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REGULAR DYNAMICS AND BOX-COUNTING DIMENSION FOR A RANDOM REACTION-DIFFUSION EQUATION ON UNBOUNDED DOMAINS*

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Abstract In this article, we study a random reaction-diffusion equation driven by a Brownian motion with a wide class of nonlinear multiple. First, it is exhibited that the weak solution mapping $L^2(\mathbb{R}^N)$ into $L^p(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$ is Hölder continuous for arbitrary space dimension $N \geq 1$, where $p > 2$ is the growth degree of the nonlinear forcing. The main idea to achieve this is the classic induction technique based on the difference equation of solutions, by using some appropriate multipliers at different stages. Second, the continuity results are applied to investigate the sample-wise regular dynamics. It is showed that the $L^2(\mathbb{R}^N)$-pullback attractor is exactly a pullback attractor in $L^p(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$, and furthermore it is attracting in $L^\delta(\mathbb{R}^N)$ for any $\delta \geq 2$, under almost identical conditions on the nonlinearity as in Wang et al [31], whose result is largely developed in this paper. Third, we consider the box-counting dimension of the attractor in $L^p(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$, and two comparison formulas with L^2-dimension are derived, which are a straightforward consequence of Hölder continuity of the systems.

Keywords Random dynamical system, pullback random attractor, Hölder continuity, higher-order attracting, box counting dimension.

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1. Introduction

In this article, we consider the regular dynamics of the following random reaction-diffusion equation driven by an unbounded stationary stochastic process with a general nonlinear multiple,

$$\frac{du}{dt} = \Delta u - \lambda u + f(t, x, u) + g(t, x) + h(t, x, u)\mathcal{G}(\vartheta_t \omega), \quad t > \tau, x \in \mathbb{R}^N, \tag{1.1}$$

with the initial condition

$$u(\tau, x) = u_\tau, \quad x \in \mathbb{R}^N, \tag{1.2}$$

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where $\lambda > 0, g \in L^2_{loc}(\mathbb{R}, L^2(\mathbb{R}^N))$, and $W(t)$ is a two-sided Browian motion over a classical Wiener probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\}$ with the compact open topology such that Ω is Polish space, $\mathcal{F}$ is its Borel σ-algebra, and $\mathbb{P}$ is the Wiener probability measure on $(\Omega, \mathcal{F})$. The Brownian motion $W(t, \omega)$ is identified as $\omega(t)$, *i.e.*, $W(t, \omega) = \omega(t)$. We define a Wiener shift $\{\vartheta_t\}_{t\in\mathbb{R}}$ over Ω defined as $\vartheta_t \omega(.) = \omega(. + t) - \omega(t)$ for every $t \in \mathbb{R}$. This shift preserves the Wiener measure and is ergodic. Thus the quadruple form $(\Omega, \mathcal{F}, \mathbb{P}, \{\vartheta_t\}_{t\in\mathbb{R}})$ forms a ergodic metric dynamical system, see [1].

For $\flat \neq 0$, let $\mathcal{G} : \Omega \to \mathbb{R}$ be a random variable such that

$$\mathcal{G}(\omega) = \frac{\omega(\flat)}{\flat}.$$

By the Wiener shift $\{\vartheta_t\}_{t\in\mathbb{R}}$ we have

$$\mathcal{G}(\vartheta_t \omega) = \frac{\omega(t + \flat) - \omega(t)}{\flat}, \quad t \in \mathbb{R}.$$

Then by the characteristics of Brownian motion, $\mathcal{G}(\vartheta_t \omega)$ is a stationary process with a normal distribution and is unbounded in t for almost all $\omega \in \Omega$. It is noted that $\mathcal{G}(\vartheta_t \omega)$ can be regarded as a discrete version of the white noise. This stationary process was used to study the chaotic behavior of random differential equations driven by a multiplicative noise of $\mathcal{G}(\vartheta_t \omega)$, see [21, 25].

The random forcing in (1.1) is an unbounded multiplicative noise by a Brownian motion and a nonlinear multiple h. We impose almost identical conditions on f and h as in Wang et al [31], where the authors proved the existence of pullback random attractor for problem (1.1)-(1.2) in $L^2(\mathbb{R}^N)$ and obtained the Wong-Zakai approximation results for the additive noise and linear multiplicative noise cases. The same results are also investigated in [20] if the state space is bounded for problem (1.1)-(1.2).

Note that we ignore the dependence of $\mathcal{G}$ on $\flat$ for our problem, since we do not consider the approximation with respect to $\flat$ as in [31]. Nevertheless, for the Wong-Zakai approximations as $|\flat| \to 0$, [36] obtain some Higher-order results in the cases of $h(t, x, u) = \phi(x)$ and $h(t, x, u) = u$.

In this paper, we study some further asymptotic dynamics of problem (1.1)-(1.2) with a general nonlinear multiple h in some high-order regular (stronger) spaces. For this purpose, we prove the Hölder continuity of solutions in $L^p(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$. The main idea comes from the spirit in [3], where the Sobolev embedding is used. However, the embedding condition $N \geq 3$ is much restrictive. To surmount this hurdle, we directly obtain an iterative relation from the nonlinearity. In particular, some appropriate multipliers at different stages are appropriately employed. By many intricate calculus estimates, we obtain the higher order integrability of difference of solutions in $L^\delta(\mathbb{R}^N)$ for arbitrary $\delta \geq 2$; see Theorem 4.1, and the Hölder continuity of solutions in $H^1(\mathbb{R}^N)$ for the initial data belonging to $L^2(\mathbb{R}^N)$; see Theorem 5.1.

Further, we use the the continuity results to explain the needed regular asymptotic dynamics. On the one hand, we show that the $L^2(\mathbb{R}^N)$-attractor derived in [31] is compact and attracting in the topology of space $L^p(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$, and furthermore is attracting in $L^\delta(\mathbb{R}^N)$ for arbitrary $\delta \geq 2$, see Theorem 6.2 and 6.3 . On the other hand, the Hölder continuity helps us to compare the box-counting dimension of the attractor in $L^p(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$ with that in $L^2(\mathbb{R}^N)$, and two comparison formulas with L^2-dimension are derived; see Theorem 7.1.

The theory of attractors feature prominently in understanding the long time behavior of deterministic or random dynamical system, see recent works [2,8,14,23,26]. The regular asymptotic dynamics in L^p and H^1 spaces are richly investigated, for instance, [27,33,34,43] for the deterministic equations, and [9,13,17–19,35,38,39] for the stochastic differential equations on bounded or unbounded domains. It is pointed out that in random cases, the random forcing in the mentioned literature was always the additive noise or the linear multiplicative noise. The related method is the well-known truncation estimate [13,39], tail estimate technique and spectral decomposition method [17]. Conversely, here the considered random forcing is driven by a multiplicative noise with a general multiple h. Furthermore, the nonlinear deterministic forcing f is different from the version used in [38–40], where a monotone condition is explicitly assumed. In this paper, the form of nonlinearity is completely borrowed from the original sources with only some small additional assumptions on the coefficient. The nonlinear structure of the forcings is an obstacle for us to obtain some higher-order estimate of difference of solutions. To solve this, we analyse the structure of the nonlinearities f and h, and a monotone property is in essence derived, which is crucial for our inductive idea.

We recall the results on the continuity of solution of partial differential equations in regular spaces. In the deterministic case of the reaction-diffusion equation, if the state space $\mathcal{O} \subset \mathbb{R}^N$ is bounded, Robinson [22, pp. 227–231] proved in 2001 that the strong solution $u : H_0^1(\mathcal{O}) \cap L^p(\mathcal{O}) \to H_0^1(\mathcal{O})$ is continuous for $N \leq 2$, where $p \geq 2$ is the order of the nonlinearity of polynomial growth. However, if $N = 3$, the proof in [22] required $p \leq 4$. The continuity in $L^p(\mathcal{O})$ was not included there. Up to 2008, Trujillo and Wang [28] proved that the strong solutions $u : H_0^1(\mathcal{O}) \cap L^p(\mathcal{O}) \to H_0^1(\mathcal{O})$ is continuous for any $N \geq 1$ and $p \geq 2$, which largely extended the result in [22]. The key point in that paper is to derive the estimates that $t\frac{du}{dt} \in L^\infty(0, T; H^2(\mathcal{O}))$ by differentiating the equation with respect to t. However, since the Browian motion is not necessary differentiable, then the method used in the deterministic cases is not applicable to the stochastic differential equations such as problem (1.1)-(1.2).

In 2015, in the random case, by using the Sobolev critical embedding that $H_0^1(\mathcal{O}) \hookrightarrow L^{\frac{2N}{N-2}}(\mathcal{O})$ and a mathematical induction method, Cao et al [3] proved the continuity of solutions from $H_0^1(\mathcal{O}) \cap L^p(\mathcal{O})$ to $H_0^1(\mathcal{O})$ with $N \geq 3$ and $p \geq 2$. This method successfully surmounted the obstacle of non-differentiability of the white noises, and was also used to obtain the higher-order integrability of attractor for stochastic p-Laplacian with multiplicative noise on unbounded domain [39]. Latter, Zhu and Zhou [43] generalized this technique to derive the the continuity of solutions from $L^2(\mathbb{R}^N)$ to $H^1(\mathbb{R}^N)$ for the deterministic reaction-diffusion equations on unbounded domain. However, since the proof in [3, 39, 43] heavily depended the Sobolev critical embedding inequalities, the dimension $N \geq 3$ for reaction-diffusion equations (rep. $N > p$ for p-Lapacian equations [39]) is required and the technique can not be applied directly to the general case $N \geq 1$, especially in unbounded domains. Recently, by an induction technique as in [38, 40], Cui et al [10] studied the strong $(L^2, L^\gamma \cap H_0^1)$-continuity for the deterministic reaction-diffusion equation on bounded domain in any space dimension, and in particular some applications to the fractional dimension was also covered.

The finite box counting dimension (fractional dimension) of attractor is of importance in the sense that if a compact subset A of a metric space X has a finite fractional dimension $\dim_X(A)$ of A such that $\dim_X(A) < \frac{N}{2}$, then A can be embedded into $\mathbb{R}^N$ by an injective mapping, see [4, 15]. In other words, the asymptotic behavior of these systems is determined by only a finite number of degree of freedom [22]. The box counting dimension of random attractors was studied in [16]. For the applications to the stochastic partial differential equations, [32, 41, 42] obtained the finite dimension of stochastic reaction-diffusion equations, FitzHugh-Nagumo system and strongly damped wave equation on bounded domains. This paper discusses the box counting dimension in regular spaces and some comparison relations are derived.

