Journal of Applied Analysis and Computation
Volume 11, Number 1, February 2021, 486–520
Website:http://www.jaac-online.com
DOI:10.11948/20200072

EXISTENCE OF SOLUTIONS TO FRACTIONAL DIFFERENTIAL EQUATIONS WITH FRACTIONAL-ORDER DERIVATIVE TERMS*

Ai Sun[1,2], Youhui Su[2,†], Qingchun Yuan[2] and Tongxiang Li[2]

Abstract The study in this paper is made on the nonlinear fractional differential equation whose nonlinearity involves the explicit fractional order $D^{\beta}_{0^+}u(t)$. The corresponding Green's function is derived first, and then the completely continuous operator is proved. Besides, based on the Schauder's fixed point theorem and the Krasnosel'skii's fixed point theorem, the sufficient conditions for at least one or two existence of positive solutions are established. Furthermore, several other sufficient conditions for at least three, n or $2n-1$ positive solutions are also obtained by applying the generalized Avery-Henderson fixed point theorem and the Avery-Peterson fixed point theorem. Finally, several simulation examples are provided to illustrate the main results of the paper. In particularly, a novel efficient iterative method is employed for simulating the examples mentioned above, that is, the interesting point of this paper is that the approximation graphics for the solutions are given by using the iterative method.

Keywords Fractional differential equation, Green's function, the fixed point theorems, numerical simulation.

MSC(2010) 34A08, 34B18, 34K37.

1. Introduction

Generally speaking, the fractional differential equation comprises integer order differential equation, and therefore, in the description of properties of various materials, fractional order models are more accurate than integer order models. Fractional calculus provides potentially the useful tools for solving differential and integral equations, and other problems [13, 15]. The fractional differential equation or fractional calculus has enjoyed considerable popularity due to their applications in various sciences, such as physics, chemistry, biology, engineering, finance and dynamical control, etc [1, 4, 11, 12, 21, 22, 25, 29]. More recent results are described in [3, 8, 14, 18–20, 23, 24, 26–28].

†The corresponding author. Email address:suyh02@163.com(Y. Su)
1College of Science, Shenyang University of Technology, Shenliao Road, Shenyang 110870, Liaoning, China
2School of Mathematics and Statistics, Xuzhou University of Technology, Lishui Road, Xuzhou 221018, Jiangsu, China
*The authors were supported by the Foundation of XZIT (XKY2020102).

In [6], Goodrich investigated the following problem

$$\begin{cases} D_{0+}^{\alpha}u(t)+f\left(t,u(t)\right)=0, & t\in(0,1)\,,\ n-1<\alpha\leq n,\\ u^{(i)}(0)=0, & i=0,1,\cdots,n-2,\\ \left[D_{0+}^{\beta}u(t)\right]_{t=1}=0, & 1\leq\beta\leq n-2, \end{cases}$$

where D_{0+}^{α} is the standard Riemann-Liouville fractional derivative of order α ($n-1<\alpha\leq n$) and f: $[0,1]\times[0,+\infty)\to[0,+\infty)$ is continuous. By applying the Krasnosel'skii's fixed point theorem, he proved the existence of at least one positive solution.

In [5], Chen et al. studied the following fractional differential equation

$$\begin{cases} D_{0+}^{\alpha}u(t)+f\left(t,u(t),u'(t)\right))=0, & t\in(0,1)\,,\ n-1<\alpha\leq n,\\ u^{(i)}(0)=0, & i=0,1,\cdots,n-2,\\ \left[D_{0+}^{\beta}u(t)\right]_{t=1}=0, & 2\leq\beta\leq n-2, \end{cases}$$

where $n>4$ ($n\in\mathbb{N}$), D_{0+}^{α} is the standard Riemann-Liouville fractional derivative of order α and $f(t,u,u'):[0,1]\times[0,+\infty)\times\mathbb{R}\to[0,+\infty)$ satisfies the Carathéodory type condition. They obtained sufficient conditions for the existence of at least one or multiple positive solutions to the above equation by using the fixed point theorems.

In [10], Luca considered the following fractional differential equation

$$\begin{cases} D_{0+}^{\alpha}u(t)+f\left(t,u(t)\right)=0, & t\in(0,1)\,,\\ u(0)=u'(0)=u''(0)=\cdots=u^{(n-2)}(0)=0,\\ D_{0+}^{p}u(1)=\int_0^1 D_{0+}^{q}u(t)dH(t), \end{cases}$$

where $\alpha\in\mathbb{R},\ \alpha\in(n-1,n],\ n\in\mathbb{N},\ n\geq 3,\ p,\ q\in\mathbb{R},\ p\in[1,n-2],\ q\in[0,p]$, D_{0+}^{k} is the standard Riemann-Liouville fractional derivative of order k ($k=\alpha,p,q$), and the nonlinearity f may change sign and may be singular in the points $t=0,1$. Luca obtained the existence and multiplicity of positive solutions for the above equation by means of the Guo-Krasnosel'skii fixed point theorem and some height functions defined on special bounded sets.

For the existence of positive solutions to the nonlinear fractional differential equation, some authors have obtained a few results, for details, see [14, 18–20, 23, 24, 26–28] and the references therein. It is also noted that the above mentioned references [5, 6, 14, 18–20, 24, 26–28] only considered the existence of positive solutions of fractional differential equation whose nonlinear terms are not involved with fractional derivative. Besides, they failed to further provide numerical simulation and comprehensive results of positive solutions to fractional differential equations. Therefore, it is quite necessary to give the numerical simulation and existence for positive solutions to fractional differential equations whose nonlinearity involves the explicit fractional order $D_{0+}^{\beta}u(t)$ in all respects.

Motivated by above-mentioned ideas, we all-sidedly consider the following fractional differential equation

$$\begin{cases} D_{0+}^{\alpha}u(t)+f\left(t,u(t),D_{0+}^{\beta}u(t)\right)=0, & t\in(0,1)\,,\ 0<\beta\leq\theta-1,\\ u^{(i)}(0)=0, & i=0,1,\cdots,n-2,\\ \left[D_{0+}^{\theta}u(t)\right]_{t=1}-\lambda u(1)=0, & 0<\lambda<\frac{\Gamma(\alpha)}{\Gamma(\alpha-\theta)},\ 0<\theta\leq n-2, \end{cases} \tag{1.1}$$

where $n > 4$ $(n \in \mathbb{N})$, $\alpha \in (n-1, n]$, D_{0+}^{α} is the standard Riemann-Liouville fractional derivative of order α and $f(t, u, D_{0+}^{\beta}u(t)) : [0,\, 1] \times [0,\, +\infty) \times \mathbb{R} \to [0,\, +\infty)$ is a the nonnegative continuous function. The numerical simulation and existence of positive solutions are discovered in the study. The corresponding Green's function is derived first, and then the completely continuous operator of equation (1.1) is proved. Besides, based on the Schauder's fixed point theorem and the Krasnosel'skii's fixed point theorem, the sufficient conditions for at least one or two existence of positive solutions to equation (1.1) are established. Furthermore, several other sufficient conditions for at least three, n or $2n-1$ positive solutions to equation (1.1) are also obtained by applying the generalized Avery-Henderson fixed point theorem and the Avery-Peterson fixed point theorem. Finally, several simulation examples are provided to illustrate the main results of the paper. In particularly, a novel efficient iterative method is employed for simulating the examples mentioned above, that is, the interesting point of this paper is that the approximation graphics for the solutions are given by using the iterative method.

This paper is organized as follows. Section 1 is the introduction of the paper. In Section 2, we state some basic definitions and technical lemmas which lay the way for the latter part. In Section 3, the corresponding Green's function and some properties for equation (1.1) are listed. In Section 4, the completely continuous operator of fractional differential equation (1.1) is derived. In Section 5, by applying the Schauder's fixed point theorem and the Krasnosel'skii's fixed point theorem, we discussed the existence of single or twin positive solutions to problem (1.1). Moreover, we provide two examples and the approximation graphics of the solutions. In Section 6, by using the generalized Avery-Henderson fixed point theorem and the Avery-Peterson fixed point theorem, the existence criteria for at least three or arbitrary n or $2n-1$ positive solutions to problem (1.1) are established. And two examples with graphics are given here to illustrate our main results.

2. Preliminaries

In this section, some basic definitions and technical lemmas are introduced first to help us understand the discussions in what follows.

Definition 2.1. The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function y: $(0,\, +\infty) \to \mathbb{R}$ is given by

$$I_{0+}^{\alpha}y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)ds,$$

provided that the right side is pointwise defined on $(0, +\infty)$.

Definition 2.2. The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $y : (0, +\infty) \to \mathbb{R}$ is given by

$$D_{0+}^{\alpha}y(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} y(s)ds,$$

provided that the right side is pointwise defined on $(0, +\infty)$, where $n = [\alpha] + 1$.

Definition 2.3. Let E be a real Banach space. A nonempty closed convex set $P \subset E$ is called a cone if for all $x \in P$ and $\lambda \geq 0$, $\lambda x \in P$ and if $x, -x \in P$ then

$x = 0$. Every cone $P \subset E$ induces an ordering in E given by $x < y$ if and only if $y - x \in P$.

Definition 2.4. A map δ is said to be a nonnegative continuous concave functional on a cone P of a real Banach space E if $\delta : P \to [0, +\infty)$ is continuous and

$$\delta(tx + (1-t)y) \geq t\delta(x) + (1-t)\delta(y)$$

for all $x, y \in P$ and $t \in [0,1]$. Similarly we say the map η is a continuous convex functional on a cone P of a real Banach space E if $\eta : P \to [0, +\infty)$ is continuous and

$$\eta(tx + (1-t)y) \leq t\eta(x) + (1-t)\eta(y)$$

for all $x, y \in P$ and $t \in [0,1]$.

Lemma 2.1 (Lemma 1, [9])**.** *If $\alpha > 0, \beta > 0, u(t) \in L[0,1]$, then*

(i) $D_{0+}^{\beta} I_{0+}^{\alpha} u(t) = I_{0+}^{\alpha-\beta} u(t), \quad \alpha > \beta;$

(ii) $D_{0+}^{\alpha} I_{0+}^{\alpha} u(t) = u(t);$

(iii) $I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + \sum_{i=1}^{n} c_i t^{\alpha-i}, \quad n-1 < \alpha \leq n,\ c_i \in \mathbb{R},\ i = 1, 2, \cdots, n,$ $D_{0+}^{\alpha} u(t) \in L[0,1];$

(iv) $D_{0+}^{\alpha} t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} t^{\beta-\alpha}, \quad \beta > -1,\ \beta > \alpha - 1,\ t > 0.$

Lemma 2.2 (Lemma 2, [9])**.** *Let $\alpha > 0$. Then the differential equation*

$$D_{0+}^{\alpha} u = 0$$

has unique solution $u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}, c_i \in \mathbb{R},\ i = 1, 2, \cdots, n,$ here $n - 1 < \alpha \leq n$.

Lemma 2.3 (Lemma 3, [7])**.** *Let P be a cone in a Banach space E. Assume Ω_1 and Ω_2 are open subsets of E with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. If A: $P \cap (\overline{\Omega}_2 \backslash \Omega_1) \to P$ is a completely continuous operator such that either*

(i) $\|Tx\| \leq \|x\|, \forall x \in P \cap \partial\Omega_1$ *and* $\|Tx\| \geq \|x\|, \forall x \in P \cap \partial\Omega_2$*, or*

(ii) $\|Tx\| \geq \|x\|, \forall x \in P \cap \partial\Omega_1$ *and* $\|Tx\| \leq \|x\|, \forall x \in P \cap \partial\Omega_2$*. Then A has a fixed point in $P \cap (\overline{\Omega_2} \backslash \Omega_1)$.*

For each $d > 0$, let $P(\nu, d) = \{x \in P : \nu(x) < d\}$, where ν is a nonnegative continuous functional on a cone P of a real Banach space E.

