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EIGENVALUE PROBLEM FOR FRACTIONAL DIFFERENCE EQUATION WITH NONLOCAL CONDITIONS*

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Abstract In this work, we investigate a class of boundary value problem for fractional difference equation with nonlocal conditions

$$\begin{cases}\Delta^{\nu}u(t)+\lambda f(t+\nu-1,u(t+\nu-1))=0, t\in\mathbb{N}_0^{b+1},\\ u(\nu-2)=h(u), \Delta u(\nu+b)=g(u),\end{cases}$$

where $1<\nu\le 2$ is a real number, $f:\mathbb{N}_{\nu-1}^{\nu+b}\times\mathbb{R}\to(0,+\infty)$ is a continuous function, g,h are given functionals, $b\ge 2$ is an integer, $\lambda>0$ is a parameter. By upper and lower solutions method, we can present the existence result of positive solutions. The eigenvalue intervals to this problem are studied by the properties of the Green function and Guo-Krasnosel'skii fixed point theorem in cones.

Keywords Boundary value problem, fractional difference equation, positive solutions, eigenvalue problem, fixed-point theorem.

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1. Introduction

In recent years, the fractional differential equation theory has gained considerable popularity due to its demonstrated applications in numerous widespread fields, such as chaotic maps, image encryption and neural networks et al. For example, some new chaotic behaviors of the logistic map ware obtained by introducing a delta fractional discrete logistic map, see Wu and Baleanu [17]. Mittag-Leffler stability analysis of fractional discrete-time neural networks was studied by using fixed point technique, see Wu etc [16]. Driven by the wide range of the applications, the boundary value problems for fractional order differential equations have been studied by more and more researchers, see [4, 6, 10–13, 19–22] and references therein.

Compared with the fruitful results in the fractional differential equations, the fractional difference equations are a particularly new topic. Recently, there appeared a number of papers on the discrete fractional calculus, which has begun to build up the theoretical foundations of this area, see [1–3, 5, 15, 18]. For example,

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Goodrich [8] explores a boundary value problem for fractional difference equation of the form

$$\begin{cases} -\Delta^{\nu} y(t) = f(t+\nu-1, y(t+\nu-1)), t \in \mathbb{N}_0^b, \\ y(\nu-2) = g(y), \\ y(\nu+b) = 0, \end{cases}$$

where $f : \mathbb{N}_{\nu-2}^{\nu+b-1} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $g : C(\mathbb{N}_{\nu-2}^{\nu+b} \to \mathbb{R})$ is given functional, and $1 < \nu \leq 2$. This problem is solved by the contraction mapping theorem, Brouwer fixed point theorem, and Guo-Krasnosel'skii fixed point theorem.

Although the boundary value problems for fractional difference equations have been studied by some authors, to the best of our knowledge, the eigenvalue problems are seldom considered. Y. Pan etc [14] study the existence and nonexistence of positive solutions to a boundary value problem for fractional difference equation with a parameter

$$\begin{cases} -\Delta^{\nu} y(t) = \lambda f(t+\nu-1, y(t+\nu-1)), t \in \mathbb{N}_0^{b+1}, \\ y(\nu-2) = y(\nu+b+1) = 0, \end{cases}$$

where $1 < \nu \leq 2$ is a real number, $f : \mathbb{N}_{\nu-1}^{\nu+b} \times \mathbb{R} \to (0, +\infty)$ is a continuous function, $b \geq 2$ is an integer, λ is a parameter. The eigenvalue intervals of boundary value problem to a nonlinear fractional difference equation are considered by the properties of the Green function and Guo-Krasnosel'skii fixed point theorem in cones. Some sufficient conditions to the nonexistence of positive solutions for the boundary value problem are obtained.

In this paper, we consider the following boundary value problem

$$\begin{cases} \Delta^{\nu} u(t) + \lambda f(t+\nu-1, u(t+\nu-1)) = 0, t \in \mathbb{N}_0^{b+1}, \\ u(\nu-2) = h(u), \Delta u(\nu+b) = g(u), \end{cases} \tag{1.1}$$

where $1 < \nu \leq 2$, $f : \mathbb{N}_{\nu-1}^{\nu+b} \times \mathbb{R} \to (0, +\infty)$ is a continuous function and $g, h : C(\mathbb{N}_{\nu-1}^{\nu+b} \to [0, +\infty))$ are given functionals, $b \geq 2$ is an integer, $\lambda > 0$ is a parameter. In this paper, the following features are worth emphasizing. We consider the problem with nonlocal conditions which include some more general forms. What's more, we investigate the intervals of parameter λ for boundary value problem to a nonlinear fractional difference equation.

This paper is organized as follows. In Section 2, we shall recall some definitions and lemmas in order to prove our main results, the corresponding Green function and some properties of the Green function. In Section 3, the existence of positive solution to problem (1.1) is considered by lower and upper solutions method which based on the monotone iterative technique and Guo-Krasnosel'skii fixed point theorem. In Section 4, we present some sufficient conditions for the nonexistence of positive solutions for boundary value problem (1.1). At last, we give an example to demonstrate applications in Section 5.

Notations. Denote $\mathbb{N}_a := \{a, a+1, a+2, \ldots\}$ and $\mathbb{N}_a^b := \{a, a+1, a+2, \ldots, b\}$, where $a, b \in \mathbb{R}$ and $b - a$ is a positive integer.

2. Preliminaries

For convenience to read, we give some necessary basic definitions and lemmas that will be important to us in what follows.

