Journal of Applied Analysis and Computation
Volume 11, Number 1, February 2021, 477–485
Website:http://www.jaac-online.com
DOI:10.11948/20200069

MONOTONE TRAVELING WAVES OF NONMONOTONE INTEGRODIFFERENCE EQUATIONS*

Shuxia Pan[1,†]

Abstract This paper deals with the existence of monotone traveling wave solutions of integrodifference equations without monotonicity. By the comparison principle in the sense of exponential ordering, we confirm the existence of monotone traveling wave solutions for integrodifference equations when the overcompensatory is weak.

Keywords Exponential ordering, overcompensatory, minimal wave speed.

MSC(2010) 35C07, 39A20, 37C65.

1. Introduction

Discrete temporal systems including difference equations are often utilized to model the species without overlapping generations. Because the overcompensatory phenomenon is very universal in ecology, many discrete time models are not monotone and may admit complex dynamics [15]. Integrodifference equations are discrete temporal models considering the spatial movement and contact distribution of individuals [13,16,22], which models that the birth and spatial diffusion occur at different stages. When the nonmonotone integrodifference equations are concerned, they may have complex propagation dynamics that is often indexed by traveling wave solutions and asymptotic spreading, see some results by [1,2,13,17,19].

In the study of traveling wave solutions, besides the threshold of wave speed modeling the invasion speed of individuals in population dynamics, the wave profile is also important to describe the front shape of invasion process. To describe the wave profile, the monotonicity and oscillation of traveling wave solutions are basic topics. Moreover, the decay behavior of traveling wave solutions is crucial to understand the propagation mode including the nonconstant entire solutions [27] and the long time behavior of initial value problems [11] . For monotone integrodifference equations, there are some sharp results on the minimal wave speed of monotone traveling wave solutions [3, 9, 12, 23]. When they are local monotone, the minimal wave speed of nontrivial traveling wave solutions was also studied [4, 8, 10, 25], and Yu and Yuan [26] studied the monotonicity, decay behavior and oscillation of traveling wave solutions.

The monotonicity of integrodifference equations often can be characterized by selecting proper parameters. A natural question is: when a nonmonotone model is a

†Email address:shxpan@yeah.net
1School of Science, Lanzhou University of Technology, Lanzhou, Gansu 730050, China
*The author was partially supported by NSF of China (11971213).

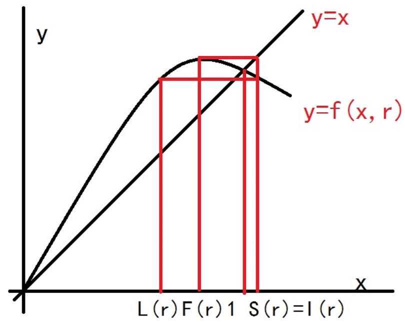

Figure 1. A graph of $f(u, r)$ and constants in (f1), (f3).

small perturbation of monotone equation in proper sense, is it possible to obtain the existence of monotone traveling wave solutions of the nonmonotone integrodifference equation? To answer the question, we consider the following integrodifference equation

$$u_{n+1}(x) = \frac{a}{2} \int_{\mathbb{R}} f(u_n(y), r) e^{-a|x-y|} dy. \tag{1.1}$$

Here, $a > 0$ is a constant and $\frac{a}{2}e^{-a|x|}, x \in \mathbb{R}$ describes the movement law, $f(u, r)$ is often called the birth function and satisfies the following conditions

(f1) there exist $r_0 > 0, I(r) < \infty, r \in [0, r_0]$ such that $I(0) = 1, I(r) \geq 1, r \geq 0$ is continuous, increasing and $f(u, r) : [0, I(r_0)] \times [0, r_0]$ has continuous partial derivatives;

(f2) $f(0, r) = 0, f(1, r) = 1, r \in [0, r_0]$;

(f3) there exist $L(r), F(r), S(r), D > 0$ with

$$0 < L(r) \leq F(r) < 1 < S(r) \leq I(r), r \in (0, r_0]$$

such that $f(u, r)$ satisfies (a) $f(u, r)$ is strictly increasing if $u \in (0, F(r))$ while it is strictly decreasing if $u \in (F(r), S(r))$; (b) $f(F(r), r) = S(r), f(S(r), r) = f(L(r), r)$; (c) $0 \leq S(r) - L(r) \leq Dr, r \in [0, r_0]$; (d) if $F(r) \leq u < v \leq S(r)$, then $F(u, r) - F(v, r) < v - u$;