Lemma 2.4 (Lemma 4, [16])**.** *Let P be a cone in a real Banach space E. Let ψ, ζ and ν be increasing, nonnegative continuous functionals on P such that for some $c > 0$ and $H > 0$, $\nu(x) \leq \zeta(x) \leq \psi(x)$ and $\|x\| \leq H\nu(x)$ for all $x \in \overline{P}(\nu, c)$. Suppose that there exist positive numbers a and b with $a < b < c$, and T: $\overline{P}(\nu, c) \to P$ is a completely continuous operator such that:*

(i) $\nu(Tx) < c$ *for all* $x \in \partial P(\nu, c)$*;*

(ii) $\zeta(Tx) > b$ *for all* $x \in \partial P(\zeta, b)$*;*

(iii) $P(\psi, a) \neq \emptyset$ *and* $\psi(Tx) < a$ *for* $x \in \partial P(\psi, a)$*.*

Then T has at least three fixed points x_1, x_2 and x_3 belonging to $\overline{P}(\nu, c)$ such that

$$0 \leq \psi(x_1) < a < \psi(x_2),\ \zeta(x_2) < b < \zeta(x_3) \text{ and } \nu(x_3) < c.$$

Lemma 2.5 (Lemma 5, [17]). *Let P be a cone in a real Banach space E. Let ψ, ζ and ν be increasing, nonnegative continuous functionals on P such that for some $c > 0$ and $H > 0$, $\nu(x) \le \zeta(x) \le \psi(x)$ and $\|x\| \le H\nu(x)$ for all $x \in \overline{P}(\nu, c)$. Suppose that there exist positive numbers a and b with $a < b < c$, and $T\colon \overline{P}(\nu, c) \to P$ is a completely continuous operator such that:*

(i) $\nu(Tx) > c$ *for all* $x \in \partial P(\nu, c)$;

(ii) $\zeta(Tx) < b$ *for all* $x \in \partial P(\zeta, b)$;

(iii) $P(\psi, a) \neq \emptyset$ *and* $\psi(Tx) > a$ *for* $x \in \partial P(\psi, a)$.

Then T has at least three fixed points x_1, x_2 and x_3 belonging to $\overline{P}(\nu, c)$ such that

$$0 \le \psi(x_1) < a < \psi(x_2),\ \zeta(x_2) < b < \zeta(x_3) \text{ and } \nu(x_3) < c.$$

Let η and φ be nonnegative continuous convex functionals on P, δ be a nonnegative continuous concave functional on P and θ be a nonnegative continuous functional on P. We define the following convex sets:

$$P(\varphi, \delta, b, d) = \{x \in P :\ b \le \delta(x),\ \varphi(x) \le d\},$$
$$P(\varphi, \eta, \delta, b, c, d) = \{x \in P :\ b \le \delta(x),\ \eta(x) \le c,\ \varphi(x) \le d\},$$

and

$$R(\varphi, \theta, a, d) = \{x \in P :\ a \le \theta(x),\ \varphi(x) \le d\}.$$

Lemma 2.6 (Lemma 6, [2]). *Let P be a cone in a real Banach space E, and η, φ, δ and θ be defined as the above. Moreover, θ satisfies $\theta(\lambda' x) \le \lambda'\theta(x)$ for $0 \le \lambda' \le 1$ such that for some positive numbers h and d,*

$$\delta(x) \le \theta(x) \text{ and } \|x\| \le h\varphi(x) \tag{2.1}$$

holds for all $x \in \overline{P(\varphi, d)}$. Suppose that $T\colon \overline{P(\varphi, d)} \to \overline{P(\varphi, d)}$ is completely continuous and there exist positive real numbers a, b, c, with $a < b$ such that:

(i) $\{x \in P(\varphi, \eta, \delta, b, c, d) : \delta(x) > b\} \neq \emptyset$ *and* $\delta(T(x)) > b$ *for* $x \in P(\varphi, \eta, \delta, b, c, d)$;

(ii) $\delta(T(x)) > b$ *for* $x \in P(\varphi, \delta, b, d)$ *with* $\eta(T(x)) > c$;

(iii) $0 \notin R(\varphi, \theta, a, d)$ *and* $\theta(T(x)) < a$ *for all* $x \in R(\varphi, \theta, a, d)$ *with* $\theta(x) = a$.

Then T has at least three fixed points $x_1, x_2, x_3 \in \overline{P(\varphi, d)}$ such that

$$\varphi(x_i) \le d \text{ for } i=1, 2, 3,\ \ b < \delta(x_1),\ a < \theta(x_2) \text{ and } \delta(x_2) < b \text{ with } \theta(x_3) < a.$$

3. The Properties of Green's Function

In this section, the corresponding Green's function and some properties of the Green's function are derived which are needed in the discussions.

Lemma 3.1 (Lemma 7). *Assume that $y(t) \in L[0, 1]$, then the following fractional differential equation*

$$\begin{cases} D^{\alpha}_{0+}u(t) + y(t) = 0, & t \in (0,1),\ n-1 < \alpha \le n,\ n > 4, \\ u^{(i)}(0) = 0, & i = 0, 1, \cdots, n-2, \\ \left[D^{\theta}_{0+}u(t)\right]_{t=1} - \lambda u(1) = 0, & 0 < \lambda < \frac{\Gamma(\alpha)}{\Gamma(\alpha-\theta)},\ 0 < \theta \le n-2, \end{cases} \tag{3.1}$$

has the unique solution

$$u(t)=\int_0^1 G(t,s)y(s)ds,$$

where

$$G(t,s)=\begin{cases}\frac{t^{\alpha-1}(1-s)^{\alpha-\theta-1}[\Gamma(\alpha)-\lambda\Gamma(\alpha-\theta)(1-s)^{\theta}]-L(t-s)^{\alpha-1}}{L\Gamma(\alpha)}, & 0\le s\le t\le 1,\\ \frac{t^{\alpha-1}(1-s)^{\alpha-\theta-1}[\Gamma(\alpha)-\lambda\Gamma(\alpha-\theta)(1-s)^{\theta}]}{L\Gamma(\alpha)}, & 0\le t\le s\le 1,\end{cases}\tag{3.2}$$

is the Green's function of problem (3.1) with $n>4$, *here* $L=\Gamma(\alpha)-\lambda\Gamma(\alpha-\theta)$.

Proof. According to Lemma 2.1, the general solution to problem (3.1) is

$$u(t)=-I_{0+}^{\alpha}y(t)+c_1t^{\alpha-1}+c_2t^{\alpha-2}+\cdots+c_nt^{\alpha-n},\tag{3.3}$$

where $n-1<\alpha\le n,\ c_i\in R,\ i=1,2,\cdots,n$. Observe that

$$u'(t)=-I_{0+}^{\alpha-1}y(t)+c_1(\alpha-1)t^{\alpha-2}+c_2(\alpha-2)t^{\alpha-3}+\cdots+c_n(\alpha-n)t^{\alpha-n-1},$$
$$u''(t)=-I_{0+}^{\alpha-2}y(t)+c_1(\alpha-1)(\alpha-2)t^{\alpha-3}+\cdots+c_n(\alpha-n)(\alpha-n-1)t^{\alpha-n-2}.$$

The boundary conditions $u^{(i)}(0)=0,\quad i=0,1,\cdots,n-2$ implies that $c_2=\cdots=c_n=0$. In view of the boundary conditions $[D_{0+}^{\theta}u(t)]_{t=1}-\lambda u(1)=0$, we get

$$\lambda u(1)=-\lambda[I_{0+}^{\alpha}y(t)]_{t=1}+\lambda c_1,$$
$$[D_{0+}^{\theta}u(t)]_{t=1}=-I_{0+}^{\alpha-\theta}y(t)]_{t=1}+c_1[D_{0+}^{\theta}t^{\alpha-1}]_{t=1},$$

and

$$\lambda[I_{0+}^{\alpha}y(t)]_{t=1}-\lambda c_1-[I_{0+}^{\alpha-\theta}y(t)]_{t=1}+c_1[D_{0+}^{\theta}t^{\alpha-1}]_{t=1}=0.$$

Based on the Definition 2.1 and the equality mentioned above, the following equality is derived

$$c_1=\int_0^1\frac{(1-s)^{\alpha-\theta-1}[\Gamma(\alpha)-\lambda\Gamma(\alpha-\theta)(1-s)^{\theta}]}{\Gamma(\alpha)[\Gamma(\alpha)-\lambda\Gamma(\alpha-\theta)]}y(s)ds.\tag{3.4}$$

Finally, take c_1 into (3.3), and with the fact that $c_i=0,\ i=2,\cdots n$, we get that

$$\begin{aligned}u(t)&=\int_0^t-\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds\\&\quad+\int_0^1\frac{t^{\alpha-1}(1-s)^{\alpha-\theta-1}[\Gamma(\alpha)-\lambda\Gamma(\alpha-\theta)(1-s)^{\theta}]}{\Gamma(\alpha)[\Gamma(\alpha)-\lambda\Gamma(\alpha-\theta)]}y(s)ds\\&=\int_0^1G(t,s)y(s)ds.\end{aligned}$$

Thus, the proof is completed. □

Lemma 3.2 (Lemma 8)**.** *Let* $G(t,s)$ *be given as* (3.2) *and* $0\le\beta^*\le\theta-1\le n-3$, *then we have*

(i) $D_t^{\beta^*}G(t,s)$ *is a continuous function on the unit square* $[0,1]\times[0,1]$;

(ii) $D_t^{\beta^*}G(t,s) \geq 0$ *for* $(t,s) \in [0,1] \times [0,1]$;

(iii) $\max_{t\in[0,1]} D_t^{\beta^*}G(t,s) = D_t^{\beta^*}G(1,s)$ *for each* $s \in [0,1]$;

(iv) there exists a constant $\gamma^* \in (0,1)$ *such that*

$$\min_{t\in[\frac{1}{2},1]} D_t^{\beta^*}G(t,s) \geq \gamma^* \max_{t\in[0,1]} D_t^{\beta^*}G(t,s) = \gamma^* D_t^{\beta^*}G(1,s),$$

where $D_t^{\beta^*}$ *denotes fractional partial differential of order* β^* *with respect to* t*, and*

$$\gamma^* = \left(\frac{1}{2}\right)^{\alpha-\beta^*-1}.$$

Proof. Note that

$$D_{0+}^{\beta^*}[t^{\alpha-1}] = \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta^*)} t^{\alpha-\beta^*-1}.$$

By Definition 2.2 and Lemma 2.1, we have

$$D_t^{\beta^*}G(t,s) = \begin{cases} \frac{t^{\alpha-\beta^*-1}(1-s)^{\alpha-\theta-1}[\Gamma(\alpha)-\lambda\Gamma(\alpha-\theta)(1-s)^{\theta}]-L(t-s)^{\alpha-\beta^*-1}}{L\Gamma(\alpha-\beta^*)}, & 0 \leq s \leq t \leq 1, \\ \frac{t^{\alpha-\beta^*-1}(1-s)^{\alpha-\theta-1}[\Gamma(\alpha)-\lambda\Gamma(\alpha-\theta)(1-s)^{\theta}]}{L\Gamma(\alpha-\beta^*)}, & 0 \leq t \leq s \leq 1. \end{cases} \tag{3.5}$$

Let

$$D_t^{\beta^*}G_1(t,s) = \frac{t^{\alpha-\beta^*-1}(1-s)^{\alpha-\theta-1}[\Gamma(\alpha)-\lambda\Gamma(\alpha-\theta)(1-s)^{\theta}]-L(t-s)^{\alpha-\beta^*-1}}{L\Gamma(\alpha-\beta^*)}, \quad 0 \leq s \leq t \leq 1,$$

and

$$D_t^{\beta^*}G_2(t,s) = \frac{t^{\alpha-\beta^*-1}(1-s)^{\alpha-\theta-1}[\Gamma(\alpha)-\lambda\Gamma(\alpha-\theta)(1-s)^{\theta}]}{L\Gamma(\alpha-\beta^*)}, \quad 0 \leq t \leq s \leq 1.$$

It is easy to see that property (i) holds since $D_t^{\beta^*}G_1$ and $D_t^{\beta^*}G_2$ are continuous on their domains and $D_t^{\beta^*}G_1(s,s) = D_t^{\beta^*}G_2(s,s)$.