Definition 2.1 ([9]). Assume $f : \mathbb{N}_a^b \to \mathbb{R}$, then we define the forward difference operator Δ by

$$\Delta f(t) := f(t+1) - f(t), t \in \mathbb{N}_a^{b-1}.$$

Definition 2.2 (Definition 2.1, [8]). We define $t^{\underline{\nu}} := \frac{\Gamma(t+1)}{\Gamma(t+1-\nu)}$ for any t and ν, for which the right-hand side is defined. We also appeal to the convention that if $t+1-\nu$ is a pole of the Gamma function and $t+1$ is not a pole, then $t^{\underline{\nu}} = 0$.

Definition 2.3. ([9]) Assume $f : \mathbb{N}_a \to \mathbb{R}$ and $\nu > 0$. Then the ν-th fractional sum of f(based at a) at the point $t \in \mathbb{N}_{a+\nu}$ is defined by

$$\Delta_a^{-\nu} f(t) := \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-s-1)^{\underline{\nu-1}} f(s).$$

Note that by convention on delta sums we can extend the domain of $\Delta_a^{-\nu} f$ to $\mathbb{N}_{a+\nu-N}$, by noting that

$$\Delta_a^{-\nu} f(t) = 0, t \in \mathbb{N}_{a+\nu-N}^{a+\nu-1}.$$

We also define the ν-th fractional difference by $\Delta_a^{\nu} f(t) := \Delta^N \Delta_a^{\nu-N} f(t)$, where $t \in \mathbb{N}_{a+N-\nu}$ and $N \in \mathbb{N}$ is chosen so that $0 \le N-1 < \nu \le N$. Particularly, if we consider $f : \mathbb{N}_0 \to \mathbb{R}$, then the ν-th fractional sum of f at the point $t \in \mathbb{N}_\nu$ is defined by

$$\Delta^{-\nu} f(t) := \frac{1}{\Gamma(\nu)} \sum_{s=0}^{t-\nu} (t-s-1)^{\underline{\nu-1}} f(s),$$

and ν-th fractional difference is defined by $\Delta^{\nu} f(t) := \Delta^N \Delta^{\nu-N} f(t)$.

Lemma 2.1 (Lemma 2.3, [8]). *Let t and ν be any numbers for which $t^{\underline{\nu}}$ and $t^{\underline{\nu-1}}$ are defined. Then*

$$\Delta t^{\underline{\nu}} = \nu t^{\underline{\nu-1}}.$$

Lemma 2.2 (Lemma 2.4, [8]). *Let $0 \le N-1 < \nu \le N$, where $N \in \mathbb{N}$ and $N-1 \ge 0$. Then*

$$\Delta^{-\nu} \Delta^{\nu} y(t) = y(t) + C_1 t^{\underline{\nu-1}} + C_2 t^{\underline{\nu-2}} + \cdots + C_N t^{\underline{\nu-N}},$$

for some $C_i \in \mathbb{R}$, with $1 \le i \le N$.

Lemma 2.3. *Let $g, h : C(\mathbb{N}_{\nu-1}^{\nu+b} \to [0, +\infty))$ and $f : \mathbb{N}_{\nu-1}^{\nu+b} \times \mathbb{R} \to (0, +\infty)$ be given. The unique solution of*

$$\Delta^{\nu} u(t) + f(t+\nu-1, u(t+\nu-1)) = 0, t \in \mathbb{N}_0^{b+1}, \tag{2.1}$$

$$u(\nu-2) = h(u), \Delta u(\nu+b) = g(u), \tag{2.2}$$

has the form

$$u(t) = \sum_{s=0}^{b+1} G(t,s) f(s+\nu-1, u(s+\nu-1)) + \alpha(t) g(u) + \beta(t) h(u), t \in \mathbb{N}_{\nu-2}^{\nu+b+1}, \tag{2.3}$$

where $G(t,s), \alpha(t), \beta(t)$ are given by

$$G(t,s) = \frac{1}{\Gamma(\nu)} \begin{cases} \frac{t^{\underline{\nu-1}}}{(\nu+b)^{\underline{\nu-2}}} (\nu+b-s-1)^{\underline{\nu-2}} - (t-s-1)^{\underline{\nu-1}}, 0 \le s < t-\nu+1 \le b+1, \\ \frac{t^{\underline{\nu-1}}}{(\nu+b)^{\underline{\nu-2}}} (\nu+b-s-1)^{\underline{\nu-2}}, t-\nu+1 \le s \le b+1, \end{cases} \tag{2.4}$$

$$\alpha(t) = \frac{t^{\underline{\nu-1}}}{(\nu-1)(\nu+b)^{\underline{\nu-2}}}, \qquad \beta(t) = \frac{t^{\underline{\nu-2}}}{\Gamma(\nu-1)} - \frac{(\nu-2)t^{\underline{\nu-1}}}{(b+3)\Gamma(\nu)}.$$

Proof. From Lemma 2.2, we find that a general solution for (2.1)–(2.2) is the function

$$u(t) = -\Delta^{-\nu} f(s+\nu-1, u(s+\nu-1)) + c_1 t^{\underline{\nu-1}} + c_2 t^{\underline{\nu-2}}, t \in \mathbb{N}_{\nu-2}^{\nu+b+1}.$$

Applying boundary condition $u(\nu-2) = h(u)$, we can get that

$$c_2 = \frac{h(u)}{\Gamma(\nu-1)}.$$

Then we have

$$u(t) = -\frac{1}{\Gamma(\nu)} \sum_{s=0}^{t-\nu} (t-s-1)^{\underline{\nu-1}} f(s+\nu-1, u(s+\nu-1)) + c_1 t^{\underline{\nu-1}} + \frac{h(u) t^{\underline{\nu-2}}}{\Gamma(\nu-1)}, t \in \mathbb{N}_{\nu-2}^{\nu+b+1}. \tag{2.5}$$

