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RANDOM PULLBACK ATTRACTOR FOR A NON-AUTONOMOUS MODIFIED SWIFT-HOHENBERG EQUATION WITH MULTIPLICATIVE NOISE*

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Abstract In this paper, we study long time behavior of a non-autonomous stochastic modified Swift-Hohenberg equation with multiplicative noise in stratonovich sense. We show that a random $\mathcal{D}$-pullback attractor exists in H_0^2 for the corresponding non-autonomous random dynamical system. Due to the stochastic term, the estimates are delicate, the Ornstein-Uhlenbeck(O-U) transformation and its properties are used to overcome the difficulty that the stochastic term brings to us.

Keywords Swift-Hohenberg equation, random $\mathcal{D}$-pullback attractor, non-autonomous random dynamical system.

MSC(2010) 35B40, 35B41, 60H15, 60H30, 60H40.

1. Introduction

We consider the asymptotic behavior of solution to the following stochastic modified Swift-Hohenberg equation with multiplicative noise:

$$du+(\triangle^2 u+2\triangle u+au+b|\nabla u|^2+u^3-g(x,t))dt=lu(t)\circ dW(t), \text{ in } D\times[\tau,\infty), \quad (1.1)$$

$$u=\frac{\partial u}{\partial \nu}=0 \text{ on } \partial D\times[\tau,\infty), \quad (1.2)$$

$$u(x,\tau)=u_\tau(x) \text{ in } D. \quad (1.3)$$

Where D is a bounded open domain in $\mathbb{R}^2$ with a smooth boundary ∂D, a, b and l are arbitrary constants, g is an external forcing term, lu is the noise intensity, $W(t)$ is a two-sided real-valued Wiener process on a probability space which will be specified later.

The Swift-Hohenberg type equations arising in the study of convective hydrodynamical, plasma confinement in toroidal and viscous film flow, was introduced by authors in [20] and the long time behavior have been investigated by several authours [14–16, 26]. If $b=0$ and we omit the noise term $lu\circ dW(t)$ and the external force $g(x,t)$, which means $b=0, l=0, g=0$, the system becomes the usual Swift-Hohenbeg equation [7, 19]. A.Doelman et al. in [5] proposed system (1.1)-(1.3) with

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$l = 0, g = 0$ for a pattern formation system with two unbounded directions that is near the onset to instability. Rencently, many authors have paid much attention to analyze the long time behavior of dynamical system generated by (1.1)-(1.3) and have made a lot of progress([9,17,18,26,27]). Polate [18] established the existence of global attractor([13,22]) for system (1.1)-(1.3) when $g \equiv 0$ and $l = 0$, Park [17] proved the existence of pullback attractor([2,8,10,12,21,23–25]) with exponential growth of the external forcing term $g(x,t)$ when $l = 0$, Xu [27] proved the existence of uniform attractors when the external term $g(x,t)$ satisfies the translation bounded condition. The existence of random term in (1.1) is more consistent in practice problems, when $l \neq 0$, the system (1.1)-(1.3) becomes a stochastic partial differential equation, to the best our knowledge, the existence of the random pullback attractors for a non-autonomous modified Swift-Hohenberg equation has not yet been considered.

In this paper, we consider the existence of random $\mathcal{D}$-pullback attractors of system (1.1)-(1.3). We let an operation $A = -\Delta$ and λ be the first eigenvalue of A. For any $t \in \mathbb{R}$, the external force $g(x,t) \in L^2(D)$, we assume that there exists $M > 0$ such that

$$\|g(x,t)\|^2 \leq Me^{\alpha|t|}, \text{ for any } t \in \mathbb{R}, \quad 0 \leq \alpha < \tfrac{\lambda}{100}. \tag{1.4}$$

The assumption is same as [10, 18], through simple calculation, for all $t \in \mathbb{R}$, we have

$$H_1(t) := \int_{-\infty}^{t} e^{\frac{\lambda}{4}s}\|g(x,s)\|^2 ds < \infty, \; H_2(t) = \int_{-\infty}^{t}\int_{-\infty}^{s} e^{\frac{\lambda}{4}y}\|g(x,y)\|^2 dy ds < \infty, \tag{1.5}$$

$$\int_{-\infty}^{t} e^{-\frac{\lambda}{2}s} H_1^3(s) ds < \infty, \quad \int_{-\infty}^{t} e^{-\lambda s} H_1^5(s) ds < \infty, \quad \text{for any } t \in \mathbb{R}. \tag{1.6}$$

An outline of this paper is as follows: In section 2, we recall some basic concepts about random $\mathcal{D}$-pullback attractor and the existence theorem of random $\mathcal{D}$-pullback attractor. In section 3, we prove that the stochastic dynamical system generated by (1.1)-(1.3) exists a random $\mathcal{D}$-pullback attractor in $H_0^2(D)$.

2. Preliminaries

In this section, we first give some basic definitions and abstract results concerning the random $\mathcal{D}$-pullback attractor for non-autonomous random dynamical system [1]. The reader is referred to [3,6,8,11,12,23–25] for more details.

Let $(X, \|\cdot\|_X)$ be a separable Banach space with Borel σ-algebra $\mathcal{B}(X)$ and $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. In this paper, the term $\mathbb{P}$-a.s.(the abbreviation for $\mathbb{P}$ almost surely) denotes that an event happens with probability one. In other words, the set of possible exception may be non-empty, but it has probability zero.

Definition 2.1([1,3,4,11,24,25]). $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ is called a metric dynamical systems if $\theta : \mathbb{R} \times \Omega \to \Omega$ is $(\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}, \mathcal{F})$-measurable, and θ_0 is the identity on Ω, $\theta_{s+t} = \theta_t \circ \theta_s$ for all $t, s \in \mathbb{R}$ and $\theta_t\mathbb{P} = \mathbb{P}$ for all $t \in \mathbb{R}$.