Now, we consider that

$$D_t^{\beta^*+1}G(t,s) = \begin{cases} \frac{t^{\alpha-\beta^*-2}(1-s)^{\alpha-\theta-1}[\Gamma(\alpha)-\lambda\Gamma(\alpha-\theta)(1-s)^{\theta}]-L(t-s)^{\alpha-\beta^*-2}}{L\Gamma(\alpha-\beta^*-1)}, & 0 \leq s \leq t \leq 1, \\ \frac{t^{\alpha-\beta^*-2}(1-s)^{\alpha-\theta-1}[\Gamma(\alpha)-\lambda\Gamma(\alpha-\theta)(1-s)^{\theta}]}{L\Gamma(\alpha-\beta^*-1)}, & 0 \leq t \leq s \leq 1. \end{cases}$$

Evidently, for $0 \leq t \leq s \leq 1$, since $0 < \lambda < \frac{\Gamma(\alpha)}{\Gamma(\alpha-\theta)}$, we observe

$$\Gamma(\alpha) - \lambda\Gamma(\alpha-\theta)(1-s)^{\theta} > \Gamma(\alpha) - \Gamma(\alpha)(1-s)^{\theta} = \Gamma(\alpha)[1-(1-s)^{\theta}] \geq 0,$$

which implies $D_t^{\beta^*+1}G_2(t,s) \geq 0$ on its domain. Similarly, for $0 \leq s \leq t \leq 1$, by $0 \leq \beta^* \leq \theta - 1$, we have

$$\begin{aligned} & t^{\alpha-\beta^*-2}(1-s)^{\alpha-\theta-1}[\Gamma(\alpha) - \lambda\Gamma(\alpha-\theta)(1-s)^{\theta}] - L(t-s)^{\alpha-\beta^*-2} \\ =& t^{\alpha-\beta^*-2}\{(1-s)^{\alpha-\theta-1}[\Gamma(\alpha) - \lambda\Gamma(\alpha-\theta)(1-s)^{\theta}] \\ & - [\Gamma(\alpha) - \lambda\Gamma(\alpha-\theta)](1-\frac{s}{t})^{\alpha-\beta^*-2}\} \end{aligned}$$

$$
\begin{aligned}
\geq & t^{\alpha-\beta^*-2}\{(1-s)^{\alpha-\theta-1}[\Gamma(\alpha)-\lambda\Gamma(\alpha-\theta)(1-s)^{\theta}] \\
& -[\Gamma(\alpha)-\lambda\Gamma(\alpha-\theta)](1-s)^{\alpha-\beta^*-2}\} \\
= & t^{\alpha-\beta^*-2}(1-s)^{\alpha-\theta-1}\{[\Gamma(\alpha)-\lambda\Gamma(\alpha-\theta)(1-s)^{\theta}] \\
& -[\Gamma(\alpha)-\lambda\Gamma(\alpha-\theta)](1-s)^{\theta-\beta^*-1}\} \\
= & t^{\alpha-\beta^*-2}(1-s)^{\alpha-\theta-1}\{\Gamma(\alpha)[1-(1-s)^{\theta-\beta^*-1}] \\
& +\lambda\Gamma(\alpha-\theta)(1-s)^{\theta-\beta^*-1}[1-(1-s)^{\beta^*+1}]\} \\
\geq & 0,
\end{aligned}
$$

which means $D_t^{\beta^*+1}G_1(t,s)\geq 0$, for each s.

It implies that $D_t^{\beta^*}G(t,s)$ is increasing on t for each s. Consequently, (iii) holds. Now observe that, for every fixed admissible s, we have that $D_t^{\beta^*}G(0,s)=0$. Then, as (iii) implies that $D_t^{\beta^*}G(t,s)$ is increasing on its domain, we get $D_t^{\beta^*}G(t,s)\geq 0$, for $(t,s)\in[0,1]\times[0,1]$. So, (ii) holds.

It follows from the property (iii) that

$$
\min_{t\in\left[\frac{1}{2},1\right]} D_t^{\beta^*}G(t,s)=D_t^{\beta^*}G\left(\frac{1}{2},s\right)=\begin{cases} D_t^{\beta^*}G_1\left(\frac{1}{2},s\right), & s\in\left(0,\frac{1}{2}\right], \\ D_t^{\beta^*}G_2\left(\frac{1}{2},s\right), & s\in\left[\frac{1}{2},1\right), \end{cases}
$$

$$
=\begin{cases} \frac{(\frac{1}{2})^{\alpha-\beta^*-1}(1-s)^{\alpha-\theta-1}[\Gamma(\alpha)-\lambda\Gamma(\alpha-\theta)(1-s)^{\theta}]-L\left(\frac{1}{2}-s\right)^{\alpha-\beta^*-1}}{L\Gamma(\alpha-\beta^*)}, & s\in\left(0,\frac{1}{2}\right], \\ \frac{(\frac{1}{2})^{\alpha-\beta^*-1}(1-s)^{\alpha-\theta-1}[\Gamma(\alpha)-\lambda\Gamma(\alpha-\theta)(1-s)^{\theta}]}{L\Gamma(\alpha-\beta^*)}, & s\in\left[\frac{1}{2},1\right). \end{cases}
$$

Let

$$
\phi(s)=\frac{D_t^{\beta^*}G\left(\frac{1}{2},s\right)}{D_t^{\beta^*}G\left(1,s\right)}. \tag{3.6}
$$

Observe that for $0<s\leq\frac{1}{2}$

$$
\begin{aligned}
\phi(s)= & \frac{(\frac{1}{2})^{\alpha-\beta^*-1}(1-s)^{\alpha-\theta-1}[\Gamma(\alpha)-\lambda\Gamma(\alpha-\theta)(1-s)^{\theta}]-L\left(\frac{1}{2}-s\right)^{\alpha-\beta^*-1}}{(1-s)^{\alpha-\theta-1}[\Gamma(\alpha)-\lambda\Gamma(\alpha-\theta)(1-s)^{\theta}]-L(1-s)^{\alpha-\beta^*-1}} \\
= & \left(\frac{1}{2}\right)^{\alpha-\beta^*-1}\frac{(1-s)^{\alpha-\theta-1}[\Gamma(\alpha)-\lambda\Gamma(\alpha-\theta)(1-s)^{\theta}]-L\left(1-2s\right)^{\alpha-\beta^*-1}}{(1-s)^{\alpha-\theta-1}[\Gamma(\alpha)-\lambda\Gamma(\alpha-\theta)(1-s)^{\theta}]-L(1-s)^{\alpha-\beta^*-1}} \\
\geq & \left(\frac{1}{2}\right)^{\alpha-\beta^*-1}\frac{(1-s)^{\alpha-\theta-1}[\Gamma(\alpha)-\lambda\Gamma(\alpha-\theta)(1-s)^{\theta}]-L\left(1-s\right)^{\alpha-\beta^*-1}}{(1-s)^{\alpha-\theta-1}[\Gamma(\alpha)-\lambda\Gamma(\alpha-\theta)(1-s)^{\theta}]-L(1-s)^{\alpha-\beta^*-1}} \\
= & \left(\frac{1}{2}\right)^{\alpha-\beta^*-1}.
\end{aligned}
$$

Now, according to (3.6) and L'Hospital's rule, we get

$$
\begin{aligned}
& \lim_{s\to 0+}\phi(t,s) \\
= & \lim_{s\to 0+}\frac{(\frac{1}{2})^{\alpha-\beta^*-1}(1-s)^{\alpha-\theta-1}[\Gamma(\alpha)-\lambda\Gamma(\alpha-\theta)(1-s)^{\theta}]-L\left(\frac{1}{2}-s\right)^{\alpha-\beta^*-1}}{(1-s)^{\alpha-\theta-1}[\Gamma(\alpha)-\lambda\Gamma(\alpha-\theta)(1-s)^{\theta}]-L(1-s)^{\alpha-\beta^*-1}} \\
= & \lim_{s\to 0+}\left(\frac{1}{2}\right)^{\alpha-\beta^*-1}\frac{\Gamma(\alpha)-\lambda\Gamma(\alpha-\theta)(1-s)^{\theta}-L\left(1-2s\right)^{\theta-\beta^*}}{\Gamma(\alpha)-\lambda\Gamma(\alpha-\theta)(1-s)^{\theta}-L(1-s)^{\theta-\beta^*}}
\end{aligned}
$$

$$
\begin{aligned}
&\overset{L'H}{=} \lim_{s\to 0+}\left(\frac{1}{2}\right)^{\alpha-\beta^*-1} \frac{\lambda\theta\Gamma(\alpha-\theta)(1-s)^{\theta-1}+2(\theta-\beta^*)L\,(1-2s)^{\theta-\beta^*-1}}{\lambda\theta\Gamma(\alpha-\theta)(1-s)^{\theta-1}+(\theta-\beta^*)L\,(1-s)^{\theta-\beta^*-1}}\\
&=\left(\frac{1}{2}\right)^{\alpha-\beta^*-1} \frac{\lambda\theta\Gamma(\alpha-\theta)+2(\theta-\beta^*)L}{\lambda\theta\Gamma(\alpha-\theta)+(\theta-\beta^*)L}\\
&\geq \left(\frac{1}{2}\right)^{\alpha-\beta^*-1}.
\end{aligned}\tag{3.7}
$$

On the other hand, by $0<\lambda<\frac{\Gamma(\alpha)}{\Gamma(\alpha-\theta)}$, for $\frac{1}{2}\leq s<1$, we have

$$
\begin{aligned}
\phi(s) =&\frac{(\frac{1}{2})^{\alpha-\beta^*-1}(1-s)^{\alpha-\theta-1}[\Gamma(\alpha)-\lambda\Gamma(\alpha-\theta)(1-s)^{\theta}]}{(1-s)^{\alpha-\theta-1}[\Gamma(\alpha)-\lambda\Gamma(\alpha-\theta)(1-s)^{\theta}]-L(1-s)^{\alpha-\beta^*-1}}\\
=&\left(\frac{1}{2}\right)^{\alpha-\beta^*-1}\frac{\Gamma(\alpha)-\lambda\Gamma(\alpha-\theta)(1-s)^{\theta}}{\Gamma(\alpha)-\lambda\Gamma(\alpha-\theta)(1-s)^{\theta}-[\Gamma(\alpha)-\lambda\Gamma(\alpha-\theta)](1-s)^{\theta-\beta^*}}\\
=&\left(\frac{1}{2}\right)^{\alpha-\beta^*-1}\frac{\Gamma(\alpha)-\lambda\Gamma(\alpha-\theta)(1-s)^{\theta}}{\Gamma(\alpha)[1-(1-s)^{\theta-\beta^*}]+\lambda\Gamma(\alpha-\theta)[(1-s)^{\theta-\beta^*}-(1-s)^{\theta}]}\\
\geq&\left(\frac{1}{2}\right)^{\alpha-\beta^*-1}\frac{\Gamma(\alpha)-\Gamma(\alpha)(1-s)^{\theta}}{\Gamma(\alpha)[1-(1-s)^{\theta-\beta^*}]+\Gamma(\alpha)[(1-s)^{\theta-\beta^*}-(1-s)^{\theta}]}\\
=&\left(\frac{1}{2}\right)^{\alpha-\beta^*-1}\frac{\Gamma(\alpha)[1-(1-s)^{\theta}]}{\Gamma(\alpha)[1-(1-s)^{\theta}]}=\left(\frac{1}{2}\right)^{\alpha-\beta^*-1}.
\end{aligned}
$$

Hence, for $\frac{1}{2}\leq s\leq 1$, by the above, let

$$
\phi(s)=\left(\frac{1}{2}\right)^{\alpha-\beta^*-1}\frac{\Gamma(\alpha)-\lambda\Gamma(\alpha-\theta)(1-s)^{\theta}}{\Gamma(\alpha)-\lambda\Gamma(\alpha-\theta)(1-s)^{\theta}-L(1-s)^{\theta-\beta^*}}\geq\left(\frac{1}{2}\right)^{\alpha-\beta^*-1}.
$$