What's more, (2.5) implies that

$$\begin{aligned}\Delta u(t) = &-\tfrac{1}{\Gamma(\nu-1)} \textstyle\sum_{s=0}^{t-\nu+1} (t-s-1)^{\underline{\nu-2}} f(s+\nu-1, u(s+\nu-1)) \\ &+(\nu-1)c_1 t^{\underline{\nu-2}} + \tfrac{(\nu-2)h(u)}{\Gamma(\nu-1)} t^{\underline{\nu-3}}, t \in \mathbb{N}_{\nu-2}^{\nu+b}.\end{aligned}$$

By applying boundary condition $\Delta u(\nu+b) = g(u)$, we obtain that

$$\begin{aligned}g(u) = &-\tfrac{1}{\Gamma(\nu-1)} \textstyle\sum_{s=0}^{b+1} (\nu+b-s-1)^{\underline{\nu-2}} f(s+\nu-1, u(s+\nu-1)) \\ &+(\nu-1)c_1(\nu+b)^{\underline{\nu-2}} + \tfrac{(\nu-2)h(u)}{\Gamma(\nu-1)} (\nu+b)^{\underline{\nu-3}},\end{aligned} \tag{2.6}$$

whence (2.6) implies that

$$\begin{aligned}c_1 = &\frac{1}{\Gamma(\nu)(\nu+b)^{\underline{\nu-2}}} \sum_{s=0}^{b+1} (\nu+b-s-1)^{\underline{\nu-2}} f(s+\nu-1, u(s+\nu-1)) \\ &+ \frac{g(u)}{(\nu-1)(\nu+b)^{\underline{\nu-2}}} - \frac{(\nu-2)h(u)}{(b+3)\Gamma(\nu)}.\end{aligned}$$

Now, taking c_1 into (2.5) gives

$$\begin{aligned}u(t) = &-\tfrac{1}{\Gamma(\nu)} \textstyle\sum_{s=0}^{t-\nu} (t-s-1)^{\underline{\nu-1}} f(s+\nu-1, u(s+\nu-1)) \\ &+t^{\underline{\nu-1}} \Big[\tfrac{1}{\Gamma(\nu)(\nu+b)^{\underline{\nu-2}}} \textstyle\sum_{s=0}^{b+1} (\nu+b-s-1)^{\underline{\nu-2}} f(s+\nu-1, u(s+\nu-1)) \\ &+ \tfrac{g(u)}{(\nu-1)(\nu+b)^{\underline{\nu-2}}} - \tfrac{(\nu-2)h(u)}{(b+3)\Gamma(\nu)} \Big] + \tfrac{h(u)}{\Gamma(\nu-1)} t^{\underline{\nu-2}}, t \in \mathbb{N}_{\nu-2}^{\nu+b+1},\end{aligned} \tag{2.7}$$

which is equivalent to (2.3).

On the other hand, by taking the first-order derivative to (2.7), we have

$$\begin{aligned}\Delta u(t) = &-\tfrac{1}{\Gamma(\nu-1)} \textstyle\sum_{s=0}^{t-\nu+1} (t-s-1)^{\underline{\nu-2}} f(s+\nu-1, u(s+\nu-1)) \\ &+(\nu-1)t^{\underline{\nu-2}} \Big[\tfrac{1}{\Gamma(\nu)(\nu+b)^{\underline{\nu-2}}} \textstyle\sum_{s=0}^{b+1} (\nu+b-s-1)^{\underline{\nu-2}} f(s+\nu-1, u(s+\nu-1)) \\ &+ \tfrac{g(u)}{(\nu-1)(\nu+b)^{\underline{\nu-2}}} - \tfrac{(\nu-2)h(u)}{(b+3)\Gamma(\nu)} \Big] + (\nu-2) \tfrac{h(u)}{\Gamma(\nu-1)} t^{\underline{\nu-3}}, t \in \mathbb{N}_{\nu-2}^{\nu+b}.\end{aligned}$$

It is easy to check that every function of the form of (2.3) is a solution of (2.1)–(2.2). This completes the proof. □

Lemma 2.4 ([7]). *The function $G(t,s)$ defined by (2.4) satisfies the following conditions:*

(*i*) $G(t,s) \geq 0$, *for* $(t,s) \in \mathbb{N}_{\nu-2}^{\nu+b+1} \times \mathbb{N}_0^{b+1}$;

(*ii*) $\max_{t\in\mathbb{N}_{\nu-2}^{\nu+b+1}} G(t,s) = G(s+\nu-1,s)$, *for* $s \in \mathbb{N}_0^{b+1}$;

(*iii*) *there exists a number* $\gamma \in (0,1)$ *such that*

$$\min_{t\in[\frac{b+\nu}{4},\frac{3(b+\nu)}{4}]} G(t,s) \geq \gamma \max_{t\in\mathbb{N}_{\nu-2}^{\nu+b+1}} G(t,s) = \gamma G(s+\nu-1,s), \quad \textit{for} \ \ s \in \mathbb{N}_0^{b+1}.$$

Lemma 2.5. *The function $\alpha(t)$ is strictly increasing in t, for $t \in \mathbb{N}_{\nu-2}^{\nu+b+1}$. In addition, $\min_{t\in\mathbb{N}_{\nu-2}^{\nu+b+1}} \alpha(t) = 0$ and $\max_{t\in\mathbb{N}_{\nu-2}^{\nu+b+1}} \alpha(t) = \frac{\nu+b+1}{\nu-1}$. On the other hand, the function $\beta(t)$ is nonincreasing in t, for $t \in \mathbb{N}_{\nu-2}^{\nu+b}$. In addition, $\min_{t\in\mathbb{N}_{\nu-2}^{\nu+b+1}} \beta(t) = \frac{(\nu+b+1)^{\underline{\nu-2}}}{\Gamma(\nu)}$ and $\max_{t\in\mathbb{N}_{\nu-2}^{\nu+b+1}} \beta(t) = 1$.*