(f4) $\lim_{u \to 0^+} f(u, r)/u := K(r) > 1$ and there exist $M > 0, m \in (0, 1)$ such that

$$\max\{K(r)u - Mu^{1+m}, 0\} \leq f(u, r) \leq K(r)u, r \in [0, r_0], u \in [0, I(r)].$$

Example 1.1. Consider the following Logistic type birth function

$$f(u, r) = (r + 2)u \left(1 - \frac{r + 1}{r + 2}u\right), r \in [0, 1], u \in \left[0, \frac{r + 2}{r + 1}\right].$$

Then it satisfies the above assumptions by taking $r_0 = 1, D = 1$ and

$$L(r) = \frac{4 - r^2}{4(r + 1)}, F(r) = \frac{r + 2}{2(r + 1)}, I(r) = S(r) = \frac{(r + 2)^2}{4(r + 1)}, K(r) = r + 2.$$

In this paper, a traveling wave solution is a special entire solution taking the form $u_n(x) = \phi(t), t = x + cn \in \mathbb{R}$, in which $c > 0$ is the wave speed while ϕ is the wave profile. So, ϕ, c must satisfy

$$\phi(t+c) = \frac{a}{2}\int_{\mathbb{R}} f(\phi(y), r)e^{-a|t-y|}dy, \quad t \in \mathbb{R}. \tag{1.2}$$

To formulate the desired evolutionary process, ϕ often satisfies proper asymptotic boundary condition, and one widely studied condition is

$$\lim_{t\to-\infty}\phi(t) = 0, \quad \lim_{t\to\infty}\phi(t) = 1, \tag{1.3}$$

which is evident when ϕ is monotone and nonconstant.

In Bourgeois [1] (also see Lutscher [13]), the author simulated several examples of nonmonotone integrodifference equations that admit monotone traveling wave solutions when $r > 0$ is small, and some linear approximation was given to analyze the precise asymptotic behavior. Evidently, the precisely asymptotic behavior can not be obtained by simulation. The purpose of this paper is to study the monotone traveling wave solutions by constructive method, which also shows the decay behavior of traveling wave solutions that can not be studied by numerical simulation.

2. Main Results

In this paper, $C(\mathbb{R}, \mathbb{R})$ is the space of uniform continuous and bounded functions from $\mathbb{R}$ to $\mathbb{R}$, then it is a Banach space equipped with the supremum norm $\|\cdot\|$, and $C_{[a,b]}(\mathbb{R}, \mathbb{R})$ with $b > a$ is defined by

$$C_{[a,b]}(\mathbb{R}, \mathbb{R}) = \{u \in C(\mathbb{R}, \mathbb{R}) : a \le u(x) \le b, x \in \mathbb{R}\}.$$

By Kot [6], ϕ can be rewritten by

$$\phi''(t) + a^2[f(\phi(t-c), r) - \phi(t)] = 0. \tag{2.1}$$

Therefore, it suffices to investigate the existence of (2.1) when $r > 0$ is small. That is, we look for a smooth solution ϕ satisfing (2.1) and $\phi'' \in C$. Clearly, if $r = 0$, then (2.1) is quasimonotone and (1.1) is monotone in $[0, 1]$ while $r > 0$ formulates the overcompensatory leading to the deficiency of quasimonotonicity [21].

Define

$$c'(r) = \sup\{c > 0 : \lambda^2 + a^2[K(r)e^{-\lambda c} - 1] > 0, \lambda \in [0, a]\}.$$

From Hsu and Zhao [4], we see that $c'(r)$ is the minimal wave speed of nontrivial traveling wave solutions. Denote

$$\Lambda(r, \lambda, c) = \lambda^2 + a^2[K(r)e^{-\lambda c} - 1].$$

Then for any fixed r, c, $\Lambda(r, \lambda, c)$ is convex in λ and satisfies the following conclusion.