Define

$$
\gamma^*(s):=\begin{cases}\frac{(\frac{1}{2})^{\alpha-\beta^*-1}(1-s)^{\alpha-\theta-1}[\Gamma(\alpha)-\lambda\Gamma(\alpha-\theta)(1-s)^{\theta}]-L\left(\frac{1}{2}-s\right)^{\alpha-\beta^*-1}}{(1-s)^{\alpha-\theta-1}[\Gamma(\alpha)-\lambda\Gamma(\alpha-\theta)(1-s)^{\theta}]-L(1-s)^{\alpha-\beta^*-1}}, & s\in\left[0,\frac{1}{2}\right],\\ \left(\frac{1}{2}\right)^{\alpha-\beta^*-1}\frac{\Gamma(\alpha)-\lambda\Gamma(\alpha-\theta)(1-s)^{\theta}}{\Gamma(\alpha)-\lambda\Gamma(\alpha-\theta)(1-s)^{\theta}-L(1-s)^{\theta-\beta^*}}, & s\in\left[\frac{1}{2},1\right].\end{cases}
$$

Let

$$
\gamma^*=\left(\frac{1}{2}\right)^{\alpha-\beta^*-1},
$$

where $\gamma^*(0)=\lim_{s\to 0+}\gamma^*(s)>0$ due to (3.7). It is obvious that $0<\gamma^*<1$. Consequently, we have

$$
\min_{t\in\left[\frac{1}{2},1\right]} D_t^{\beta^*}G(t,s)\geq\gamma^*\max_{t\in[0,1]} D_t^{\beta^*}G(t,s)=\gamma^* D_t^{\beta^*}G(1,s).
$$

□

Remark 3.1. In equality (3.5), considering the special case $\beta^*=0$, we have

(i) $G(t,s)$ is a continuous function on the unit square $[0,1]\times[0,1]$;

(ii) $G(t,s)\geq 0$ for $(t,s)\in[0,1]\times[0,1]$;

(iii) $\max_{t\in[0,1]} G(t,s) = G(1,s)$ for each $s \in [0,1]$;

(iv) there exists a constant $\gamma_0 \in (0,1)$ such that

$$\min_{t\in\left[\frac{1}{2},1\right]} G(t,s) \geq \gamma_0 \max_{t\in[0,1]} G(t,s) = \gamma_0 G(1,s),$$

where

$$\gamma_0 = \left(\frac{1}{2}\right)^{\alpha-1}.$$

Remark 3.2. In equality (3.5), choosing $\beta > 0$, by Lemma 3.2, we have

$$\min_{t\in\left[\frac{1}{2},1\right]} D_t^{\beta} G(t,s) \geq \overline{\gamma} \max_{t\in[0,1]} D_t^{\beta} G(t,s) = \overline{\gamma} D_t^{\beta} G(1,s),$$

where

$$\overline{\gamma} = \left(\frac{1}{2}\right)^{\alpha-\beta-1}.$$

Then,

$$\min\{\overline{\gamma}, \gamma_0\} = \gamma_0.$$

Lemma 3.3 (Lemma 9)**.** *Let $G(t,s)$ be given as* (3.2)*, $0 < \beta \leq \theta - 1 \leq n-3$, and $n-1 < \alpha \leq n$, and then we have*

$$D_t^{\beta} G(1,s) \leq \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)} G(1,s),$$

for all $s \in [0,1]$.

Proof. According to $0 < \beta < \alpha$, there is

$$\begin{aligned}
\frac{D_t^{\beta} G(1,s)}{G(1,s)} &= \begin{cases} \frac{(1-s)^{\alpha-\theta-1}[\Gamma(\alpha)-\lambda\Gamma(\alpha-\theta)(1-s)^{\theta}]-L(1-s)^{\alpha-\beta-1}}{(1-s)^{\alpha-\theta-1}[\Gamma(\alpha)-\lambda\Gamma(\alpha-\theta)(1-s)^{\theta}]-L(1-s)^{\alpha-1}} \frac{L\Gamma(\alpha)}{L\Gamma(\alpha-\beta)}, & 0 \leq s \leq t \leq 1, \\ \frac{(1-s)^{\alpha-\theta-1}[\Gamma(\alpha)-\lambda\Gamma(\alpha-\theta)(1-s)^{\theta}]}{(1-s)^{\alpha-\theta-1}[\Gamma(\alpha)-\lambda\Gamma(\alpha-\theta)(1-s)^{\theta}]} \frac{L\Gamma(\alpha)}{L\Gamma(\alpha-\beta)}, & 0 \leq t \leq s \leq 1, \end{cases} \\
&\leq \begin{cases} \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}, & 0 \leq s \leq t \leq 1, \\ \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}, & 0 \leq t \leq s \leq 1. \end{cases}
\end{aligned}$$

Thus, for any $s \in [0,1]$, we can conclude

$$D_t^{\beta} G(1,s) \leq \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)} G(1,s).$$

□

4. The Completely Continuous Operator

In this section, we establish the completely continuous operator for equation (1.1), and then we obtain that finding the solution to fractional differential equation (1.1) is equivalent to finding the fixed points of the associated completely continuous operator.

Suppose that $E = C^1([0,1], \mathbb{R})$, then E is a Banach space endowed with norm

$$\|u\| = \left(\|u\|_1^2 + \|u\|_2^2\right)^{\frac{1}{2}},$$

where

$$\|u\|_1 = \max_{t\in[0,1]} |u(t)|,$$

and

$$\|u\|_2 = \max_{t\in[0,1]} \left|D_{0+}^{\beta} u(t)\right|.$$

The cone $P \subset E$ is defined by

$$P = \left\{ u \in E : u(t) \geq 0, \min_{t\in\left[\frac{1}{2},1\right]} u(t) \geq \gamma_0 \|u\|_1, \ \min_{t\in\left[\frac{1}{2},1\right]} D_{0+}^{\beta} u(t) \geq \gamma_0 \|u\|_2, \ t \in [0,1] \right\}. \tag{4.1}$$

Assume that $y(t) = f\left(t, u(t), D_{0+}^{\beta} u(t)\right)$, then it follows from Lemma 3.1 that the solutions of fractional differential equation (1.1) are the corresponding fixed points of the operator T: $E \to E$, which is defined by

$$\begin{aligned} Tu &= \int_0^1 G(t,s) f\left(s, u(s), D_{0+}^{\beta} u(s)\right) ds \qquad (4.2)\\ &= \int_0^1 \frac{t^{\alpha-1}(1-s)^{\alpha-\theta-1}[\Gamma(\alpha) - \lambda\Gamma(\alpha-\theta)(1-s)^{\theta}] f\left(s, u(s), D_{0+}^{\beta} u(s)\right) ds}{\Gamma(\alpha)[\Gamma(\alpha) - \lambda\Gamma(\alpha-\theta)]} \\ &\quad - \int_0^t \frac{(t-s)^{\alpha-1} f\left(s, u(s), D_{0+}^{\beta} u(s)\right) ds}{\Gamma(\alpha)}. \end{aligned}$$

For our later convenience, we denote

$$\begin{aligned} M &= \int_0^1 G(1,s)ds, \quad N = \int_0^1 D_t^{\beta} G(1,s)ds, \\ S &= \int_{\frac{1}{2}}^1 G(1,s)ds, \quad T = \int_{\frac{1}{2}}^1 D_t^{\beta} G(1,s)ds. \end{aligned}$$

Lemma 4.1 (Lemma 10). *Suppose that $f \in C([0,1] \times [0,+\infty) \times \mathbb{R}, \ [0,+\infty))$, then the operator T: $P \to P$ is completely continuous.*

Proof. Firstly, we prove that T: $P \to P$ is continuous.

Let $u \in P$, from Lemma 3.2 and Remark 3.1, we know $u(t) \in P$, $u(t) \geq 0$, and $D_{0+}^{\beta} u(t) \geq 0$. Also, since f is nonnegative continuous function, we have

$$\begin{aligned} \min_{t\in\left[\frac{1}{2},1\right]} Tu &= \min_{t\in\left[\frac{1}{2},1\right]} \int_0^1 G(t,s) f\left(s, u(s), D_{0+}^{\beta} u(s)\right) ds \\ &= \int_0^1 \min_{t\in\left[\frac{1}{2},1\right]} G(t,s) f\left(s, u(s), D_{0+}^{\beta} u(s)\right) ds \\ &\geq \int_0^1 \gamma_0 \max_{t\in[0,1]} G(t,s) f\left(s, u(s), D_{0+}^{\beta} u(s)\right) ds \end{aligned}$$

$$=\gamma_0\|Tu\|_1,$$

and

$$\begin{aligned}\min_{t\in\left[\frac{1}{2},1\right]} T(D^{\beta}_{0+}u) =& \min_{t\in\left[\frac{1}{2},1\right]} \int_0^1 D^{\beta}_{t} G(t,s) f\left(s,u(s),D^{\beta}_{0+}u(s)\right)ds\\ =& \int_0^1 \min_{t\in\left[\frac{1}{2},1\right]} D^{\beta}_{t} G(t,s) f\left(s,u(s),D^{\beta}_{0+}u(s)\right)ds\\ \geq& \overline{\gamma}\|Tu\|_2 \geq \gamma_0\|Tu\|_2.\end{aligned}$$

It implies $TP \subset P$. Hence, T: $P \to P$ is well defined. Since $f \in C([0,1] \times [0,+\infty) \times \mathbb{R},\ [0,+\infty))$, $G(t,s) \in C([0,1]\times[0,1])$ and $D^{\beta}_{t}G(t,s) \in C([0,1]\times[0,1])$, the continuity of T is obvious.

Secondly, we show that T is compact. Let $\Omega \subset P$ be bounded, i.e. there exists a constant $r > 0$ such that for each $u \in \Omega_r = \{u \in P : \|u\| \leq r\}$. By virtue of the continuity and nonnegativity of f, there exists a $K > 0$ such that $\left|f\left(s,u(s),D^{\beta}_{0+}u(s)\right)\right| = f\left(s,u(s),D^{\beta}_{0+}u(s)\right) \leq K$, for each $(t,u,D^{\beta}_{0+}u) \in [0,1] \times [0,r] \times [-r,r]$. By the definition of $\|u\|$, we have

$$\begin{aligned}\|Tu\|_1 =& \max_{t\in[0,1]} |Tu(t)|\\ =& \max_{t\in[0,1]} \int_0^1 G(t,s) f\left(s,u(s),D^{\beta}_{0+}u(s)\right)ds\\ \leq& K \max_{t\in[0,1]} \int_0^1 G(t,s)ds\\ \leq& K \int_0^1 \max_{t\in[0,1]} G(t,s)ds\\ \leq& K \int_0^1 G(1,s)ds\\ =& KM,\end{aligned}\tag{4.3}$$

and

$$\begin{aligned}\|Tu\|_2 =& \max_{t\in[0,1]} \left|TD^{\beta}_{0+}u(t)\right|\\ =& \max_{t\in[0,1]} \int_0^1 D^{\beta}_{t}G(t,s) f\left(s,u(s),D^{\beta}_{0+}u(s)\right)ds\\ \leq& K \int_0^1 \max_{t\in[0,1]} D^{\beta}_{t}G(t,s)ds\\ \leq& K \int_0^1 D^{\beta}_{t}G(1,s)ds\\ =& KN.\end{aligned}\tag{4.4}$$

Hence, we get

$$\|Tu\| = \left(\|Tu\|_1^2 + \|Tu\|_2^2\right)^{\frac{1}{2}} \leq K\left(M^2+N^2\right)^{\frac{1}{2}}.$$

Thirdly, we prove that T maps bounded sets into equicontinuous sets of P. From Lemma 3.2 and Remark 3.1, we know $D_t^{\beta}G(t,s)$ and $G(t,s)$ are continuous in $[0,1]\times[0,1]$. Then $D_t^{\beta}G(t,s)$ and $G(t,s)$ are uniformly continuous in $[0,1]\times[0,1]$. Take $t_1,t_2\in[0,1]$. For any $\varepsilon>0$, there exists $\delta>0$, whenever $|t_1-t_2|<\delta$, we have

$$|G(t_2,s)-G(t_1,s)|<\tfrac{\varepsilon}{K},$$

and

$$\left|D_t^{\beta}G(t_2,s)-D_t^{\beta}G(t_1,s)\right|<\tfrac{\varepsilon}{K}.$$

Without loss of generality, we assume $t_1<t_2$. Using (4.3) and (4.4), for any $u\in\Omega_r$, we have

$$\begin{aligned}\|Tu(t_2)-Tu(t_1)\|_1 &\le \max_{t\in[0,1]}\left|\int_0^1 [G(t_2,s)-G(t_1,s)]\, f\left(s,u(s),D_{0+}^{\beta}u(s)\right)ds\right|\\ &\le \max_{t\in[0,1]}\int_0^1 |G(t_2,s)-G(t_1,s)|\left|f\left(s,u(s),D_{0+}^{\beta}u(s)\right)\right|ds\\ &<\varepsilon,\end{aligned}$$

and

$$\begin{aligned}&\|Tu(t_2)-Tu(t_1)\|_2\\ \le&\max_{t\in[0,1]}\left|\int_0^1\left[D_t^{\beta}G(t_2,s)-D_t^{\beta}G(t_1,s)\right]f\left(s,u(s),D_{0+}^{\beta}u(s)\right)ds\right|\\ \le&\max_{t\in[0,1]}\int_0^1\left|D_t^{\beta}G(t_2,s)-D_t^{\beta}G(t_1,s)\right|\left|f\left(s,u(s),D_{0+}^{\beta}u(s)\right)\right|ds\\ <&\varepsilon.\end{aligned}$$

Therefore, we get

$$\|Tu(t_2)-Tu(t_1)\|=\left[\|Tu(t_2)-Tu(t_1)\|_1^2+\|Tu(t_2)-Tu(t_1)\|_2^2\right]^{\frac{1}{2}}<\sqrt{2}\varepsilon.$$

It follows from Arzela-Ascoli theorem that the operator T: $P\to P$ is completely continuous. The proof is complete. □

5. Existence of One or Two Solutions

In this section, the existence of single or twin positive solutions to problem (1.1) are discussed.