Definition 2.2([8,12,23–25]). A non-autonomous random dynamical system (NRDS) (φ, θ) on X over a metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ is a mapping

$$\varphi(t, \tau, \omega) : X \to X, \quad (t, \tau, \omega, x) \to \varphi(t, \tau, \omega)x,$$

which represents the dynamics in the state space X and satisfies the properties

(i) $\varphi(\tau,\tau,\omega)$ is the identity on X;

(ii) $\varphi(t,\tau,\omega)=\varphi(t,s,\theta_{s-\tau}\omega)\varphi(s,\tau,\omega)$ for all $\tau\le s\le t$;

(iii) $\omega\to\varphi(t,\tau,\omega)x$ is $\mathcal{F}$-measurable for all $t\ge\tau$ and $x\in X$.

A NRDS (φ,θ) is called a continuous random dynamical system if $\varphi(t,\tau,\omega):X\to X$ is continuous for all $t\ge\tau$ and $\omega\in\Omega$. A NRDS (φ,θ) is called a norm-to-weak continuous random dynamical system if $x_n\to x, \varphi(t,\tau,\omega)x_n\rightharpoonup\varphi(t,\tau,\omega)x$ for all $t\ge\tau$, and $\omega\in\Omega$. Obviously, a continuous NRDS is also a norm-to-weak NRDS.

In the sequel, we use $\mathcal{D}$ to denote a collection of some families of nonempty bounded subsets of X:

$$D'\in\mathcal{D}, D'=\{D(t,\omega)\in\mathcal{B}(X):t\in\mathbb{R},\omega\in\Omega\}.$$

Definition 2.3([8,12,23–25])**.** A set $B'\in\mathcal{D}$ is called a random $\mathcal{D}$-pullback bounded absorbing set for NRDS (φ,θ) if for any $t\in\mathbb{R}$ and any $D'\in\mathcal{D}$, there exists $\tau_0(t,D')$ such that $\varphi(t,\tau,\theta_{\tau-t}\omega)D(\tau,\theta_{\tau-t}\omega)\subset B(t,\omega)$ for any $\tau\le\tau_0$.

Definition 2.4.([8, 12, 23–25])**.** A set $\mathcal{A}=\{A(t,\omega):t\in\mathbb{R},\omega\in\Omega\}$ is called a random $\mathcal{D}$-pullback attractor for (φ,θ) if the following hold:

(i) $A(t,\omega)$ is a random compact set;

(ii) $\mathcal{A}$ is invariant; that is, for $\mathbb{P}$-a.s. $\omega\in\Omega$, and $\tau\le t$, $\varphi(t,\tau,\omega)A(\tau,\omega)=A(t,\theta_{t-\tau}\omega)$;

(iii) $\mathcal{A}$ attracts all sets in $\mathcal{D}$, that is, for all $B'\in\mathcal{D}$ and $\mathbb{P}$-a.s. $\omega\in\Omega$,

$$\lim_{\tau\to-\infty}d(\varphi(t,\tau,\theta_{\tau-t}\omega)B(\tau,\theta_{\tau-t}\omega),A(t,\omega))=0,$$

where d is the Hausdorff semimetric given by $dist(B,\mathcal{A})=\sup_{b\in B}\inf_{a\in\mathcal{A}}\parallel b-a\parallel_X$.

Definition 2.5([8,12])**.** A NRDS (φ,θ) on a Banach space X is said to be pullback flattening if for every random bounded set $B'=\{B(t,\omega):t\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D}$, for any $\varepsilon>0$ and $\omega\in\Omega$ there exist a $T(B',\varepsilon,\omega)<t$ and a finite dimensional subspace X_ϵ such that

(i) $P(\bigcup_{\tau\le T_\varepsilon}\varphi(t,\tau,\theta_{\tau-t}\omega)B(\tau,\theta_{\tau-t}\omega))$ is bounded, and

(ii)$||(I-P)(\bigcup_{\tau\le T_\varepsilon}\varphi(t,\tau,\theta_{\tau-t}\omega)B(\tau,\theta_{\tau-t}\omega))||_X<\varepsilon$,

where $P:X\to X_\varepsilon$ is a bounded projector.

Theorem 2.1([12])**.** *Suppose that (φ,θ) is a norm-to-weak continuous NRDS on a uniformly convex Banach space X. If (φ,θ) possesses a random $\mathcal{D}$-pullback bounded absorbing sets $B'=\{B(t,\omega):t\in\mathbb{R},\omega\in\Omega\}$ and (φ,θ) is pullback flattening, then there exists a random $\mathcal{D}$-pullback attractor $\mathcal{A}=\{A(t,\omega):t\in\mathbb{R},\omega\in\Omega\}$ and*

$$A(t,\omega)=\bigcap_{s\le t}\overline{\bigcup_{\tau\le s}\varphi(t,\tau,\theta_{\tau-t}\omega)B(\tau,\theta_{\tau-t}\omega)}.$$

3. Random pullback attractor for modified Swift-Hohenberg

In this section, we will use abstract theory in section 2 to obtain the random $\mathcal{D}$-pullback attractor for equation (1.1)-(1.3). We introduce an Ornstein-Uhlenbeck process

$$z(\theta_t(\omega)):=-\int_{-\infty}^{0}e^{\tau}(\theta_t\omega)(\tau)d\tau, t\in\mathbb{R}.$$

We known from [4], it is the solution of Langevin equation

$$dz + zdt = dW(t).$$

$W(t)$ is a two-sided real-valued Wiener process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where

$$\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\},$$

$\mathcal{F}$ is the Borel algebra induced by the compact open topology of Ω, and $\mathbb{P}$ is the corresponding Wiener measure on $\{\Omega, \mathcal{F}\}$. We identify $\omega(t)$ with $W(t)$, i.e.,

$$W(t) = W(t, \omega) = \omega(t), t \in \mathbb{R}.$$

Define the Wiener time shift by

$$\theta_t \omega(s) = \omega(s + t) - \omega(t), \omega \in \Omega, t, s \in \mathbb{R}.$$

Then $(\Omega, \mathcal{F}, \mathbb{P}, \theta_t)$ is an ergodic metric dynamical system.