Lemma 2.1. *Assume that r is fixed. Then $c \in (0, c'(r))$ implies $\Lambda(r, \lambda, c) > 0, \lambda \in [0, a]$ while $c > c'(r)$ implies that $\Lambda(r, \lambda, c) = 0$ has two positive roots $\lambda_1(r, c) < \lambda_2(r, c) < a$ such that $\Lambda(r, \lambda, c) < 0, \lambda \in (\lambda_1(r, c), \lambda_2(r, c))$. If $c = c'(r)$, then there exists $\lambda'(r) > 0$ such that $\Lambda(r, \lambda'(r), c'(r)) = 0$ and $2\lambda'(r) - c'(r)a^2K(r)e^{-\lambda'(r)c'(r)} = 0$.*

By these constants, we present the following main result of this paper.

Theorem 2.1. *There exists $r' > 0$ such that for each $r \in (0, r']$, (2.1) has a monotone solution $\phi(t)$ satisfying (1.3) if $c \in [c'(r), c'(r)+1]$ while $c < c'(r)$ implies the nonexistence of such a solution. Moreover, if $c \in (c'(r), c'(r)+1]$, then $\phi(t)$ satisfies*

$$\lim_{t \to -\infty} \phi(t) e^{-\lambda_1(r,c)t} > 0 \tag{2.2}$$

and $c = c'(r)$ implies

$$\lim_{t \to -\infty} \phi(t) e^{-\lambda'(r)t}/t < 0. \tag{2.3}$$

As for our main conclusion, we make the following remarks.

Remark 2.1. When $c < c'(r)$, Hsu and Zhao [4] obtained the nonexistence of traveling wave solutions without monotone requirement, so the nonexistence in Theorem 2.1 is evident and we shall focus on the existence of monotone traveling wave solutions.

The above theorem answers the existence of traveling wave solutions if c is close to $c'(r)$. When c is large, we further make the following remark, of which the proof is similar to that of Theorem 2.1.

Remark 2.2. For any fixed c, we may obtain the existence of monotone traveling wave solutions of (1.1) if $r > 0$ is small enough and $c > c'(r)$.

Remark 2.3. If $c > c'(r)$, then the limit behavior has been proved by Yu and Yuan [26, Theorem 1.2] for general traveling wave solutions. When $c = c'(r)$, our results imply different decay property as $t \to -\infty$.

3. Proof of Main Results

In this part, we prove Theorem 2.1 motivated by the idea in [5, 24]. We first show the following comparison principle motivated by exponential ordering [21].

Lemma 3.1. *Fix $c > 0$. Then there exist constants $r_1 > 0, \beta > 0$ such that*

$$a^2[f(\phi(t-c), r) - f(\psi(t-c), r)] - a^2[\phi(t) - \psi(t)] + \beta[\phi(t) - \psi(t)] \geq 0 \tag{3.1}$$

for any $r \in [0, r_1], \phi(t), \psi(t) \in C(\mathbb{R}, \mathbb{R})$ satisfying

(i) $0 \leq \psi(t) \leq \phi(t) \leq I(r)$, $t \in \mathbb{R}$*;*

(ii) $e^{\beta t}[\phi(t) - \psi(t)]$ *is nondecreasing with respect to* $t \in \mathbb{R}$.

Proof. If $0 \leq \psi(t-c) \leq L(r)$, then (3.1) holds by taking $\beta \geq a^2$. Otherwise, (f3) leads to

$$\begin{aligned}
&a^2[f(\phi(t-c), r) - f(\psi(t-c), r)] - a^2[\phi(t) - \psi(t)] \\
\geq\ & -a^2 Dr[\phi(t-c) - \psi(t-c)] - a^2[\phi(t) - \psi(t)] \\
=\ & -a^2 Dr e^{\beta c} e^{-\beta c}[\phi(t-c) - \psi(t-c)] - a^2[\phi(t) - \psi(t)] \\
\geq\ & -a^2 Dr e^{\beta c}[\phi(t) - \psi(t)] - a^2[\phi(t) - \psi(t)] \\
=\ & -a^2\left[Dr e^{\beta c} + 1\right][\phi(t) - \psi(t)].
\end{aligned}$$

For any given $\beta > a^2$, there exists $r > 0$ small enough such that

$$a^2\left[Dre^{\beta c}+1\right] \le \beta \tag{3.2}$$

and (3.1) is true. The proof is complete. □

Remark 3.1. Fix $\beta \ge a^2, c > 0$, we may select $r_1 > 0$ small enough such that (3.2) holds for all $r \in [0, r_1]$. Moreover, if c is bounded, then we may fix β, r_1 for all c.