The structure of this paper is as follows. Section 2 is concerned with the notion of pullback random attractor and existence result of bi-spatial attractor for non-autonomous random dynamics systems. In Section 3, we present the assumptions on the nonlinearities. In Section 4, we prove the Hölder continuity of solutions in $L^p(\mathbb{R}^N)$ with respect to initial data in $L^2(\mathbb{R}^N)$ and high-order integrability of difference in $L^\delta(\mathbb{R}^N)$ for arbitrary $\delta > 2$. Section 5 studies the Hölder continuity of solutions in $H^1(\mathbb{R}^N)$. Section 6 and 7 are the applications of the Hölder continuity results to obtain pullback random attractor in $L^p(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$ and two comparison formulas about box counting dimension with L^2-dimension.

2. Non-autonomous random attractors and random dynamical systems

In this section, we recall some basic notions on the pullback random attractor [29, 30] and the existence theorem of bi-spatial attractor [18, 37] for non-autonomous random dynamical systems. The reader is referred to the monographs [1, 5] for a comprehensive information on the random dynamical systems, and to [6, 7, 24] for the original work.

Let $(X, \mathcal{B}(X))$ and $(Y, \mathcal{B}(Y))$ be two completely separable metric spaces, where X serves as the initial space, and Y as the regular space, satisfying $Y \subset X$ (Y has stronger topology than X generally). Let $(\Omega, \mathcal{F}, P, \{\vartheta_t\}_{t\in\mathbb{R}})$ be a metric dynamical system (briefly, MDS ϑ), $\mathbb{R}^+ = \{t \in \mathbb{R} : t \geq 0\}$, and 2^X be the collection of all subsets of X. Let $\mathfrak{D}$ be a collection of some families of nonempty subsets of the initial space X, which serves as the universe of sets.

Definition 2.1. A family of single-valued mappings $\varphi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times X \mapsto X, (t, \tau, \omega, x) \mapsto \varphi(t, \tau, \omega, x)$ is called a random cocycle on X over an MDS ϑ if for all $s, t \in \mathbb{R}^+, \tau \in \mathbb{R}$ and $\omega \in \Omega$, the following statements are satisfied:

- $\varphi(., \tau, ., .) : \mathbb{R}^+ \times \Omega \times X \mapsto X$ is $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(X), \mathcal{B}(X))$-measurable;
- $\varphi(0, \tau, \omega, .)$ is the identity on X;
- $\varphi(t + s, \tau, \omega, .) = \varphi(t, \tau + s, \vartheta_s\omega, \varphi(s, \tau, \omega, .))$.

Definition 2.2. Let φ be a random cocycle on X over an MDS ϑ such that $\varphi(t, \tau, \omega, .)$ maps X into Y for every $t > 0, \tau \in \mathbb{R}$ and $\omega \in \Omega$. A random cocycle φ is said to be continuous in X if the mapping $\varphi(t, \tau, \omega, .) : X \mapsto X$ is continuous for each $t \in \mathbb{R}^+, \tau \in \mathbb{R}$ and $\omega \in \Omega$. A random cocycle φ is said to be continuous from X to Y if the mapping $\varphi(t, \tau, \omega, .) : X \mapsto Y$ is continuous for each $t > 0, \tau \in \mathbb{R}$ and $\omega \in \Omega$.

Definition 2.3. Let $D : \mathbb{R} \times \Omega \to 2^X\backslash\emptyset; D : (\tau, \omega) \mapsto D(\tau, \omega) \in 2^X$ be a set-valued mapping with closed images. We say $D : (\tau, \omega) \mapsto D(\tau, \omega)$ is measurable with respect to $\mathcal{F}$ (briefly, $\mathcal{F}$-measurable) in X if for every fixed $x \in X$ and $\tau \in \mathbb{R}$, the mapping

$$\omega \mapsto d_X(x, D(\tau, \omega)) = \inf_{z \in D(\tau,\omega)} d_X(x, z)$$

is $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$-measurable. If D is measurable, then the family of its images $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ is also called a random set.

Definition 2.4. Let φ be a random cocycle on X over an MDS ϑ. A family of sets $\mathcal{A} = \{\mathcal{A}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathfrak{D}$ is called a pullback attractor in X for φ if the following statements hold:

- $\mathcal{A}$ is a random set in X and $\mathcal{A}(\tau, \omega)$ is compact in X for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$;
- $\mathcal{A}$ is invariant, that is, for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$, $\varphi(t, \tau, \omega, \mathcal{A}(\tau, \omega)) = \mathcal{A}(\tau + t, \vartheta_t\omega), \forall\, t \geq 0$;
- $\mathcal{A}$ is attracting in X, namely, for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$ and $D \in \mathfrak{D}$,

$$\lim_{t\to\infty} \mathrm{dist}_X(\varphi(t, \tau - t, \vartheta_{-t}\omega, D(\tau - t, \vartheta_{-t}\omega)), \mathcal{A}(\tau, \omega)) = 0.$$

Suppose further that $\varphi(t, \tau, \omega, .)$ maps X into Y for every $t > 0, \tau \in \mathbb{R}$ and $\omega \in \Omega$. Then a family of sets $\mathcal{A} = \{\mathcal{A}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathfrak{D}$ is called a (X, Y)-pullback attractor for φ if there hold:

- $\mathcal{A}$ is a random set in Y, $\mathcal{A}(\tau, \omega)$ is compact in Y for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$;
- $\mathcal{A}$ is attracting in Y, namely, for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$ and $D \in \mathfrak{D}$,

$$\lim_{t\to\infty} \mathrm{dist}_Y(\varphi(t, \tau - t, \vartheta_{-t}\omega, D(\tau - t, \vartheta_{-t}\omega)), \mathcal{A}(\tau, \omega)) = 0,$$

where $dist_Y$ is the Hausdorff semi-metric in 2^Y with

$$\mathrm{dist}_Y(A, B) = \sup_{x\in A} \inf_{y\in B} d_Y(x, y).$$

We present the following results for the regularity of pullback random attractor for non-autonomous dynamical systems from [18, 37], where the measurability of attractor in regular space is from [11, Theorem 19]. The result in the initial space X is adapted from [29, 30].

Theorem 2.1. *Suppose that φ is a continuous random cocycle on X (over an MDS ϑ) and $\mathfrak{D}$ is inclusion closed universe in the initial space X. Suppose that*

(i) φ has a closed $\mathfrak{D}$-pullback random absorbing set $K = \{K(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathfrak{D}$ in X, i.e., K is a closed random set and for every $\tau \in \mathbb{R}, \omega \in \Omega$ and $D \in \mathfrak{D}$, there exists an absorbing time $T = T(\tau, \omega, D) > 0$ such that for all $t \geq T$,

$$\varphi(t, \tau - t, \vartheta_{-t}\omega, D(\tau - t, \vartheta_{-t}\omega)) \subseteq K(\tau, \omega);$$

(ii) φ is $\mathfrak{D}$-pullback asymptotically compact in X, i.e., for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$, the sequence

$$\{\varphi(t_n, \tau - t_n, \vartheta_{-t_n}\omega, x_n)\}_{n=1}^{\infty} \text{ is precompact in } X,$$

whenever $t_n \to \infty, x_n \in D(\tau - t_n, \vartheta_{-t_n}\omega)$ with $D \in \mathfrak{D}$;

Then the random cocycle φ possesses a unique $\mathfrak{D}$-pullback random attractor $\mathcal{A} = \{\mathcal{A}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathfrak{D}$ such that for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\mathcal{A}(\tau, \omega) = \cap_{s>0} \overline{\cup_{t\geq s}\varphi(t, \tau - t, \vartheta_{-t}\omega, K(\tau - t, \vartheta_{-t}\omega))}^{X}. \tag{2.1}$$

Suppose further that $Y \subset X$ and $\varphi(t, \tau, \omega, .)$ maps X into Y for every $t > 0, \tau \in \mathbb{R}$ and $\omega \in \Omega$ and φ is $\mathfrak{D}$-pullback asymptotically compact from X to Y i.e., for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$, the sequence

$$\{\varphi(t_n, \tau - t_n, \vartheta_{-t_n}\omega, x_n)\}_{n=1}^{\infty} \text{ is precompact in } Y,$$

whenever $t_n \to \infty, x_n \in D(\tau - t_n, \vartheta_{-t_n}\omega)$ with $D \in \mathfrak{D}$.

Then the $\mathfrak{D}$-pullback random attractor $\mathcal{A}$ defined as (2.1) is also a (X, Y)-pullback random attractor, which can be structured by the Y-metric, namely, for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\mathcal{A}(\tau, \omega) = \cap_{s>0}\overline{\cup_{t \geq s}\varphi(t, \tau - t, \vartheta_{-t}\omega, K(\tau - t, \vartheta_{-t}\omega))}^{Y}.$$

3. Mathematical Background on the equation

In this section, we give the conditions on the nonlinearity f and h, which is totally borrowed from [31]. The nonlinear function f in (1.1) is continuous on $\mathbb{R} \times \mathbb{R}^N \times \mathbb{R}$ and satisfies the following conditions: for all $t, s \in \mathbb{R}$ and $x \in \mathbb{R}^N$ and

$$f(t, x, s)s \leq -\alpha_1|s|^p + \psi_1(t, x), \tag{3.1}$$

$$|f(t, x, s)| \leq \alpha_2|s|^{p-1} + \psi_2(t, x), \tag{3.2}$$

$$\frac{\partial}{\partial s}f(t, x, s) \leq -\alpha_3|s|^{p-2} + \psi_3(t, x), \tag{3.3}$$

where $p > 2, \alpha_1, \alpha_2$ and α_3 are positive constants, $\psi_1 \in L^1_{loc}(\mathbb{R}, L^1(\mathbb{R}^N)) \cap L^{\frac{p}{2}}_{loc}(\mathbb{R}, L^{\frac{p}{2}}(\mathbb{R}^N)), \psi_2 \in L^2_{loc}(\mathbb{R}, L^2(\mathbb{R}^N)) \cap L^{p_1}_{loc}(\mathbb{R}, L^{p_1}(\mathbb{R}^N))$ with $\frac{1}{p_1} + \frac{1}{p} = 1$, and $\psi_3 \in L^{\infty}_{loc}(\mathbb{R}, L^{\infty}(\mathbb{R}^N))$.