Proof. Note that

$$\begin{aligned}
\Delta_t\beta(t) &= \Delta_t(\tfrac{t^{\underline{\nu-2}}}{\Gamma(\nu-1)}) - \Delta_t(\tfrac{(\nu-2)t^{\underline{\nu-1}}}{(b+3)\Gamma(\nu)}) \\
&= \tfrac{t^{\underline{\nu-3}}(\nu-2)}{\Gamma(\nu-1)} - \tfrac{t^{\underline{\nu-2}}(\nu-2)}{(b+3)\Gamma(\nu-1)} \\
&= \tfrac{(\nu-2)\Gamma(t+1)}{(b+3)\Gamma(\nu-1)\Gamma(t+4-\nu)}(b+\nu-t).
\end{aligned}$$

By $1 < \nu \leq 2, t \in \mathbb{N}_{\nu-2}^{\nu+b+1}$, we know that

$$\frac{(\nu-2)\Gamma(t+1)}{(b+3)\Gamma(\nu-1)\Gamma(t+4-\nu)} \leq 0.$$

Therefore, it is obvious that $\Delta_t\beta(t) \leq 0$, for all $t \in \mathbb{N}_{\nu-2}^{\nu+b}$ and $\Delta_t\beta(t) \geq 0$, for $t = \nu+b+1$. So, the first claim about $\beta(t)$ holds. On the other hand, we have $\beta(\nu-2) = 1$ and $\beta(\nu+b) = \beta(\nu+b+1) = \frac{(\nu+b+1)^{\underline{\nu-2}}}{\Gamma(\nu)}$, it follows that the second claim about $\beta(t)$ holds. □

It may be shown in a similar way that $\alpha(t)$ satisfies the properties given in the statement of this lemma. We omit the details, however. The proof is complete.

Lemma 2.6 (Lemma 4.1, [3]). *Let B be a Banach space and let $P \subseteq B$ be a cone. Assume that Ω_1 and Ω_2 are open subsets contained in B such that $0 \in \Omega_1$ and $\overline{\Omega_1} \subseteq \Omega_2$. Assume, further, that $T : P \cap (\overline{\Omega_2} \setminus \Omega_1) \to P$ is a completely continuous operator. If either*
(1)$\|Tu\| \leq \|u\|$ for $u \in P \cap \partial\Omega_1$ and $\|Tu\| \geq \|u\|$ for $u \in P \cap \partial\Omega_2$; or
(2)$\|Tu\| \geq \|u\|$ for $u \in P \cap \partial\Omega_1$ and $\|Tu\| \leq \|u\|$ for $u \in P \cap \partial\Omega_2$.
Then T has at least one fixed point in $P \cap (\overline{\Omega_2} \setminus \Omega_1)$.

Define the Banach space E by

$$E = \{u : \mathbb{N}_{\nu-2}^{\nu+b+1} \to \mathbb{R}\}$$

with norm $\|u\| = \max\{|u(t)|, t \in \mathbb{N}_{\nu-2}^{\nu+b+1}\}$. Let $I =: [\frac{b+\nu}{4}, \frac{3(b+\nu)}{4}]$. There are two constants $M_\alpha, M_\beta \in (0,1)$ such that $\min_{t\in I} \alpha(t) = M_\alpha\|\alpha\|$ and $\min_{t\in I} \beta(t) = M_\beta\|\beta\|$.

Define the cones

$$P_1 = \{u \in E \mid u(t) \geq 0, t \in \mathbb{N}_{\nu-2}^{\nu+b+1}\},$$
$$P_2 = \left\{u \in E \mid u(t) \geq 0, t \in \mathbb{N}_{\nu-2}^{\nu+b+1}, \min_{t\in I} u(t) \geq \widetilde{\gamma}\|u\|\right\}, \quad (2.8)$$

where $\widetilde{\gamma} = \min\{\gamma, M_\alpha, M_\beta\}$. From Lemma 2.3, we know that u is a unique solution of problem (1.1) if u is a fixed point of the operator $T_\lambda : E \to E$ defined by

$$T_\lambda u(t) = \sum_{s=0}^{b+1} G(t,s)\lambda f(s+\nu-1, u(s+\nu-1)) + \alpha(t)g(u) + \beta(t)h(u), t \in \mathbb{N}_{\nu-2}^{\nu+b+1}. \quad (2.9)$$

Lemma 2.7. *Let T_λ be defined as in* (2.9) *and P_2 as in* (2.8). *Then $T_\lambda : P_2 \to P_2$ is completely continuous.*

Proof. Note that T_λ is a summation operator on a discrete finite set, so T_λ is completely continuous. For all $u \in P_2$, we have that

$$\begin{aligned}
\min_{t\in I}(T_\lambda u)(t) &= \min_{t\in I} \sum_{s=0}^{b+1} G(t,s)\lambda f(s+\nu-1, u(s+\nu-1)) \\
&\quad + \min_{t\in I} \alpha(t)g(u) + \min_{t\in I} \beta(t)h(u) \\
&\geq \widetilde{\gamma} \sum_{s=0}^{b+1} \max_{t\in\mathbb{N}_{\nu-2}^{\nu+b+1}} G(t,s)\lambda f(s+\nu-1, u(s+\nu-1)) + \widetilde{\gamma}\|\alpha\|g(u) + \widetilde{\gamma}\|\beta\|h(u) \\
&= \widetilde{\gamma}\|T_\lambda u\|.
\end{aligned}$$