From [3,4,11], it is known that the random variable $z(\omega)$ is tempered and there exists a θ_t-invariant set of full measure $\widetilde{\Omega} \subset \Omega$ such that for all $\omega \in \widetilde{\Omega}$:

$$\lim_{t \to \pm\infty} \frac{|z(\theta_t \omega)|}{|t|} = 0, \quad \lim_{t \to \pm\infty} \frac{1}{t} \int_0^t z(\theta_s \omega) ds = 0, \tag{3.1}$$

and for any $\varepsilon > 0$, there exists $\rho(\varepsilon) > 0$, such that

$$|z(\theta_t \omega)| \leq \rho(\omega) + \varepsilon |t|, \quad |\int_0^t z(\theta_s \omega) ds| \leq \rho(\omega) + \varepsilon |t|. \tag{3.2}$$

Let $v(s, \tau, \theta_{s-t}\omega), v_\tau) = e^{-lz(\theta_{s-\tau}\omega)} u(s), \tau \leq s \leq t$, then $dv = -le^{-lz(\theta_{s-t}\omega)} u(s) dz + e^{-lz(\theta_{s-t}\omega)} du$. Using Langevin equation, combined with the original equation (1.1), we get

$$\begin{aligned} &\frac{dv}{ds} + \Delta^2 v + 2\Delta v + (a - lz)v + be^{lz(\theta_{s-t}\omega)} |\nabla v|^2 + e^{2lz(\theta_{s-t}\omega)} v^3 \\ &= e^{-lz(\theta_{s-t}\omega)} g(x, s), \quad \text{in} \;\; D \times [\tau, t], \end{aligned} \tag{3.3}$$

and

$$v = \frac{\partial v}{\partial \nu} = 0 \;\; \text{on} \;\; \partial D \times [\tau, t], \tag{3.4}$$

$$v(x, \tau) = v_\tau = e^{-lz(\theta_{s-t}\omega)} u_\tau(x) \;\; \text{in} \; D. \tag{3.5}$$

Equation (1.1)-(1.3) is equivalent to equation (3.3)-(3.5), by a standard Faedo-Galerkin approximation approache, it can be proved that the problem (3.3)-(3.5) is well posed in $H_0^2(D)$, that is, for every $\tau \in \mathbb{R}$ and $v_\tau \in H_0^2(D)$, there exists a unique solution $v \in C([\tau, \infty), H_0^2(D))$ (see e.g. [18, 22]). Furthermore, the solution is continuous with respect to the initial condition v_τ in $H_0^2(D)$. To construct a non-autonomous random dynamical system $\{V(t, \tau, \omega)\}$ for problem (3.3)-(3.5), we define $V(t, \tau, \omega) : H_0^2(D) \to H_0^2(D)$ by $V(t, \tau, \omega) v_\tau$. Then the system $\{V(t, \tau, \omega)\}$ is a non-autonomous random dynamical system in $H_0^2(D)$.

We now apply Theorem 2.1 in section 2 to obtain the random $\mathcal{D}$-pullback attractors for non-autonomous modified Swift-Hohenberg equation, according to equivalence, we only consider the random $\mathcal{D}$-pullback attractor of equation (3.3)-(3.5).

For convenience, the $L^p(D)$ norm will be denoted by $\|\cdot\|_p$, $H = L^2(D)$ with a scalar product $(u,v) = \int_D u(x)v(x)dx$ and the norm of Sobolev spaces $W_p^k(D)$ by $\|\cdot\|_{k,p}$, we regard the space $H_0^2(D)$ endowed with the norm $\|u\|_{2,2} = \|\triangle u\|$, c or $c(\omega)$ denote the arbitrary positive constants, which only depend on ω and may be different from line to line and even in the same line.

For our purpose that the following Gagliardo-Nirenberg inequality will be used.

Lemma 3.1 (Gagliardo-Nirenberg Inequality)**.** *Let D be an open, bounded domain of the Lipschitz class in $\mathbb{R}^n$. Assume that $1 \le p \le \infty, 1 \le q \le \infty, 1 \le r, 0 < \theta \le 1$ and let*

$$k - \frac{n}{p} \le \theta(m - \frac{n}{q}) - (1-\theta)\frac{n}{r}.$$

Then the following inequality holds:

$$\|u\|_{k,p} \le c(D)\|u\|_r^{1-\theta}\|u\|_{m,q}^{\theta}$$

Here, we need to point out an essential error in literature [13,14], the sign in the middle on the right side of the G-N inequality is a minus sign, but literature [13,14] writes it as a plus sign and proved the main results.

