbb{N}$.

Proof. Clearly, all solutions of (2.1) with initial condition (1.4) remain nonnegative for all $n \in \mathbb{N}$. For each i, denote

$$F_{i_n} = S_{i_n} + I_{i_n} + A_{i_n} + \Delta t \sum_{j=1}^{m} \sum_{k=n-\ell_i}^{n-1} S_{i_{k+1}} G_{ij}^{k} e^{-\Delta t \mu_i (n-k)} + \Delta t \nu_i S_{i_{n-l_i+1}} e^{-\mu_i \tau_i}, \tag{2.4}$$

and a calculation shows that

$$\begin{aligned}
F_{i_{n+1}} - F_{i_n} =& S_{i_{n+1}} - S_{i_n} + I_{i_{n+1}} - I_{i_n} + A_{i_{n+1}} - A_{i_n} \\
&+ \Delta t \sum_{j=1}^{m} \sum_{k=n-\ell_i+1}^{n} S_{i_{k+1}} G_{ij}^{k} e^{-\Delta t \mu_i (n-k+1)} \\
&- \Delta t \sum_{j=1}^{m} \sum_{k=n-\ell_i}^{n-1} S_{i_{k+1}} G_{ij}^{k} e^{-\Delta t \mu_i (n-k)} \\
&+ \Delta t \nu_i S_{i_{n-l_i+2}} e^{-\mu_i \tau_i} - \Delta t \nu_i S_{i_{n-l_i+1}} e^{-\mu_i \tau_i} \\
=& \Delta t \Big[q_i \Lambda_i - (\mu_i + \nu_i) S_{i_{n+1}} - \sum_{j=1}^{m} S_{i_{n+1}} G_{ij}^{n} + (p_i \Lambda_i + \nu_i S_{i_{n-l_i+1}}) e^{-\mu_i \tau_i} \\
&+ \sum_{j=1}^{m} S_{i_{n-\ell_i+1}} G_{ij}^{n-\ell_i} e^{-\mu_i \sigma_i} - \eta_i I_{i_{n+1}} - h_i A_{i_{n+1}} \Big] \\
&+ \Delta t \sum_{j=1}^{m} \sum_{k=n-\ell_i+1}^{n} S_{i_{k+1}} G_{ij}^{k} e^{-\Delta t \mu_i (n-k+1)} \\
&- \Delta t \sum_{j=1}^{m} \sum_{k=n-\ell_i}^{n-1} S_{i_{k+1}} G_{ij}^{k} e^{-\Delta t \mu_i (n-k)} \\
&+ \Delta t \nu_i S_{i_{n-l_i+2}} e^{-\mu_i \tau_i} - \Delta t \nu_i S_{i_{n-l_i+1}} e^{-\mu_i \tau_i} \\
=& \Delta t \Big[(q_i + p_i e^{-\mu_i \tau_i}) \Lambda_i - (\mu_i + \nu_i) S_{i_{n+1}} - \eta_i I_{i_{n+1}} - h_i A_{i_{n+1}} \\
&+ (1 - e^{\Delta t \mu_i}) \sum_{j=1}^{m} \sum_{k=n-\ell_i+1}^{n} S_{i_{k+1}} G_{ij}^{k} e^{-\Delta t \mu_i (n-k+1)} + \nu_i S_{i_{n-l_i+2}} e^{-\mu_i \tau_i} \Big] \\
\leq& \Delta t [\aleph_i - \hbar_i F_{i_{n+1}}],
\end{aligned} \tag{2.5}$$

where $\aleph_i = (q_i + p_i e^{-\mu_i \tau_i}) \Lambda_i$ and $\hbar_i = \min \Big\{ \mu_i + \nu_i, \eta_i, h_i, \frac{e^{\Delta t \mu_i} - 1}{\Delta t}, \frac{1}{\Delta t} \Big\}$.

By (2.5), we have

$$F_{i_{n+1}} \leq \frac{1}{1+\Delta t\hbar_i} F_{i_n} + \frac{\Delta t \aleph_i}{1+\Delta t\hbar_i}.$$

We can further obtain

$$F_{i_n} \leq \Big(\frac{1}{1+\Delta t\hbar_i}\Big)^n F_{i_1} + \frac{\aleph_i}{\hbar_i}\Big[1-\Big(\frac{1}{1+\Delta t\hbar_i}\Big)^n\Big]. \tag{2.6}$$

Then $\limsup_{n\to\infty} F_{i_n} \leq \aleph_i/\hbar_i$. This proves the boundedness of the solution. □

It follows from model (1.3) that there is a disease-free equilibrium(DFE) $E_0=(S_1^0, 0, 0, \dots, S_m^0, 0, 0)$, where $S_i^0 = (q_i + p_i e^{-\mu_i\tau_i})\Lambda_i/[\mu_i + (1-e^{-\mu_i\tau_i})\nu_i]$. As in Ref. [9], we know the basic reproduction number $R = \rho(\mathcal{M}_0)$, ρ is the spectral radius and

$$\mathcal{M}_0 = \left(e^{-\mu_i\sigma_i}\beta_{ij}S_i^0\Big(\frac{\rho_i}{\eta_i}+\frac{\kappa_i\varrho_i}{h_i}\Big)\right)_{m\times m}. \tag{2.7}$$

Here β_{ij} are defined in (1.2). For convenience, we denote the matrix $B = (\beta_{ij})_{m\times m}$.

3. Global stability of the disease-free equilibrium

Theorem 3.1. *For model* (1.3)*, assume that B is irreducible. If $R \leq 1$, then the unique DFE E_0 is globally asymptotically stable (GAS); If $R > 1$, then the DFE E_0 is unstable.*

Proof. Since B is irreducible, we can get the irreducibility of $\mathcal{M}_0$ directly. Further, by Perron-Frobenius theorem, there is a strictly positive left eigenvector $\omega = (\omega_1, \omega_2, \dots, \omega_m)$ corresponding to the eigenvalue $\rho(\mathcal{M}_0)$, which means $\omega\rho(\mathcal{M}_0) = \omega\mathcal{M}_0$. Assign

$$\pi_i = \left(\frac{\rho_i}{\eta_i}+\frac{\kappa_i\varrho_i}{h_i}\right)^{-1}, \quad c_i = \frac{\omega_i}{\pi_i}e^{-\mu_i\sigma_i}. \tag{3.1}$$

We construct the following Lyapunov function

$$V_n = \sum_{i=1}^m c_i\Big[U_n + X_n + \sum_{j=1}^m (Y_n + Z_n)\Big],$$

where

$$\begin{aligned}
U_n &= \frac{1}{\Delta t}\left\{S_i^0\Psi\Big(\frac{S_{i_n}}{S_i^0}\Big) + \pi_i e^{\mu_i\sigma_i}\Big[\Big(\frac{1}{\eta_i}+\Delta t\Big)I_{i_n} + \kappa_i\Big(\frac{1}{h_i}+\Delta t\Big)A_{i_n}\Big]\right\},\\
X_n &= \nu_i S_i^0 e^{-\mu_i\tau_i}\sum_{k=n-l_i}^{n-1}\Psi\Big(\frac{S_{i_{k+1}}}{S_i^0}\Big),\\
Y_n &= \sum_{k=n-\ell_i}^{n-1} S_{i_{k+1}} g_{ij}(I_{jk}),\\
Z_n &= \sum_{k=n-\ell_i}^{n-1} S_{i_{k+1}} \mathcal{G}_{ij}(A_{jk}).
\end{aligned} \tag{3.2}$$

Calculation $U_{n+1}-U_n$ along the solution of model (1.3) yields

$$
\begin{aligned}
&U_{n+1}-U_n\\
=&\frac{1}{\Delta t}\bigg[S_{i_{n+1}}-S_{i_n}+S_i^0\ln\frac{S_{i_n}}{S_{i_{n+1}}}+\pi_i e^{\mu_i\sigma_i}\Big(\frac{1}{\eta_i}+\Delta t\Big)(I_{i_{n+1}}-I_{i_n})\\
&+\pi_i\kappa_i e^{\mu_i\sigma_i}\Big(\frac{1}{h_i}+\Delta t\Big)(A_{i_{n+1}}-A_{i_n})\bigg]\\
\le&\frac{1}{\Delta t}\bigg[(S_{i_{n+1}}-S_{i_n})\Big(1-\frac{S_i^0}{S_{i_{n+1}}}\Big)+\pi_i e^{\mu_i\sigma_i}\Big(\frac{1}{\eta_i}+\Delta t\Big)(I_{i_{n+1}}-I_{i_n})\\
&+\pi_i\kappa_i e^{\mu_i\sigma_i}\Big(\frac{1}{h_i}+\Delta t\Big)(A_{i_{n+1}}-A_{i_n})\bigg]\\
=&\Big(1-\frac{S_i^0}{S_{i_{n+1}}}\Big)\bigg[q_i\Lambda_i-(\mu_i+\nu_i)S_{i_{n+1}}-\sum_{j=1}^m S_{i_{n+1}}G_{ij}^n+(p_i\Lambda_i+\nu_i S_{i_{n-l_i+1}})e^{-\mu_i\tau_i}\bigg]\\
&+\pi_i e^{\mu_i\sigma_i}\Big(\frac{1}{\eta_i}+\Delta t\Big)\bigg[\rho_i\sum_{j=1}^m S_{i_{n-\ell_i+1}}G_{ij}^{n-\ell_i}e^{-\mu_i\sigma_i}-(\mu_i+\delta_i+\gamma_i)I_{i_{n+1}}\bigg]\\
&+\pi_i\kappa_i e^{\mu_i\sigma_i}\Big(\frac{1}{h_i}+\Delta t\Big)\bigg[\varrho_i\sum_{j=1}^m S_{i_{n-\ell_i+1}}G_{ij}^{n-\ell_i}e^{-\mu_i\sigma_i}-(\mu_i+\lambda_i+r_i)A_{i_{n+1}}\bigg]\\
=&\Big(1-\frac{S_i^0}{S_{i_{n+1}}}\Big)\bigg[(\mu_i+\nu_i)S_i^0\Big(1-\frac{S_{i_{n+1}}}{S_i^0}\Big)-\sum_{j=1}^m S_{i_{n+1}}G_{ij}^n+\nu_i S_i^0 e^{-\mu_i\tau_i}\Big(\frac{S_{i_{n+1}}}{S_i^0}-1\Big)\\
&+\nu_i S_i^0 e^{-\mu_i\tau_i}\Big(\frac{S_{i_{n-l_i+1}}}{S_i^0}-\frac{S_{i_{n+1}}}{S_i^0}\Big)\bigg]+\sum_{j=1}^m S_{i_{n-\ell_i+1}}G_{ij}^{n-\ell_i}-\pi_i e^{\mu_i\sigma_i}(I_{i_n}+\kappa_i A_{i_n})\\
=&[(\mu_i+(1-e^{-\mu_i\tau_i})\nu_i]S_i^0\Big(2-\frac{S_{i_{n+1}}}{S_i^0}-\frac{S_i^0}{S_{i_{n+1}}}\Big)+\sum_{j=1}^m S_i^0G_{ij}^n-\sum_{j=1}^m S_{i_{n+1}}G_{ij}^n\\
&+\nu_i S_i^0 e^{-\mu_i\tau_i}\Big(\frac{S_{i_{n-l_i+1}}}{S_i^0}-\frac{S_{i_{n-l_i+1}}}{S_{i_{n+1}}}-\frac{S_{i_{n+1}}}{S_i^0}+1\Big)+\sum_{j=1}^m S_{i_{n-\ell_i+1}}G_{ij}^{n-\ell_i}\\
&-\pi_i e^{\mu_i\sigma_i}(I_{i_n}+\kappa_i A_{i_n}).
\end{aligned}\tag{3.3}
$$