It is obvious that $(T_\lambda u)(t) \geq 0$ whenever $u \in P_2$, thus, $T_\lambda : P_2 \to P_2$ as desired. □

We introduce the following additional conditions on f, g and h

(H_1) $f(t,v) \leq f(t,w), 0 \leq v \leq w, t \in \mathbb{N}_{\nu-1}^{\nu+b}$;

(A_1) $\lim_{u\to 0^+} \frac{\max_{t\in\mathbb{N}_{\nu-1}^{\nu+b}} f(t,u)}{u} = 0$, $\lim_{u\to+\infty} \frac{\min_{t\in\mathbb{N}_{\nu-1}^{\nu+b}} f(t,u)}{u} = +\infty$;

(A_2) $f_0 = \lim_{u\to 0^+} \frac{\max_{t\in\mathbb{N}_{\nu-1}^{\nu+b}} f(t,u)}{u} < +\infty$, $f_\infty = \lim_{u\to+\infty} \frac{\max_{t\in\mathbb{N}_{\nu-1}^{\nu+b}} f(t,u)}{u} < +\infty$;

(A_3) $f_0^* = \lim_{u\to 0^+} \frac{\min_{t\in\mathbb{N}_{\nu-1}^{\nu+b}} f(t,u)}{u} > 0$, $f_\infty^* = \lim_{u\to+\infty} \frac{\min_{t\in\mathbb{N}_{\nu-1}^{\nu+b}} f(t,u)}{u} > 0$;

(A_4) $g(u), h(u)$ are increasing in u and $g(u) \leq \frac{u}{3A}$, $h(u) \leq \frac{u}{3B}$, for $0 \leq u < +\infty$.

3. Existence of positive solution

Next, we present the method of lower and upper solutions and existence theorems of positive solutions for boundary value problem (1.1) based on the monotone iterative technique.

Definition 3.1. Let $v \in C(\mathbb{N}_{\nu-2}^{\nu+b+1}, \mathbb{E})$. We say that v is a lower solution of boundary value problem (1.1), if

$$\begin{cases} \Delta^\nu v(t) \geq -\lambda f(t+\nu-1, v(t+\nu-1)), t \in \mathbb{N}_0^{b+1}, \\ v(\nu-2) \leq h(v), \\ \Delta v(\nu+b) \leq g(v). \end{cases}$$

Let $w \in C(\mathbb{N}_{\nu-2}^{\nu+b+1}, \mathbb{E})$. We say that w is a upper solution of boundary value problem (1.1), if

$$\begin{cases} \Delta^{\nu} w(t) \leq -\lambda f(t+\nu-1, w(t+\nu-1)), t \in \mathbb{N}_0^{b+1}, \\ w(\nu-2) \geq h(w), \\ \Delta w(\nu+b) \geq g(w). \end{cases}$$

We denote $v \preceq w$ if and only if $w - v \in P_1$, for $v, w \in E$.

Theorem 3.1. *Assume that* (H_1) *and* (A_4) *hold. Boundary value problem* (1.1) *has a lower solution* $v_0 \in P_1$ *and an upper solution* $w_0 \in P_1$ *such that* $v_0 \preceq w_0$. *Then boundary value problem* (1.1) *has the maximal lower solution* v^* *and the minimal upper solution* w^* *on* $[v_0, w_0] \in P_1$, *both* v^* *and* w^* *are positive solutions of boundary value problem* (1.1). *Furthermore,*

$$0 \leq v_0(t) \leq v^*(t) \leq w^*(t) \leq w_0(t), t \in \mathbb{N}_{\nu-2}^{\nu+b+1}.$$

Proof. We will divide our proof into three steps.

Step 1. We will obtain the lower solution sequence $\{v_k\}$ and the upper solution sequence $\{w_k\}$.

Similar to Lemma 2.3, for the given $v_0 \in P_1$, the following boundary value problem

$$\begin{cases} \Delta^{\nu} v_1(t) = -\lambda f(t+\nu-1, v_0(t+\nu-1)), t \in \mathbb{N}_0^{b+1}, \\ v_1(\nu-2) = h(v_0), \Delta v_1(\nu+b) = g(v_0), \end{cases} \tag{3.1}$$

has a unique solution v_1. Since v_0 is a lower solution of boundary value problem (1.1), then

$$\begin{cases} \Delta^{\nu} v_0(t) \geq -\lambda f(t+\nu-1, v_0(t+\nu-1)), t \in \mathbb{N}_0^{b+1}, \\ v_0(\nu-2) \leq h(v_0), \\ \Delta v_0(\nu+b) \leq g(v_0), \end{cases} \tag{3.2}$$

(3.1) minus (3.2), and we can get that

$$\begin{cases} \Delta^{\nu} v_1(t) - \Delta^{\nu} v_0(t) \leq 0, t \in \mathbb{N}_0^{b+1}, \\ v_1(\nu-2) - v_0(\nu-2) \geq 0, \\ \Delta v_1(\nu+b) - \Delta v_0(\nu+b) \geq 0. \end{cases}$$

Let $\Delta^{\nu} v_1(t) - \Delta^{\nu} v_0(t) = \Delta^{\nu}(v_1(t) - v_0(t)) := \sigma(t)$ and $v_1(\nu-2) - v_0(\nu-2) := a$, $\Delta(v_1(\nu+b) - v_0(\nu+b)) := e$. Then we obtain the following boundary value problem

$$\begin{cases} \Delta^{\nu}(v_1(t) - v_0(t)) = \sigma(t) \leq 0, t \in \mathbb{N}_0^{b+1}, \\ v_1(\nu-2) - v_0(\nu-2) = a \geq 0, \\ \Delta(v_1(\nu+b) - v_0(\nu+b)) = e \geq 0. \end{cases}$$