Lemma 3.2. *Assume that $|b|c(D) < 1$, then for all $t \ge \tau$, the following inequalities hold:*

$$\|v(t,\tau,\theta_{\tau-t}\omega)\|^2 \le c(\omega)(e^{-\frac{\lambda}{4}(t-\tau)}\|v_\tau\|^2 + 1 + e^{-\frac{\lambda}{4}t}H_1(t)), \tag{3.6}$$

and

$$\begin{aligned}&\int_\tau^t e^{\frac{\lambda}{2}(s-t)+2l\int_s^t dr}\|\triangle v\|^2 ds\\ \le& c(\omega)((1+(t-\tau))e^{-\frac{\lambda}{4}(t-\tau)}\|v_\tau\|^2 + 1 + e^{-\frac{\lambda}{4}t}H_1(t) + e^{-\frac{\lambda}{4}t}H_2(t)).\end{aligned} \tag{3.7}$$

Proof. Let $v(s) = v(s,\tau,\theta_{s-t}\omega)$ denotes the solution of equation (3.3)-(3.5). Taking the inner product of equation (3.3) with v, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{ds}\|v\|^2 + \|\Delta v\|^2 = 2\|\nabla v\|^2 + (lz-a)\|v\|^2 - e^{2lz(\theta_{s-t}\omega)}\|v\|_4^4\\ -be^{lz(\theta_{s-t}\omega)}\int_D |\nabla v|^2 v dx + e^{-lz(\theta_{s-t}\omega)}(g(x,s),v).\end{aligned} \tag{3.8}$$

By the Hölder inequality and the Young inequality, we have

$$2\|\nabla v\|^2 = 2\int_D |v||\Delta v|dx \le 4\|\Delta v\|\|v\| \le \frac{1}{4}\|\Delta v\|^2 + 4\|v\|^2, \tag{3.9}$$

and

$$e^{-lz(\theta_{s-t}\omega)}|(g(x,s),v)| \le \frac{\lambda}{2}\|v\|^2 + \frac{1}{2\lambda}e^{-2lz(\theta_{s-t}\omega)}\|g(x,s)\|^2. \tag{3.10}$$

Applying the Hölder inequality and the Gagliardo-Nirenberg inequality with $k=1, p=\frac{8}{3}, r=4, n=m=q=2, \theta=\frac{1}{2}$, we have

$$\begin{aligned} |b|e^{lz(\theta_{s-t}\omega)}\int_D |\nabla v|^2|v|dx &\le |b|e^{lz(\theta_{s-t}\omega)}\|\nabla v\|^2_{\frac{8}{3}}\|v\|_4 \\ &\le |b|c(D)e^{lz(\theta_{s-t}\omega)}\|\Delta v\|\|v\|_4^2 \\ &\le \frac{1}{4}\|\Delta v\|^2 + b^2c^2(D)e^{2lz(\theta_{s-t}\omega)}\|v\|_4^4 \end{aligned} \tag{3.11}$$

By the Poincaré inequality $\lambda\|v\|^2 \le \|\Delta v\|^2$, and (3.9)-(3.11), we get

$$\begin{aligned} &\frac{d}{ds}\|v\|^2 + \frac{1}{2}\|\Delta v\|^2 - 2lz\|v\|^2 \\ &\le -2e^{2lz(\theta_{s-t}\omega)}\|v\|_4^4 + c\|v\|^2 + 2b^2c^2(D)e^{2lz(\theta_{s-t}\omega)}\|v\|_4^4 + ce^{-2lz(\theta_{s-t}\omega)}\|g(x,s)\|^2 \\ &= e^{-2lz(\theta_{s-t}\omega)}(-2(1-b^2c^2(D))\|u(s)\|_4^4 + c\|u\|^2 + c\|g(x,s)\|^2). \end{aligned} \tag{3.12}$$

Since $b^2c^2(D) < 1$, there exists $M > 0$ such that

$$(-2(1-b^2c^2(D))\|u(s)\|_4^4 + c\|u\|^2) \le M.$$

Thus we arrive at

$$\frac{d}{ds}\|v\|^2 + \frac{1}{2}\|\Delta v\|^2 - 2lz\|v\|^2 \le ce^{-2lz(\theta_{s-t}\omega)}(1+\|g(x,s)\|^2). \tag{3.13}$$

By the Poincaré inequality, we get

$$\frac{d}{ds}\|v\|^2 + (\frac{\lambda}{2} - 2lz)\|v\|^2 \le ce^{-2lz(\theta_{s-t}\omega)}(1+\|g(x,s)\|^2). \tag{3.14}$$

Multiplying (3.14) by $e^{\frac{\lambda}{2}s - 2l\int_\tau^s z(\theta_{r-t}\omega)dr}$ and integrating it over (τ, t), we obtain

$$\begin{aligned} \|v(t)\|^2 \le\ & e^{-\frac{\lambda}{2}(t-\tau) + 2l\int_\tau^t z(\theta_{r-t}\omega)dr}\|v_\tau\|^2 \\ &+ c\int_\tau^t e^{\frac{\lambda}{2}(s-t) + 2l\int_s^t z(\theta_{r-t}\omega)dr - 2lz(\theta_{s-t}\omega)}(1+\|g(x,s)\|^2)ds. \end{aligned} \tag{3.15}$$

Using (3.2), we get

$$e^{-\frac{\lambda}{2}(t-\tau) + 2l\int_\tau^t z(\theta_{r-t}\omega)dr} \le c(\omega)e^{-\frac{\lambda}{4}(t-\tau)}, \text{ for any } \tau \le t, \tag{3.16}$$

$$e^{\frac{\lambda}{2}s + 2l\int_s^0 z(\theta_r\omega)dr - 2lz(\theta_s\omega)} \le c(\omega)e^{-\frac{\lambda}{4}s}, \text{ for any } s \le 0, \tag{3.17}$$

and

$$\begin{aligned} &\int_\tau^t e^{\frac{\lambda}{2}(s-t) + 2l\int_s^t z(\theta_{r-t}\omega)dr - 2lz(\theta_{s-t}\omega)}(1+\|g(x,s)\|^2)ds \\ &= \int_{\tau-t}^0 e^{\frac{\lambda}{2}s + 2l\int_s^0 z(\theta_r\omega)dr - 2lz(\theta_s\omega)}(1+\|g(x,s+t)\|^2)ds \\ &\le e^{\rho(\omega)}\int_{\tau-t}^0 e^{\lambda s/4}(1+|g(x,s+t)|^2)ds \end{aligned}$$

$$= e^{\rho(\omega)} \int_\tau^t e^{\lambda(s-t)/4}(1 + \|g(x,s)\|^2)ds$$
$$\leq c(\omega)(1 + e^{-\frac{\lambda t}{4}} H_1(t)). \tag{3.18}$$