Let $h : \mathbb{R} \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be continuous such that for all $t, s \in \mathbb{R}$ and $x \in \mathbb{R}^N$

$$|h(t, x, s)| \leq \beta_1(t, x)|s|^{q-1} + \beta_2(t, x), \tag{3.4}$$

$$\Big|\frac{\partial}{\partial s}h(t, x, s)\Big| \leq \beta_3(t, x)|s|^{q-2} + \beta_4(t, x), \tag{3.5}$$

where $2 \leq q < p, \beta_1 \in L^{\frac{p}{p-q}}_{loc}(\mathbb{R}, L^{\frac{p}{p-q}}(\mathbb{R}^N)) \cap L^{\frac{2p-2}{p-q}}_{loc}(\mathbb{R}, L^{\frac{2p-2}{p-q}}(\mathbb{R}^N))$ and $\beta_2 \in L^2_{loc}(\mathbb{R}, L^2(\mathbb{R}^N)) \cap L^{\frac{2p-2}{p-1}}_{loc}(\mathbb{R}, L^{\frac{2p-2}{p-1}}(\mathbb{R}^N))$ and $\beta_3, \beta_4 \in L^{\infty}_{loc}(\mathbb{R}, L^{\infty}(\mathbb{R}^N))$.

Remark 3.1. The conditions on f and h are the same as in [21, 31]. For our purpose, we need to split f into $f = f_1 + f_2$ with

$$f_1 = -\frac{\alpha_3}{2(p-1)}|s|^{p-2}s, \tag{3.6}$$

and $f_2 = f - f_1$. Then f_1 is monotonous, i.e., there exists a positive constant c_1 such that

$$(f_1(t, x, s_1) - f_1(t, x, s_2))(s_1 - s_2) \leq -c_1|s_1 - s_2|^p, \tag{3.7}$$

where $c_1 = c_1(\alpha_3, p, N)$. Since $\frac{\partial}{\partial s}f_1(t, x, s) = -\frac{\alpha_3}{2}|s|^{p-2}$ for $p > 2$, then it is easy to see that f_2 satisfies

$$\frac{\partial}{\partial s}f_2(t, x, s) \leq -\frac{\alpha_3}{2}|s|^{p-2} + \psi_3(t, x), \tag{3.8}$$

where α_3 and ψ_3 are as in (3.3).

Remark 3.2. For the function h, it is easy to check that condition (3.5) implies that

$$|h(t,x,s_1) - h(t,x,s_2)| \leq \beta_5(t,x)|s_1 - s_2|(1 + |s_1|^{q-2} + |s_2|^{q-2}). \tag{3.9}$$

for some $\beta_5 \in L^{\infty}_{loc}(\mathbb{R}, L^{\infty}(\mathbb{R}^N))$.

For the non-autonomous term g and ψ_1, we will assume that for any $\tau \in \mathbb{R}$,

$$\int_{-\infty}^{\tau} e^{\lambda s}(\|g(s)\|^2 + \|\psi_1(s)\|_{L^1})ds < +\infty. \tag{3.10}$$

As for the existence of the tempered random attractor, we need further to assume that the non-autonomous term satisfies for any $c > 0$,

$$\lim_{t \to -\infty} e^{ct} \int_{-\infty}^{0} e^{\lambda s}(\|g(s+t)\|^2 + \|\psi_1(s+t)\|_{L^1})ds = 0. \tag{3.11}$$

Throughout the paper, the letter c is a generic positive constant that may change its value from line to line, and $\|.\|_p$ denote the norm in $L^p(\mathbb{R}^N)$. For $p = 2$ we write $\|.\|_2 = \|.\|$.

4. Hölder continuity of solutions in $L^p(\mathbb{R}^N)$ and high-order integrability of difference in $L^\delta(\mathbb{R}^N)$ for arbitrary $\delta \geq 2$

In this section, based on the difference equation of solutions to problem (1.1)-(1.2), we prove the Hölder continuity of solutions in $L^p(\mathbb{R}^N)$ and high-order integrability of difference in $L^\delta(\mathbb{R}^N)$ for arbitrary $\delta \in [2, +\infty)$. It is note that the existence and continuity of solution in $L^2(\mathbb{R}^N)$ for problem (1.1)-(1.2) have been proved in [31].

Given $\tau \in \mathbb{R}$ and $\omega \in \Omega$, let $u_i(t,\tau,\omega,u_{\tau,i})$ be the solutions to equations (1.1)-(1.2) with the initial data $u_{\tau,i}, i = 1,2$. Then we get the difference equation with respect to $U = u_1 - u_2$:

$$\frac{dU}{dt} = \Delta U - \lambda U + f(t,x,u_1) - f(t,x,u_2) + (h(t,x,u_1) - h(t,x,u_2))\mathcal{G}(\vartheta_t\omega), \tag{4.1}$$

with the initial condition $U_\tau = u_{\tau,1} - u_{\tau,2}$.

For our purpose, we need to give an equivalent form of (4.1) by rewriting the nonlinearities in (4.1). Let

$$\gamma_{f_2}(t) = \int_0^1 \frac{\partial f_2}{\partial s}(t,x,su_1 + (1-s)u_2)ds, \tag{4.2}$$

and

$$\gamma_h(t) = \int_0^1 \frac{\partial h}{\partial s}(t,x,su_1 + (1-s)u_2)ds, \tag{4.3}$$

where f_1 and f_2 are in (3.6). Then (4.1) is equivalent to

$$\frac{dU}{dt} = \Delta U - \lambda U + (f_1(t,x,u_1) - f_1(t,x,u_2)) + \gamma_{f_2}(t)U + \mathcal{G}(\vartheta_t\omega)\gamma_h(t)U, \tag{4.4}$$

with the initial condition $U_\tau = u_{\tau,1} - u_{\tau,2}$.

The following lemma shows the higher-order integrability of difference and Hölder continuity of solutions to problem (1.1)-(1.2) with f satisfying (3.1)-(3.3) and h satisfying (3.4) and (3.5).

Theorem 4.1. *Suppose that (3.1)-(3.5) hold, $p > q \geq 2$ and the space dimension $N \geq 1$. Take $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $T > 0$. Let $u_i(t) = u_i(t, \tau, \omega, u_{\tau,i})$ be the unique weak solution of Eq.(1.1)-(1.2) corresponding to the initial value $u_{\tau,i}$, $i = 1, 2$. Then for every $k \in \mathbb{N}$, there exists a positive deterministic constant $c^{(k)}$ depending only on k, τ, ω, T such that for every $t \in (\tau, \tau + T]$,*

$$(t-\tau)\|(t-\tau)^{b_k}(u_1(t,\tau,\omega,u_{\tau,1}) - u_2(t,\tau,\omega,u_{\tau,2}))\|_{a_k}^{a_k} \leq c^{(k)}\|u_{\tau,1} - u_{\tau,2}\|^2, \quad (A_k)$$

and

$$\int_\tau^t \|(s-\tau)^{b_{k+1}}(u_1(s,\tau,\omega,u_{\tau,1}) - u_2(s,\tau,\omega,u_{\tau,2}))\|_{a_{k+1}}^{a_{k+1}} ds \leq c^{(k)}\|u_{\tau,1} - u_{\tau,2}\|^2, \ (B_k)$$

where $a_k = kp - 2(k-1), k = 1, 2, ...;\quad b_1 = 1, b_k = \frac{2p+2k-4}{kp-2(k-1)},\quad k = 2, 3, ...$.

Proof. We prove the result by the induction scheme. First by (3.7), it is clear that for $m \geq 2$,

$$\int_{\mathbb{R}^N} (f_1(t,x,u_1) - f_1(t,x,u_2))|U|^{m-2}U dx \leq -c_1\|U(t)\|_{m+p-2}^{m+p-2}. \tag{4.5}$$

Using the test function $|U|^{m-2}U$ in (4.4) with $m \geq 2$, we obtain

$$\begin{aligned}
&\frac{1}{m}\frac{d}{dt}\|U(t)\|_m^m + \lambda\|U(t)\|_m^m + c_0\|\nabla|U(t)|^{\frac{m}{2}}\|^2 + c_1\|U(t)\|_{m+p-2}^{m+p-2} \\
&\leq \int_{\mathbb{R}^N} \gamma_{f_2}(t)|U(t)|^m dx + |\mathcal{G}(\vartheta_t\omega)| \int_{\mathbb{R}^N} \gamma_h(t)|U(t)|^m dx.
\end{aligned} \tag{4.6}$$

By (4.2) and (3.8), the first term on the right hand side of (4.6) is estimated as

$$\begin{aligned}
\int_{\mathbb{R}^N} \gamma_{f_2}(t)|U(t)|^m dx &\leq -\frac{\alpha_3}{2}\int_{\mathbb{R}^N}\Big(\int_0^1 |su_1 + (1-s)u_2|^{p-2}ds + \psi_3(t,x)\Big)|U(t)|^m dx \\
&\leq -\frac{\alpha_3}{2}\int_{\mathbb{R}^N}\int_0^1 |su_1 + (1-s)u_2|^{p-2}ds|U(t)|^m dx + c\int_{\mathbb{R}^N}|U(t)|^m dx,
\end{aligned} \tag{4.7}$$

since $\psi_3 \in L^\infty_{loc}(\mathbb{R}, L^\infty(\mathbb{R}^N))$. On the other hand, when $2 < q < p$, by (4.3) and (3.5) and using Young inequality, the second term on the right hand side of (4.6) is bounded by, for every $t \in (\tau, \tau + T]$,

$$\begin{aligned}
&|\mathcal{G}(\vartheta_t\omega)| \int_{\mathbb{R}^N} \gamma_h(t)|U(t)|^m dx \\
\leq &|\mathcal{G}(\vartheta_t\omega)| \int_{\mathbb{R}^N} \Big(\beta_3(t,x)\int_0^1 |su_1 + (1-s)u_2|^{q-2}ds + \beta_4(t,x)\Big)|U(t)|^m dx \\
= &|\mathcal{G}(\vartheta_t\omega)| \int_{\mathbb{R}^N} \beta_3(t,x)\int_0^1 |su_1 + (1-s)u_2|^{q-2}ds|U(t)|^m dx
\end{aligned}$$

$$
\begin{aligned}
&+|\mathcal{G}(\vartheta_t\omega)|\int_{\mathbb{R}^N}\beta_4(t,x)|U(t)|^m dx\\
\leq&\frac{\alpha_3}{2}\int_{\mathbb{R}^N}\int_0^1|su_1+(1-s)u_2|^{p-2}ds|U(t)|^m dx\\
&+c|\mathcal{G}(\vartheta_t\omega)|^{\frac{p-2}{p-q}}\int_{\mathbb{R}^N}|\beta_3(t,x)|^{\frac{p-2}{p-q}}|U(t)|^m dx\\
&+|\mathcal{G}(\vartheta_t\omega)|\int_{\mathbb{R}^N}\beta_4(t,x)|U(t)|^m dx\\
\leq&\frac{\alpha_3}{2}\int_{\mathbb{R}^N}\int_0^1|su_1+(1-s)u_2|^{p-2}ds|U(t)|^m dx+c\int_{\mathbb{R}^N}|U(t)|^m dx,
\end{aligned}
\tag{4.8}
$$