Similar to (2.1)–(2.2), and by Lemmas 2.3 and 2.5, we have

$$v_1(t) - v_0(t) = \beta(t)a + \alpha(t)e - \sum_{s=0}^{b+1} G(t,s)\sigma(s) \geq 0, t \in \mathbb{N}_{\nu-2}^{\nu+b+1}.$$

So, we can get that $v_0 \preceq v_1$. From conditions (H_1) and (A_4), we get

$$\begin{cases} \Delta^\nu v_1(t) = -\lambda f(t+\nu-1, v_0(t+\nu-1)) \geq -\lambda f(t+\nu-1, v_1(t+\nu-1)), t \in \mathbb{N}_0^{b+1}, \\ v_1(\nu-2) = h(v_0) \leq h(v_1), \Delta v_1(\nu+b) = g(v_0) \leq g(v_1). \end{cases}$$

Then $v = v_1$ is a lower solution of boundary value problem (1.1).

Starting from the initial function v_0, by the following iterative scheme

$$\begin{cases} \Delta^\nu v_k(t) = -\lambda f(t+\nu-1, v_{k-1}(t+\nu-1)), t \in \mathbb{N}_0^{b+1}, \\ v_k(\nu-2) = h(v_{k-1}), \Delta v_k(\nu+b) = g(v_{k-1}), \end{cases} \tag{3.3}$$

we can obtain the sequence $\{v_k\}$, where $v = v_k(t)$ are lower solutions of boundary value problem (1.1), and $v_{k-1} \preceq v_k$, so that $\{v_k\}$ is monotonically increasing.

Starting from the initial function w_0, by the following iterative scheme

$$\begin{cases} \Delta^\nu w_k(t) = -\lambda f(t+\nu-1, w_{k-1}(t+\nu-1)), t \in \mathbb{N}_0^{b+1}, \\ w_k(\nu-2) = h(w_{k-1}), \Delta w_k(\nu+b) = g(w_{k-1}), \end{cases} \tag{3.4}$$

we can get the sequence $\{w_k\}$, where $w = w_k(t)$ are upper solutions of boundary value problem (1.1), and $w_k \preceq w_{k-1}$, so that $\{w_k\}$ is monotonically decreasing.

Step 2. We prove that $v_k \preceq w_k$, if $v_{k-1} \preceq w_{k-1}, k = 1, 2, ...$

Since $v_{k-1} \preceq w_{k-1}$, $v_{k-1}(t) \leq w_{k-1}(t)$, and from (H_1), $f(t+\nu-1, v_{k-1}(t+\nu-1)) \leq f(t+\nu-1, w_{k-1}(t+\nu-1))$. By (3.3)–(3.4), for $t \in \mathbb{N}_0^{b+1}$, we can get

$$\begin{cases} \Delta^\nu w_k(t) - \Delta^\nu v_k(t) = -\lambda[f(t+\nu-1, w_{k-1}(t+\nu-1)) - f(t+\nu-1, v_{k-1}(t+\nu-1))] \leq 0, \\ w_k(\nu-2) - v_k(\nu-2) = h(w_{k-1}) - h(v_{k-1}) \geq 0, \\ \Delta w_k(\nu+b) - \Delta v_k(\nu+b) = g(w_{k-1}) - g(v_{k-1}) \geq 0. \end{cases}$$

Similarly, we can show that $v_k \preceq w_k$, in the same way as the above.

Therefore,

$$v_0 \preceq v_1 \preceq \cdots \preceq v_k \preceq \cdots \preceq \cdots \preceq w_k \preceq \cdots \preceq w_1 \preceq w_0.$$

Since P_1 is a normal cone on E, the $\{v_k\}$ is uniformly bounded. Because f is continuous, we can easily get that $\{v_k\}$ is equicontinuous. Hence, the $\{v_k\}$ is relatively compact. In the same way, we can get the $\{w_k\}$ is relatively compact, too. Then there exist $\{v^*\}$ and $\{w^*\}$ such that

$$\lim_{k\to\infty} \|v_k - v^*\| = 0, \lim_{k\to\infty} \|w_k - w^*\| = 0, \tag{3.5}$$

which imply that v^* is the maximum lower solution, w^* is the minimal upper solution of boundary value problem (1.1) in $[v_0, w_0] \subset P_1$, and $v^* \preceq w^*$.

Step 3. We prove that both v^* and w^* are the solutions of boundary value problem (1.1).

Similar to Lemma 2.3 and (3.3), we can get that for $t \in \mathbb{N}_{\nu-2}^{\nu+b+1}$,

$$v_k(t) = \sum_{s=0}^{b+1} G(t,s)\lambda f(s+\nu-1, v_{k-1}(s+\nu-1)) + \alpha(t)g(v_{k-1}) + \beta(t)h(v_{k-1}).$$

From (3.5), and by the continuity of f, g, h and Lebesgue dominated convergence theorem, we have for $t \in \mathbb{N}_{\nu-2}^{\nu+b+1}$,

$$v^*(t) = \sum_{s=0}^{b+1} G(t,s)\lambda f(s+\nu-1, v^*(s+\nu-1)) + \alpha(t)g(v^*) + \beta(t)h(v^*),$$

which implies that v^* is a solution of boundary value problem (1.1).