Hence
$$\|v(t)\|^2 \leq c(\omega)(e^{-\frac{\lambda}{4}(t-\tau)}\|v_\tau\|^2 + 1 + e^{-\frac{\lambda t}{4}} H_1(t)). \tag{3.19}$$

Thus we get the desired result (3.6).

Multiplying (3.15) by $e^{-\frac{\lambda}{2}(t-\tau)+2l\int_\tau^t z(\theta_{r-t}\omega)dr}$ and integrating it in (τ, t), and using (3.15),(3.16), we obtain

$$\begin{aligned}&\int_\tau^t e^{\frac{\lambda}{2}(s-t)+2l\int_s^t z(\theta_{r-t}\omega)dr}\|v(s)\|^2 ds\\ \leq& c(\omega)((t-\tau)e^{-\frac{\lambda}{4}(t-\tau)}\|v_\tau\|^2 + 1 + H_2(t)).\end{aligned} \tag{3.20}$$

By (3.13), we get

$$\begin{aligned}&\int_\tau^t e^{\frac{\lambda}{2}s-2l\int_\tau^s z(\theta_{r-t}\omega)dr} d\|v(s)\|^2\\ =&e^{\frac{\lambda}{2}t-2l\int_\tau^t z(\theta_{r-t}\omega)dr}\|v(t)\|^2 - e^{\frac{\lambda}{2}\tau}\|v_\tau\|^2 - \int_\tau^t (\frac{\lambda}{2} - 2lz)e^{\frac{\lambda}{2}s-2l\int_\tau^s z(\theta_{r-t}\omega)dr}\|v(s)\|^2 ds\\ \leq& \int_\tau^t e^{\frac{\lambda}{2}s-2l\int_\tau^s z(\theta_{r-t}\omega)dr}(-(\frac{1}{2}\|\triangle v\|^2 - 2lz\|v\|^2) + ce^{-2lz(\theta_{s-t}\omega)}(1 + \|g(x,s)\|^2))ds.\end{aligned} \tag{3.21}$$

By (3.2) and (3.20), we obtain

$$\begin{aligned}&\frac{1}{2}\int_\tau^t e^{\frac{\lambda}{2}(s-t)+2l\int_s^t dr}\|\triangle v\|^2 ds\\ \leq& e^{-\frac{\lambda}{2}(t-\tau)+2l\int_\tau^t z(\theta_{r-t}\omega)dr}\|v_\tau\|^2 + \frac{\lambda}{2}\int_\tau^t e^{\frac{\lambda}{2}(s-t)+2l\int_s^t z(\theta_{r-t}\omega)dr}\|v(s)\|^2 ds\\ &+ c\int_\tau^t e^{\frac{\lambda}{2}(s-t)+2l\int_s^t z(\theta_{r-t}\omega)dr - 2lz(\theta_{s-t}\omega)}(1 + \|g(x,s)\|^2)ds\\ \leq& c(\omega)((1 + (t-\tau))e^{-\frac{\lambda}{4}(t-\tau)}\|v_\tau\|^2 + 1 + e^{-\frac{\lambda}{4}t}H_1(t) + e^{-\frac{\lambda}{4}t}H_2(t)).\end{aligned} \tag{3.22}$$

The proof is complete. □

Lemma 3.3. *Assume that $|b|c(D) < 1$, then for all $t \geq \tau$, the following inequality holds:*

$$\begin{aligned}\|\Delta v(t,\tau,\theta_{\tau-t}\omega)\|^2 \leq& c(\omega)((1 + t - \tau + \frac{1}{t-\tau})e^{-\frac{\lambda}{4}(t-\tau)}(\|v_\tau\|^2 + \|v_\tau\|^{10}) + 1\\ &+ e^{-\frac{\lambda}{4}t}((1 + \frac{1}{t-\tau})H_1(t) + (1 + \frac{1}{t-\tau})H_2(t)\\ &+ \int_{-\infty}^t e^{-\lambda s}H_1^5(s)ds)).\end{aligned} \tag{3.23}$$

Proof. Multiplying (3.3) by $\Delta^2 v$, we have

$$\frac{1}{2}\frac{d}{ds}\|\Delta v\|^2 + \|\Delta^2 v\|^2 + 2(\triangle v, \Delta^2 v) + (a - lz)\|\triangle v\|^2$$

$$+ be^{lz(\theta_{s-t}\omega)}(|\nabla v|^2, \triangle^2 v) + e^{2lz(\theta_{s-t}\omega)}(v^3, \triangle^2 v)$$
$$= e^{-lz(\theta_{s-t}\omega)}(g(x,s), \triangle^2 v). \tag{3.24}$$