We first consider that $c \in (c'(r), c'(r)+1]$ is fixed. Using the above constants, we now define two continuous functions as follows

$$\varphi(t) = \min\{e^{\lambda_1(r,c)t}, S(r)\}, \psi(t) = \max\{0, e^{\lambda_1(r,c)t} - qe^{\eta\lambda_1(r,c)t}\}, t \in \mathbb{R},$$

where $q = \frac{-a^2M}{\Lambda(r,\eta\lambda_1(r,c),c)} + 1(>1)$, $\eta \in (1, 1+m)$ such that $\eta\lambda_1(r,c) < \lambda_2(r,c)$.

Lemma 3.2. *Assume that $c \in (c'(r), c'(r)+1]$ is fixed. Then $\psi(t), \varphi(t)$ satisfy*

(1) $0 \le \psi(t) < \varphi(t), t \in \mathbb{R}$;

(2) if $\varphi'(t)(\psi'(t))$ does not exists, then $\varphi'(t+) \le \varphi'(t-)(\psi'(t+) \ge \psi'(t-))$;

(3) there exists $\beta > 0$ such that $e^{\beta t}[\varphi(t) - \psi(t)]$ is strict decreasing;

(4) if they are differentiable, then

$$\varphi''(t) + a^2[f(\varphi(t-c), r) - \varphi(t)] \le 0, \tag{3.3}$$
$$\psi''(t) + a^2[f(\psi(t-c), r) - \psi(t)] \ge 0. \tag{3.4}$$

Proof. It suffices to verify the forth item. If $\varphi(t) = S(r) < e^{\lambda_1(r,c)t}$, then (3.3) is clear. When $\varphi(t) = e^{\lambda_1(r,c)t} < S(r)$, then (f4) implies

$$\begin{aligned}
&\varphi''(t) + a^2[f(\varphi(t-c), r) - \varphi(t)] \\
\le\; &\varphi''(t) + a^2[K(r)\varphi(t-c) - \varphi(t)] \\
=\; &e^{\lambda_1(r,c)t}\Lambda(r, \lambda_1(r,c), c) = 0,
\end{aligned}$$

which completes the verification of (3.3).

If $\psi(t) = 0 > e^{\lambda_1(r,c)t} - qe^{\eta\lambda_1(r,c)t}$, then (3.4) is clear. Since $q > 1$ such that $t < 0$, when $\psi(t) = e^{\lambda_1(r,c)t} - qe^{\eta\lambda_1(r,c)t} > 0$, we have

$$\begin{aligned}
&\psi''(t) + a^2[f(\psi(t-c), r) - \psi(t)] \\
\ge\; &\psi''(t) + a^2[K(r)\psi(t-c) - \psi(t)] - a^2M\psi^{1+m}(t-c) \\
\ge\; &\psi''(t) + a^2[K(r)\psi(t-c) - \psi(t)] - a^2M\varphi^{1+m}(t-c) \\
\ge\; &\psi''(t) + a^2[K(r)\psi(t-c) - \psi(t)] - a^2M\varphi^{1+m}(t) \\
=\; &e^{\lambda_1(r,c)t}[\lambda_1^2(r,c) + a^2(K(r)e^{-\lambda_1(r,c)c} - 1)] \\
&-qe^{\eta\lambda_1(r,c)t}[\eta^2\lambda_1^2(r,c) + a^2(K(r)e^{-\eta\lambda_1(r,c)c} - 1)] - a^2Me^{(1+m)\lambda_1(r,c)t} \\
=\; &-qe^{\eta\lambda_1(r,c)t}\Lambda(r, \eta\lambda_1(r,c), c) - a^2Me^{(1+m)\lambda_1(r,c)t} \\
=\; &e^{\eta\lambda_1(r,c)t}\left\{-q\Lambda(r, \eta\lambda_1(r,c), c) - a^2Me^{(1+m-\eta)\lambda_1(r,c)t}\right\} \\
\ge\; &e^{\eta\lambda_1(r,c)t}\left\{-q\Lambda(r, \eta\lambda_1(r,c), c) - a^2M\right\} \\
\ge\; &0,
\end{aligned}$$

which completes the proof. □

Remark 3.2. Due to (3.3) and (3.4), φ is an upper solution of (2.1) while ψ is a lower solution of (2.1).