In the same way, we can show that w^* is a solution of boundary value problem (1.1), too. Furthermore, $0 \le v_0(t) \le v_*(t) \le w_*(t) \le w_0(t), t \in \mathbb{N}_{\nu-2}^{\nu+b+1}$. The proof is completed. □

Then, we investigate the existence of positive solutions for boundary problem (1.1) by Guo-Krasnosel'skii fixed point theorem in cones. For convenience in what follows, we denote that $L = \lfloor \frac{3(\nu+b)}{4} - \nu + 1 \rfloor + 1 - \lceil \frac{\nu+b}{4} - \nu + 1 \rceil$, $K = \max G(t,s)$, for $(t,s) \in \mathbb{N}_{\nu-2}^{\nu+b+1} \times \mathbb{N}_0^{b+1}$, $A = \max_{t\in\mathbb{N}_{\nu-2}^{\nu+b+1}} \alpha(t)$ and $B = \max_{t\in\mathbb{N}_{\nu-2}^{\nu+b+1}} \beta(t)$.

Theorem 3.2. *Suppose that conditions* (A_1) *and* (A_4) *hold. If there exist a small positive constant* M *and a sufficient large constant* N *such that* $M(b+2) < \gamma NL$ *holds for each* $\lambda \in ((\gamma KNL)^{-1}, (KM(b+2))^{-1})$, *then the boundary value problem* (1.1) *has at least one positive solution.*

Proof. By condition (A_1), there exist $r_1 > 0$ and a sufficient small constant $M > 0$ such that

$$f(t,u) \le \frac{Mu}{3}, 0 < u \le r_1, t \in \mathbb{N}_{\nu-1}^{\nu+b}. \tag{3.6}$$

So for $u \in P_2$ with $\|u\| = r_1$, by condition (A_4) and (3.6), we have for all $t \in \mathbb{N}_{\nu-2}^{\nu+b+1}$,

$$\begin{aligned}
\|T_\lambda u\| &= \max_{t\in\mathbb{N}_{\nu-2}^{\nu+b+1}} \left|\sum_{s=0}^{b+1} G(t,s)\lambda f(s+\nu-1, u(s+\nu-1)) + \alpha(t)g(u) + \beta(t)h(u)\right| \\
&= \max_{t\in\mathbb{N}_{\nu-2}^{\nu+b+1}} \sum_{s=0}^{b+1} G(t,s)\lambda f(s+\nu-1, u(s+\nu-1)) + \max_{t\in\mathbb{N}_{\nu-2}^{\nu+b+1}} \alpha(t)g(u) \\
&\quad + \max_{t\in\mathbb{N}_{\nu-2}^{\nu+b+1}} \beta(t)h(u) \\
&\le \tfrac{1}{3}\lambda KMr_1(b+2) + Ag(u) + Bh(u) \\
&\le \tfrac{1}{3}r_1 + \tfrac{1}{3}r_1 + \tfrac{1}{3}r_1 = r_1 = \|u\|.
\end{aligned} \tag{3.7}$$

Thus, if we choose $\Omega_1 = \{u \in E : \|u\| < r_1\}$, then (3.7) implies that

$$\|T_\lambda u\| \le \|u\|, u \in P_2 \cap \partial\Omega_1.$$

What's more, condition (A_1) implies that there exist a number $0 < r_1 < r_2$ and a sufficient large constant N such that

$$f(t,u) \ge Nu, u \ge r_2, t \in \mathbb{N}_{\nu-1}^{\nu+b}. \tag{3.8}$$

We set $r_2^* = \frac{r_2}{\widetilde{\gamma}}$, where $\widetilde{\gamma} < 1$, then $r_2^* > r_2$. Then, $u \in P_2$ and $\|u\| = r_2^*$ implies $\min\limits_{t\in I} u(t) \ge \widetilde{\gamma}\|u\| = r_2$, thus $u(t) \ge r_2$, for all $t \in I$. Therefore, if we choose

$\Omega_2 = \{u \in E : \|u\| < r_2^*\}$, for all $t \in \mathbb{N}_{\nu-2}^{\nu+b}$, by (3.8), we have that

$$\begin{aligned}(T_\lambda u)(t) &= \textstyle\sum_{s=0}^{b+1} G(t,s)\lambda f(s+\nu-1, u(s+\nu-1)) + \alpha(t)g(u) + \beta(t)h(u)\\ &\geq \textstyle\sum_{s=\lceil \frac{\nu+b}{4}-\nu+1\rceil}^{\lfloor \frac{3(\nu+b)}{4}-\nu+1\rfloor} G(t,s)\lambda f(s+\nu-1, u(s+\nu-1))\\ &\geq \lambda\gamma K N r_2 L \geq r_2 = \|u\|,\end{aligned}$$

which means that

$$\|Tu\| \geq \|u\|, u \in P \cap \partial\Omega_2.$$

Finally, from Lemma 2.6, we get that T_λ has at least one fixed point u in P_2 such that $r_1 \leq \|u\| \leq r_2^*$, This fixed point is positive solution of (1.1) and the proof is complete. □

4. Nonexistence

Now, we present some sufficient conditions for the nonexistence of positive solution to the problem (1.1).