By the Young inequality, we get

$$
\begin{aligned}
&-2(\triangle v, \Delta^2 v) \le \frac{1}{8}\|\triangle^2 v\|^2 + 8\|\triangle v\|^2;\\
&-be^{lz(\theta_{s-t}\omega)}(|\nabla v|^2, \triangle^2 v) \le \frac{1}{8}\|\triangle^2 v\|^2 + 2b^2 e^{2lz(\theta_{s-t}\omega)}\|\nabla v\|_4^4;\\
&-e^{2lz(\theta_{s-t}\omega)}(v^3, \triangle^2 v) \le \frac{1}{8}\|\triangle^2 v\|^2 + 2e^{4lz(\theta_{s-t}\omega)}\|v\|_6^6;\\
&e^{-lz(\theta_{s-t}\omega)}(g(x,s), \triangle^2 v) \le \frac{1}{8}\|\triangle^2 v\|^2 + 2e^{-2lz(\theta_{s-t}\omega)}\|g(x,s)\|^2.
\end{aligned}
$$

Using the above inequalities to (3.24), we obtain

$$
\begin{aligned}
&\frac{d}{ds}\|\Delta v\|^2 + \|\triangle^2 v\|^2 - 2lz\|\triangle v\|^2\\
&\le (16-2a)\|\triangle v\|^2 + c(e^{2lz(\theta_{s-t}\omega)}\|\nabla v\|_4^4 + e^{4lz(\theta_{s-t}\omega)}\|v\|_6^6 + e^{-2lz(\theta_{s-t}\omega)}\|g(x,s)\|^2).
\end{aligned} \tag{3.25}
$$

The Gagliardo-Nirenberg inequality with $k=1, p=4, m=4, n=q=r=2, \theta=\frac{3}{8}$ gives

$$ce^{2lz(\theta_{s-t}\omega)}\|\nabla v\|_4^4 \le ce^{2lz(\theta_{s-t}\omega)}\|v(t)\|^{\frac{5}{2}}\|\triangle^2 v\|^{\frac{3}{2}} \le \frac{1}{4}\|\Delta^2 v\|^2 + ce^{8lz(\theta_{s-t}\omega)}\|v(t)\|^{10}.$$

The Gagliardo-Nirenberg inequality with $k=0, p=6, m=4, n=q=r=2, \theta=\frac{1}{6}$ yields

$$ce^{4lz(\theta_{s-t}\omega)}\|v\|_6^6 \le ce^{4lz(\theta_{s-t}\omega)}\|v\|^5\|\triangle^2 v\| \le \frac{1}{4}\|\Delta^2 v\|^2 + ce^{8lz(\theta_{s-t}\omega)}\|v\|^{10}.$$

Using the Poincaré inequality $\lambda\|\Delta v\|^2 \le \|\triangle^2 v\|^2$ and the above inequality, we get

$$
\begin{aligned}
&\frac{d}{ds}\|\Delta v\|^2 + (\tfrac{\lambda}{2} - 2lz)\|\triangle v\|^2\\
&\le c(\|\triangle v\|^2 + e^{8lz(\theta_{s-t}\omega)}\|v\|^{10} + e^{-2lz(\theta_{s-t}\omega)}\|g(x,s)\|^2).
\end{aligned} \tag{3.26}
$$

Multiplying this by $(s-\tau)e^{\frac{\lambda}{2}s - 2l\int_\tau^s z(\theta_{r-t}\omega)dr}$ and integrating it over(τ, t), we have

$$
\begin{aligned}
&(t-\tau)e^{\frac{\lambda}{2}t - 2l\int_\tau^t z(\theta_{r-t}\omega)dr}\|\triangle v\|^2\\
&\le \int_\tau^t (1 + c(s-\tau))e^{\frac{\lambda}{2}s - 2l\int_\tau^s z(\theta_{r-t}\omega)dr}\|\triangle v\|^2 ds\\
&\quad + \int_\tau^t (s-\tau)e^{\frac{\lambda}{2}s + 8l\int_\tau^s z(\theta_{r-t}\omega)dr}\|v\|^{10} ds\\
&\quad + \int_\tau^t (s-\tau)e^{\frac{\lambda}{2}s - 2l\int_\tau^s z(\theta_{r-t}\omega)dr - 2lz(\theta_{s-t}\omega)}\|g(x,s)\|^2 ds,
\end{aligned} \tag{3.27}
$$

and

$$\|\triangle v\|^2 \le c(1 + \frac{1}{t-\tau})\int_\tau^t e^{\frac{\lambda}{2}(s-t) + 2l\int_s^t z(\theta_{r-t}\omega)dr}\|\triangle v\|^2 ds$$

$$
\begin{aligned}
&+\int_\tau^t e^{\frac{\lambda}{2}(s-t)+2l\int_s^t z(\theta_{r-t}\omega)dr+8lz(\theta_{s-t}\omega)}\|v\|^{10}ds \\
&+\int_\tau^t e^{\frac{\lambda}{2}(s-t)+2l\int_s^t z(\theta_{r-t}\omega)dr-2lz(\theta_{s-t}\omega)}\|g(x,s)\|^2ds \\
&=: I_1+I_2+I_3. \qquad (3.28)
\end{aligned}
$$

By (3.7), we get

$$
\begin{aligned}
I_1 &= (1+\frac{1}{t-\tau})\int_\tau^t e^{\frac{\lambda}{2}(s-t)+2l\int_s^t z(\theta_{r-t}\omega)dr}\|\triangle v\|^2ds \\
&\le (1+\frac{1}{t-\tau})((1+t-\tau)e^{-\frac{\lambda}{4}(t-\tau)}\|v_\tau\|^2+1+e^{-\frac{\lambda}{4}t}H_1(t)+e^{-\frac{\lambda}{4}t}H_2(t)). \quad (3.29)
\end{aligned}
$$