Define $H : C_{[0,S(r)]}(\mathbb{R},\mathbb{R}) \to C(\mathbb{R},\mathbb{R})$ by

$$H(\phi)(t) = a^2[f(\phi(t-c)) - \phi(t)] + \beta\phi(t), t \in \mathbb{R},$$

and $F : C_{[0,S(r)]}(\mathbb{R},\mathbb{R}) \to C(\mathbb{R},\mathbb{R})$ by

$$F(\phi)(t) = \frac{1}{2\sqrt{\beta}} \int_{\mathbb{R}} e^{-\sqrt{\beta}|t-s|} H(\phi)(s)ds, t \in \mathbb{R}.$$

Evidently, if $\phi \in C_{[0,S(r)]}(\mathbb{R},\mathbb{R})$, then $F(\phi) \in C(\mathbb{R},\mathbb{R})$ satisfies

$$(F(\phi))''(t) - \beta F(\phi)(t) + H(\phi)(t) = 0, t \in \mathbb{R}.$$

Thus, a fixed point of F is a solution of (2.1). Moreover, let $\beta = a^2$, then we see that $\phi(t+c) = \frac{a}{2}\int_{\mathbb{R}} f(\phi(y), r)e^{-a|t-y|}dy$ if ϕ satisfies (2.1).

We now define the following potential wave profile set

$$\Gamma(\varphi,\psi) = \left\{ \phi(t) \in C_{[0,S(r)]}(\mathbb{R},\mathbb{R}) : \begin{array}{l} (i)\ \psi(t) \le \phi(t) \le \varphi(t); \\ (ii)\ \phi(t) \text{ is nondecreasing;} \\ (iii)\ e^{\beta t}(\varphi(t) - \phi(t)) \text{ and } e^{\beta t}(\phi(t) - \psi(t)) \\ \text{are nondecreasing for all } t \in \mathbb{R}; \\ (iv) \text{ for every } s > 0,\ e^{\beta t}(\phi(t+s) - \phi(t)) \\ \text{is nondecreasing for all } t \in \mathbb{R}. \end{array} \right\}$$

Because $1/(1 + e^{-\lambda_1(r,c)t}) \in \Gamma$ for some fixed $\beta > a^2$, we may further fix r' satisfying (3.1) for all $c \in (c'(r), c'(r)+1], r \in (0, r']$. Therefore, with such a constant r', Γ is not empty if $r \in (0, r'], c \in (c'(r), c'(r)+1]$. In what follows, we shall prove that F has a fixed point in Γ by the idea in [5,14,24]. Define a Banach space equipped with the decay norm as follows

$$B_\mu(\mathbb{R},\mathbb{R}) = \left\{ \Phi \in C : \sup_{x\in\mathbb{R}}\{\|\Phi(x)\|e^{-\mu|x|}\} < \infty \right\}$$

and

$$|\Phi|_\mu = \sup_{x\in\mathbb{R}}\{\|\Phi(x)\|e^{-\mu|x|}\},$$

where $\mu > 0$ is a constant. Similar to that in [5,14,24], we have the following result.

Lemma 3.3. *Assume that* $2\mu < \sqrt{\beta}$. *Then* $F : \Gamma \to \Gamma$ *is completely continuous in the sense of* $|\cdot|_\mu$

Due to the fixed point theorem and upper and lower solutions that we have constructed, we obtain the following existence result.

Lemma 3.4. *If* $r > 0$ *is small and* $c \in (c'(r), c'(r)+1]$, *then* (2.1) *has a nonconstant monotone solution* $\phi(t)$ *satisfying* (1.3) *and*

$$\lim_{t\to-\infty} \phi(t)e^{-\lambda_1(r,c)t} > 0.$$

When $c = c'(r)$, we have the following existence result.

Lemma 3.5. *If $r > 0$ is small and $c = c'(r)$, then (2.1) has a monotone solution $\phi(t)$ satisfying (1.3) and*

$$\lim_{t \to -\infty} \phi(t) e^{-\lambda'(r)t}/(-t) > 0.$$

Proof. Similar to what have done, it suffices to construct proper upper and lower solutions. For convenience, we denote $\lambda'(r)$ by λ' in the proof. Let $K > 0$ be a constant such that $K(t) = -Kte^{\lambda' t}$ satisfying (i). $K(t)$ is monotone increasing if $t < -c'(r) - 1/\lambda' := T$, (ii). $K(2T) > S(r)$. Let $q > 1$ be a constant clarified later and construct continuous functions

$$\varphi(t) = \begin{cases} \min\{-Kte^{\lambda' t}, S(r)\}, t < T, \\ S(r), t \geq T, \end{cases}$$

$$\psi(t) = \begin{cases} \max\{(-Kt - q\sqrt{t})e^{\lambda' t}, 0\}, t < 0, \\ 0, t \geq 0. \end{cases}$$