We can also derive that

$$
\begin{aligned}
&X_{n+1}-X_n\\
=&\nu_i S_i^0 e^{-\mu_i\tau_i}\bigg[\sum_{k=n-l_i+1}^{n}\Big(\frac{S_{i_{k+1}}}{S_i^0}-1-\ln\frac{S_{i_{k+1}}}{S_i^0}\Big)-\sum_{k=n-l_i}^{n-1}\Big(\frac{S_{i_{k+1}}}{S_i^0}-1-\ln\frac{S_{i_{k+1}}}{S_i^0}\Big)\bigg]\\
=&\nu_i S_i^0 e^{-\mu_i\tau_i}\bigg[\frac{S_{i_{n+1}}}{S_i^0}-1-\ln\frac{S_{i_{n+1}}}{S_i^0}-\frac{S_{i_{n-l_i+1}}}{S_i^0}+1+\ln\frac{S_{i_{n-l_i+1}}}{S_i^0}\bigg]\\
=&\nu_i S_i^0 e^{-\mu_i\tau_i}\bigg[\frac{S_{i_{n+1}}}{S_i^0}-\frac{S_{i_{n-l_i+1}}}{S_i^0}+\ln\frac{S_{i_{n-l_i+1}}}{S_{i_{n+1}}}\bigg].
\end{aligned}\tag{3.4}
$$

Similarly, we have

$$
\begin{aligned}
Y_{n+1}-Y_n=&\sum_{k=n-\ell_i+1}^{n} S_{i_{k+1}} g_{ij}(I_{jk})-\sum_{k=n-\ell_i}^{n-1} S_{i_{k+1}} g_{ij}(I_{jk})\\
=&S_{i_{n+1}} g_{ij}(I_{jn})-S_{i_{n-\ell_i+1}} g_{ij}(I_{j_{n-\ell_i}})
\end{aligned}
\tag{3.5}
$$

and

$$
\begin{aligned}
Z_{n+1}-Z_n=&\sum_{k=n-\ell_i+1}^{n} S_{i_{k+1}} \mathcal{G}_{ij}(A_{jk})-\sum_{k=n-\ell_i}^{n-1} S_{i_{k+1}} \mathcal{G}_{ij}(A_{jk})\\
=&S_{i_{n+1}} \mathcal{G}_{ij}(A_{jn})-S_{i_{n-\ell_i+1}} \mathcal{G}_{ij}(A_{j_{n-\ell_i}}).
\end{aligned}
\tag{3.6}
$$

Consequently

$$
\begin{aligned}
V_{n+1}-V_n=\sum_{i=1}^{m} c_i\Big\{&-[(\mu_i+(1-e^{-\mu_i\tau_i})\nu_i]\frac{(S_{i_{n+1}}-S_i^0)^2}{S_{i_{n+1}}}+\sum_{j=1}^{m} S_i^0 G_{ij}^n\\
&-\nu_i S_i^0 e^{-\mu_i\tau_i}\Psi\Big(\frac{S_{i_{n-l_i+1}}}{S_{i_{n+1}}}\Big)-\pi_i e^{\mu_i\sigma_i}(I_{i_n}+\kappa_i A_{i_n})\Big\}.
\end{aligned}
\tag{3.7}
$$

Recalling (2.2), $g_{ij}(I_{jn})\le g'_{ij}(0)I_{jn}=\beta_{ij}I_{jn}$ and $\mathcal{G}_{ij}(A_{jn})\le \mathcal{G}'_{ij}(0)A_{jn}=\kappa_j\beta_{ij}A_{jn}$ hold. Then we can obtain that

$$
\begin{aligned}
V_{n+1}-V_n\le&\sum_{i=1}^{m}\omega_i\Big[e^{-\mu_i\sigma_i}\Big(\frac{\rho_i}{\eta_i}+\frac{\kappa_i\varrho_i}{h_i}\Big)\sum_{j=1}^{m}\beta_{ij}S_i^0(I_{j_n}+\kappa_j A_{j_n})-(I_{i_n}+\kappa_i A_{i_n})\Big]\\
=&(\omega_1,\omega_2,\ldots,\omega_n)(\mathcal{M}_0 L-L)\\
\le&(\rho(\mathcal{M}_0)-1)(\omega_1,\omega_2,\ldots,\omega_n)L,
\end{aligned}
\tag{3.8}
$$

where $L:=(I_1+\kappa_1A_1, I_2+\kappa_2A_2,\ldots,I_n+\kappa_nA_n)^T$. By using similar arguments with the proof of Theorem 4.2 in Ref. [23], if $\rho(\mathcal{M}_0)\le 1$, the disease-free equilibrium E_0 is globally asymptotically stable. □

Assume that $\rho(\mathcal{M}_0)\ge 1$ and $L\ne 0$. Since

$$
\mathcal{M}_0=\Bigg(e^{-\mu_i\sigma_i}S_i^0\Big(\frac{\rho_i}{\eta_i}\lim_{I_{j_n}\to 0^+}\frac{g_{ij}(I_{j_n})}{I_{j_n}}+\frac{\varrho_i}{h_i}\lim_{A_{j_n}\to 0^+}\frac{\kappa_i g_{ij}(A_{j_n})}{A_{j_n}}\Big)\Bigg)_{m\times m},
$$

we have $\omega(\mathcal{M}_0L-L)=\omega(\rho(\mathcal{M}_0)-1)L>0$. It follows form the continuity of g_{ij} that $V_{n+1}-V_n>0$ in a neighbourhood of E_0, namely, if $\rho(\mathcal{M}_0)>1$, then E_0 is unstable.

4. Global stability of the endemic equilibrium

If $R > 1$ and B is irreducible, it is obvious that (1.3) has at least an endemic equilibrium (EE), denoted by E^*, which is determined by the following equations:

$$\begin{cases} (q_i + p_i e^{-\mu_i \tau_i})\Lambda_i = (\mu_i + \nu_i)S_i^* + \sum\limits_{j=1}^{m} S_i^* G_{ij}^* - \nu_i S_i^* e^{-\mu_i \tau_i}, \\ \eta_i = \dfrac{\rho_i}{I_i^*} e^{-\mu_i \sigma_i} \sum\limits_{j=1}^{m} S_i^* G_{ij}^*, \\ h_i = \dfrac{\varrho_i}{A_i^*} e^{-\mu_i \sigma_i} \sum\limits_{j=1}^{m} S_i^* G_{ij}^*, \\ i \in \mathbb{N}^+ = \{1, 2, \dots m\}, \end{cases} \tag{4.1}$$

where $G_{ij}^* = g_{ij}(I_j^*) + \mathcal{G}_{ij}(A_j^*)$, $\rho_i + \varrho_i = 1$.

Theorem 4.1. *For model (1.3), suppose that B is irreducible. If $R > 1$, then E^* is GAS and is thus the unique EE.*

Proof. In the sequel, we finish the proof of Theorem 4.1 into two case: $m = 1$ and $m \geq 2$.
Case 1: $m = 1$. For convenience, we denote $g_{ij}(I_{j_n})$ as $g(I_n)$. Choosing Lyapunov function

$$\mathbb{V}_n = \mathbb{U}_n + \mathbb{X}_n + \mathbb{Y}_n + \mathbb{Z}_n,$$

where

$$\begin{aligned} \mathbb{U}_n =& \frac{1}{\Delta t}\Bigg[S^* \Psi\Big(\frac{S_n}{S^*}\Big) + \Big(\frac{g(I^*)I^*}{\rho e^{-\mu\sigma} G^*} + \Delta t S^* g(I^*)\Big)\Psi\Big(\frac{I_n}{I^*}\Big) \\ &+ \Big(\frac{\mathcal{G}(A^*)A^*}{\varrho e^{-\mu\sigma} G^*} + \Delta t S^* \mathcal{G}(A^*)\Big)\Psi\Big(\frac{A_n}{A^*}\Big)\Bigg], \\ \mathbb{X}_n =& \nu S^* e^{-\mu\tau} \sum_{k=n-l}^{n-1} \Psi\Big(\frac{S_{k+1}}{S^*}\Big), \\ \mathbb{Y}_n =& S^* g(I^*) \sum_{k=n-\ell}^{n-1} \Psi\Big(\frac{S_{k+1} g(I_k)}{S^* g(I^*)}\Big), \\ \mathbb{Z}_n =& S^* \mathcal{G}(A^*) \sum_{k=n-\ell}^{n-1} \Psi\Big(\frac{S_{k+1} \mathcal{G}(A_k)}{S^* \mathcal{G}(A^*)}\Big). \end{aligned} \tag{4.2}$$