By (3.6) and (3.2), we have

$$
\begin{aligned}
I_2 &= \int_\tau^t e^{\frac{\lambda}{2}(s-t)+2l\int_s^t z(\theta_{r-t}\omega)dr+8lz(\theta_{s-t}\omega)}\|v\|^{10}ds \\
&\le c(\omega)(e^{-\frac{\lambda}{4}(t-\tau)}\|v_\tau\|^{10}+1+e^{-\frac{\lambda}{4}t}\int_{-\infty}^t e^{-\lambda s}(H_1(s))^5ds), \qquad (3.30)
\end{aligned}
$$

and

$$
I_3 = \int_\tau^t e^{\frac{\lambda}{2}(s-t)+2l\int_s^t z(\theta_{r-t}\omega)dr-2lz(\theta_{s-t}\omega)}\|g(x,s)\|^2ds \le c(\omega)e^{-\frac{\lambda}{4}t}H_1(t). \qquad (3.31)
$$

Combing these estimates with (3.28), we complete the proof of Lemma 3.3. □

Let $\mathcal{R}$ be the sets of all functions $r:\mathbb{R}\to(0,+\infty)$ such that $\lim\limits_{t\to-\infty} te^{\frac{\lambda}{4}t}r^{10}(t)=0$ and denote by $\mathcal{D}$ the class of all families $\hat{D}=\{D(t): t\in\mathbb{R}\}$ such that $D(t)\subset \bar{B}(r(t))$ for some $r(t)\in\mathcal{R}$, $\bar{B}(r(t))$ denote the closed ball in $H_0^2(D)$ with radius $r(t)$. Let

$$
r_1^2(t) = 2c(\omega)(1+e^{-\frac{\lambda}{4}t}(H_1(t)+H_2(t)+\int_{-\infty}^t e^{-\lambda s}H_1^5(s)ds)) \qquad (3.32)
$$

By lemma 3.3 for any $\hat{D}\in\mathcal{D}$ and $t\in\mathbb{R}$, there exists $\tau_0(\hat{D},t,\omega)<t$ such that

$$
\|\Delta v(t,\tau,\theta_{\tau-t}\omega)\|\le r_1(t),\ \text{for any } \tau<\tau_0. \qquad (3.33)
$$

Since $0\le\alpha<\frac{\lambda}{100}$, simple calculation implies that $r_1(t)\in\mathcal{R}$, which means that the $\overline{B}(r_1(t))$ be a family of random $\mathcal{D}$-pullback bounded absorbing sets in $H_0^2(D)$ and $\{\overline{B}(r_1(t))\}\in\mathcal{D}$.

Theorem 3.1. *Assume that $|b|c(D)<1$, then the non-autonomous random dynamical system to problem (1.1)-(1.3) possesses a unique random $\mathcal{D}$-pullback attractor in $H_0^2(D)$.*

Proof. By Lemma 3.3, we know that the dynamical system generated by (3.3)-(3.5) exists a random $\mathcal{D}$-pullback bounded absorbing sets $\{\overline{B}(r_1(t))\}$ in $H_0^2(D)$, we need only prove that the system satisfies the pullback flattening condition. Since A^{-1} is a continuous compact operator in H, by the classical spectral theorem, there exists a sequence $\{\lambda_j\}_{j=1}^\infty$ satisfing

$$
0<\lambda_1\le\lambda_2\le\cdots\lambda_j\le\cdots,\lambda_j\to+\infty, \text{as } j\to+\infty,
$$

and a family of elements $\{e_j\}_{j=1}^{\infty}$ of H_0^1 which are orthonormal in H such that

$$Ae_j = \lambda_j e_j, \ \forall j \in \mathbb{N}^+.$$

Let $H_m = span\{e_1, e_2, \cdots, e_m\}$ in H and $P_m : H \to H_m$ be an orthogonal projector. For any $v \in H$ we write

$$v = P_m v + (I - P_m)v = v_1 + v_2.$$

Taking inner product of 3.3 with $\Delta^2 v_2$ in H, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{ds}\|\Delta v_2\|^2 + \|\Delta^2 v_2\|^2 + 2(\triangle v, \Delta^2 v_2) + (a - lz)\|\triangle v_2\|^2 \\
&+ be^{lz(\theta_{s-t}\omega)}(|\nabla v|^2, \triangle^2 v_2) + e^{2lz(\theta_{s-t}\omega)}(v^3, \triangle^2 v_2) \\
=&e^{-lz(\theta_{s-t}\omega)}(g(x,s), \triangle^2 v_2).
\end{aligned} \tag{3.34}$$

Using a similar method of proof (3.25) and the Poincaré inequality $\lambda_n\|\Delta v_2\|^2 \leq \|\Delta^2 v_2\|^2$, we have

$$\begin{aligned}
&\frac{d}{ds}\|\Delta v_2\|^2 + (\lambda_n - 2lz)\|\triangle v_2\|^2 \\
\leq& c(\|\triangle v\|^2 + e^{2lz(\theta_{s-t}\omega)}\|\nabla v\|_4^4 + e^{4lz(\theta_{s-t}\omega)}\|v\|_6^6 + e^{-2lz(\theta_{s-t}\omega)}\|g(x,s)\|^2) \\
\leq& c(\|\triangle v\|^2 + e^{2lz(\theta_{s-t}\omega)}\|\Delta v\|^4 + e^{4lz(\theta_{s-t}\omega)}\|\Delta v\|^6 + e^{-2lz(\theta_{s-t}\omega)}\|g(x,s)\|^2).
\end{aligned} \tag{3.35}$$