Theorem 4.1. *Suppose that conditions* (A_2) *and* (A_4) *hold. Then there exists a* λ_0 *such that for all* $0 < \lambda < \lambda_0 := (3Km(b+2))^{-1}$*, the boundary value problem* (1.1) *has no positive solution.*

Proof. Since $f_0 < +\infty$ and $f_\infty < +\infty$, there exist positive numbers m_1, m_2, d_1, d_2 such that $d_1 < d_2$ and $f(t,u) \leq m_1 u$, for $(t,u) \in \mathbb{N}_0^{b+1} \times [0, d_1]$, $f(t,u) \leq m_2 u$, for $(t,u) \in \mathbb{N}_0^{b+1} \times [d_2, +\infty)$. Let $m = \max\{m_1, m_2, \max_{(t,u)\in\mathbb{N}_0^{b+1}\times[d_1,d_2]} \frac{f(t,u)}{u}\}$. Then we have $f(t,u) \leq mu$, for $(t,u) \in \mathbb{N}_0^{b+1} \times [0, +\infty)$. Suppose u_0 is a positive solution of (1.1). Then we will show that this leads to a contradiction for $0 < \lambda < \lambda_0 := (3Km(b+2))^{-1}$. Since $T_\lambda u_0(t) = u_0(t)$, for $t \in \mathbb{N}_{\nu-2}^{\nu+b+1}$, from the conditions (A_2) and (A_4), we obtain

$$\begin{aligned}u_0(t) = Tu_0(t) &= \textstyle\sum_{s=0}^{b+1} G(t,s)\lambda f(s+\nu-1, u_0(s+\nu-1)) + \alpha(t)g(u_0) + \beta(t)h(u_0)\\ &\leq \lambda K m u_0(b+2) + Ag(u_0) + Bh(u_0)\\ &< \tfrac{1}{3}u_0(t) + \tfrac{1}{3}u_0(t) + \tfrac{1}{3}u_0(t)\\ &= u_0(t).\end{aligned}$$

So, we get $\|u_0\| < \|u_0\|$, which is a contradiction. Therefore, (1.1) has no positive solution. This completes the proof. □

Theorem 4.2. *Suppose that condition* (A_3) *holds. Then there exists a* λ_1 *such that for all* $\lambda > \lambda_1 := (\gamma K m' L)^{-1}$*, the boundary value problem* (1.1) *has no positive solution.*

Proof. Since $f_0^* > 0$ and $f_\infty^* > 0$, there exist positive numbers m_3, m_4, n_1, n_2 such that $n_1 < n_2$ and $f(t,u) \geq m_3 u$, for $(t,u) \in \mathbb{N}_0^{b+1} \times [0, n_1]$, $f(t,u) \geq m_4 u$, for $(t,u) \in \mathbb{N}_0^{b+1} \times [n_2, +\infty)$. Let $m' = \min\{m_3, m_4, \min_{(t,u)\in\mathbb{N}_0^{b+1}\times[n_1,n_2]} \frac{f(t,u)}{u}\}$. Then we have $f(t,u) \geq m'u$, for $(t,u) \in \mathbb{N}_0^{b+1} \times [0, +\infty)$. Suppose u_1 is a positive solution of (1.1). Then we will show that this leads to a contradiction for $\lambda > \lambda_1 :=$

$(\gamma K m' L)^{-1}$. Since $T_\lambda u_1(t) = u_1(t)$, for $t \in \mathbb{N}_{\nu-2}^{\nu+b+1}$, from the conditions (A_3), we obtain

$$\begin{aligned} u_1(t) = Tu_1(t) &= \sum_{s=0}^{b+1} G(t,s)\lambda f(s+\nu-1, u_1(s+\nu-1)) + \alpha(t)g(u_1) + \beta(t)h(u_1) \\ &\geq \lambda\gamma L K m' u_1(t) \\ &> u_1(t). \end{aligned}$$

So, we get $\|u_1\| > \|u_1\|$, which is a contradiction. Therefore, (1.1) has no positive solution. This completes the proof. □

5. Example

In this section, we will present an example to illustrate main results.

Example 5.1. Suppose that $\nu = \frac{3}{2}, b = 9$. Let $f(t,u) = u^2$ and $g(u) = h(u) = 0$. Take $M = \frac{1}{16}$ and $N = 2$. Then problem (1.1) becomes

$$\begin{cases} \Delta^{\frac{3}{2}} u(t) + \lambda u^2 = 0, t \in \mathbb{N}_0^{b+1}, \\ u(\frac{-1}{2}) = \Delta u(\frac{21}{2}) = 0. \end{cases} \tag{5.1}$$

By calculation, we have $K = \max_{(t,s)\in[-\frac{1}{2},\frac{23}{2}]_{\mathbb{N}_0}\times[0,10]_{\mathbb{N}_0}} G(t,s) \approx 0.3364$, $L = \lfloor \frac{3(\nu+b)}{4} - \nu + 1 \rfloor + 1 - \lceil \frac{\nu+b}{4} - \nu + 1 \rceil = 7$, $\gamma = 0.518$, and $\lim_{u\to 0^+} \frac{u^2}{u} = 0$, $\lim_{u\to+\infty} \frac{u^2}{u} = +\infty$. Then $M(b+2) = \frac{11}{16}$ and $\gamma N L = 7.252$. It is easy to see that $M(b+2) < \gamma N L$. So, the conditions of Theorem 3.2 are satisfied. Then the boundary value problem (5.1) has at least one positive solution for each $\lambda \in (0.4099, 0.4893)$.

The numerical results of the proposed problem (5.1) are given in Figure 1 with $\lambda = 0.45$.

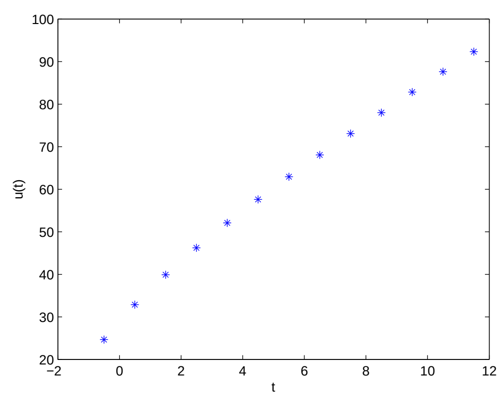

Figure 1. The solution of problem (5.1) with $\lambda = 0.45$.

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