Together with model (1.3) and equilibrium condition (4.1), we can deduce that

$$\begin{aligned} &\mathbb{U}_{n+1} - \mathbb{U}_n \\ =& \frac{1}{\Delta t}\Big(S_{n+1} - S_n + S^* \ln \frac{S_n}{S_{n+1}}\Big) \\ &+ \frac{1}{\Delta t}\Big(\frac{g(I^*)I^*}{\rho e^{-\mu\sigma} G^*} + \Delta t S^* g(I^*)\Big)\Big(\frac{I_{n+1}}{I^*} - \frac{I_n}{I^*} + \ln \frac{I_n}{I_{n+1}}\Big) \\ &+ \frac{1}{\Delta t}\Big(\frac{\mathcal{G}(A^*)A^*}{\varrho e^{-\mu\sigma} G^*} + \Delta t S^* \mathcal{G}(A^*)\Big)\Big(\frac{A_{n+1}}{A^*} - \frac{A_n}{A^*} + \ln \frac{A_n}{A_{n+1}}\Big) \end{aligned}$$

$$
\begin{aligned}
&\le \frac{S_{n+1}-S_n}{\Delta t}\Big(1-\frac{S^*}{S_{n+1}}\Big)+\frac{g(I^*)}{\rho e^{-\mu\sigma}G^*}\frac{I_{n+1}-I_n}{\Delta t}\Big(1-\frac{I^*}{I_{n+1}}\Big)\\
&\quad+S^*g(I^*)\Big(\frac{I_{n+1}}{I^*}-\frac{I_n}{I^*}+\ln\frac{I_n}{I_{n+1}}\Big)+\frac{\mathcal{G}(A^*)}{\varrho e^{-\mu\sigma}G^*}\frac{A_{n+1}-A_n}{\Delta t}\Big(1-\frac{A^*}{A_{n+1}}\Big)\\
&\quad+S^*\mathcal{G}(A^*)\Big(\frac{A_{n+1}}{A^*}-\frac{A_n}{A^*}+\ln\frac{A_n}{A_{n+1}}\Big)\\
&=\Big(1-\frac{S^*}{S_{n+1}}\Big)\Big[q\Lambda-(\mu+\nu)S_{n+1}-S_{n+1}G^n+(p\Lambda+\nu S_{n-l+1})e^{-\mu\tau}\Big]\\
&\quad+\frac{g(I^*)}{\rho e^{-\mu\sigma}G^*}\Big(1-\frac{I^*}{I_{n+1}}\Big)\Big(\rho S_{n-\ell+1}G^{n-\ell}e^{-\mu\sigma}-\eta I_{n+1}\Big)\\
&\quad+S^*g(I^*)\Big(\frac{I_{n+1}}{I^*}-\frac{I_n}{I^*}+\ln\frac{I_n}{I_{n+1}}\Big)\\
&\quad+\frac{\mathcal{G}(A^*)}{\varrho e^{-\mu\sigma}G^*}\Big(1-\frac{A^*}{A_{n+1}}\Big)\Big(\varrho S_{n-\ell+1}G^{n-\ell}e^{-\mu\sigma}-hA_{n+1}\Big)\\
&\quad+S^*\mathcal{G}(A^*)\Big(\frac{A_{n+1}}{A^*}-\frac{A_n}{A^*}+\ln\frac{A_n}{A_{n+1}}\Big)\\
&=\Big(1-\frac{S^*}{S_{n+1}}\Big)\Big[(\mu+\nu)S^*\Big(1-\frac{S_{n+1}}{S^*}\Big)+S^*G^*-S_{n+1}G^n\\
&\quad+\nu S^*e^{-\mu\tau}\Big(\frac{S_{n+1}}{S^*}-1\Big)+\nu S^*e^{-\mu\tau}\Big(\frac{S_{n-l+1}}{S^*}-\frac{S_{n+1}}{S^*}\Big)\Big]\\
&\quad+\frac{g(I^*)}{\rho e^{-\mu\sigma}G^*}\Big(1-\frac{I^*}{I_{n+1}}\Big)\Big(\rho S_{n-\ell+1}G^{n-\ell}e^{-\mu\sigma}-\frac{I_{n+1}}{I^*}\rho S^*G^*e^{-\mu\sigma}\Big)\\
&\quad+S^*g(I^*)\Big(\frac{I_{n+1}}{I^*}-\frac{I_n}{I^*}+\ln\frac{I_n}{I_{n+1}}\Big)\\
&\quad+\frac{\mathcal{G}(A^*)}{\varrho e^{-\mu\sigma}G^*}\Big(1-\frac{A^*}{A_{n+1}}\Big)\Big(\varrho S_{n-\ell+1}G^{n-\ell}e^{-\mu\sigma}-\frac{A_{n+1}}{A^*}\varrho S^*G^*e^{-\mu\sigma}\Big)\\
&\quad+S^*\mathcal{G}(A^*)\Big(\frac{A_{n+1}}{A^*}-\frac{A_n}{A^*}+\ln\frac{A_n}{A_{n+1}}\Big)\\
&=[\mu+(1-e^{-\mu\tau})\nu]S^*\Big(2-\frac{S_{n+1}}{S^*}-\frac{S^*}{S_{n+1}}\Big)+S^*G^*-S_{n+1}G^n\\
&\quad-\frac{S^*}{S_{n+1}}S^*G^*+S^*G^n+\nu S^*e^{-\mu\tau}\Big(\frac{S_{n-l+1}}{S^*}-\frac{S_{n-l+1}}{S_{n+1}}-\frac{S_{n+1}}{S^*}+1\Big)\\
&\quad+\frac{g(I^*)}{G^*}S_{n-\ell+1}G^{n-\ell}-\frac{g(I^*)I^*}{G^*I_{n+1}}S_{n-\ell+1}G^{n-\ell}-S^*g(I^*)\frac{I_{n+1}}{I^*}\\
&\quad+S^*g(I^*)+S^*g(I^*)\Big(\frac{I_{n+1}}{I^*}-\frac{I_n}{I^*}+\ln\frac{I_n}{I_{n+1}}\Big)\\
&\quad+\frac{\mathcal{G}(A^*)}{G^*}S_{n-\ell+1}G^{n-\ell}-\frac{\mathcal{G}(A^*)A^*}{G^*A_{n+1}}S_{n-\ell+1}G^{n-\ell}-S^*\mathcal{G}(A^*)\frac{A_{n+1}}{A^*}\\
&\quad+S^*\mathcal{G}(A^*)+S^*\mathcal{G}(A^*)\Big(\frac{A_{n+1}}{A^*}-\frac{A_n}{A^*}+\ln\frac{A_n}{A_{n+1}}\Big)\\
&=[\mu+(1-e^{-\mu\tau})\nu]S^*\Big(2-\frac{S_{n+1}}{S^*}-\frac{S^*}{S_{n+1}}\Big)\\
&\quad+\nu S^*e^{-\mu\tau}\Big(\frac{S_{n-l+1}}{S^*}-\frac{S_{n-l+1}}{S_{n+1}}-\frac{S_{n+1}}{S^*}+1\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+ S^*g(I^*)\Big(1-\frac{S^*}{S_{n+1}}+\frac{g(I_n)}{g(I^*)}-\frac{I_n}{I^*}+\ln\frac{I_n}{I_{n+1}}\Big)\\
&+ S^*\mathcal{G}(A^*)\Big(1-\frac{S^*}{S_{n+1}}+\frac{\mathcal{G}(A_n)}{\mathcal{G}(A^*)}-\frac{A_n}{A^*}+\ln\frac{A_n}{A_{n+1}}\Big)\\
&+ S^*g(I^*)\Big(1-\frac{I^*}{S^*G^*I_{n+1}}S_{n-\ell+1}G^{n-\ell}\Big)\\
&+ S^*\mathcal{G}(A^*)\Big(1-\frac{A^*}{S^*G^*A_{n+1}}S_{n-\ell+1}G^{n-\ell}\Big)\\
&- S_{n+1}G^n+S_{n-\ell+1}G^{n-\ell}.
\end{aligned}
\tag{4.3}
$$

To proceed, we can derive that

$$
\begin{aligned}
&\mathbb{X}_{n+1}-\mathbb{X}_n\\
=&\nu S^*e^{-\mu\tau}\Big[\sum_{k=n-l+1}^{n}\Big(\frac{S_{k+1}}{S^*}-1-\ln\frac{S_{k+1}}{S^*}\Big)-\sum_{k=n-l}^{n-1}\Big(\frac{S_{k+1}}{S^*}-1-\ln\frac{S_{k+1}}{S^*}\Big)\Big]\\
=&\nu S^*e^{-\mu\tau}\Big(\frac{S_{n+1}}{S^*}-\frac{S_{n-l+1}}{S^*}+\ln\frac{S_{n-l+1}}{S_{n+1}}\Big),
\end{aligned}
\tag{4.4}
$$

and

$$
\begin{aligned}
\mathbb{Y}_{n+1}-\mathbb{Y}_n =&S^*g(I^*)\Big[\sum_{k=n-\ell+1}^{n}\Big(\frac{S_{k+1}g(I_k)}{S^*g(I^*)}-1-\ln\frac{S_{k+1}g(I_k)}{S^*g(I^*)}\Big)\\
&-\sum_{k=n-\ell}^{n-1}\Big(\frac{S_{k+1}g(I_k)}{S^*g(I^*)}-1-\ln\frac{S_{k+1}g(I_k)}{S^*g(I^*)}\Big)\Big]\\
=&S_{n+1}g(I_n)-S_{n-\ell+1}g(I_{n-\ell})+S^*g(I^*)\ln\frac{S_{n-\ell+1}g(I_{n-\ell})}{S_{n+1}g(I_n)},
\end{aligned}
\tag{4.5}
$$

together with

$$
\begin{aligned}
&\mathbb{Z}_{n+1}-\mathbb{Z}_n = S^*\mathcal{G}(A^*)\Big[\sum_{k=n-\ell+1}^{n}\Big(\frac{S_{k+1}\mathcal{G}(A_k)}{S^*\mathcal{G}(A^*)}-1-\ln\frac{S_{k+1}\mathcal{G}(A_k)}{S^*\mathcal{G}(A^*)}\Big),\\
&-\sum_{k=n-\ell}^{n-1}\Big(\frac{S_{k+1}\mathcal{G}(A_k)}{S^*\mathcal{G}(A^*)}-1-\ln\frac{S_{k+1}\mathcal{G}(A_k)}{S^*\mathcal{G}(A^*)}\Big)\Big] =S_{n+1}\mathcal{G}(A_n)-S_{n-\ell+1}\mathcal{G}(A_{n-\ell})\\
&\qquad+S^*\mathcal{G}(A^*)\ln\frac{S_{n-\ell+1}\mathcal{G}(A_{n-\ell})}{S_{n+1}\mathcal{G}(A_n)}.
\end{aligned}
\tag{4.6}
$$