Multiplying this by $(s-\tau)e^{\lambda_n s - 2l\int_\tau^s z(\theta_{r-t}\omega)dr}$ and integrating from τ to t, we obtain

$$\begin{aligned}
&(t-\tau)e^{\lambda_n t - 2l\int_\tau^t z(\theta_{r-t}\omega)dr}\|\Delta v_2\|^2 \\
\leq& \int_\tau^t e^{\lambda_n s - 2l\int_\tau^s z(\theta_{r-t}\omega)dr}\|\Delta v\|^2 ds \\
&+ c\int_\tau^t e^{\lambda_n s - 2l\int_\tau^s z(\theta_{r-t}\omega)dr}(\|\Delta v\|^2 + e^{2lz(\theta_{s-t}\omega)}\|\Delta v\|^4 + e^{4lz(\theta_{s-t}\omega)}\|\Delta v\|^6 \\
&+ e^{-2lz(\theta_{s-t}\omega)}\|g(x,s)\|^2)ds.
\end{aligned} \tag{3.36}$$

and

$$\begin{aligned}
\|\Delta v_2\|^2 \leq& \frac{1}{t-\tau}\int_\tau^t e^{\lambda_n(s-t) + 2l\int_s^t z(\theta_{r-t}\omega)dr}\|\Delta v\|^2 ds \\
&+ c\int_\tau^t e^{\lambda_n(s-t) + 2l\int_s^t z(\theta_{r-t}\omega)dr}(\|\Delta v\|^2 + e^{2lz(\theta_{s-t}\omega)}\|\Delta v\|^4 + e^{4lz(\theta_{s-t}\omega)}\|\Delta v\|^6 \\
&+ e^{-2lz(\theta_{s-t}\omega)}\|g(x,s)\|^2)ds.
\end{aligned} \tag{3.37}$$

(3.33) implies that

$$\begin{aligned}
\|\Delta v_2\|^2 \leq& \frac{1}{t-\tau}\int_\tau^t e^{\lambda_n(s-t) + 2l\int_s^t z(\theta_{r-t}\omega)dr} r_1^2(s) ds \\
&+ c\int_\tau^t e^{\lambda_n(s-t) + 2l\int_s^t z(\theta_{r-t}\omega)dr}(r_1^2(s) + e^{2lz(\theta_{s-t}\omega)} r_1^4(s) + e^{4lz(\theta_{s-t}\omega)} r_1^6(s)
\end{aligned}$$

$$+ e^{-2lz(\theta_{s-t}\omega)}\|g(x,s)\|^2)ds. \tag{3.38}$$

By (3.2), we obtain

$$\begin{aligned}\|\Delta v_2\|^2 \leq & c(\omega)(\frac{1}{t-\tau}\int_\tau^t e^{(\lambda_n-\lambda)(s-t)} r_1^2(s)ds \\ & + \int_\tau^t e^{(\lambda_n-\lambda)(s-t)}(r_1^2(s)+r_1^4(s)+r_1^6(s)+\|g(x,s)\|^2)ds).\end{aligned} \tag{3.39}$$

Let $G(t) = H_1(t) + H_2(t) + \int_{-\infty}^t e^{-\lambda s} H_1^5(s)ds$, we get

$$r_1^2(s) \leq 2c(\omega)(1 + e^{-\frac{\lambda}{4}s} G(t)) \quad \text{for any} \quad \tau \leq t.$$

By simple calculation, we get

$$\begin{aligned}
&\int_\tau^t e^{(\lambda_n-\lambda)(s-t)} r_1^2(s)ds \leq c(\omega)(\frac{1}{\lambda_n-\lambda} + \frac{e^{-\frac{\lambda}{4}t}}{\lambda_n - \frac{5}{4}\lambda} G(t)) \to 0 \quad \text{as} \quad n \to \infty, \\
&\frac{1}{t-\tau}\int_\tau^t e^{(\lambda_n-\lambda)(s-t)} r_1^2(s)ds \leq c(\omega)(\frac{1}{t-\tau}(\frac{1}{\lambda_n-\lambda} + \frac{e^{-\frac{\lambda}{4}t}}{\lambda_n - \frac{5}{4}\lambda} G(t))) \to 0 \quad \text{as} \quad n \to \infty, \\
&\int_\tau^t e^{(\lambda_n-\lambda)(s-t)} r_1^4(s)ds \leq c(\omega)(\frac{1}{\lambda_n-\lambda} + \frac{e^{-\frac{\lambda}{2}t}}{\lambda_n - \frac{3}{2}\lambda} G^2(t)) \to 0 \quad \text{as} \quad n \to \infty, \\
&\int_\tau^t e^{(\lambda_n-\lambda)(s-t)} r_1^6(s)ds \leq c(\omega)(\frac{1}{\lambda_n-\lambda} + \frac{e^{-\frac{3\lambda}{4}t}}{\lambda_n - \frac{7}{4}\lambda} G^3(t)) \to 0 \quad \text{as} \quad n \to \infty.
\end{aligned}$$

Since $\lambda_n \to +\infty$, there exists N, for any $n > N$, we have

$$\int_\tau^t e^{(\lambda_n-\lambda)(s-t)}\|g(x,s)\|^2 ds \leq e^{-\frac{\lambda}{4}t}\int_\tau^t e^{\frac{\lambda}{4}s}\|g(x,s)\|^2 ds,$$

and

$$e^{(\lambda_n-\lambda)(s-t)}\|g(x,s)\|^2 \to 0 \quad \text{as} \quad n \to \infty \quad \text{for any} \quad s < t$$

by Lebesgue dominated convergent theorem, we get

$$\int_\tau^t e^{(\lambda_n-\lambda)(s-t)}\|g(x,s)\|^2 ds \to 0 \quad \text{as} \quad n \to \infty.$$

In summary, we obtain that the terms on the right hand of inequality (3.39) tend to 0 as $n \to \infty$, which means that $\|\Delta v_2(t,\tau,\theta_{\tau-t}\omega)\| \to 0$, i.e., the random dynamical system (3.3)-(3.5) satisfies pullback flattening in Theorem 2.1. □

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