Theorem 5.1. *Suppose that $f\in C([0,1]\times[0,+\infty)\times\mathbb{R},\ [0,+\infty))$ and it meets the growth condition: there exists a nonnegative function $m(t)\in L(0,1)\cap C[0,1]$ such that*

$$|f(t,u,v)|\le m(t)+\lambda_1|u|^{\sigma_1}+\lambda_2|v|^{\sigma_2},\ \lambda_i>0,\ 0<\sigma_i<1,\ i=1,2.$$

Then there exists one positive solution to the problem (1.1).

Proof. Take $c \geq \max\left\{3\tau,\ (3\lambda_1 M)^{\frac{1}{1-\sigma_1}},\ (3\lambda_2 N)^{\frac{1}{1-\sigma_2}}\right\}$. Let

$$\overline{P}_c = \{u \in E : \|u\| < c\},$$

where $\tau = \int_0^1 G(1,s)m(s)ds + \int_0^1 D_t^\beta G(1,s)m(s)ds$. Then, it follows from the above that $\tau \leq \frac{c}{3},\ \lambda_1 M \leq \frac{c^{1-\sigma_1}}{3},\ \lambda_1 N \leq \frac{c^{1-\sigma_2}}{3}$.

Now we show that $T : \overline{P}_c \to \overline{P}_c$. If $u \in \overline{P}_c$, then we have

$$\begin{aligned} 0 \leq & u(t) \leq \max_{t\in[0,1]} |u(t)| \leq \|u\| < c, \\ 0 \leq & |D_{0+}^\beta u(t)| \leq \max_{t\in[0,1]} \left|D_{0+}^\beta u(t)\right| \leq \|u\| < c. \end{aligned}$$

So,

$$|f(t,u,v)| \leq m(t) + \lambda_1 |c|^{\sigma_1} + \lambda_2 |c|^{\sigma_2},\ \lambda_i > 0,\ 0 < \sigma_i < 1,\ i = 1,2.$$

Also,

$$\begin{aligned} \|Tu\|_1 =& \max_{t\in[0,1]} \int_0^1 G(t,s) f\left(s, u(s), D_{0+}^\beta u(s)\right) ds \\ \leq & (\lambda_1 |c|^{\sigma_1} + \lambda_2 |c|^{\sigma_2}) \int_0^1 G(1,s)ds + \int_0^1 G(1,s)m(s)ds \\ =& (\lambda_1 |c|^{\sigma_1} + \lambda_2 |c|^{\sigma_2})M + \int_0^1 G(1,s)m(s)ds, \end{aligned}$$

and

$$\begin{aligned} \|Tu\|_2 =& \max_{t\in[0,1]} \int_0^1 D_t^\beta G(t,s) f\left(s, u(s), D_{0+}^\beta u(s)\right) ds \\ \leq & (\lambda_1 |c|^{\sigma_1} + \lambda_2 |c|^{\sigma_2}) \int_0^1 D_t^\beta G(1,s)ds + \int_0^1 D_t^\beta G(1,s)m(s)ds \\ =& (\lambda_1 |c|^{\sigma_1} + \lambda_2 |c|^{\sigma_2})N + \int_0^1 D_t^\beta G(1,s)m(s)ds. \end{aligned}$$

Thus,

$$\begin{aligned} \|Tu\| &= \left(\|Tu\|_1^2 + \|Tu\|_2^2\right)^{\frac{1}{2}} \\ &\leq \|Tu\|_1 + \|Tu\|_2 \leq \tau + (\lambda_1 |c|^{\sigma_1} + \lambda_2 |c|^{\sigma_2})(M+N) \leq \frac{c}{3} + \frac{c}{3} + \frac{c}{3} = c. \end{aligned}$$

Consequently, $T : \overline{P}_c \to \overline{P}_c$. From Lemma 4.1, we know $T : \overline{P}_c \to \overline{P}_c$ is completely continuous. Also, by Schauder's fixed point theorem, there exists a solution to problem (1.1), which completes the proof. $\square$

We now provide the following example to illustrate the our theoretical result.

Example 5.1. Consider the following the fractional equation problem

$$\begin{cases} D_{0+}^{\frac{11}{3}} u(t) + f\left(t, u(t), D_{0+}^{\frac{1}{2}} u(t)\right) = 0, & t \in (0,1), \\ u^{(i)}(0) = 0, & i = 0,1,2, \\ D_{0+}^{\frac{5}{3}} u(t)_{t=1} - 2u(1) = 0, & t \in (0,1), \end{cases} \tag{5.1}$$

where $v = D_{0^+}^{\frac{1}{2}} u(t)$ and

$$f\,(t, u, v) = t + 2u^{\frac{1}{2}} + v^{\frac{2}{3}}.$$

Since $|f| \le t + 2u^{\frac{1}{2}} + |v|^{\frac{2}{3}}$, it is easy to see that the conditions of the Theorem 5.1 are satisfied. Then, applying the Theorem 5.1, this problem has one positive solution. To obtain the approximation of the solution, we now use the iterative method which is similar to what Wei et al. proposed in [23]. Firstly, given $\Psi \in C[0,1]$, let

$$u(t) = \int_0^1 G(t,s)\Psi(s)ds,\ \ v(t) = \int_0^1 D_t^{\frac{1}{2}} G(t,s)\Psi(s)ds.$$

Define an operator $A : C[0,1] \to C[0,1]$, by

$$(A\Psi)(t) = f\left(t, \int_0^1 G(t,s)\Psi(s)ds, \int_0^1 D_t^{\frac{1}{2}} G(t,s)\Psi(s)ds\right).$$

According to the continuity of $G(t,s)$ and $\Psi(s)$, it is easy to prove that A is continuous operator. By the definition of the Green's function, it is easy to see that if $\Psi(s)$ is a fixed point of the operator A, then

$$u(t) = \int_0^1 G(t,s)\Psi(s)ds$$

is a solution to problem (5.1). And the vice versa.

Let

$$\begin{aligned}
&\Psi_0(t) = f(t,0,0) \ \text{ for } \ t \in [0,1],\\
&\Psi_{k+1}(t) = f\left(t, \int_0^1 G(t,s)\Psi_k(s)ds, \int_0^1 D_t^{\frac{1}{2}} G(t,s)\Psi_k(s)ds\right),
\end{aligned}$$

where $k = 0, 1, \cdots$.

From the proof of Wei et al. in [23], we know the operator A is contraction and this iterative method converges with the rate of geometric progression. Therefore, $u(t) = \int_0^1 G(t,s)\Psi^*(s)ds$ is a solution of problem (5.1), where Ψ^* is a fixed point of operator A.

Now we give the the numerical simulation of solution to problem (5.1) , that is, the approximation of the solution to problem (5.1) is given in Figure 1.

Theorem 5.2. *Suppose that $f \in C([0,1] \times [0,+\infty) \times \mathbb{R},\ [0,+\infty))$ and the following conditions hold:*

(A1) there exists a constant $a > 0$ such that $f\,(t,u,v) \le a\Lambda_1$ for $(t,u,v) \in [0,1] \times [0,a] \times [-a,a]$, where $\Lambda_1 = \min\left\{(\sqrt{2}M)^{-1}, (\sqrt{2}N)^{-1}\right\}$;

(A2) there exists a constant $b > 0$ such that $f\,(t,u,v) \ge b\Lambda_2$ for $(t,u,v) \in [\frac{1}{2},1] \times [0,b] \times [-b,b]$, where $\Lambda_2 = \max\left\{(\sqrt{2}S)^{-1}, (\sqrt{2}T)^{-1}\right\}$, and $a \ne b$.

Then problem (1.1) has at least one positive solution u such that $\|u\|$ lies between a and b.

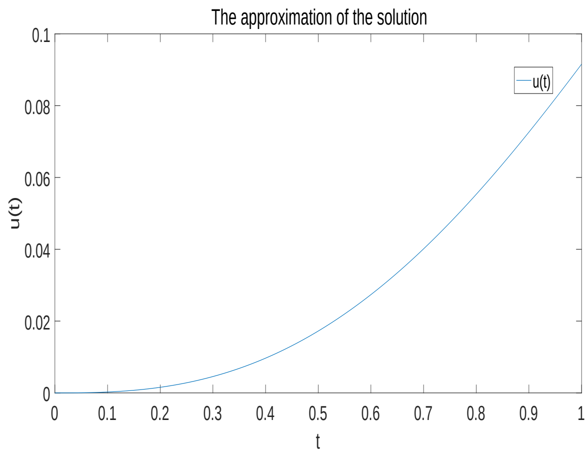

Figure 1. the approximation of the solution to problem (5.1)

Proof. Without loss of generality, we assume that $a < b$. Let

$$\Omega_a = \{u \in E : \|u\| < a\}.$$

For any $u \in P \cap \partial\Omega_a$, there is

$$\max_{t\in[0,1]} |u(t)| \leq \|u\| < a \quad \text{and} \quad \max_{t\in[0,1]} \left|D_{0+}^{\beta} u(t)\right| \leq \|u\| < a.$$

It follows from Lemma 3.2 and condition $(A1)$ that

$$\begin{aligned}
\|Tu\|_1 = &\max_{t\in[0,1]} \int_0^1 G(t,s) f\left(s, u(s), D_{0+}^{\beta} u(s)\right) ds \\
\leq &\int_0^1 G(1,s) a\Lambda_1 ds \\
= &a\Lambda_1 \int_0^1 G(1,s) ds \\
= &aM \min\left\{(\sqrt{2}M)^{-1}, (\sqrt{2}N)^{-1}\right\} \\
\leq &\frac{a}{\sqrt{2}},
\end{aligned}$$

and

$$\begin{aligned}
\|Tu\|_2 = &\max_{t\in[0,1]} \int_0^1 D_t^{\beta} G(t,s) f\left(s, u(s), D_{0+}^{\beta} u(s)\right) ds \\
\leq &a\Lambda_1 \int_0^1 D_t^{\beta} G(1,s) ds \\
= &aN \min\left\{(\sqrt{2}M)^{-1}, (\sqrt{2}N)^{-1}\right\} \\
\leq &\frac{a}{\sqrt{2}}.
\end{aligned}$$

Hence, we get

$$\|Tu\| = \left(\|Tu\|_1^2 + \|Tu\|_2^2\right)^{\frac{1}{2}} \leq a.$$

which implies that

$$\|Tu\| \leq \|u\| \text{ for } u \in P \cap \partial\Omega_a. \tag{5.2}$$

Define

$$\Omega_b = \{u \in E : \|u\| < b\},$$

for arbitrary $u \in P \cap \partial\Omega_b$, we have

$$\max_{t\in[0,1]} |u(t)| \leq \|u\| < b \text{ and } \max_{t\in[0,1]} \left|D_{0+}^{\beta} u(t)\right| \leq \|u\| < b.$$