Combining with the above formulas, we have

$$
\begin{aligned}
\mathbb{V}_{n+1}-\mathbb{V}_n =&[\mu+(1-e^{-\mu\tau})\nu]S^*\Big(2-\frac{S_{n+1}}{S^*}-\frac{S^*}{S_{n+1}}\Big)\\
&+\nu S^*e^{-\mu\tau}\Big(-\frac{S_{n-l+1}}{S_{n+1}}+1+\ln\frac{S_{n-l+1}}{S_{n+1}}\Big)\\
&+S^*g(I^*)\Big(1-\frac{S^*}{S_{n+1}}+\frac{g(I_n)}{g(I^*)}-\frac{I_n}{I^*}+\ln\frac{I_n}{I_{n+1}}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+S^*\mathcal{G}(A^*)\Big(1-\frac{S^*}{S_{n+1}}+\frac{\mathcal{G}(A_n)}{\mathcal{G}(A^*)}-\frac{A_n}{A^*}+\ln\frac{A_n}{A_{n+1}}\Big)\\
&+S^*g(I^*)\Big(1-\frac{I^*}{S^*G^*I_{n+1}}S_{n-\ell+1}G^{n-\ell}\Big)\\
&+S^*\mathcal{G}(A^*)\Big(1-\frac{A^*}{S^*G^*A_{n+1}}S_{n-\ell+1}G^{n-\ell}\Big)\\
&+S^*g(I^*)\ln\frac{S_{n-\ell+1}g(I_{n-\ell})}{S_{n+1}g(I_n)}+S^*\mathcal{G}(A^*)\ln\frac{S_{n-\ell+1}\mathcal{G}(A_{n-\ell})}{S_{n+1}\mathcal{G}(A_n)}.
\end{aligned}\tag{4.7}
$$

Since

$$
\begin{aligned}
\ln\frac{S_{n-\ell+1}g(I_{n-\ell})}{S_{n+1}g(I_n)}&=\ln\frac{S^*}{S_{n+1}}-\ln\frac{g(I_n)}{g(I^*)}+\ln\frac{I_{n+1}}{I^*}+\ln\frac{S_{n-\ell+1}G^{n-\ell}I^*}{S^*G^*I_{n+1}}+\ln\frac{G^*g(I_{n-\ell})}{G^{n-\ell}g(I^*)},\\
\ln\frac{S_{n-\ell+1}\mathcal{G}(A_{n-\ell})}{S_{n+1}\mathcal{G}(A_n)}&=\ln\frac{S^*}{S_{n+1}}-\ln\frac{\mathcal{G}(A_n)}{\mathcal{G}(A^*)}+\ln\frac{A_{n+1}}{A^*}\\
&\quad+\ln\frac{S_{n-\ell+1}G^{n-\ell}A^*}{S^*G^*A_{n+1}}+\ln\frac{G^*\mathcal{G}(A_{n-\ell})}{G^{n-\ell}\mathcal{G}(A^*)},
\end{aligned}
$$

(4.7) can be rewritten as

$$
\begin{aligned}
&\mathbb{V}_{n+1}-\mathbb{V}_n\\
=&-[\mu+(1-e^{-\mu\tau})\nu]\frac{(S_{n+1}-S^*)^2}{S_{n+1}}+\nu S^*e^{-\mu\tau}\Big(-\frac{S_{n-l+1}}{S_{n+1}}+1+\ln\frac{S_{n-l+1}}{S_{n+1}}\Big)\\
&+S^*G^*\Big(1-\frac{S^*}{S_{n+1}}+\ln\frac{S^*}{S_{n+1}}\Big)+S^*g(I^*)\Big(\frac{g(I_n)}{g(I^*)}-\ln\frac{g(I_n)}{g(I^*)}-\frac{I_n}{I^*}+\ln\frac{I_n}{I^*}\Big)\\
&+S^*\mathcal{G}(A^*)\Big(\frac{\mathcal{G}(A_n)}{\mathcal{G}(A^*)}-\ln\frac{\mathcal{G}(A_n)}{\mathcal{G}(A^*)}-\frac{A_n}{A^*}+\ln\frac{A_n}{A^*}\Big)\\
&+S^*g(I^*)\Big(1-\frac{I^*S_{n-\ell+1}G^{n-\ell}}{S^*G^*I_{n+1}}+\ln\frac{I^*S_{n-\ell+1}G^{n-\ell}}{S^*G^*I_{n+1}}\Big)\\
&+S^*\mathcal{G}(A^*)\Big(1-\frac{A^*S_{n-\ell+1}G^{n-\ell}}{S^*G^*A_{n+1}}+\ln\frac{A^*S_{n-\ell+1}G^{n-\ell}}{S^*G^*A_{n+1}}\Big)\\
&+S^*g(I^*)\Big(1-\frac{G^*g(I_{n-\ell})}{G^{n-\ell}g(I^*)}+\ln\frac{G^*g(I_{n-\ell})}{G^{n-\ell}g(I^*)}\Big)\\
&+S^*\mathcal{G}(A^*)\Big(1-\frac{G^*\mathcal{G}(A_{n-\ell})}{G^{n-\ell}\mathcal{G}(A^*)}+\ln\frac{G^*\mathcal{G}(A_{n-\ell})}{G^{n-\ell}\mathcal{G}(A^*)}\Big),
\end{aligned}\tag{4.8}
$$

where

$$
\begin{aligned}
&S^*g(I^*)\Big(1-\frac{G^*g(I_{n-\ell})}{G^{n-\ell}g(I^*)}\Big)+S^*\mathcal{G}(A^*)\Big(1-\frac{G^*\mathcal{G}(A_{n-\ell})}{G^{n-\ell}\mathcal{G}(A^*)}\Big)\\
=&S^*g(I^*)+S^*\mathcal{G}(A^*)-\Big(\frac{S^*G^*g(I_{n-\ell})}{G^{n-\ell}}+\frac{S^*G^*\mathcal{G}(A_{n-\ell})}{G^{n-\ell}}\Big)\\
=&S^*G^*-\Big(\frac{S^*G^*(g(I_{n-\ell})+\mathcal{G}(A_{n-\ell}))}{G^{n-\ell}}\Big)=0.
\end{aligned}
$$

Notice that

$$
\frac{g(I_n)}{g(I^*)}-\ln\frac{g(I_n)}{g(I^*)}-\frac{I_n}{I^*}+\ln\frac{I_n}{I^*}=\Psi\Big(\frac{g(I_n)}{g(I^*)}\Big)-\Psi\Big(\frac{I_n}{I^*}\Big)=\Gamma(I_n),
$$

$$\frac{\mathcal{G}(A_n)}{\mathcal{G}(A^*)} - \ln\frac{\mathcal{G}(A_n)}{\mathcal{G}(A^*)} - \frac{A_n}{A^*} + \ln\frac{A_n}{A^*} = \Psi\Big(\frac{\mathcal{G}(A_n)}{\mathcal{G}(A^*)}\Big) - \Psi\Big(\frac{A_n}{A^*}\Big) = \Gamma(A_n).$$

Hence

$$\begin{aligned}\mathbb{V}_{n+1} - \mathbb{V}_n =& -[\mu + (1-e^{-\mu\tau})\nu]\frac{(S_{n+1}-S^*)^2}{S_{n+1}} - \nu S^* e^{-\mu\tau}\Psi\Big(\frac{S_{n-l+1}}{S_{n+1}}\Big)\\ &- S^*G^*\Psi\Big(\frac{S^*}{S_{n+1}}\Big) + S^*g(I^*)\Gamma(I_n) + S^*\mathcal{G}(A^*)\Gamma(A_n)\\ &- S^*g(I^*)\Psi\Big(\frac{I^*S_{n-\ell+1}G^{n-\ell}}{S^*G^*I_{n+1}}\Big) - S^*\mathcal{G}(A^*)\Psi\Big(\frac{A^*S_{n-\ell+1}G^{n-\ell}}{S^*G^*A_{n+1}}\Big)\\ &- S^*g(I^*)\Psi\Big(\frac{G^*g(I_{n-\ell})}{G^{n-\ell}g(I^*)}\Big) - S^*\mathcal{G}(A^*)\Psi\Big(\frac{G^*\mathcal{G}(A_{n-\ell})}{G^{n-\ell}\mathcal{G}(A^*)}\Big). \end{aligned} \tag{4.9}$$

That is to say, $\mathbb{V}_{n+1} - \mathbb{V}_n \le 0$ and "=" holds if and only if $S_{n-l} = S_n = S^*$, $I_{n-\ell} = I_n = I^*$, $A_{n-\ell} = A_n = A^*$ for almost all $l, \ell > 0$, thus $\{\mathbb{V}_n\}$ is a monotone decreasing sequence. Since $\mathbb{V}_n > 0$, we can derive that $\lim_{n\to\infty}(\mathbb{V}_{n+1} - \mathbb{V}_n) \ge 0$. Then similar to Ref. [9], the only compact invariant subset of $\{\lim_{n\to\infty}(\mathbb{V}_{n+1} - \mathbb{V}_n) = 0\}$ is the singleton $\{E^*\}$. And it follows from LaSalle's invariance principle that E^* is globally asymptotically stable.