We now verify that $\varphi(t)(\psi(t))$ satisfies (3.3) ((3.4)). For (3.3), the result is evident if $\varphi(t) = S(r)$. When $\varphi(t) = -Kte^{\lambda' t}$, we have

$$\varphi'(t) = \lambda' \varphi(t) - Ke^{\lambda' t}, \varphi''(t) = \lambda'^2 \varphi(t) - 2K\lambda' e^{\lambda' t},$$

and (3.3) is clear.

Furthermore, (3.4) is true if $\psi(t) = 0$. Otherwise,

$$\psi'(t) = \lambda' \psi(t) - Ke^{\lambda' t} + \frac{q}{2\sqrt{-t}} e^{\lambda' t},$$

$$\psi''(t) = \lambda'^2 \psi(t) - 2K\lambda' e^{\lambda' t} + \frac{q\lambda'}{\sqrt{-t}} e^{\lambda' t} + \frac{q}{4\sqrt{(-t)^3}} e^{\lambda' t}.$$

Fix $\lambda \in (0, \lambda')$ such that

$$\lambda(1 + m) > \lambda'.$$

Let $q_1 > K + 1$ be a constant such that $q > q_1$ implies

$$\varphi(t) < e^{\lambda t} \text{ when } t < -q^2/K^2.$$

Then $\psi(t)$ satisfies (3.4) if

$$\begin{aligned}
&\psi''(t) + a^2[f(\psi(t - c), r) - \psi(t)] \\
\geq\; &\psi''(t) + a^2[K(r)\psi(t - c) - \psi(t)] - a^2 M\varphi^{1+m}(t) \\
\geq\; &\lambda'^2 \left[-Kte^{\lambda' t} - q\sqrt{-t}e^{\lambda' t}\right] - 2K\lambda' e^{\lambda' t} + \frac{q\lambda'}{\sqrt{-t}} e^{\lambda' t} + \frac{q}{4\sqrt{(-t)^3}} e^{\lambda' t} - a^2 Me^{(1+m)\lambda t} \\
&+ a^2 \left[K(r)\left(-K(t - c)e^{\lambda'(t-c)} - q\sqrt{-(t - c)}e^{\lambda'(t-c)}\right) - \left(-Kte^{\lambda' t} - q\sqrt{-t}e^{\lambda' t}\right)\right] \\
=\; &(c - 2\lambda')Ke^{\lambda' t} + \frac{q\lambda'}{\sqrt{-t}} e^{\lambda' t} + \frac{q}{4\sqrt{(-t)^3}} e^{\lambda' t} - a^2 Me^{(1+m)\lambda t} \\
&- a^2 \left[q\sqrt{-(t - c)}e^{\lambda'(t-c)} - q\sqrt{-t}e^{\lambda'(t-c)}\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{q\lambda'}{\sqrt{-t}} e^{\lambda' t} + \frac{q}{4\sqrt{(-t)^3}} e^{\lambda' t} - a^2 M e^{(1+m)\lambda t} \\
&\quad -a^2 \left[q\sqrt{-(t-c)} e^{\lambda'(t-c)} - q\sqrt{-t} e^{\lambda'(t-c)} \right] \\
&\geq \frac{q\lambda'}{\sqrt{-t}} e^{\lambda' t} + \frac{q}{4\sqrt{(-t)^3}} e^{\lambda' t} - a^2 M e^{(1+m)\lambda t} \\
&\geq 0,
\end{aligned}
$$

which is equivalent to

$$
\frac{q\lambda'}{\sqrt{-t}} + \frac{q}{4\sqrt{(-t)^3}} \geq a^2 M e^{\left[(1+m)\lambda - \lambda'\right] t}. \tag{3.5}
$$

Let $q > 0$ be large, then $t < -q^2/K^2$ implies that (3.5) is true. The proof is complete. □

References

[1] A. Bourgeois, *Spreading speeds and travelling waves in integrodifference equations with overcompensatory dynamics*, Master Thesis, University of Ottawa, 2016.