using $\beta_3,\beta_4\in L^\infty_{loc}(\mathbb{R},L^\infty(\mathbb{R}^N))$. Combine (4.7) and (4.8) to get that for every $t\in(\tau,\tau+T]$,

$$
\int_{\mathbb{R}^N}\gamma_{f_2}(t)|U(t)|^m dx+|\mathcal{G}(\vartheta_t\omega)|\int_{\mathbb{R}^N}\gamma_h(t)|U(t)|^m dx\leq c\int_{\mathbb{R}^N}|U(t)|^m dx, \tag{4.9}
$$

where $c=c(m,\tau,\omega,T)$. If $q=2$, then it is easy to show that (4.9) also holds. Therefore by (4.6) and (4.9), we find that there exist positive constants c_1 and c_2 such that for every $t\in(\tau,\tau+T]$, $2\leq q<p$ and $m\geq 2$,

$$
\frac{d}{dt}\|U(t)\|_m^m+c_0\|\nabla|U(t)|^{\frac{m}{2}}\|^2+c_1\|U(t)\|_{m+p-2}^{m+p-2}\leq c_2\|U(t)\|_m^m, \tag{4.10}
$$

which obviously gives

$$
\frac{d}{dt}\|U(t)\|_m^m+c_1\|U(t)\|_{m+p-2}^{m+p-2}\leq c_2\|U(t)\|_m^m, \tag{4.11}
$$

where $c_i=c_i(m,\tau,\omega,T)$ for $i=1,2$.

We first show (A_k) and (B_k) hold true for $k=1$.

Given $m=2$ in (4.10), we have

$$
\frac{d}{dt}\|U(t)\|^2+c_0\|\nabla U(t)\|^2+c_1\|U(t)\|_p^p\leq c_2\|U(t)\|^2. \tag{4.12}
$$

Multiplying (4.12) with e^{-c_2t} and then replacing t by s and integrating from τ to t for $t\in(\tau,\tau+T]$, we find

$$
\|U(t)\|^2\leq e^{c_2(t-\tau)}\|U(\tau)\|^2,\qquad t\in(\tau,\tau+T]. \tag{4.13}
$$

By (4.12) and (4.13), we obtain that for every $t\in(\tau,\tau+T]$,

$$
\int_\tau^t(\|\nabla U(s)\|^2+\|U(s)\|_p^p)ds\leq c\|U_\tau\|^2. \tag{4.14}
$$

Given $b_1=1$, from (4.14) it is easy to see that for every $t\in(\tau,\tau+T]$,

$$
\int_\tau^t\|(s-\tau)^{b_1}U(s)\|_p^p ds\leq c\|U_\tau\|^2. \tag{4.15}
$$

Letting $m=p$ in (4.11), we have

$$
\frac{d}{dt}\|U(t)\|_p^p+c_1\|U(t)\|_{2p-2}^{2p-2}\leq c_2\|U(t)\|_p^p. \tag{4.16}
$$

Observer that

$$(t-\tau)^{b_1 p}\frac{d}{dt}\|U(t)\|_p^p = \frac{d}{dt}\|(t-\tau)^{b_1}U(t)\|_p^p - b_1 p(t-\tau)^{b_1 p-1}\|U(t)\|_p^p. \qquad (4.17)$$

Then multiplying (4.16) with $(t-\tau)^{b_1 p+1}$ and using (4.17), we see that for every $t \in (\tau, \tau+T]$,

$$\begin{aligned}
&(t-\tau)\frac{d}{dt}\|(t-\tau)^{b_1}U(t)\|_p^p + c_1\|(t-\tau)^{\frac{p+1}{2p-2}}U(t)\|_{2p-2}^{2p-2}\\
\le& c_2(t-\tau)^{b_1 p+1}\|U(t)\|_p^p + b_1 p(t-\tau)^{b_1 p}\|U(t)\|_p^p\\
=& c_2(t-\tau)\|(t-\tau)^{b_1}U(t)\|_p^p + b_1 p\|(t-\tau)^{b_1}U(t)\|_p^p\\
\le& c\|(t-\tau)^{b_1}U(t)\|_p^p, \qquad (4.18)
\end{aligned}$$

from which it follows that for every $t \in (\tau, \tau+T]$,

$$(t-\tau)\frac{d}{dt}\|(t-\tau)^{b_1}U(t)\|_p^p \le c\|(t-\tau)^{b_1}U(t)\|_p^p. \qquad (4.19)$$

Replacing t by s in (4.19) and integrating from τ to t for $t \in (\tau, \tau+T]$, by integration by parts, we find

$$\begin{aligned}
\int_\tau^t (s-\tau)\frac{d}{ds}\|(s-\tau)^{b_1}U(s)\|_p^p ds &= (t-\tau)\|(t-\tau)^{b_1}U(t)\|_p^p - \int_\tau^t \|(s-\tau)^{b_1}U(s)\|_p^p ds\\
&\le c\int_\tau^t \|(s-\tau)^{b_1}U(s)\|_p^p ds. \qquad (4.20)
\end{aligned}$$

By (4.15) and (4.20), we get that for every $t \in (\tau, \tau+T]$,

$$(t-\tau)\|(t-\tau)^{b_1}U(t)\|_p^p \le (c+1)\int_\tau^t \|(s-\tau)^{b_1}U(s)\|_p^p ds \le c^{(1)}\|U_\tau\|^2, \qquad (4.21)$$

where $c^{(1)} = c^{(1)}(\tau, \omega, T)$. Multiply (4.18) with $(t-\tau)^{p-1}$ and along with (4.21) to yield

$$\begin{aligned}
&(t-\tau)^p\frac{d}{dt}\|(t-\tau)^{b_1}U(t)\|_p^p + c_1\|(t-\tau)^{\frac{2p}{2p-2}}U(t)\|_{2p-2}^{2p-2}\\
\le& c(t-\tau)^{p-1}\|(t-\tau)^{b_1}U(t)\|_p^p \le c\|U_\tau\|^2, \qquad (4.22)
\end{aligned}$$

for all $t \in (\tau, \tau+T]$. Integrating (4.22) from τ to t, by integration by parts, and utilizing (4.15), we find that for every $t \in (\tau, \tau+T]$,

$$\begin{aligned}
c_1\int_\tau^t \|(s-\tau)^{b_2}U(s)\|_{2p-2}^{2p-2}ds &\le cT\|U_\tau\|^2 + p\int_\tau^t (s-\tau)^{p-1}\|(s-\tau)^{b_1}U(s)\|_p^p ds\\
&\le cT\|U_\tau\|^2 + pT^{p-1}\int_\tau^t \|(s-\tau)^{b_1}U(s)\|_p^p ds\\
&\le c^{(1)}\|U_\tau\|^2, \qquad (4.23)
\end{aligned}$$

where $b_2 = \frac{2p}{2p-2}$. Therefore by (4.21) and (4.23) we claim that (A_k) and (B_k) hold true for $k=1$. Letting $m = 2p-2$ in (4.11), choosing the multipliers first $(t-\tau)^{2p+1}$

and then $(t-\tau)$, by an almost similar procedure as (4.16)-(4.23), we can show that (A_2) and (B_2) hold, respectively.

Step two. We now prove that (A_{k+1}) and (B_{k+1}) hold if (A_k) and (B_k) hold for some $k \geq 1$. To this end, given $m = a_{k+1}$ in (4.11), we obtain

$$\frac{d}{dt}\|U(t)\|_{a_{k+1}}^{a_{k+1}} + c_1\|U(t)\|_{a_{k+1}+p-2}^{a_{k+1}+p-2} \leq c_2\|U(t)\|_{a_{k+1}}^{a_{k+1}}, \tag{4.24}$$

where $c_i = c_i(k,\tau,\omega,T)$, $i=1,2$. Note that

$$\begin{aligned}(t-\tau)^{b_{k+1}a_{k+1}}\frac{d}{dt}\|U(t)\|_{a_{k+1}}^{a_{k+1}} =&\frac{d}{dt}\|(t-\tau)^{b_{k+1}}U(t)\|_{a_{k+1}}^{a_{k+1}}\\ &- b_{k+1}a_{k+1}(t-\tau)^{b_{k+1}a_{k+1}-1}\|U(t)\|_{a_{k+1}}^{a_{k+1}}.\end{aligned} \tag{4.25}$$

Multiplying (4.24) with $(t-\tau)^{b_{k+1}a_{k+1}+1}$ and using (4.25), we deduce that for all $t \in (\tau, \tau+T]$,

$$\begin{aligned}&(t-\tau)\frac{d}{dt}\|(t-\tau)^{b_{k+1}}U(t)\|_{a_{k+1}}^{a_{k+1}} + c_1(t-\tau)^{b_{k+1}a_{k+1}+1}\|U(t)\|_{a_{k+1}+p-2}^{a_{k+1}+p-2}\\ \leq&(c_2(t-\tau)+b_{k+1}a_{k+1})\|(t-\tau)^{b_{k+1}}U(t)\|_{a_{k+1}}^{a_{k+1}}\\ \leq&c\|(t-\tau)^{b_{k+1}}U(t)\|_{a_{k+1}}^{a_{k+1}}.\end{aligned} \tag{4.26}$$

Replacing t of (4.26) by s, then integrating from τ to t, by integration by parts and along with assumption (B_k), we get that there exists a constant $c^{(k+1)} = c^{(k+1)}(k,\tau,\omega,T) > 0$ such that

$$\begin{aligned}&(t-\tau)\|(t-\tau)^{b_{k+1}}U(t)\|_{a_{k+1}}^{a_{k+1}}\\ &\leq c\int_\tau^t \|(s-\tau)^{b_{k+1}}U(s)\|_{a_{k+1}}^{a_{k+1}}ds + \int_\tau^t \|(s-\tau)^{b_{k+1}}U(s)\|_{a_{k+1}}^{a_{k+1}}ds\\ &\leq c^{(k+1)}\|U_\tau\|^2, \quad \forall\, t \in (\tau,\tau+T],\end{aligned} \tag{4.27}$$

which gives that (A_{k+1}) holds true for arbitrary $k \in \mathbb{N}$.