Case 2: $m \ge 2$. Define

$$\varpi_{ij} = S_i^* G_{ij}^*, \quad i, j \in \mathbb{N}^+$$

and

$$\varpi = \begin{pmatrix} \sum\limits_{j\neq 1}\varpi_{1j} & -\varpi_{21} & \cdots & -\varpi_{m1} \\ -\varpi_{12} & \sum\limits_{j\neq 2}\varpi_{2j} & \cdots & -\varpi_{m2} \\ \vdots & \vdots & \vdots & \vdots \\ -\varpi_{1m} & -\varpi_{2m} & \cdots & \sum\limits_{j\neq m}\varpi_{mj} \end{pmatrix},$$

where ϖ is the Laplacian version of the matrix $(\varpi_{ij})_{m\times m}$. It follows form the irreducibility of $(\beta_{ij})_{m\times m}$ that $(\varpi_{ij})_{m\times m}$ and ϖ are irreducible. It follows from Lemma 2.1 in [5] that the solution space $\mathcal{C}$ of system $\varpi\xi = 0$ has dimension 1. Let $a_{ii}, 1 \le i \le m$, be the co-factor of the ith diagonal entry of ϖ. Then the base ξ of $\mathcal{C}$ is in the form of $\xi = (\xi_1, \xi_2, \ldots, \xi_m)^T = (a_{11}, a_{22}, \ldots, a_{mm})^T$. Whence $\xi_i > 0$.

Define the following Lyapunov function

$$\mathcal{V}_n = \sum_{i=1}^m \xi_i\Big[\mathcal{U}_n + \mathcal{X}_n + \sum_{j=1}^m(\mathcal{Y}_n + \mathcal{Z}_n)\Big],$$

where

$$
\begin{aligned}
\mathcal{U}_n &= \frac{1}{\Delta t}\Bigg[S_i^*\Psi\Big(\frac{S_{i_n}}{S_i^*}\Big) + \Big(\sum_{j=1}^m \frac{g_{ij}(I_j^*)I_i^*}{\rho_i e^{-\mu_i\sigma_i}G_{ij}^*} + \Delta t\sum_{j=1}^m S_i^* g_{ij}(I_j^*)\Big)\Psi\Big(\frac{I_{i_n}}{I_i^*}\Big) \\
&\quad + \Big(\sum_{j=1}^m \frac{\mathcal{G}(A_j^*)A_i^*}{\varrho_i e^{-\mu_i\sigma_i}G_{ij}^*} + \Delta t\sum_{j=1}^m S_i^*\mathcal{G}_{ij}(A_j^*)\Big)\Psi\Big(\frac{A_{i_n}}{A_i^*}\Big)\Bigg], \\
\mathcal{X}_n &= \nu_i S_i^* e^{-\mu_i\tau_i}\sum_{k=n-l_i}^{n-1}\Psi\Big(\frac{S_{i_{k+1}}}{S_i^*}\Big), \\
\mathcal{Y}_n &= S_i^* g_{ij}(I_j^*)\sum_{k=n-\ell_i}^{n-1}\Psi\Big(\frac{S_{i_{k+1}} g_{ij}(I_{j_k})}{S_i^* g_{ij}(I_j^*)}\Big), \\
\mathcal{Z}_n &= S_i^*\mathcal{G}_{ij}(A_j^*)\sum_{k=n-\ell_i}^{n-1}\Psi\Big(\frac{S_{i_{k+1}}\mathcal{G}_{ij}(A_{j_k})}{S_i^*\mathcal{G}_{ij}(A_j^*)}\Big).
\end{aligned}
\tag{4.10}
$$

The difference of $\mathcal{U}_n$ satisfies

$$
\begin{aligned}
&\mathcal{U}_{n+1} - \mathcal{U}_n \\
=&\frac{1}{\Delta t}\Big(S_{i_{n+1}} - S_{i_n} + S_i^*\ln\frac{S_{i_n}}{S_{i_{n+1}}}\Big) \\
&+ \frac{1}{\Delta t}\Big(\sum_{j=1}^m \frac{g_{ij}(I_j^*)I_i^*}{\rho_i e^{-\mu_i\sigma_i}G_{ij}^*} + \Delta t\sum_{j=1}^m S_i^* g_{ij}(I_j^*)\Big)\Big(\frac{I_{i_{n+1}}}{I_i^*} - \frac{I_{i_n}}{I_i^*} + \ln\frac{I_{i_n}}{I_{i_{n+1}}}\Big) \\
&+ \frac{1}{\Delta t}\Big(\sum_{j=1}^m \frac{\mathcal{G}_{ij}(A_j^*)A_i^*}{\varrho_i e^{-\mu_i\sigma_i}G_{ij}^*} + \Delta t\sum_{j=1}^m S_i^*\mathcal{G}_{ij}(A_j^*)\Big)\Big(\frac{A_{i_{n+1}}}{A_i^*} - \frac{A_{i_n}}{A_i^*} + \ln\frac{A_{i_n}}{A_{i_{n+1}}}\Big) \\
\le&\frac{S_{i_{n+1}} - S_{i_n}}{\Delta t}\Big(1 - \frac{S_i^*}{S_{i_{n+1}}}\Big) + \Big(\sum_{j=1}^m \frac{g_{ij}(I_j^*)}{\rho_i e^{-\mu_i\sigma_i}G_{ij}^*}\Big)\frac{I_{i_{n+1}} - I_{i_n}}{\Delta t}\Big(1 - \frac{I_i^*}{I_{i_{n+1}}}\Big) \\
&+ \sum_{j=1}^m S_i^* g_{ij}(I_j^*)\Big(\frac{I_{i_{n+1}}}{I_i^*} - \frac{I_{i_n}}{I_i^*} + \ln\frac{I_{i_n}}{I_{i_{n+1}}}\Big) \\
&+ \Big(\sum_{j=1}^m \frac{\mathcal{G}_{ij}(A_j^*)}{\varrho_i e^{-\mu_i\sigma_i}G_{ij}^*}\Big)\frac{A_{i_{n+1}} - A_{i_n}}{\Delta t}\Big(1 - \frac{A_i^*}{A_{i_{n+1}}}\Big) \\
&+ \sum_{j=1}^m S_i^*\mathcal{G}_{ij}(A_j^*)\Big(\frac{A_{i_{n+1}}}{A_i^*} - \frac{A_{i_n}}{A_i^*} + \ln\frac{A_{i_n}}{A_{i_{n+1}}}\Big) \\
=&\Big(1 - \frac{S_i^*}{S_{i_{n+1}}}\Big)\Big[q_i\Lambda_i - (\mu_i + \nu_i)S_{i_{n+1}} - \sum_{j=1}^m S_{i_{n+1}}G_{ij}^n + (p_i\Lambda_i + \nu_i S_{i_{n-l_i+1}})e^{-\mu_i\tau_i}\Big] \\
&+ \Big(\sum_{j=1}^m \frac{g_{ij}(I_j^*)}{\rho_i e^{-\mu_i\sigma_i}G_{ij}^*}\Big)\Big(1 - \frac{I_i^*}{I_{i_{n+1}}}\Big)\Big(\rho_i\sum_{j=1}^m S_{i_{n-\ell_i+1}}G_{ij}^{n-\ell_i}e^{-\mu_i\sigma_i} - \eta_i I_{i_{n+1}}\Big) \\
&+ \Big(\sum_{j=1}^m \frac{\mathcal{G}_{ij}(A_j^*)}{\varrho_i e^{-\mu_i\sigma_i}G_{ij}^*}\Big)\Big(1 - \frac{A_i^*}{A_{i_{n+1}}}\Big)\Big(\varrho_i\sum_{j=1}^m S_{i_{n-\ell_i+1}}G_{ij}^{n-\ell_i}e^{-\mu_i\sigma_i} - h_i A_{i_{n+1}}\Big) \\
&+ \sum_{j=1}^m S_i^* g_{ij}(I_j^*)\Big(\frac{I_{i_{n+1}}}{I_i^*} - \frac{I_{i_n}}{I_i^*} + \ln\frac{I_{i_n}}{I_{i_{n+1}}}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{j=1}^{m} S_i^*\mathcal{G}_{ij}(A_j^*)\Big(\frac{A_{i_{n+1}}}{A_i^*}-\frac{A_{i_n}}{A_i^*}+\ln\frac{A_{i_n}}{A_{i_{n+1}}}\Big)\\
=&\Big(1-\frac{S_i^*}{S_{i_{n+1}}}\Big)\Big[(\mu_i+\nu_i)S_i^*\Big(1-\frac{S_{i_{n+1}}}{S_i^*}\Big)-\sum_{j=1}^{m} S_{i_{n+1}}G_{ij}^n-\sum_{j=1}^{m} S_i^*G_{ij}^*\\
&+\nu_i S_i^* e^{-\mu_i\tau_i}\Big(\frac{S_{i_{n+1}}}{S_i^*}-1\Big)+\nu_i S_i^* e^{-\mu_i\tau_i}\Big(\frac{S_{i_{n-l_i+1}}}{S_i^*}-\frac{S_{i_{n+1}}}{S_i^*}\Big)\Big]\\
&+\Big(\sum_{j=1}^{m}\frac{g_{ij}(I_j^*)}{\rho_i e^{-\mu_i\sigma_i}G_{ij}^*}\Big)\Big(1-\frac{I_i^*}{I_{i_{n+1}}}\Big)\\
&\times\Big(\rho_i\sum_{j=1}^{m} S_{i_{n-\ell_i+1}}G_{ij}^{n-\ell_i}e^{-\mu_i\sigma_i}-\frac{I_{i_{n+1}}}{I_i^*}\rho_i\sum_{j=1}^{m} S_i^*G_{ij}^*e^{-\mu_i\sigma_i}\Big)\\
&+\Big(\sum_{j=1}^{m}\frac{\mathcal{G}_{ij}(A_j^*)}{\varrho_i e^{-\mu_i\sigma_i}G_{ij}^*}\Big)\Big(1-\frac{A_i^*}{A_{i_{n+1}}}\Big)\\
&\times\Big(\varrho_i\sum_{j=1}^{m} S_{i_{n-\ell_i+1}}G_{ij}^{n-\ell_i}e^{-\mu_i\sigma_i}-\frac{A_{i_{n+1}}}{A_i^*}\varrho_i\sum_{j=1}^{m} S_i^*G_{ij}^*e^{-\mu_i\sigma_i}\Big)\\
&+\sum_{j=1}^{m} S_i^* g_{ij}(I_j^*)\Big(\frac{I_{i_{n+1}}}{I_i^*}-\frac{I_{i_n}}{I_i^*}+\ln\frac{I_{i_n}}{I_{i_{n+1}}}\Big)\\
&+\sum_{j=1}^{m} S_i^*\mathcal{G}_{ij}(A_j^*)\Big(\frac{A_{i_{n+1}}}{A_i^*}-\frac{A_{i_n}}{A_i^*}+\ln\frac{A_{i_n}}{A_{i_{n+1}}}\Big)\\
\le&[\mu_i+(1-e^{-\mu_i\tau_i})\nu_i]S_i^*\Big(2-\frac{S_{i_{n+1}}}{S_i^*}-\frac{S_i^*}{S_{i_{n+1}}}\Big)+\sum_{j=1}^{m} S_i^*G_{ij}^*\\
&-\sum_{j=1}^{m} S_{i_{n+1}}G_{ij}^n-\sum_{j=1}^{m}\frac{S_i^*}{S_{i_{n+1}}}S_i^*G_{ij}^*+\sum_{j=1}^{m} S_i^*G_{ij}^n\\
&+\nu_i S_i^* e^{-\mu_i\tau_i}\Big(\frac{S_{i_{n-l_i+1}}}{S_i^*}-\frac{S_{i_{n-l_i+1}}}{S_{i_{n+1}}}-\frac{S_{i_{n+1}}}{S_i^*}+1\Big)\\
&+\sum_{j=1}^{m}\frac{g_{ij}(I_j^*)}{G_{ij}^*}\sum_{j=1}^{m} S_{i_{n-\ell_i+1}}G_{ij}^{n-\ell_i}-\sum_{j=1}^{m}\frac{g_{ij}(I_j^*)I_i^*}{G_{ij}^*I_{i_{n+1}}}S_{n-\ell_i+1}G^{n-\ell_i}\\
&-\sum_{j=1}^{m} S_i^* g_{ij}(I_j^*)\frac{I_{i_{n+1}}}{I_i^*}+\sum_{j=1}^{m}\frac{g_{ij}(I_j^*)}{G_{ij}^*}\sum_{j=1}^{m} S_i^*G_{ij}^*\\
&+\sum_{j=1}^{m}\frac{\mathcal{G}_{ij}(A_j^*)}{G_{ij}^*}\sum_{j=1}^{m} S_{i_{n-\ell_i+1}}G_{ij}^{n-\ell_i}-\sum_{j=1}^{m}\frac{\mathcal{G}_{ij}(A_i^*)A_i^*}{G_{ij}^*A_{i_{n+1}}}S_{i_{n-\ell_i+1}}G_{ij}^{n-\ell_i}\\
&-\sum_{j=1}^{m} S_i^*\mathcal{G}_{ij}(A_j^*)\frac{A_{i_{n+1}}}{A_i^*}+\sum_{j=1}^{m}\frac{\mathcal{G}_{ij}(A_i^*)}{G_{ij}^*}\sum_{j=1}^{m} S_i^*G_{ij}^*\\
&+\sum_{j=1}^{m} S_i^* g_{ij}(I_j^*)\Big(\frac{I_{i_{n+1}}}{I_i^*}-\frac{I_{i_n}}{I_i^*}+\ln\frac{I_{i_n}}{I_{i_{n+1}}}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{j=1}^{m} S_i^* \mathcal{G}_{ij}(A_j^*)\Big(\frac{A_{i_{n+1}}}{A_i^*}-\frac{A_{i_n}}{A_i^*}+\ln\frac{A_{i_n}}{A_{i_{n+1}}}\Big)\\
=&[\mu_i+(1-e^{-\mu_i\tau_i})\nu_i]S_i^*\Big(2-\frac{S_{i_{n+1}}}{S_i^*}-\frac{S_i^*}{S_{i_{n+1}}}\Big)\\
&+\nu_i S_i^* e^{-\mu_i\tau_i}\Big(\frac{S_{i_{n-l_i+1}}}{S_i^*}-\frac{S_{i_{n-l_i+1}}}{S_{i_{n+1}}}-\frac{S_{i_{n+1}}}{S_i^*}+1\Big)\\
&+\sum_{j=1}^{m} S_i^* g_{ij}(I_j^*)\Big(1-\frac{S_i^*}{S_{i_{n+1}}}+\frac{g_{ij}(A_{j_n})}{g_{ij}(A_j^*)}-\frac{I_{i_n}}{I_i^*}+\ln\frac{I_{i_n}}{I_{i_{n+1}}}\Big)\\
&+\sum_{j=1}^{m} S_i^* \mathcal{G}_{ij}(A_j^*)\Big(1-\frac{S_i^*}{S_{i_{n+1}}}+\frac{\mathcal{G}_{ij}(A_{j_n})}{\mathcal{G}_{ij}(A_j^*)}-\frac{A_{i_n}}{A_i^*}+\ln\frac{A_{i_n}}{A_{i_{n+1}}}\Big)\\
&+\sum_{j=1}^{m} S_i^* g_{ij}(I_j^*)\Big(1-\frac{I_i^*}{S_i^* G_{ij}^* I_{i_{n+1}}}S_{i_{n-\ell_i+1}}G_{ij}^{n-\ell_i}\Big)\\
&+\sum_{j=1}^{m} S_i^* \mathcal{G}_{ij}(A_j^*)\Big(1-\frac{A_i^*}{S_i^* G_{ij}^* A_{i_{n+1}}}S_{i_{n-\ell_i+1}}G_{ij}^{n-\ell_i}\Big)\\
&-\sum_{j=1}^{m} S_{i_{n+1}}G_{ij}^{n}+\sum_{j=1}^{m} S_{i_{n-\ell_i+1}}G_{ij}^{n-\ell_i}.
\end{aligned}\tag{4.11}
$$