On the other hand, it follows from Lemma 3.2 and condition $(A2)$ that

$$\begin{aligned}
\|Tu\|_1 &= \max_{t\in[0,1]} \int_0^1 G(t,s) f\left(s, u(s), D_{0+}^{\beta} u(s)\right) ds \\
&= \max_{t\in[0,1]} \left[\int_0^{\frac{1}{2}} G(t,s) f\left(s, u(s), D_{0+}^{\beta} u(s)\right) ds + \int_{\frac{1}{2}}^1 G(t,s) f\left(s, u(s), D_{0+}^{\beta} u(s)\right) ds\right] \\
&\geq \max_{t\in[0,1]} \int_{\frac{1}{2}}^1 G(t,s) f\left(s, u(s), D_{0+}^{\beta} u(s)\right) ds \\
&= \int_{\frac{1}{2}}^1 \max_{t\in[0,1]} G(t,s) f\left(s, u(s), D_{0+}^{\beta} u(s)\right) ds \\
&\geq \int_{\frac{1}{2}}^1 G(1,s) b\Lambda_2 ds \\
&= b\Lambda_2 \int_{\frac{1}{2}}^1 G(1,s) ds \\
&= bS \max\left\{(\sqrt{2}S)^{-1}, (\sqrt{2}T)^{-1}\right\} \\
&\geq \frac{b}{\sqrt{2}},
\end{aligned}$$

and

$$\begin{aligned}
\|Tu\|_2 &= \max_{t\in[0,1]} \int_0^1 D_t^{\beta} G(t,s) f\left(s, u(s), D_{0+}^{\beta} u(s)\right) ds \\
&\geq \max_{t\in[0,1]} \int_{\frac{1}{2}}^1 D_t^{\beta} G(t,s) f\left(s, u(s), D_{0+}^{\beta} u(s)\right) ds \\
&\geq \int_{\frac{1}{2}}^1 D_t^{\beta} G(1,s) b\Lambda_2 ds \\
&= b\Lambda_2 \int_{\frac{1}{2}}^1 D_t^{\beta} G(1,s) ds \\
&= bT \max\left\{(\sqrt{2}S)^{-1}, (\sqrt{2}T)^{-1}\right\} \\
&\geq \frac{b}{\sqrt{2}}.
\end{aligned}$$

Hence, we get

$$\|Tu\| = \left(\|Tu\|_1^2 + \|Tu\|_2^2\right)^{\frac{1}{2}} \geq b,$$

which implies that

$$\|Tu\| \geq \|u\| \text{ for } u \in P \cap \partial\Omega_b. \tag{5.3}$$

Using (5.2), (5.3) and Lemma 2.3, problem (1.1) has a positive solution u in $P \cap (\overline{\Omega}_b \backslash \Omega_a)$. The proof is complete. □

For $u, v \in P$, we denote

$$g^0 = \lim_{(u,v)\to(0,0)} \sup_{t\in[0,1]} \frac{f(t,u,v)}{u}, \qquad h^0 = \lim_{(u,v)\to(0,0)} \sup_{t\in[0,1]} \frac{f(t,u,v)}{|v|},$$

$$f^0 = \lim_{(u,v)\to(0,0)} \sup_{t\in[0,1]} \frac{f(t,u,v)}{u+|v|}, \qquad g^\infty = \lim_{(u,v)\to(\infty,\infty)} \sup_{t\in[0,1]} \frac{f(t,u,v)}{u},$$

$$h^\infty = \lim_{(u,v)\to(\infty,\infty)} \sup_{t\in[0,1]} \frac{f(t,u,v)}{|v|}, \qquad f^\infty = \lim_{u+|v|\to\infty} \sup_{t\in[0,1]} \frac{f(t,u,v)}{u+|v|},$$

$$f_0 = \lim_{(u,v)\to(0,0)} \inf_{t\in[0,1]} \frac{f(t,u,v)}{u+|v|}, \qquad f_\infty = \lim_{u+|v|\to\infty} \inf_{t\in[0,1]} \frac{f(t,u,v)}{u+|v|}.$$

Now, we prove the following results.

Theorem 5.3. *Suppose that $f \in C([0,1]\times[0,+\infty)\times\mathbb{R},\ [0,+\infty))$ and the following conditions hold:*

(i) $h^0 \in [0,\Lambda_1)$;

(ii) $f_\infty \in \left(\frac{\Lambda_2}{\gamma_0}, \infty\right) \cup \{\infty\}$.

Then problem (1.1) has at least one positive solution.

Proof. According to $h^0 < \Lambda_1$, for any $\varepsilon_1 > 0$, there exists a constant $r_1 > 0$ such that for $0 < u < r_1$, there is

$$f\left(t,u,D_{0+}^\beta u\right) \leq (h^0+\varepsilon_1)|D_{0+}^\beta u| \leq \Lambda_1|D_{0+}^\beta u| \leq \Lambda_1 r_1 \tag{5.4}$$

for $\left(t,u,D_{0+}^\beta u\right) \in [0,1]\times[0,r_1]\times[-r_1,r_1]$. Let

$$\Omega_{r_1} = \{u \in E : \|u\| < r_1\}.$$

By (5.4) and Theorem 5.2, we have

$$\|Tu\| \leq \|u\| \text{ for } u \in P \cap \partial\Omega_{r_1}.$$

On the other hand, using (4.1), we get

$$u + |D_{0+}^\beta u| \geq \gamma_0(\|u\|_1 + \|u\|_2) \geq \gamma_0(\|u\|_1^2 + \|u\|_2^2)^{\frac{1}{2}} = \gamma_0\|u\|.$$

Also, it follows from $f_\infty > \frac{\Lambda_2}{\gamma_0}$ that there exists an $H > 2r_1$ such that

$$f\left(t,u,D_{0+}^\beta u\right) \geq \frac{\Lambda_2}{\gamma_0}(u+|D_{0+}^\beta u|) \geq \frac{\Lambda_2}{\gamma_0}\gamma_0\|u\| = \Lambda_2\|u\|, \tag{5.5}$$

for all $t \in [0,1]$ with $u + \left|D_{0+}^{\beta} u\right| \geq H$. Set

$$\Omega_H = \left\{ u \in E : u + \left|D_{0+}^{\beta} u\right| < H \right\},$$

then we see that $\overline{\Omega}_{r_1} \subset \Omega_H$.

For any $u \in P \cap \partial\Omega_H$, we have $u + \left|D_{0+}^{\beta} u\right| = H$. Using (5.5), we get

$$\begin{aligned}
\|Tu\|_1 &= \max_{t\in[0,1]} \int_0^1 G(t,s) f\left(s, u(s), D_{0+}^{\beta} u(s)\right) ds \\
&\geq \max_{t\in[0,1]} \int_{\frac{1}{2}}^1 G(t,s) f\left(s, u(s), D_{0+}^{\beta} u(s)\right) ds \\
&= \int_{\frac{1}{2}}^1 \max_{t\in[0,1]} G(t,s) f\left(s, u(s), D_{0+}^{\beta} u(s)\right) ds \\
&\geq \int_{\frac{1}{2}}^1 G(1,s) \Lambda_2 \|u\| ds \\
&\geq \Lambda_2 \int_{\frac{1}{2}}^1 G(1,s) ds \|u\| \geq \frac{\|u\|}{\sqrt{2}},
\end{aligned}$$

and

$$\begin{aligned}
\|Tu\|_2 &= \max_{t\in[0,1]} \int_0^1 D_t^{\beta} G(t,s) f\left(s, u(s), D_{0+}^{\beta} u(s)\right) ds \\
&\geq \max_{t\in[0,1]} \int_{\frac{1}{2}}^1 D_t^{\beta} G(t,s) f\left(s, u(s), D_{0+}^{\beta} u(s)\right) ds \\
&\geq \Lambda_2 \int_{\frac{1}{2}}^1 D_t^{\beta} G(1,s) ds \|u\| \geq \frac{\|u\|}{\sqrt{2}}.
\end{aligned}$$

Thus, we obtain

$$\|Tu\| \geq \|u\| \text{ for } u \in P \cap \partial\Omega_{r_1}.$$

Consequently, by Lemma 2.3, we conclude that problem (1.1) has a positive solution u in $P \cap (\overline{\Omega}_H \backslash \Omega_{r_1})$. □

We now provide the following example to illustrate the our theoretical result.

Example 5.2. consider the following the fractional equation problem

$$\begin{cases}
D_{0+}^{\alpha} u(t) + f\left(t, u(t), D_{0+}^{\beta} u(t)\right) = 0, & t \in (0,1)\,,\ n-1 < \alpha \leq n,\ 0 < \beta \leq \theta - 1, \\
u^{(i)}(0) = 0, & i = 0, 1, \cdots, n-2, \\
D_{0+}^{\theta} u(t)_{t=1} - \lambda u(1) = 0, & 0 < \lambda < \frac{\Gamma(\alpha)}{\Gamma(\alpha-\theta)},\ 0 < \theta \leq n-2,
\end{cases}$$

where $v = D_{0+}^{\frac{1}{2}} u(t)$ and

$$f\left(t, u(t), D_{0+}^{\beta} u(t)\right) = e^{u+v} u^2 v^2.$$

We can easily verify that $h^0 = 0$ and $f_\infty = \infty$. This implies that all the conditions of the Theorem 5.3 are satisfied. According to the Theorem 5.3, this problem has at least one positive solution.

Theorem 5.4. *Suppose that $f \in C([0,1]\times[0,+\infty)\times\mathbb{R},\ [0,+\infty))$ and the following conditions hold:*

(i) $f_0 \in \left(\frac{\Lambda_2}{\gamma_0}, \infty\right) \cup \{\infty\}$;

(ii) $h^\infty \in [0, \Lambda_1)$.

Then problem (1.1) has at least one positive solution.

Proof. Since $f_0 > \frac{\Lambda_2}{\gamma_0}$, choose $\varepsilon_2 = f_0 - \frac{\Lambda_2}{\gamma_0}$ (> 0), there exists a constant $r_2 > 0$ such that for $0 < u < r_2$, it holds

$$f\left(t, u, D_{0+}^\beta u\right) \geq (f_0 - \varepsilon_2)(u + |D_{0+}^\beta u|) = \frac{\Lambda_2}{\gamma_0}(u + |D_{0+}^\beta u|) \geq \frac{\Lambda_2}{\gamma_0}\gamma_0\|u\| = \Lambda_2\|u\|$$

for $\left(t, u, D_{0+}^\beta u\right) \in [0,1]\times[0,r_2]\times[-r_2,r_2]$.