Multiply (4.26) with $(t-\tau)$ to show that

$$\begin{aligned}&(t-\tau)^2\frac{d}{dt}\|(t-\tau)^{b_{k+1}}U(t)\|_{a_{k+1}}^{a_{k+1}} + c_1(t-\tau)^{b_{k+1}a_{k+1}+2}\|U(t)\|_{a_{k+1}+p-2}^{a_{k+1}+p-2}\\ \leq&c(t-\tau)\|(t-\tau)^{b_{k+1}}U(t)\|_{a_{k+1}}^{a_{k+1}}.\end{aligned} \tag{4.28}$$

Since $a_{k+2} = a_{k+1}+p-2$ and $b_{k+2} = \frac{2p+2k}{(k+2)p-2(k+1)} = \frac{a_{k+1}b_{k+1}+2}{a_{k+1}+p-2}$, then we have

$$(t-\tau)^{b_{k+1}a_{k+1}+2}\|U(t)\|_{a_{k+1}+p-2}^{a_{k+1}+p-2} = \|(t-\tau)^{b_{k+2}}U(t)\|_{a_{k+2}}^{a_{k+2}}. \tag{4.29}$$

Hence by a combination of (4.28) and (4.29), and along with (4.27), it produces that for $t \in (\tau, \tau+T]$,

$$(t-\tau)^2\frac{d}{dt}\|(t-\tau)^{b_{k+1}}U(t)\|_{a_{k+1}}^{a_{k+1}} + c_1\|(t-\tau)^{b_{k+2}}U(t)\|_{a_{k+2}}^{a_{k+2}} \leq c\|U_\tau\|^2, \tag{4.30}$$

for all $t \in (\tau, \tau+T]$. Replacing t by s and integrating (4.30) from τ to t, by integration by parts using (B_k), we obtain

$$c_1\int_\tau^t \|(s-\tau)^{b_{k+2}}U(s)\|_{a_{k+2}}^{a_{k+2}}ds \leq 2\int_\tau^t (s-\tau)\|(s-\tau)^{b_{k+1}}U(s)\|_{a_{k+1}}^{a_{k+1}}ds + cT\|U_\tau\|^2$$

$$\leq c^{(k+1)}\|U_\tau\|^2,$$

for all $t \in (\tau, \tau+T]$. This shows that (B_{k+1}) holds true for arbitrary $k \in \mathbb{N}$, which concludes the total proof. □

Remark 4.1. The method used for the difference U can not be carried out for the single solution u to the problem (1.1)-(1.2) when the forcing $g \in L^2_{loc}(\mathbb{R}, L^2(\mathbb{R}^N))$, in which case we can only show that the solution $u(t,\tau,\omega,u_\tau) \in L^p(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$ for every $t > \tau \in \mathbb{R}$ and $\omega \in \Omega$, see Remark 5.1 in the following. Of course, if $g \in L^\infty_{loc}(\mathbb{R}, L^\infty(\mathbb{R}^N))$, by a similar technique as the above theorem we can prove the solution $u(t,\tau,\omega,u_\tau) \in L^r(\mathbb{R}^N)$ for every $t > \tau \in \mathbb{R}, \omega \in \Omega$ and any $r \geq p$.

Remark 4.2. If $k = 1$, we have

$$\|(u_1(t,\tau,\omega,u_{\tau,1}) - u_2(t,\tau,\omega,u_{\tau,2}))\|_p \leq \frac{c}{(t-\tau)^{\frac{p+1}{p}}}\|u_{\tau,1} - u_{\tau,2}\|^{\frac{2}{p}},$$

i.e., the solution is $\frac{2}{p}$-Hölder continuous from $L^2(\mathbb{R}^N)$ to $L^p(\mathbb{R}^N)$ for $t > \tau$ and $p > 2$.

5. Hölder continuity of solutions in $H^1(\mathbb{R}^N)$

In this section, the decomposition of the nonlinearity f is unavailable for us to establish the difference estimates in $H^1(\mathbb{R}^N)$, and thus we directly cope with (4.1), for which the following additional condition on f is needed: there exists a $\psi_4 \in L^\infty_{loc}(\mathbb{R}, L^\infty(\mathbb{R}^N))$ such that for all $t, s \in \mathbb{R}$ and $x \in \mathbb{R}^N$,

$$\Big|\frac{\partial}{\partial s} f(t,x,s)\Big| \leq \psi_4(t,x)(1 + |s|^{p-2}). \tag{5.1}$$

The following lemma, which is useful in what follows, is adapted from [40].

Lemma 5.1. *[40, Lemma 4.2] Let $\xi(t), g$ and h be tree nonnegative and locally integrable functions on $\mathbb{R}$ such that $\frac{d\xi}{dt}$ is also locally integrable and*

$$\frac{d\xi(t)}{dt} + \nu\xi(t) + g(t) \leq h(t), \quad t \in \mathbb{R},$$

for some constant $\nu \geq 0$. Then

(i) for arbitrary $a > 0$ and $\tau \in \mathbb{R}$, $\xi(\tau) \leq \frac{e^{-\nu\tau}}{a}\int_{\tau-a}^{\tau} e^{\nu s}\xi(s)ds + e^{-\nu\tau}\int_{\tau-a}^{\tau} e^{\nu s}h(s)ds$.

(ii) for arbitrary $a, \epsilon > 0$ and $\sigma \in [\tau - a, \tau]$,

$$\begin{aligned}\xi(\sigma) + e^{-\nu\tau}\int_{\tau-a}^{\tau} e^{\nu s}g(s)ds \leq &\frac{(e^{\nu a}+1)e^{-\nu\tau}}{\epsilon}\int_{\tau-a-\epsilon}^{\tau} e^{\nu s}\xi(s)ds \\ &+ (e^{\nu a}+2)e^{-\nu\tau}\int_{\tau-a-\epsilon}^{\tau} e^{\nu s}h(s)ds.\end{aligned}$$

In particular, (i) and (ii) hold for $\nu = 0$.

Lemma 5.2. *Suppose that (3.1)-(3.5) hold. Take $\tau \in \mathbb{R}, \omega \in \Omega$ and $T > 0$. Then there exist a constant $c = c(\tau,\omega,T) > 0$ such that the solution u of problem (1.1)* *satisfies for* $t \in (\tau, \tau + T]$,

$$\begin{aligned} &\|u(t,\tau,\omega,u_\tau)\|^2 + \int_\tau^t (\|\nabla u(s,\tau,\omega,u_\tau)\|^2 + \|u(s,\tau,\omega,u_\tau)\|_p^p) ds \\ \le & c(1 + \int_\tau^{\tau+T} \|g(s,.)\|^2 ds + \|u_\tau\|^2), \end{aligned} \tag{5.2}$$

$$\begin{aligned} &(t-\tau)\|u(t,\tau,\omega,u_\tau)\|_p^p + (t-\tau)\int_{\frac{t+\tau}{2}}^t \|u(s,\tau,\omega,u_\tau)\|_{2p-2}^{2p-2} ds \\ \le & c(1 + \int_\tau^{\tau+T} \|g(s,.)\|^2 ds + \|u_\tau\|^2), \end{aligned} \tag{5.3}$$

$$(t-\tau)\|\nabla u(t,\tau,\omega,u_\tau)\|^2 \le c(1 + \int_\tau^{\tau+T} \|g(s,.)\|^2 ds + \|u_\tau\|^2). \tag{5.4}$$

Proof. Using the test function u in (1.1), we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u\|^2 + \lambda\|u\|^2 + \|\nabla u\|^2 = & \int_{\mathbb{R}^N} f(t,x,u)u dx + \int_{\mathbb{R}^N} g(t,x)u dx \\ & + \mathcal{G}(\vartheta_t\omega)\int_{\mathbb{R}^N} h(t,x,u)u dx, \end{aligned} \tag{5.5}$$

where by Young inequality we have

$$\Big|\int_{\mathbb{R}^N} g(t,x)u dx\Big| \le \frac{\lambda}{4}\|u\|^2 + \frac{1}{\lambda}\|g(t,.)\|^2, \tag{5.6}$$

and by (3.1), we get

$$\int_{\mathbb{R}^N} f(t,x,u)u dx \le -\alpha_1\|u\|_p^p + \int_{\mathbb{R}^N} \psi_1(t,x) dx, \tag{5.7}$$

and by (3.4), it follows that

$$\begin{aligned} |\mathcal{G}(\vartheta_t\omega)|\Big|\int_{\mathbb{R}^N} h(t,x,u)u dx\Big| \le & |\mathcal{G}(\vartheta_t\omega)|\int_{\mathbb{R}^N} (\beta_1(t,x)|u|^{q-1} + \beta_2(t,x))|u| dx \\ \le & \frac{\alpha_1}{2}\int_{\mathbb{R}^N} |u|^p dx + c|\mathcal{G}(\vartheta_t\omega)|^{\frac{p}{p-q}}\|\beta_1(t,.)\|_{\frac{p}{p-q}}^{\frac{p}{p-q}} \\ & + c|\mathcal{G}(\vartheta_t\omega)|^{p_1}\|\beta_2(t,.)\|_{p_1}^{p_1}. \end{aligned} \tag{5.8}$$

Since $\beta_2 \in L^2_{loc}(\mathbb{R}, L^2(\mathbb{R}^N)) \cap L_{loc}^{\frac{2p-2}{p-1}}(\mathbb{R}, L^{\frac{2p-2}{p-1}}(\mathbb{R}^N))$ then we have $\beta_2 \in L^{p_1}_{loc}(\mathbb{R}, L^{p_1}(\mathbb{R}^N))$. By a combination of (5.5)-(5.8), we find that there exists a constant $c = c(\tau, \omega, T) > 0$ such that for all $t \in [\tau, \tau + T]$,

$$\frac{d}{dt}\|u\|^2 + \|\nabla u\|^2 + \frac{\alpha_1}{2}\|u\|_p^p \le c(\|g(t,.)\|^2 + 1). \tag{5.9}$$

Integrate (5.9) from τ to t to yield that for $t \in [\tau, \tau + T]$,

$$\|u(t)\|^2 + \int_\tau^t (\|\nabla u(s)\|^2 + \frac{\alpha_1}{2}\|u(s)\|_p^p) ds \le c\int_\tau^t (\|g(s,.)\|^2 + 1) ds + \|u_\tau\|^2$$

$$\leq c(1+\int_{\tau}^{\tau+T}\|g(s,.)\|^2ds)+\|u_\tau\|^2, \tag{5.10}$$

which proves (5.2).