Also, we have

$$
\begin{aligned}
&\mathcal{X}_{n+1}-\mathcal{X}_n\\
=&\nu_i S_i^* e^{-\mu_i\tau_i}\Big[\sum_{k=n-l_i+1}^{n}\Big(\frac{S_{i_{k+1}}}{S_i^*}-1-\ln\frac{S_{i_{k+1}}}{S_i^*}\Big)-\sum_{k=n-l_i}^{n-1}\Big(\frac{S_{i_{k+1}}}{S_i^*}-1-\ln\frac{S_{i_{k+1}}}{S_i^*}\Big)\Big]\\
=&\nu_i S_i^* e^{-\mu_i\tau_i}\Big(\frac{S_{i_{n+1}}}{S_i^*}-\frac{S_{i_{n-l_i+1}}}{S_i^*}+\ln\frac{S_{i_{n-l_i+1}}}{S_{i_{n+1}}}\Big),
\end{aligned}\tag{4.12}
$$

and

$$
\begin{aligned}
&\mathcal{Y}_{n+1}-\mathcal{Y}_n\\
=&S_i^* g_{ij}(I_j^*)\Big[\sum_{k=n-\ell_i+1}^{n}\Big(\frac{S_{i_{k+1}}g_{ij}(I_{j_k})}{S_i^* g_{ij}(I_j^*)}-1-\ln\frac{S_{i_{k+1}}g_{ij}(I_{j_k})}{S_i^* g_{ij}(I_j^*)}\Big)\\
&-\sum_{k=n-\ell_i}^{n-1}\Big(\frac{S_{i_{k+1}}g_{ij}(I_{j_k})}{S_i^* g_{ij}(I_j^*)}-1-\ln\frac{S_{i_{k+1}}g_{ij}(I_{j_k})}{S_i^* g_{ij}(I_j^*)}\Big)\Big]\\
=&S_{i_{n+1}}g_{ij}(I_{j_n})-S_{i_{n-\ell_i+1}}g_{ij}(I_{j_{n-\ell_i}})+S_i^* g_{ij}(I_j^*)\ln\frac{S_{i_{n-\ell_i+1}}g_{ij}(I_{j_{n-\ell_i}})}{S_{i_{n+1}}g_{ij}(I_{j_n})},
\end{aligned}\tag{4.13}
$$

and

$$
\begin{aligned}
&\mathcal{Z}_{n+1}-\mathcal{Z}_n\\
=&S_i^* \mathcal{G}_{ij}(A_j^*)\Big[\sum_{k=n-\ell_i+1}^{n}\Big(\frac{S_{i_{k+1}}\mathcal{G}_{ij}(A_{j_k})}{S_i^* \mathcal{G}_{ij}(A_j^*)}-1-\ln\frac{S_{i_{k+1}}\mathcal{G}_{ij}(A_{j_k})}{S_i^* \mathcal{G}_{ij}(A_j^*)}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&\quad - \sum_{k=n-\ell_i}^{n-1} \Big(\frac{S_{i_{k+1}}\mathcal{G}_{ij}(A_{j_k})}{S_i^*\mathcal{G}_{ij}(A_j^*)} - 1 - \ln \frac{S_{i_{k+1}}\mathcal{G}_{ij}(A_{j_k})}{S_i^*\mathcal{G}_{ij}(A_j^*)} \Big) \Big] \\
&= S_{i_{n+1}}\mathcal{G}_{ij}(A_{j_n}) - S_{i_{n-\ell_i+1}}\mathcal{G}_{ij}(A_{j_{n-\ell_i}}) + S_i^*\mathcal{G}_{ij}(A_j^*) \ln \frac{S_{i_{n-\ell_i+1}}\mathcal{G}_{ij}(A_{j_{n-\ell_i}})}{S_{i_{n+1}}\mathcal{G}_{ij}(A_{j_n})}. \qquad (4.14)
\end{aligned}
$$