From the above inequality, for $\left(t, u, D_{0+}^\beta u\right) \in \left[\frac{1}{2},1\right]\times[0,r_2]\times[-r_2,r_2]$, we get

$$f\left(t, u, D_{0+}^\beta u\right) \geq \Lambda_2\|u\|. \tag{5.6}$$

Let

$$\Omega_{r_2} = \{u \in E : \|u\| < r_2\}.$$

Taking (5.6) into account, by exactly the same way as in the proof of Theorem 5.3, we can get

$$\|Tu\| \geq \|u\| \ \text{ for } u \in P \cap \partial\Omega_{r_2}.$$

Let $\varepsilon_3 = \Lambda_1 - h^\infty$ (> 0). Since $h^\infty < \Lambda_1$, there exists a r_3 $(> r_2)$ such that

$$f\left(t, u, D_{0+}^\beta u\right) \leq (h^\infty + \varepsilon_3)\,|D_{0+}^\beta u| = \Lambda_1|D_{0+}^\beta u| \tag{5.7}$$

for $\left(t, u, D_{0+}^\beta u\right) \in [0,1]\times[r_3,+\infty)\times(-\infty,-r_3]\cup[r_3,+\infty)$. Considering that

$$f \in C([0,1]\times[0,+\infty)\times\mathbb{R},\ [0,+\infty)),$$

there exists a $C_3 > 0$ satisfying

$$f\left(t, u, D_{0+}^\beta u\right) \leq C_3 \ \text{ for } \left(t, u, D_{0+}^\beta u\right) \in [0,1]\times[0,r_3]\times[-r_3,r_3]. \tag{5.8}$$

By (5.7) and (5.8), we have

$$f\left(t, u, D_{0+}^\beta u\right) \leq \max\left\{C_3, \Lambda_1|D_{0+}^\beta u|\right\} \ \text{ for } \left(t, u, D_{0+}^\beta u\right) \in [0,1]\times[0,+\infty)\times\mathbb{R}.$$

Take $r_3 > \max\{C_3/\Lambda_1, 2r_2\}$. Let $\Omega_{r_3} = \{u \in E : \|u\| < r_3\}$, if $u \in P \cap \partial\Omega_{r_3}$, we have $\|u\| = r_3$,

$$\|Tu\|_1 = \max_{t\in[0,1]} \int_0^1 G(t,s) f\left(s, u(s), D_{0+}^\beta u(s)\right) ds$$

$$
\begin{aligned}
&\leq \int_0^1 \max_{t\in[0,1]} G(t,s) \max\left\{C_3, \Lambda_1 |D_{0+}^{\beta} u|\right\} ds \\
&\leq \Lambda_1 r_3 \int_0^1 G(1,s) ds \\
&= \Lambda_1 r_3 M \\
&\leq \frac{r_3}{\sqrt{2}},
\end{aligned}
$$

and

$$
\begin{aligned}
\|Tu\|_2 = &\max_{t\in[0,1]} \int_0^1 D_t^{\beta} G(t,s) f\left(s, u(s), D_{0+}^{\beta} u(s)\right) ds \\
&\leq \int_0^1 \max_{t\in[0,1]} D_t^{\beta} G(t,s) \max\left\{C_3, \Lambda_1 |D_{0+}^{\beta} u|\right\} ds \\
&\leq \Lambda_1 r_3 \int_0^1 D_t^{\beta} G(1,s) ds \\
&= \Lambda_1 r_3 N \\
&\leq \frac{r_3}{\sqrt{2}}.
\end{aligned}
$$

Hence,

$$
\|Tu\| = \left(\|Tu\|_1^2 + \|Tu\|_2^2\right)^{\frac{1}{2}} \leq r_3.
$$

This implies that problem (1.1) has at least one positive solution. □

Theorem 5.5. *Assume that one of the following two conditions hold:*

(i) $g^0 \in [0, \Lambda_1)$, *and* $f_\infty \in \left(\frac{\Lambda_2}{\gamma_0}, \infty\right) \cup \{\infty\}$;

(ii) $f_0 \in \left(\frac{\Lambda_2}{\gamma_0}, \infty\right) \cup \{\infty\}$, *and* $g^\infty \in [0, \Lambda_1)$.

Then problem (1.1) has at least one positive solution.

Proof. Under the above assumptions, by using the similar way, it is easy to prove that the conditions in Theorem 5.2 or Theorem 5.3 are satisfied. So we omit the proof. □

Theorem 5.6. *Assume that* $f \in C([0,1] \times [0,+\infty) \times \mathbb{R},\ [0,+\infty))$ *and the following conditions hold:*

(i) $f^0 \in \left[0, \frac{1}{2}\Lambda_1\right)$;

(ii) $f_\infty \in \left(\frac{\Lambda_2}{\gamma_0}, \infty\right) \cup \{\infty\}$.

Then problem (1.1) has at least one positive solution.

Proof. Since $f^0 \in \left[0, \frac{1}{2}\Lambda_1\right)$, there exists a $r_4 > 0$, such that

$$
f\left(t, u, D_{0+}^{\beta} u\right) \leq \frac{1}{2}\Lambda_1 (u + |D_{0+}^{\beta} u|) \leq \frac{1}{2}\Lambda_1 2\|u\| = \Lambda_1 \|u\| \tag{5.9}
$$

for $\left(t, u, D_{0+}^{\beta} u\right) \in [0,1] \times [0, r_4] \times [-r_4, r_4]$.

Let $\Omega_{r_4} = \{u \in E : \|u\| < r_4\}$, in view of (5.9), we have

$$f\left(t, u, D_{0+}^{\beta} u\right) \leq \Lambda_1 \|u\| \leq \Lambda_1 r_4,$$

which implies that the condition $(A1)$ of Theorem 5.2 holds. Thus,

$$\|Tu\| \leq \|u\| \text{ for } u \in P \cap \partial\Omega_{r_4}.$$

Notice that the condition (ii), in exactly the same way as in the proof of Theorem 5.2, we can get

$$\|Tu\| \geq \|u\| \text{ for } u \in P \cap \partial\Omega_{r_5},$$

where $r_5 > r_4$, and $\Omega_{r_5} = \{u \in E : \|u\| < r_5\}$. So, we have completed the proof. □

Theorem 5.7. *Assume that* $f \in C([0,1] \times [0,+\infty) \times \mathbb{R},\ [0,+\infty))$ *and the following conditions hold:*

(i) $f_0 \in \left(\frac{\Lambda_2}{\gamma_0}, \infty\right) \cup \{\infty\}$;

(ii) $f^{\infty} \in \left[0, \frac{1}{2}\Lambda_1\right)$.

Then problem (1.1) has at least one positive solution.

Proof. Firstly, since $f_0 \in \left(\frac{\Lambda_2}{\gamma_0}, \infty\right) \cup \{\infty\}$, in view of condition (ii) of Theorem 5.4, we have

$$\|Tu\| \geq \|u\| \text{ for } u \in P \cap \partial\Omega_{r_2}.$$

Secondly, according to $f^{\infty} \in \left[0, \frac{1}{2}\Lambda_1\right)$, there exists a r_6 $(> r_2)$ such that

$$f\left(t, u, D_{0+}^{\beta} u\right) \leq \tfrac{1}{2}\Lambda_1 |u + D_{0+}^{\beta} u| \leq \tfrac{1}{2}\Lambda_1 2\|u\| = \Lambda_1 \|u\| \tag{5.10}$$

for $\left(t, u, D_{0+}^{\beta} u\right) \in [0,1] \times [r_6, +\infty) \times (-\infty, -r_6] \cup [r_6, +\infty)$.

Considering (5.10), and using the technique similar to the second part of the proof in Theorem 5.4, we obtain that

$$\|Tu\| \leq \|u\| \text{ for } u \in P \cap \partial\Omega_{r_6},$$

where $\Omega_{r_6} = \{u \in E : \|u\| < r_6\}$. Therefore, fractional differential equation (1.1) has at least one positive solution. □

Next, we investigate the existence of at least two distinct positive solutions to problem (1.1).

Theorem 5.8. *Assume that* $f \in C([0,1] \times [0,+\infty) \times \mathbb{R},\ [0,+\infty))$. *If* $h^0 = 0,\ h^{\infty} = 0$ *and the condition* (A2) *in* Theorem 5.2 *is satisfied, then problem (1.1) has at least two distinct positive solutions* $u_1,\ u_2 \in P$.

Proof. Since $h^0 = 0$, take $0 < \varepsilon_1 < \Lambda_1$, then there exists a $\rho_1 > 0$ such that

$$f\left(t, u, D_{0+}^{\beta} u\right) \leq \varepsilon_1 |D_{0+}^{\beta} u| \leq \Lambda_1 |D_{0+}^{\beta} u| \leq \Lambda_1 \rho_1$$

for $\left(t, u, D_{0+}^{\beta} u\right) \in [0,1] \times [0, \rho_1] \times [-\rho_1, \rho_1]$, which implies that the condition $(A1)$ of Theorem 5.2 holds. Hence, we can get

$$\|Tu\| \leq \|u\| \text{ for } u \in P \cap \partial\Omega_{\rho_1},$$

where $\Omega_{\rho_1} = \{u \in E : \|u\| < \rho_1\}$.

Further, using the condition $(A2)$ in Theorem 5.2, we can get that

$$\|Au\| \geq \|u\| \text{ for } u \in P \cap \partial\Omega_b,$$

where $\Omega_b = \{u \in E : \|u\| < b\}$. Then we see that $\overline{\Omega}_{\rho_1} \subset \Omega_b$. By virtue of Lemma 2.3, fractional differential equation (1.1) has at least a single positive solution $u_1 \in P \cap (\overline{\Omega}_b \backslash \Omega_{\rho_1})$.

Finally, taking $0 < \varepsilon_2 < \Lambda_1$, it follows from $h^\infty = 0$ that there exists an $H_1 > b$ such that

$$f\left(t, u, D_{0+}^{\beta} u\right) \leq \varepsilon_2 |D_{0+}^{\beta} u| \leq \Lambda_1 |D_{0+}^{\beta} u| \leq \Lambda_1 \|u\| \tag{5.11}$$

for $\left(t, u, D_{0+}^{\beta} u\right) \in [0, 1] \times [H_1, +\infty) \times (-\infty, H_1] \cup [H_1, +\infty)$.

Let $\Omega_{H_1} = \{u \in E : \|u\| < H_1\}$. Then we see that $\overline{\Omega}_b \subset \Omega_{H_1}$. For any $u \in P \cap \partial\Omega_{H_1}$, we have $\|u\| = H_1$. By (5.11), we get

$$f\left(t, u, D_{0+}^{\beta} u\right) \leq \Lambda_1 H_1,$$

which indicates that condition $(A1)$ of Theorem 5.2 is satisfied. Thus, we obtain that

$$\|Tu\| \leq \|u\| \text{ for } u \in P \cap \partial\Omega_{H_1}.$$

Consequently, it follows from Lemma 2.3 that problem (1.1) has at least a single positive solution u_2 in $P \cap (\overline{\Omega}_{H_1} \backslash \Omega_b)$ with

$$b \leq \|u_2\| \text{ and } |u_2| + |D_{0+}^{\beta} u_2| \leq H_1.$$

Evidently, u_1 and u_2 are distinct. □

In a way closely similar to the above, we can obtain the following result.

Theorem 5.9. *Assume that $f \in C([0, 1] \times [0, +\infty) \times \mathbb{R},\ [0, +\infty))$ and one of the following two conditions hold:*

(i) $f_0 = \infty$, $f_\infty = \infty$, *and the condition* (A1) *in* Theorem 5.2 *is satisfied;*

(ii) $g^0 = 0$, $g^\infty = 0$, *and the condition* (A2) *in* Theorem 5.2 *is satisfied.*

Then problem (1.1) has at least two distinct positive solutions u_1, $u_2 \in P$.

6. Existence of Triple or Multiple Solutions

In this section, we will further discuss the existence of at least 3, n or $2n - 1$ positive solutions to fractional differential equation (1.1) by using different fixed point theorems in cone.