Using the test function $|u|^{p-2}u$ in (1.1), we obtain

$$\frac{1}{p}\frac{d}{dt}\|u\|_p^p+\lambda\|u\|_p^p+(-\Delta u,|u|^{p-2}u)=\int_{\mathbb{R}^N}f(t,x,u)|u|^{p-2}udx+\int_{\mathbb{R}^N}g(t,x)|u|^{p-2}udx$$
$$+\mathcal{G}(\vartheta_t\omega)\int_{\mathbb{R}^N}h(t,x,u)|u|^{p-2}udx, \tag{5.11}$$

where it is clear that $(-\Delta u,|u|^{p-2}u)\geq 0$ for $p>2$. By (3.1), we get for $t\in(\tau,\tau+T]$,

$$\int_{\mathbb{R}^N}f(t,x,u)|u|^{p-2}udx\leq-\alpha_1\int_{\mathbb{R}^N}|u|^{2p-2}dx+\int_{\mathbb{R}^N}\psi_1(t,x)|u|^{p-2}dx$$
$$\leq-\alpha_1\int_{\mathbb{R}^N}|u|^{2p-2}dx+\frac{\lambda}{2}\int_{\mathbb{R}^N}|u|^pdx+\frac{1}{2\lambda}\int_{\mathbb{R}^N}|\psi_1(t,x)|^{\frac{p}{2}}dx. \tag{5.12}$$

From (5.8) we have

$$\begin{aligned}&\mathcal{G}(\vartheta_t\omega)\int_{\mathbb{R}^N}h(t,x,u)|u|^{p-2}udx\\&\leq\frac{\alpha_1}{2}\int_{\mathbb{R}^N}|u|^{2p-2}dx+\int_{\mathbb{R}^N}\psi_1(t,x)|u|^{p-2}dx\\&\leq\frac{\alpha_1}{2}\int_{\mathbb{R}^N}|u|^{2p-2}dx+c|\mathcal{G}(\vartheta_t\omega)|^{\frac{p}{p-q}}\int_{\mathbb{R}^N}|\beta_1(t,x)|^{\frac{p}{p-q}}|u|^{p-2}dx\\&\quad+c|\mathcal{G}(\vartheta_t\omega)|^{p_1}\int_{\mathbb{R}^N}|\beta_2(t,x)|^{p_1}|u|^{p-2}dx.\end{aligned} \tag{5.13}$$

The Young inequality implies that

$$\begin{aligned}&c|g(\vartheta_t\omega)|^{\frac{p}{p-q}}\int_{\mathbb{R}^N}|\beta_1(t,x)|^{\frac{p}{p-q}}|u|^{p-2}dx\\&\leq\frac{\alpha_1}{8}\int_{\mathbb{R}^N}|u|^{2p-2}dx+c|\mathcal{G}(\vartheta_t\omega)|^{\frac{2p-2}{p-q}}\int_{\mathbb{R}^N}|\beta_1(t,x)|^{\frac{2p-2}{p-q}}dx,\end{aligned} \tag{5.14}$$

and

$$\begin{aligned}&c|g(\vartheta_t\omega)|^{p_1}\int_{\mathbb{R}^N}|\beta_2(t,x)|^{p_1}|u|^{p-2}dx\\&\leq\frac{\alpha_1}{8}\int_{\mathbb{R}^N}|u|^{2p-2}dx+c|\mathcal{G}(\vartheta_t\omega)|^{\frac{2p-2}{p-1}}\int_{\mathbb{R}^N}|\beta_2(t,x)|^{\frac{2p-2}{p-1}}dx.\end{aligned} \tag{5.15}$$

Combine (5.14) and (5.15) to (5.13) to find that for $t\in(\tau,\tau+T]$,

$$\mathcal{G}(\vartheta_t\omega)\int_{\mathbb{R}^N}h(t,x,u)|u|^{p-2}udx\leq\frac{3\alpha_1}{4}\int_{\mathbb{R}^N}|u|^{2p-2}dx+c. \tag{5.16}$$

On the other hand, for the forcing term, we have

$$\Big|\int_{\mathbb{R}^N}g(t,x)|u|^{p-2}udx\Big|\leq\frac{\alpha_1}{8}\int_{\mathbb{R}^N}|u|^{2p-2}dx+c\|g(t,.)\|^2. \tag{5.17}$$

Thus by a combination of (5.12), (5.16) and (5.17) into (5.11), we obtain that for $t \in (\tau, \tau + T]$,

$$\frac{d}{dt}\|u\|_p^p + \frac{\alpha_1}{8}\|u\|_{2p-2}^{2p-2} \leq c(1 + \|g(t,.)\|^2). \tag{5.18}$$

Applying Lemma 5.1 (ii) to (5.18) over the interval $[\frac{\tau+t}{2}, t]$ with $\nu = 0, a = \epsilon = \frac{t-\tau}{2}$, we get

$$\|u(t)\|_p^p + \frac{\alpha_1}{8}\int_{\frac{\tau+t}{2}}^t \|u(s)\|_{2p-2}^{2p-2} ds \leq \frac{4}{t-\tau}\int_\tau^t \|u(s)\|_p^p ds + c(1+\int_\tau^{\tau+T} \|g(s,.)\|^2 ds). \tag{5.19}$$

Thus by (5.2) and (5.19), we get for $t \in (\tau, \tau + T]$,

$$(t-\tau)\|u(t)\|_p^p + \frac{\alpha_1}{8}(t-\tau)\int_{\frac{\tau+t}{2}}^t \|u(s)\|_{2p-2}^{2p-2} ds \leq c(1+\int_\tau^{\tau+T} \|g(s,.)\|^2 ds + \|u_\tau\|^2), \tag{5.20}$$

which shows (5.3).

Using the test function $-\Delta u$ in (1.1), we see that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2 + \|\Delta u\|^2 \\
\leq &\int_{\mathbb{R}^N} f(x,t,u)\Delta u dx + \int_{\mathbb{R}^N} g(t,x)\Delta u dx + \mathcal{G}(\vartheta_t\omega)\int_{\mathbb{R}^N} h(t,x,u)\Delta u dx \\
\leq &\frac{1}{2}\|\Delta u\|^2 + c\int_{\mathbb{R}^N}(|\psi_2(t,x)|^2 + |u|^{2p-2})dx + c\int_{\mathbb{R}^N}|g(t,x)|^2 dx \\
&+ |\mathcal{G}(\vartheta_t\omega)|^2\int_{\mathbb{R}^N}(|\beta_1(t,x)|^2|u|^{2q-2} + |\beta_2(t,x)|^2)dx \\
\leq &\frac{1}{2}\|\Delta u\|^2 + c\int_{\mathbb{R}^N}(|\psi_2(t,x)|^2 + |u|^{2p-2})dx + c\int_{\mathbb{R}^N}|g(t,x)|^2 dx \\
&+ \int_{\mathbb{R}^N}|u|^{2p-2}dx + \int_{\mathbb{R}^N}|\mathcal{G}(\vartheta_t\omega)\beta_1(t,x)|^{\frac{2p-2}{p-q}}dx + \int_{\mathbb{R}^N}|\mathcal{G}(\vartheta_t\omega)\beta_2(t,x)|^2 dx,
\end{aligned} \tag{5.21}$$

which clearly gives that for $t \in (\tau, \tau + T]$,

$$\frac{d}{dt}\|\nabla u\|^2 + \|\Delta u\|^2 \leq c(\|u\|_{2p-2}^{2p-2} + \|g(t,.)\|^2 + 1). \tag{5.22}$$

Applying Lemma (ii) to (5.22) over the interval $[\frac{\tau+3t}{4}, t]$ with $\nu = 0$ and $a = \epsilon = \frac{t-\tau}{4}$, we get that for $t \in (\tau, \tau + T]$,

$$\begin{aligned}
&(t-\tau)\|\nabla u(t)\|^2 + (t-\tau)\int_{\frac{3t+\tau}{4}}^t \|\Delta u(s)\|^2 ds \\
\leq &8\int_\tau^t \|\nabla u(s)\|^2 ds + c(t-\tau)\int_{\frac{t+\tau}{2}}^t \|u(s)\|_{2p-2}^{2p-2} ds + c\int_\tau^t (\|g(s,.)\|^2 + 1)ds,
\end{aligned}$$

from which and (5.3), we infer that (5.4) holds true. □

Remark 5.1. This lemma in fact shows that the solution $u(t, \tau, \omega, u_\tau) \in L^p(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$ for every $t > \tau \in \mathbb{R}$ and $\omega \in \Omega$ when the initial datum $u_\tau \in L^2(\mathbb{R}^N)$, which makes sense for us to discuss the Hölder continuity and attractors in $L^p(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$.

The following result is concern about the Hölder continuity of solutions in $H^1(\mathbb{R}^N)$.

Theorem 5.1. *Suppose that (3.1)-(3.5) and (5.1) hold. Let $\tau \in \mathbb{R}$, $\omega \in \Omega$, $T > 0$. Suppose further that $\|u_{\tau,1} - u_{\tau,2}\| \leq 1$. Then there exists a constant $c = c(k, \tau, \omega, T, \|u_{\tau,1}\|, \|u_{\tau,2}\|) > 0$, such that the difference of solutions of problems (1.1)-(1.2) satisfies for all $t \in (\tau, \tau + T]$,*

$$(\frac{t-\tau}{2})^{\frac{3p-2}{p-1}} \|u_1(t,\tau,\omega,u_{\tau,1}) - u_1(t,\tau,\omega,u_{\tau,2})\|_{H^1}^2 \leq c\|u_{\tau,1} - u_{\tau,2}\|^{\frac{2}{p-1}}.$$

Proof. We begin with some estimates for the nonlinearities. First, by (5.1) we infer that for all $t \in (\tau, \tau + T]$,

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^N} (f(t,x,u_1) - f(t,x,u_2))U_t dx\Big| \\
\leq& c\int_{\mathbb{R}^N} (1 + |u_1|^{p-2} + |u_2|^{p-2})|U||U_t|dx \\
\leq& \frac{1}{4}\|U_t\|^2 + c\|U\|^2 + c\int_{\mathbb{R}^N} (|u_1|^{2p-4} + |u_2|^{2p-4})|U|^2 dx \\
\leq& \frac{1}{4}\|U_t\|^2 + c\|U\|^2 + c(\|u_1\|_{2p-2}^{2p-4} + \|u_2\|_{2p-2}^{2p-4})\|U\|_{2p-2}^2,
\end{aligned} \tag{5.23}$$

where U_t is the derivative of U with respect to the time t. On the other hand, since $2 \leq q < p$, then by (3.9) we have for all $t \in (\tau, \tau + T]$,