As a consequence

$$
\begin{aligned}
&\mathcal{V}_{n+1} - \mathcal{V}_n \\
&= \sum_{i=1}^m \xi_i \Bigg\{ [\mu_i + (1 - e^{-\mu_i\tau_i})\nu_i] S_i^* \Big(2 - \frac{S_{i_{n+1}}}{S_i^*} - \frac{S_i^*}{S_{i_{n+1}}} \Big) \\
&\quad + \nu_i S_i^* e^{-\mu_i\tau_i} \Big(- \frac{S_{i_{n-l_i+1}}}{S_{i_{n+1}}} + 1 + \ln \frac{S_{i_{n-l_i+1}}}{S_{i_{n+1}}} \Big) \\
&\quad + \sum_{j=1}^m S_i^* g_{ij}(I_j^*) \Big(1 - \frac{S_i^*}{S_{i_{n+1}}} + \frac{g_{ij}(I_{j_n})}{g_{ij}(I_j^*)} - \frac{I_{i_n}}{I_i^*} + \ln \frac{I_{i_n}}{I_{i_{n+1}}} \Big) \\
&\quad + \sum_{j=1}^m S_i^* \mathcal{G}_{ij}(A_j^*) \Big(1 - \frac{S_i^*}{S_{i_{n+1}}} + \frac{\mathcal{G}_{ij}(A_{j_n})}{\mathcal{G}_{ij}(A_j^*)} - \frac{A_{i_n}}{A_i^*} + \ln \frac{A_{i_n}}{A_{i_{n+1}}} \Big) \\
&\quad + \sum_{j=1}^m S_i^* g_{ij}(I_j^*) \Big(1 - \frac{I_i^*}{S_i^* G_{ij}^* I_{i_{n+1}}} S_{i_{n-\ell_i+1}} G_{ij}^{n-\ell_i} \Big) \\
&\quad + \sum_{j=1}^m S_i^* \mathcal{G}_{ij}(A_j^*) \Big(1 - \frac{A_i^*}{S_i^* G_{ij}^* A_{i_{n+1}}} S_{i_{n-\ell_i+1}} G_{ij}^{n-\ell_i} \Big) \\
&\quad + \sum_{j=1}^m S_i^* g_{ij}(I_j^*) \ln \frac{S_{i_{n-\ell_i+1}} g_{ij}(I_{j_{n-\ell_i}})}{S_{i_{n+1}} g_{ij}(I_{j_n})} + \sum_{j=1}^m S_i^* \mathcal{G}_{ij}(A_j^*) \ln \frac{S_{i_{n-\ell_i+1}} \mathcal{G}_{ij}(A_{j_{n-\ell_i}})}{S_{i_{n+1}} \mathcal{G}_{ij}(A_{j_n})} \Bigg\}. \\
&\qquad (4.15)
\end{aligned}
$$

In view of

$$
\begin{aligned}
&\ln \frac{S_{i_{n-\ell_i+1}} g_{ij}(I_{j_{n-\ell_i}})}{S_{i_{n+1}} g_{ij}(I_{j_n})} \\
&= \ln \frac{S_i^*}{S_{i_{n+1}}} - \ln \frac{g_{ij}(I_{j_n})}{g_{ij}(I_j^*)} + \ln \frac{I_{j_n}}{I_j^*} - \ln \frac{I_{j_n} I_i^*}{I_j^* I_{i_{n+1}}} + \ln \frac{S_{i_{n-\ell_i+1}} G_{ij}^{n-\ell_i} I_i^*}{S_i^* G_{ij}^* I_{i_{n+1}}} + \ln \frac{G_{ij}^* g_{ij}(I_{j_{n-\ell_i}})}{G_{ij}^{n-\ell_i} g_{ij}(I_j^*)}, \\
&\ln \frac{S_{i_{n-\ell_i+1}} \mathcal{G}_{ij}(A_{j_{n-\ell_i}})}{S_{i_{n+1}} \mathcal{G}_{ij}(A_{j_n})} \\
&= \ln \frac{S_i^*}{S_{i_{n+1}}} - \ln \frac{\mathcal{G}_{ij}(A_{j_n})}{\mathcal{G}_{ij}(A_j^*)} + \ln \frac{A_{j_n}}{A_j^*} - \ln \frac{A_{j_n+} A_i^*}{A_j^* A_{i_{n+1}}} + \ln \frac{S_{i_{n-\ell_i+1}} G_{ij}^{n-\ell_i} A_j^*}{S_i^* G_{ij}^* A_{j_{n+1}}} \\
&\quad + \ln \frac{G_{ij}^* \mathcal{G}_{ij}(A_{j_{n-\ell_i}})}{G_{ij}^{n-\ell_i} \mathcal{G}_{ij}(A_j^*)}
\end{aligned}
$$

and

$$
\sum_{j=1}^m S_i^* g_{ij}(I_j^*) \Big(1 - \frac{G_{ij}^* g_{ij}(I_{j_{n-\ell_i}})}{G_{ij}^{n-\ell_i} g_{ij}(I_j^*)} \Big) + \sum_{j=1}^m S_i^* \mathcal{G}_{ij}(A_j^*) \Big(1 - \frac{G_{ij}^* \mathcal{G}_{ij}(A_{j_{n-\ell_i}})}{G_{ij}^{n-\ell_i} \mathcal{G}_{ij}(A_j^*)} \Big)
$$

$$=\sum_{j=1}^{m} S_i^* g_{ij}(I_j^*)+\sum_{j=1}^{m} S_i^*\mathcal{G}_{ij}(A_j^*)-\sum_{j=1}^{m}\Big(\frac{S^*G_{ij}^* g_{ij}(I_{j_{n-\ell_i}})}{G_{ij}^{n-\ell_i}}+\frac{S^*G_{ij}^*\mathcal{G}_{ij}(A_{j_{n-\ell_i}})}{G_{ij}^{n-\ell_i}}\Big)$$
$$=\sum_{j=1}^{m} S_i^* G_{ij}^*-\sum_{j=1}^{m}\Big(\frac{S^*G_{ij}^*(g_{ij}(I_{j_{n-\ell_i}})+\mathcal{G}_{ij}(A_{j_{n-\ell_i}}))}{G_{ij}^{n-\ell_i}}\Big)=0,$$

we can obtain that

$$\begin{aligned}
&\mathcal{V}_{n+1}-\mathcal{V}_n\\
&=\sum_{i=1}^{m}\xi_i\Bigg\{[\mu_i+(1-e^{-\mu_i\tau_i})\nu_i]S_i^*\Big(2-\frac{S_{i_{n+1}}}{S_i^*}-\frac{S_i^*}{S_{i_{n+1}}}\Big)+\nu_i S_i^* e^{-\mu_i\tau_i}\\
&\quad\times\Big(-\frac{S_{i_{n-l_i+1}}}{S_{i_{n+1}}}+1+\ln\frac{S_{i_{n-l_i+1}}}{S_{i_{n+1}}}\Big)+\sum_{j=1}^{m} S_i^* G_{ij}^*\Big(1-\frac{S_i^*}{S_{i_{n+1}}}-\ln\frac{S_i^*}{S_{i_{n+1}}}\Big)\\
&\quad+\sum_{j=1}^{m} S_i^* g_{ij}(I_j^*)\Big(\frac{g_{ij}(I_{j_n})}{g_{ij}(I_j^*)}-\frac{I_{i_n}}{I_i^*}-\ln\frac{g_{ij}(I_{j_n})}{g_{ij}(I_j^*)}+\ln\frac{I_{j_n}}{I_j^*}-\ln\frac{I_{j_n}I_i^*}{I_j^*I_{i_n}}\Big)\\
&\quad+\sum_{j=1}^{m} S_i^*\mathcal{G}_{ij}(A_j^*)\Big(\frac{\mathcal{G}_{ij}(A_{j_n})}{\mathcal{G}_{ij}(A_j^*)}-\frac{A_{i_n}}{A_i^*}-\ln\frac{\mathcal{G}_{ij}(A_{j_n})}{\mathcal{G}_{ij}(A_j^*)}+\ln\frac{A_{j_n}}{A_j^*}-\ln\frac{A_{j_n}A_i^*}{A_j^*A_{i_n}}\Big)\\
&\quad+\sum_{j=1}^{m} S_i^* g_{ij}(I_j^*)\Big(1-\frac{S_{i_{n-\ell_i+1}}G_{ij}^{n-\ell_i}I_i^*}{S_i^*G_{ij}^*I_{i_{n+1}}}+\ln\frac{S_{i_{n-\ell_i+1}}G_{ij}^{n-\ell_i}I_i^*}{S_i^*G_{ij}^*I_{i_{n+1}}}\Big)\\
&\quad+\sum_{j=1}^{m} S_i^*\mathcal{G}_{ij}(A_j^*)\Big(1-\frac{S_{i_{n-\ell_i+1}}G_{ij}^{n-\ell_i}A_i^*}{S_i^*G_{ij}^*A_{i_{n+1}}}+\ln\frac{S_{i_{n-\ell_i+1}}G_{ij}^{n-\ell_i}A_i^*}{S_i^*G_{ij}^*A_{i_{n+1}}}\Big)\\
&\quad+\sum_{j=1}^{m} S_i^* g_{ij}(I_j^*)\Big(1-\frac{G_{ij}^* g_{ij}(I_{j_{n-\ell_i}})}{G_{ij}^{n-\ell_i}g_{ij}(I_j^*)}+\ln\frac{G_{ij}^* g_{ij}(I_{j_{n-\ell_i}})}{G_{ij}^{n-\ell_i}g_{ij}(I_j^*)}\Big)\\
&\quad+\sum_{j=1}^{m} S_i^*\mathcal{G}_{ij}(A_j^*)\Big(1-\frac{G_{ij}^*\mathcal{G}_{ij}(A_{j_{n-\ell_i}})}{G_{ij}^{n-\ell_i}\mathcal{G}_{ij}(A_j^*)}+\ln\frac{G_{ij}^*\mathcal{G}_{ij}(A_{j_{n-\ell_i}})}{G_{ij}^{n-\ell_i}\mathcal{G}_{ij}(A_j^*)}\Big)\Bigg\}.
\end{aligned}\tag{4.16}$$

Assign

$$\mathcal{H}(I)=\frac{g_{ij}(I_{j_n})}{g_{ij}(I_j^*)}-\frac{I_{i_n}}{I_i^*}-\ln\frac{g_{ij}(I_{j_n})}{g_{ij}(I_j^*)}+\ln\frac{I_{j_n}}{I_j^*}-\ln\frac{I_{j_n}I_i^*}{I_j^*I_{i_n}},$$
$$\mathcal{H}(A)=\frac{\mathcal{G}_{ij}(A_{j_n})}{\mathcal{G}_{ij}(A_j^*)}-\frac{A_{i_n}}{A_i^*}-\ln\frac{\mathcal{G}_{ij}(A_{j_n})}{\mathcal{G}_{ij}(A_j^*)}+\ln\frac{A_{j_n}}{A_j^*}-\ln\frac{A_{j_n}A_i^*}{A_j^*A_{i_n}}$$

and

$$F_i(x_{i_n})=-\frac{x_{i_n}}{x_i^*}+\ln\frac{x_{i_n}}{x_i^*},\quad\text{for } x_{i_n},\ x_i^*>0. \tag{4.17}$$