[2] A. Bourgeois, V. LeBlanc and F. Lutscher, *Spreading phenomena in integrodifference equations with nonmonotone growth functions,* SIAM J. Appl. Math., 2018, 78, 2950–2972.

[3] J. Fang and X. Zhao, *Traveling waves for monotone semiflows with weak compactness,* SIAM J. Math. Anal., 2014, 46, 3678–3704.

[4] S. B. Hsu and X. Zhao, *Spreading speeds and traveling waves for nonmonotone integrodifference equations,* SIAM J. Math. Anal., 2008, 40, 776–789.

[5] J. Huang and X. Zou, *Existence of travelling wavefronts of delayed reaction-diffusion systems without monotonicity,* Discrete Conti. Dyn. Sys., 2003, 9, 925–936.

[6] M. Kot, *Discrete-time travelling waves: Ecological examples,* J. Math. Biol., 1992, 30, 413–436.

[7] M. Kot and W. M. Schaffer, *Discrete-time growth-dispersal models,* Math. Biosci., 1986, 80, 109–136.

[8] B. Li, M. A. Lewis and H. F. Weinberger, *Existence of traveling waves for integral recursions with nonmonotone growth functions,* J. Math. Biol., 2009, 58, 323–338.

[9] X. Liang and X. Zhao, *Asymptotic speeds of spread and traveling waves for monotone semiflows with applications,* Comm. Pure Appl. Math., 2007, 60, 1–40.

[10] G. Lin, *Travelling wave solutions for intefro-difference systems,* J. Differential Equations, 2015, 258, 2908–2940.

[11] G. Lin, W. Li and S. Ruan, *Asymptotic stability of monostable wavefronts in discrete-time integral recursions,* Sci. China Math., 2010, 53, 1185–1194.

[12] R. Lui, *Biological growth and spread modeled by systems of recursions. I Mathematical theory,* Math. Biosci., 1989, 93, 269–295.

[13] F. Lutscher, *Integrodifference Equations in Spatial Ecology,* Springer, Cham, 2019.

[14] S. Ma, *Traveling wavefronts for delayed reaction-diffusion systems via a fixed point theorem,* J. Differential Equations, 2001, 171, 294–314.

[15] R. M. May, *Biological populations with nonoverlapping generations: stable points, stable cycles, and chaos,* Science, 1974, 186, 645–647.

[16] D. Mollison, *Spatial contact models for ecological and epidemic spread,* J. Roy. Statist. Soc. Ser. B, 1977, 39, 283–326.

[17] M. Neubert, M. Kot and M. A. Lewis, *Dispersal and pattern formation in a discrete-time predator-prey model,* Theoret. Popul. Biol., 1995, 48, 7–43.

[18] Y. Pan, *New methods for the existence and uniqueness of traveling waves of non-monotone integro-difference equations with applications,* J. Differential Equations, 2020, 268, 6319–6349.

[19] Y. Pan, J. Fang and J. Wei, *Seasonal influence on age-structured invasive species with yearly generation,* SIAM J. Appl. Math., 2018, 78, 1842–1862.

[20] M. Slatkin, *Gene flow and selection in a cline,* Genetics, 1973, 75, 733–756.

[21] H. L. Smith, *Monotone Dynamical Systems: An Introduction to the Theory of Competitive and Cooperative Systems,* AMS, Providence, RI, 1995.

[22] H. F. Weinberger, *Long-time behavior of a class of biological model,* SIAM J. Math. Anal., 1982, 13, 353–396.

[23] H. F. Weinberger, M. A. Lewis and B. Li, *Analysis of linear determinacy for spread in cooperative models,* J. Math. Biol., 2002, 45, 183–218.

[24] J. Wu and X. Zou, *Traveling wave fronts of reaction-diffusion systems with delay,* J. Dynam. Diff. Eqns., 2001, 13, 651–687.

[25] T. Yi, Y. Chen and J. Wu, *Unimodal dynamical systems: Comparison principles, spreading speeds and travelling waves,* J. Differential Equations, 2013, 254, 3538–3572.

[26] Z. Yu and R. Yuan, *Properties of traveling waves for integrodifference equations with nonmonotone growth functions,* Z. Angew. Math. Phys., 2012, 63, 249–259.

[27] L. Zhang and S. Pan, *Entire solutions of integrodifference equations,* J. Difference Equ. Appl., 2019, 25, 504–515.