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^N} (h(t,x,u_1) - h(t,x,u_2))\mathcal{G}(\vartheta_t\omega)U_t dx\Big| \\
\leq& |\mathcal{G}(\vartheta_t\omega)|\int_{\mathbb{R}^N} |\beta_5(t,x)|(1 + |u_1|^{q-2} + |u_2|^{q-2})|U||U_t|dx \\
\leq& \frac{1}{4}\|U_t\|^2 + c\|U\|^2 + |\mathcal{G}(\vartheta_t\omega)|^2\int_{\mathbb{R}^N} |\beta_5(t,x)|^2(|u_1|^{2q-4} + |u_2|^{2q-4})|U|^2 dx \\
\leq& \frac{1}{4}\|U_t\|^2 + c\|U\|^2 + c\int_{\mathbb{R}^N} (|u_1|^{2p-4} + |u_2|^{2p-4})|U|^2 dx \\
&+ \int_{\mathbb{R}^N} |\beta_5(t,x)\mathcal{G}(\vartheta_t\omega)|^{\frac{2p-4}{p-q}}|U|^2 dx \\
\leq& \frac{1}{4}\|U_t\|^2 + c\|U\|^2 + c(\|u_1\|_{2p-2}^{2p-4} + \|u_2\|_{2p-2}^{2p-4})\|U\|_{2p-2}^2.
\end{aligned} \tag{5.24}$$

Taking the inner product of (4.1) in $L^2(\mathbb{R}^N)$ with U_t, along with (5.23) and (5.24), we deduce that, with t replaced by s, for all $s \in [\tau, \tau + T]$,

$$\frac{d}{ds}\|\nabla U(s)\|^2 \leq c\|U(s)\|^2 + c(\|u_1(s)\|_{2p-2}^{2p-4} + \|u_2(s)\|_{2p-2}^{2p-4})\|U(s)\|_{2p-2}^2. \tag{5.25}$$

By multiplying (5.25) by $(s - \frac{t+\tau}{2})^{\frac{3p-2}{p-1}}$ for $s \in [\frac{t+\tau}{2}, t]$ with $t \in (\tau, \tau + T]$, we get

$$\begin{aligned}
&(s - \frac{t+\tau}{2})^{\frac{3p-2}{p-1}} \frac{d}{ds}\|\nabla U(s)\|^2 \\
\leq& c(s - \frac{t+\tau}{2})^{\frac{3p-2}{p-1}}\|U(s)\|^2 + c(s - \frac{t+\tau}{2})^{\frac{3p-2}{p-1}}(\|u_2(s)\|_{2p-2}^{2p-4} + \|u_1(s)\|_{2p-2}^{2p-4})\|U(s)\|_{2p-2}^2.
\end{aligned} \tag{5.26}$$

By integrating (5.26) over the intervals $[\frac{t+\tau}{2}, t]$, we deduce that for all $t \in (\tau, \tau+T]$,

$$\begin{aligned}
&(\frac{t-\tau}{2})^{\frac{3p-2}{p-1}} \|\nabla U(t)\|^2 \\
\leq & c\int_{\frac{t+\tau}{2}}^t \|\nabla U(s)\|^2 ds + c\int_{\frac{t+\tau}{2}}^t \|U(s)\|^2 ds \\
&+ c(t-\tau)^{\frac{p-2}{p-1}} \int_{\frac{t+\tau}{2}}^t (s-\frac{t+\tau}{2})^{\frac{2p}{p-1}} (\|u_1(s)\|_{2p-2}^{2p-4} + \|u_2(s)\|_{2p-2}^{2p-4}) \|U(s)\|_{2p-2}^2 ds.
\end{aligned} \tag{5.27}$$

We now estimate every term on the right hand side of (5.27). By (4.13) and (4.14) we have for all $t \in (\tau, \tau+T]$,

$$c\int_{\frac{t+\tau}{2}}^t \|\nabla U(s)\|^2 ds + c\int_{\frac{t+\tau}{2}}^t \|U(s)\|^2 ds \leq c\|u_{\tau,1} - u_{\tau,2}\|^2. \tag{5.28}$$

For the last term on the right hand side of (5.27), by using Hölder inequality, we deduce that

$$\begin{aligned}
&c(t-\tau)^{\frac{p-2}{p-1}} \int_{\frac{t+\tau}{2}}^t (s-\frac{t+\tau}{2})^{\frac{2p}{p-1}} (\|u_1(s)\|_{2p-2}^{2p-4} + \|u_2(s)\|_{2p-2}^{2p-4}) \|U(s)\|_{2p-2}^2 ds \\
\leq & c\Big((t-\tau)\int_{\frac{t+\tau}{2}}^t (\|u_1(s)\|_{2p-2}^{2p-2} + \|u_2(s)\|_{2p-2}^{2p-2}) ds\Big)^{\frac{p-2}{p-1}} \\
&\times \Big(\int_{\frac{t+\tau}{2}}^t (s-\frac{t+\tau}{2})^{2p} \|U(s)\|_{2p-2}^{2p-2} ds\Big)^{\frac{1}{p-1}}.
\end{aligned} \tag{5.29}$$

By Lemma 4.1 (B_1) with $k=1$ and (5.3) it follows from (5.29) that

$$\begin{aligned}
&c(t-\tau)^{\frac{p-2}{p-1}} \int_{\frac{t+\tau}{2}}^t (s-\frac{t+\tau}{2})^{\frac{2p}{p-1}} (\|u_1(s)\|_{2p-2}^{2p-4} + \|u_2(s)\|_{2p-2}^{2p-4}) \|U(s)\|_{2p-2}^2 ds \\
\leq & c\|u_{\tau,1} - u_{\tau,2}\|^{\frac{2}{p-1}}.
\end{aligned} \tag{5.30}$$

Hence by a combination of (5.28) and (5.30) into (5.27) to yield that for all $t \in (\tau, \tau+T]$,

$$(\frac{t-\tau}{2})^{\frac{3p-2}{p-1}} \|\nabla U(t)\|^2 \leq c_1(\|u_{\tau,1} - u_{\tau,2}\|^2 + \|u_{\tau,1} - u_{\tau,2}\|^{\frac{2}{p-1}}). \tag{5.31}$$

This along with (4.13) concludes the total proof. □

Remark 5.2. According to Theorem 5.1, the solution $u(t, \tau, \omega, .)$ is $\frac{1}{p-1}$-Hölder continuous from $L^2(\mathbb{R}^N)$ to $H^1(\mathbb{R}^N)$ for every $t > \tau \in \mathbb{R}$ and $\omega \in \Omega$.

Remark 5.3. The condition (5.1) on f is also assumed in [20, 31], which can be replaced by the following weaker version: there exists a $\psi_4 \in L^\infty_{loc}(\mathbb{R}, L^\infty(\mathbb{R}^N))$ such that for all $t, s \in \mathbb{R}$ and $x \in \mathbb{R}^N$,

$$|f(t,x,s_1) - f(t,x,s_2)| \leq \psi_4(t,x)|s_1 - s_2|(2 + |s_1|^{p-2} + |s_2|^{p-2}). \tag{5.32}$$

Remark 5.4. From Theorem 4.1 and 5.1, we know that the solution of problem (1.1)-(1.2) is Hölder continuous from $L^2(\mathbb{R}^N)$ to $L^p(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$ with respect to the initial data. However this continuity is not uniform with respect to t in a small neighborhood of the initial time $t = \tau$, because the left-hand side of inequalities (5.31) and (4.21) may vanish when $t \downarrow \tau$, even though the initial data belong to $L^p(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$.

6. Regular dynamics of Pullback attractors

This section is to generalize the pullback attractor derived in [31] to the higher regular spaces $L^p(\mathbb{R}^N)$ and $H^1(\mathbb{R}^N)$ with almost the same conditions on f and h. For this purpose, we need to show the $\mathfrak{D}$-pullback asymptotically compact of φ from $L^2(\mathbb{R}^N)$ to $L^p(\mathbb{R}^N)$ and $H^1(\mathbb{R}^N)$, respectively. First we recall some known results in the literature.

Let u be a solution of problem (1.1)-(1.2). Given $t \in \mathbb{R}^+, \tau \in \mathbb{R}$ and $\omega \in \Omega$, the cocycle in $L^2(\mathbb{R}^N)$ is defined as

$$\varphi(t, \tau, \omega, u_\tau) = u(t + \tau, \tau, \theta_{-\tau}\omega, u_\tau). \tag{6.1}$$

Then φ is continuous on $L^2(\mathbb{R}^N)$ over the MDS $(\Omega, \mathcal{F}, \mathbb{P}, \{\vartheta_t\}_{t\in\mathbb{R}})$ which is introduced in the introduction.

For the universe of sets, we suppose that $\mathfrak{D}$ is a collection of all families of tempered subsets of $L^2(\mathbb{R}^N)$, i.e.,

$$\mathfrak{D} = \{D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} : D \text{ is tempered in } L^2(\mathbb{R}^N)\}. \tag{6.2}$$

Then it is obvious that $\mathfrak{D}$ is inclusion closed.

Lemma 6.1. *[31, Corollary 3.1] Suppose that (3.1)-(3.5), (3.10) and (3.11) hold. Then the continuous cocycle φ of problem (1.1)-(1.2) has a closed measurable $\mathfrak{D}$-pullback absorbing set $K = \{K(\tau, \omega) : \tau \in \epsilon\, \mathbb{R}, \omega \in \Omega\} \in \mathfrak{D}$.*

Lemma 6.2. *[31, Lemma 3.6] Suppose that (3.1)-(3.5), (3.10) and (3.11) hold. Then the continuous cocycle φ of problem (1.1)-(1.2) is $\mathfrak{D}$-pullback asymptotically compact in $L^2(\mathbb{R}^N)$.*

Theorem 6.1. *[31, Theorem 3.1] Suppose that (3.1)-(3.5), (3.10) and (3.11) hold. Then the continuous cocycle φ of problem (1.1)-(1.2) admits a unique $\mathfrak{D}$-pullback random attractor, which is characterized by, for each $\tau \in \mathbb{R}$ and $\omega \in \Omega$*

$$\begin{aligned} \mathcal{A}(\tau, \omega) &= \cap_{s\geq 0}\overline{\cup_{t\geq s}\varphi\left(t, \tau - t, \vartheta_{-t}\omega, K\left(\tau - t, \vartheta_{-t}\omega\right)\right)}^{L^2(\mathbb{R}^N)} \\ &= \{\varrho(0, \tau, \omega) : \varrho(., \tau, \omega) \textit{is a } \mathfrak{D} - \textit{complete orbit of } \varphi\}. \end{aligned} \tag{6.3}$$

Lemma 6.3. *[38, Lemma 2.10] Suppose that φ is a random cocycle on X over $(\Omega, \mathcal{F}, P, \{\vartheta_t\}_{t\in\mathbb{R}})$ and further $\varphi(t, \tau, \omega, .) : X \mapsto Y$ is continuous for every $t > 0, \tau \in \mathbb{R}$ and $\omega \in \Omega$. Assume that φ is $\mathfrak{D}$-pullback asymptotically compact in X. Then φ is $\mathfrak{D}$-pullback asymptotically compact from X to Y, i.e., for each $\tau \in \mathbb{R}, \omega \in \Omega$ and $D \in \mathfrak{D}$, the sequence $\{\varphi(t_n, \tau - t_n, \vartheta_{-t_n}\omega, x_n)\}_{n=1}^\infty$ has a convergent subsequence in Y whenever $t_n \to \infty$ and $x_n \in D\left(\tau - t_n, \vartheta_{-t_n}\omega\right)$.*

We now address the $\mathfrak{D}$-pullback asymptotical compactness of φ from $L^2(\mathbb{R}^N)$ to $L^p(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$.