Then $\mathcal{H}(I)=\Gamma(I_{jn})+F_i(I_{i_n})-F_j(I_{j_n})$, $\mathcal{H}(A)=\Gamma(A_{jn})+F_i(A_{i_n})-F_j(A_{j_n})$, it follows that

$$\begin{aligned}
&\mathcal{V}_{n+1}-\mathcal{V}_n\\
&=\sum_{i=1}^{m}\xi_i\Bigg\{-[\mu_i+(1-e^{-\mu_i\tau_i})\nu_i]\frac{(S_{i_{n+1}}-S_i^*)^2}{S_{i_{n+1}}}-\nu_i S_i^* e^{-\mu_i\tau_i}\Psi\Big(\frac{S_{i_{n-l_i+1}}}{S_{i_{n+1}}}\Big)
\end{aligned}$$

$$
\begin{aligned}
& -\sum_{j=1}^{m} S_i^* G_{ij}^* \Psi\Big(\frac{S_i^*}{S_{i_{n+1}}}\Big) + \sum_{j=1}^{m} S_i^* g_{ij}(I_j^*)\Big(\Gamma(I_{jn}) + F_i(I_{i_n}) - F_j(I_{j_n})\Big) \\
& + \sum_{j=1}^{m} S_i^* \mathcal{G}_{ij}(A_j^*)\Big(\Gamma(A_{jn}) + F_i(A_{i_n}) - F_j(A_{j_n})\Big) \\
& - \sum_{j=1}^{m} S_i^* g_{ij}(I_j^*)\Psi\Big(\frac{S_{i_{n-\ell_i+1}} G_{ij}^{n-\ell_i} I_i^*}{S_i^* G_{ij}^* I_{i_{n+1}}}\Big) - \sum_{j=1}^{m} S_i^* \mathcal{G}_{ij}(A_j^*)\Psi\Big(\frac{S_{i_{n-\ell_i+1}} G_{ij}^{n-\ell_i} A_i^*}{S_i^* G_{ij}^* A_{i_{n+1}}}\Big) \\
& - \sum_{j=1}^{m} S_i^* g_{ij}(I_j^*)\Psi\Big(\frac{G_{ij}^* g_{ij}(I_{j_{n-\ell_i}})}{G_{ij}^{n-\ell_i} g_{ij}(I_j^*)}\Big) - \sum_{j=1}^{m} S_i^* \mathcal{G}_{ij}(A_j^*)\Psi\Big(\frac{G_{ij}^* \mathcal{G}_{ij}(A_{j_{n-\ell_i}})}{G_{ij}^{n-\ell_i} \mathcal{G}_{ij}(A_j^*)}\Big)\Bigg\} \\
\le & \sum_{i=1}^{m}\sum_{j=1}^{m} \xi_i \varpi_{ij}\Bigg\{\max\Big\{\Gamma(I_{jn}), \Gamma(A_{jn})\Big\} + \max\Big\{F_i(I_{i_n}) - F_j(I_{j_n}), F_i(A_{i_n}) - F_j(A_{j_n})\Big\}\Bigg\} \\
\le & \sum_{i=1}^{m}\sum_{j=1}^{m} \xi_i \varpi_{ij}\Bigg\{\max\Big\{F_i(I_{i_n}) - F_j(I_{j_n}), F_i(A_{i_n}) - F_j(A_{j_n})\Big\}\Bigg\}. \qquad (4.18)
\end{aligned}
$$

According to Theorem 3.1 and Corollary 3.3 in Ref. [10], we can derive that $\mathcal{V}_{n+1} - \mathcal{V}_n \le 0$. Similar to the proof in Case 1, we get that E^* is GAS. □

5. Numerical simulations

In this section, we will perform two numerical examples to illustrate our theoretical results. With selection of $m = 2$, $g_{ij}(I_j) = \frac{\beta_{ij} I_j}{1+I_j}$ and the following parameters

Table 1. Parameters

parameter	q_1	q_2	Λ_1	Λ_2	μ_1	μ_2	ν_1	ν_2	κ_1	κ_2	p_1	p_2
value	0.2	0.1	6.45	2.4	0.1	0.05	0.2	0.4	0.2	0.6	0.8	0.9
parameter	l_1^1	l_2^1	ρ_1	ρ_2	ϱ_1	ϱ_2	l_1^2	l_2^2	δ_1	δ_2	γ_1	γ_2
value	1	2	0.7	0.75	0.3	0.25	2	1	0.2	0.15	0.3	0.2
parameter	λ_1	λ_2	r_1	r_2								
value	0.15	0.1	0.05	0.25								

Choose $B = \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix} = \begin{pmatrix} 0.006 & 0.004 \\ 0.007 & 0.005 \end{pmatrix}$. A calculation shows $R = 0.2372$. It follows from Theorem 3.1, the disease-free equilibrium $E_0 = [50, 0, 0, 25, 0, 0]$ of model (1.3) is globally asympotically stable (see Fig.1).

We consider $B = \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix} = \begin{pmatrix} 0.06 & 0.04 \\ 0.07 & 0.05 \end{pmatrix}$. Then $R = 2.372$. The endemic equilibrium $E^* = [28.2107, 2.4844, 2.1295; 11.0236, 2.6239, 1.0496]$ of model (1.3) is globally asympotically stable (see Fig.2) by Theorem 4.1.

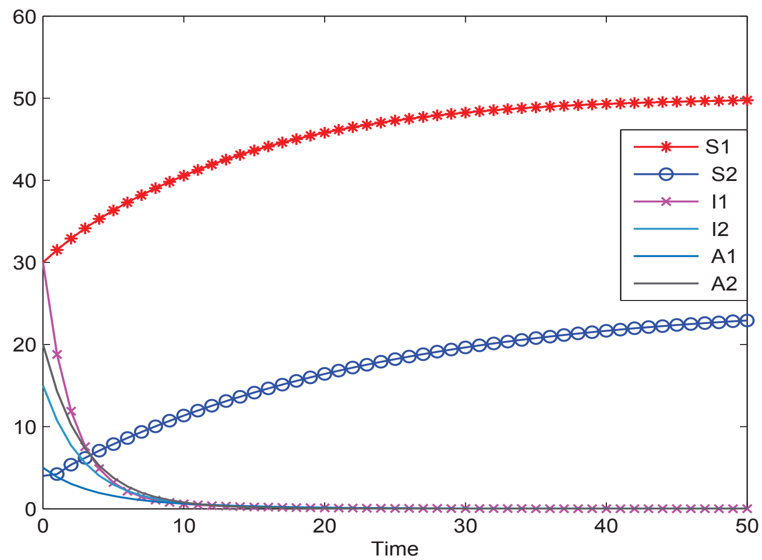

Figure 1.

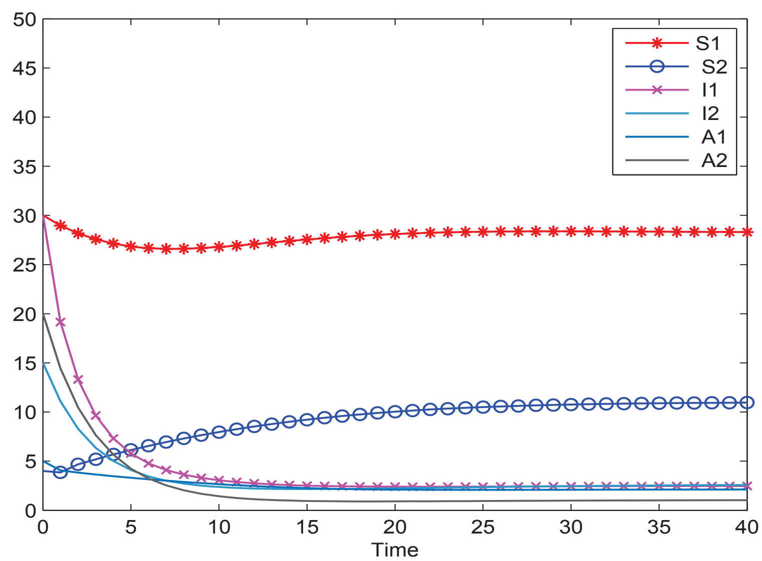

Figure 2.

6. Discussions

In [9], we developed a multi-group nonlinear epidemic model with delays of vaccination, latency, and asymptomatic infection. Due to the complexity of the model and technical shortcomings, we failed to give the numerical simulations of the above continuous differential model. However, the discrete models can provide efficient computational models of the continuous models for numerical simulations. As a continuation of the work of our differential model in Ref. [9], a reasonable idea is to discretize the continuous model in Ref. [9]. To this end, we apply NSFD scheme to discretize the above continuous model (1.1) and obtain its discrete analog model (1.3). Recalling our paper, it is easy to see that there exists correspondence between our discrete results (Theorems 3.1 and 4.1) and the continuous results (Theorem 5.1) in Ref. [9]. That is, the basic reproductive number R fully determines the extinction and persistence of the epidemic of model (1.3). The disease-free equilibrium is globally asymptotically stable if R is not more than one, while the unique endemic equilibrium exists and is globally asymptotically stable when R is great than one. This is the result obtained by constructing the appropriate Lyapunov functions, and the construction in this paper is different from Ref. [9], for details to see the discrete Lyapunov functions V_n (page 5), $\mathbb{V}_n$ (page 8) and $\mathcal{V}_n$ (page 13) in this paper and the continuous Lyapunov functions $\mathbf{U}_1$ (page 9), $\mathbf{U}_2$ (page 11) and W (page 14) in Ref. [9].

Meanwhile, in terms of the NSFD scheme, it is worth highlighting that there are few research results on global dynamics of multi-group epidemic model with delay (e.g., Ref. [22]). Our initial thought is to use the NSFD scheme to discretize the main model proposed in the Ref. [9], but it was found that the model is too complicated to solve the problem. Therefore, we have to start with a relatively simple goal and choose one of the cases (i.e., Case I in Ref. [9]). Fortunately, we finished the investigation in this relatively simple case and obtained some theoretical results (see Theorems 3.1 and 4.1). Until now, we focus on the above continuous version in Ref. [9] and look forward to achieve better results in the future.

Acknowledgments

The authors thank the editor and referees for their valuable comments and careful reading, which have significantly improved the presentation of this paper.

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