Lemma 6.4. *Suppose that (3.1)-(3.5), (5.1), (3.10) and (3.11) hold. Then the cocycle φ for problem (1.1)-(1.2) is $\mathfrak{D}$-pullback asymptotically compact from $L^2(\mathbb{R}^N)$ to $L^p(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$.*

Proof. By Theorem 4.1 (with $k = 1$) and Theorem 5.1, we know that $\varphi(t, \tau, \omega, .) : L^2(\mathbb{R}^N) \mapsto L^p(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$ is continuous for every $t > \tau \in \mathbb{R}$ and $\omega \in \Omega$. And by Lemma 6.2 and Lemma 6.3, we get that φ is $\mathfrak{D}$-pullback asymptotically compact from $L^2(\mathbb{R}^N)$ to $L^p(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$. □

Remark 6.1. Assumption (5.1) is only need to obtain the $\mathfrak{D}$-pullback asymptotical compactness from $L^2(\mathbb{R}^N)$ to $H^1(\mathbb{R}^N)$.

The following results are concerned with the existence of pullback attractor in $L^p(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$. In particular, some new dynamics of problem (1.1)-(1.2) in $L^\delta(\mathbb{R}^N)$ for arbitrary $\delta \geq 2$ are rigorously demonstrated.

Theorem 6.2. *Suppose that (3.1)-(3.5), (3.10) and (3.11) hold. Then the $\mathfrak{D}$-pullback random attractor $\mathcal{A}$ derive by Theorem 6.1 is also a $(L^2(\mathbb{R}^N), L^2(\mathbb{R}^N) \cap L^p(\mathbb{R}^N))$-pullback random attractor. Furthermore,*

(i) $\mathcal{A}$ is attracting in the the space $L^\delta\left(\mathbb{R}^N\right)$, i.e., for every $\tau \in \mathbb{R}, \omega \in \Omega$ and $D \in \mathfrak{D}$

$$\lim_{t\to\infty} \operatorname{dist}_{L^\delta}\left(\varphi\left(t, \tau - t, \vartheta_{-t}\omega, D\left(\tau - t, \vartheta_{-t}\omega\right)\right), \mathcal{A}(\tau, \omega)\right) = 0,$$

for any $\delta \in [2, \infty)$.

(ii) $\mathcal{A}$ is difference bounded in $L^\delta\left(\mathbb{R}^N\right)$, i.e., for every $\tau \in \mathbb{R}, \omega \in \Omega$, there exists a positive constant $M = M(\tau, \omega)$ such that

$$\sup_{x_i \in \mathcal{A}(\tau,\omega), i=1,2} \|x_1(\tau, \omega) - x_2(\tau, \omega)\|_\delta \leq M(\tau, \omega).$$

(iii) φ is translation absorbing in $L^\delta(\mathbb{R}^N)$, i.e., for every $\tau \in \mathbb{R}, \omega \in \Omega$, there exist positive constants $T = T(\tau, \omega, D, \varrho)$ and $M = M(\tau, \omega)$ such that for all $t \geq T$,

$$\sup_{u_0 \in D(\tau - t, \vartheta_{-t}\omega)} \|\varphi(t, \tau - t, \vartheta_{-t}\omega, u_0) - \varrho(0, \tau, \omega)\|_\delta \leq M(\tau, \omega),$$

for every $\mathfrak{D}$-complete orbit $\{\varrho(., \tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ of φ and $D \in \mathfrak{D}$.

Proof. By Lemma 6.1, 6.2 and 6.4, along with Theorem 2.1, we know that $\mathcal{A}$ is $(L^2(\mathbb{R}^N), L^2(\mathbb{R}^N) \cap L^p(\mathbb{R}^N))$-pullback random attractor. The measurability of $\mathcal{A}$ in $L^2(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$ is from [11, Theorem 19]. The properties (i)-(iii) can be proved by a similar procedure as in [38, Theorem 7.1-7.2] and here we omit the proof. □

Theorem 6.3. *Suppose that (3.1)-(3.5), (5.1), (3.10) and (3.11) hold. Then the $\mathfrak{D}$-pullback random attractor $\mathcal{A}$ derive by Theorem 6.1 is also a $(L^2(\mathbb{R}^N), H^1(\mathbb{R}^N))$-pullback random attractor.*

Proof. The result is followed by Lemma 6.1, 6.2 and 6.4, along with Theorem 2.1. □

7. Boxing counting dimension of attractors in $L^p(\mathbb{R}^N)$ and $H^1(\mathbb{R}^N)$

In this section, we investigate the boxing counting dimension (fractal dimension) of A, which is based on the number of closed balls of a fixed radius ε needed to cover

A. Denote by $N(A,\varepsilon)$ the minimum number of balls that cover A (the center of the covering may not belong to A). Then we define

Definition 7.1. If A is a compact subset of X, the boxing counting dimension of A, $\dim(A)$, is defined as

$$\dim(A) = \limsup_{\varepsilon\to 0} \log_{\frac{1}{\varepsilon}} N(A,\varepsilon) = \limsup_{\varepsilon\to 0} \frac{\log N(A,\varepsilon)}{-\log\varepsilon}.$$

This shows that the minimum number $N(A,\varepsilon) \sim \varepsilon^{-\dim_B(A)}$. For more information on the boxing counting dimension, we refer to [4,22]. For the boxing counting dimension in L^p for $p>2$, the authors in [19,35] derived the relation of dimension between L^p and L^2, which gives that

$$\frac{p}{2(p-1)} \dim_{L^2}(A) \le \dim_{L^p}(A) \le (p-1)\dim_{L^2}(A). \tag{7.1}$$

In this paper, for the problem (1.1)-(1.2) we will find the optimal bound for the dimension of attractor in $L^p(\mathbb{R}^N)$. Furthermore, we compare the $H^1(\mathbb{R}^N)$-dimension with $L^2(\mathbb{R}^N)$-dimension of attractor.

We first present a lemma on the property of boxing counting dimension.

Lemma 7.1. *[4, Lemma 4.2] Let (X,d_X) and (Y,d_Y) be two metric spaces. If $\phi: X\mapsto Y$ is Hölder continuous with exponent $\theta(0<\theta<1)$, i.e., there exists a constants $c>0$ such that,*

$$d_Y(\phi(x_1),\phi(x_2)) \le c d_X(x_1,x_2)^\theta,$$

for $x_1,x_2\in X$ with $d_X(x_1,x_2)\le 1$, then

$$\dim_Y(\phi(A)) \le \dim_X(A)/\theta.$$

Then we have

Theorem 7.1. *Let $\mathcal{A}$ be the pullback attractor defined by (6.3). Then for every $\tau\in\mathbb{R}$ and $\omega\in\Omega$,*

(i) $\frac{p}{2(p-1)}\dim_{L^2}(\mathcal{A}(\tau,\omega)) \le \dim_{L^p}(\mathcal{A}(\tau,\omega)) \le \frac{p}{2}\dim_{L^2}(\mathcal{A}(\tau-1,\vartheta_{-1}\omega))$,

(ii) $\dim_{L^2}(\mathcal{A}(\tau,\omega)) \le \dim_{H^1}(\mathcal{A}(\tau,\omega)) \le (p-1)\dim_{L^2}(\mathcal{A}(\tau-1,\vartheta_{-1}\omega))$.

Proof. The first inequality of (i) is followed from (7.1). We now check the second part. By Theorem 4.1, we know that φ is $\frac{2}{p}$-Hölder continuous from $L^2(\mathbb{R}^N)$ to $L^p(\mathbb{R}^N)$ for the solution to problem (1.1)-(1.2). Then replacing t by τ, $\tau-1$ by τ and ω by $\vartheta_{-1}\omega$ we get for $\tau\in\mathbb{R}$ and $\omega\in\Omega$,

$$\|\varphi(1,\tau-1,\vartheta_{-1}\omega,x_1)-\varphi(1,\tau-1,\vartheta_{-1}\omega,x_2)\|_p \le c\|x_1-x_2\|^{\frac{2}{p}},$$

for every $x_1,x_2\in L^2(\mathbb{R}^N)$. Thus by Lemma 7.1, we get that for $\tau\in\mathbb{R}$ and $\omega\in\Omega$,

$$\dim_{L^p}(\varphi(1,\tau-1,\vartheta_{-1}\omega,\mathcal{A}(\tau-1,\vartheta_{-1}\omega))) \le \frac{p}{2}\dim_{L^2}(\mathcal{A}(\tau-1,\vartheta_{-1}\omega)).$$

Consider that by the invariance, $(\varphi(t-1,\tau-t,\vartheta_t\omega,\mathcal{A}(\tau-t,\vartheta_{-t}\omega))) = \mathcal{A}(\tau-1,\vartheta_{-1}\omega))$. Then by the cocycle property, we immediately find that for every $\tau\in\mathbb{R}$ and $\omega\in\Omega$,

$$\dim_{L^p}(\mathcal{A}(\tau,\omega)) \le \frac{p}{2}\dim_{L^2}(\mathcal{A}(\tau-1,\vartheta_{-1}\omega)).$$

The first inequality of (ii) is obvious. The second part can be analogously proved by Theorem 5.1 and Lemma 7.1. $\square$

Remark 7.1. In this paper, by proving the Hölder continuity of solutions, we obtained the regular dynamics of random reaction-diffusion driven by a multiplicative noise with a general nonlinear multiple. This technique is applicable to some other random partial differential equations driven by such type noise, such as the random FitzHugh-Nagumo systems and p-Laplacian equations; moreover, we can invetigate the high-order Wang-Zakai approximations in some regular spaces, which will be the forthcoming work.

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