6.1. Three Solutions

Now, we define the nonnegative, continuous, increasing functionals ψ, ν, ζ as follows,

$$\psi(u) = \max_{t \in [0,1]} u(t) + \max_{t \in [0,1]} |D_{0+}^{\beta} u(t)| = \|u\|_1 + \|u\|_2,$$

$$\nu(u)=\zeta(u)=\max\left\{\min_{t\in\left[\frac{1}{2},1\right]}|D^{\beta}_{0+}u(t)|,\ \min_{t\in\left[\frac{1}{2},1\right]}u(t)\right\}.$$

We have

$$\nu(u)=\zeta(u)\le\psi(u)\quad\text{for each}\quad u\in P,$$

and

$$\|u\|\le\psi(u)\le 2\|u\|.$$

Moreover, we know $u(t)=\int_0^1 G(t,s)y(s)ds$, by Lemma 3.3, we have

$$\max_{t\in[0,1]}|D^{\beta}_{0+}u(t)|\le\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}\max_{t\in[0,1]}u(t). \tag{6.1}$$

Thus,

$$\|u\|_2\le\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}\|u\|_1.$$

Since $\|u\|=\left(\|u\|_1^2+\|u\|_2^2\right)^{\frac{1}{2}}$, we get

$$\begin{aligned}\|u\|&\le\left(\|u\|_1^2+\frac{\Gamma^2(\alpha)}{\Gamma^2(\alpha-\beta)}\|u\|_1^2\right)^{\frac{1}{2}}\\&=\frac{\sqrt{\Gamma(\alpha-\beta)^2+\Gamma(\alpha)^2}}{\Gamma(\alpha-\beta)}\|u\|_1\le\frac{\sqrt{\Gamma(\alpha-\beta)^2+\Gamma(\alpha)^2}}{\gamma_0\Gamma(\alpha-\beta)}\min_{t\in\left[\frac{1}{2},1\right]}|u(t)|.\end{aligned}$$

Consequently,

$$\|u\|\le\frac{\sqrt{\Gamma(\alpha-\beta)^2+\Gamma(\alpha)^2}}{\gamma_0\Gamma(\alpha-\beta)}\nu(u)=\frac{\sqrt{\Gamma(\alpha-\beta)^2+\Gamma(\alpha)^2}}{\gamma_0\Gamma(\alpha-\beta)}\zeta(u). \tag{6.2}$$

Theorem 6.1. *Assume that there exist real numbers $a,\ b,\ c$ such that $0<a<b<\frac{\gamma_0}{l}c$, where $l=\frac{\sqrt{\Gamma(\alpha-\beta)^2+\Gamma(\alpha)^2}}{\Gamma(\alpha-\beta)}$. In addition, if $f\in C([0,1]\times[0,+\infty)\times\mathbb{R},\ [0,+\infty))$, and $f(t,u,v)$ satisfies the following conditions:*

(H1) $f(t,u,v)<\frac{c}{\max\{M,N\}}$ for $(t,u,v)\in[0,1]\times\left[0,\frac{l}{\gamma_0}c\right]\times\left[-\frac{l}{\gamma_0}c,\frac{l}{\gamma_0}c\right]$;

(H2) $f(t,u,v)>\frac{b}{\gamma_0\max\{S,T\}}$ for $(t,u,v)\in\left[\frac{1}{2},1\right]\times\left[0,\frac{l}{\gamma_0}b\right]\times\left[-\frac{l}{\gamma_0}b,\frac{l}{\gamma_0}b\right]$;

(H3) $f(t,u,v)<\frac{a}{M+N}$ for $(t,u,v)\in[0,1]\times[0,a]\times[-a,a]$.

Then problem (1.1) has at least three distinct positive solutions $u_1,u_2,u_3\in\overline{P}(\nu,c)$ such that

$$\begin{gathered}0<\|u_1\|_1+\|u_1\|_2<a<\|u_2\|_1+\|u_2\|_2,\\ \max\left\{\min_{t\in\left[\frac{1}{2},1\right]}|D^{\beta}_{0+}u_2|,\ \min_{t\in\left[\frac{1}{2},1\right]}u_2\right\}<b,\\ b<\max\left\{\min_{t\in\left[\frac{1}{2},1\right]}|D^{\beta}_{0+}u_3|,\ \min_{t\in\left[\frac{1}{2},1\right]}u_3\right\}<c.\end{gathered}$$

Proof. Using a way similar to the first part of the proof of Lemma 4.1, it is easy to see that $T:\overline{P(\nu,c)}\to P$. Now, we show that the conditions of Lemma 2.4 are satisfied.

Firstly, according to (6.2), for any $u\in\partial P(\nu,c)$, we have that $\nu(u)=c$ and $\|u\|\leq\frac{\sqrt{\Gamma(\alpha-\beta)^2+\Gamma(\alpha)^2}}{\gamma_0\Gamma(\alpha-\beta)}c$. This implies that

$$0\leq u(t)\leq\frac{l}{\gamma_0}c\quad\text{and}\quad -\frac{l}{\gamma_0}c\leq D^{\beta}_{0+}u(t)\leq\frac{l}{\gamma_0}c\quad\text{for } t\in[0,1],$$

where $l=\frac{\sqrt{\Gamma(\alpha-\beta)^2+\Gamma(\alpha)^2}}{\Gamma(\alpha-\beta)}$. By condition $(H1)$, we have

$$\begin{aligned}
\nu(Tu)&=\max\left\{\min_{t\in\left[\frac{1}{2},1\right]}|TD^{\beta}_{0+}u(t)|,\ \min_{t\in\left[\frac{1}{2},1\right]}Tu(t)\right\}\\
&=\max\left\{\min_{t\in\left[\frac{1}{2},1\right]}\int_0^1 D^{\beta}_tG(t,s)f\left(s,u(s),D^{\beta}_{0+}u(s)\right)ds\right.,\\
&\quad\left.\min_{t\in\left[\frac{1}{2},1\right]}\int_0^1 G(t,s)f\left(s,u(s),D^{\beta}_{0+}u(s)\right)ds\right\}\\
&\leq\max\left\{\int_0^1 D^{\beta}_tG(1,s)f\left(s,u(s),D^{\beta}_{0+}u(s)\right)ds,\int_0^1 G(1,s)f\left(s,u(s),D^{\beta}_{0+}u(s)\right)ds\right\}\\
&<\max\left\{\frac{c}{\max\{M,N\}}\int_0^1 D^{\beta}_tG(1,s)ds,\ \frac{c}{\max\{M,N\}}\int_0^1 G(1,s)ds\right\}<c.
\end{aligned}$$

This implies condition (ii) of Lemma 2.4 is true.

Secondly, for arbitrary $u\in\partial P(\zeta,b)$, we have $\zeta(u)=\nu(u)=b$ and $\|u\|\leq\frac{l}{\gamma_0}b$. Similarly, there is

$$0\leq u(t)\leq\frac{l}{\gamma_0}b\quad\text{and}\quad -\frac{l}{\gamma_0}b\leq D^{\beta}_{0+}u(t)\leq\frac{l}{\gamma_0}b\quad\text{for } t\in\left[\frac{1}{2},1\right].$$

According to condition $(H2)$, we obtain

$$\begin{aligned}
\zeta(Tu)=&\nu(Tu)=\max\left\{\min_{t\in\left[\frac{1}{2},1\right]}|TD^{\beta}_{0+}u(t)|,\ \min_{t\in\left[\frac{1}{2},1\right]}Tu(t)\right\}\\
=&\max\left\{\min_{t\in\left[\frac{1}{2},1\right]}\int_0^1 D^{\beta}_tG(t,s)f\left(s,u(s),D^{\beta}_{0+}u(s)\right)ds,\right.\\
&\left.\min_{t\in\left[\frac{1}{2},1\right]}\int_0^1 G(t,s)f\left(s,u(s),D^{\beta}_{0+}u(s)\right)ds\right\}\\
=&\max\left\{\int_0^1\min_{t\in\left[\frac{1}{2},1\right]}D^{\beta}_tG(t,s)f\left(s,u(s),D^{\beta}_{0+}u(s)\right)ds,\right.\\
&\left.\int_0^1\min_{t\in\left[\frac{1}{2},1\right]}G(t,s)f\left(s,u(s),D^{\beta}_{0+}u(s)\right)ds\right\}\\
\geq&\gamma_0\max\left\{\int_{\frac{1}{2}}^1 D^{\beta}_tG(1,s)f\left(s,u,D^{\beta}_{0+}u\right)ds,\int_{\frac{1}{2}}^1 G(1,s)f\left(s,u,D^{\beta}_{0+}u\right)ds\right\}
\end{aligned}$$

$$
\begin{aligned}
&>\gamma_0 \max\left\{\frac{b}{\gamma_0 \max\{S,T\}}\int_{\frac{1}{2}}^{1} D_t^{\beta}G(1,s)ds,\ \frac{b}{\gamma_0 \max\{S,T\}}\int_{\frac{1}{2}}^{1} G(1,s)ds\right\}\\
&>b,
\end{aligned}
$$

which means condition (i) of Lemma 2.4 is true.

Thirdly, for each $u \in \partial P(\psi, a)$, we have that $\psi(u) = a$ and $\|u\| \leq a$. Hence, we have

$$
0 \leq u(t) \leq a \quad \text{and} \quad -a \leq D_{0+}^{\beta}u(t) \leq a \quad \text{for } t \in [0,1].
$$

Since $\frac{a}{3} \in \partial P(\psi, a)$, then $P(\psi, a) \neq \emptyset$. Further, it follows from condition $(H3)$ that

$$
\begin{aligned}
\psi(Tu) &= \max_{t\in[0,1]} |D_{0+}^{\beta}Tu(t)| + \max_{t\in[0,1]} Tu(t)\\
&= \max_{t\in[0,1]} \int_0^1 D_t^{\beta}G(t,s) f\left(s,u,D_{0+}^{\beta}u\right) ds + \max_{t\in[0,1]} \int_0^1 G(t,s) f\left(s,u,D_{0+}^{\beta}u\right) ds\\
&= \int_0^1 \max_{t\in[0,1]} D_t^{\beta}G(t,s) f\left(s,u,D_{0+}^{\beta}u\right) ds + \int_0^1 \max_{t\in[0,1]} G(t,s) f\left(s,u,D_{0+}^{\beta}u\right) ds\\
&= \int_0^1 D_t^{\beta}G(1,s) f\left(s,u,D_{0+}^{\beta}u\right) ds + \int_0^1 G(1,s) f\left(s,u,D_{0+}^{\beta}u\right) ds\\
&< \frac{a}{M+N}\left[\int_0^1 D_t^{\beta}G(1,s)ds + \int_0^1 G(1,s)ds\right] < a.
\end{aligned}
$$

Therefore, condition (iii) of Lemma 2.4 is satisfied and the operator T has at least three distinct positive solutions $u_1, u_2, u_3 \in \overline{P}(\nu, c)$ such that

$$
\begin{aligned}
&0 < \|u_1\|_1 + \|u_1\|_2 < a < \|u_2\|_1 + \|u_2\|_2,\\
&\max\left\{\min_{t\in\left[\frac{1}{2},1\right]} |D_{0+}^{\beta}u_2|,\ \min_{t\in\left[\frac{1}{2},1\right]} u_2\right\} < b,\\
&b < \max\left\{\min_{t\in\left[\frac{1}{2},1\right]} |D_{0+}^{\beta}u_3|,\ \min_{t\in\left[\frac{1}{2},1\right]} u_3\right\} < c.
\end{aligned}
$$

□

We now provide the following example to illustrate the our main result and give the numerical simulation of the solution.

Example 6.1. consider the following fractional equation problem

$$
\begin{cases}
D_{0^+}^{\frac{11}{3}}u(t) + f\left(t, u(t), D_{0^+}^{\frac{1}{2}}u(t)\right) = 0, & t \in (0,1),\\
u^{(i)}(0) = 0, & i = 0,1,2,\\
D_{0^+}^{\frac{5}{3}}u(t)_{t=1} - 2u(1) = 0, & t \in (0,1),
\end{cases}
\tag{6.3}
$$

where $v = D_{0^+}^{\frac{1}{2}}u(t)$ and

$$
f(t,u,v) = \begin{cases}
t^2 + 160u^{10} + \left(\frac{v}{600}\right)^2, & u \leq 1,\\
t^2 + 160 + \left(\frac{v}{600}\right)^2, & u > 1.
\end{cases}
$$

Notice that $\alpha = \frac{11}{3},\ \beta = \frac{1}{2},\ \theta = \frac{5}{3},\ \lambda = 2$. By calculation, we get that

$$
\begin{aligned}
M &= \int_0^1 G(1,s)ds = \int_0^1 \frac{(1-s)-(1-s)^{\frac{8}{3}}}{\Gamma(\frac{11}{3})-2\Gamma(2)} ds \approx 0.1129,\\
N &= \int_0^1 \frac{\Gamma(\frac{11}{3})[(1-s)-(1-s)^{\frac{13}{6}}]+2\Gamma(2)[(1-s)^{\frac{13}{6}}-(1-s)^{\frac{8}{3}}]}{\Gamma(\frac{19}{6})(\Gamma(\frac{11}{3})-2\Gamma(2))} ds \approx 0.1749,\\
S &= \int_{\frac{1}{2}}^1 G(1,s)ds = \int_{\frac{1}{2}}^1 \frac{(1-s)-(1-s)^{\frac{8}{3}}}{\Gamma(\frac{11}{3})-2\Gamma(2)} ds \approx 0.0514, \qquad (6.4)\\
T &= \int_{\frac{1}{2}}^1 \frac{\Gamma(\frac{11}{3})[(1-s)-(1-s)^{\frac{13}{6}}]+2\Gamma(2)[(1-s)^{\frac{13}{6}}-(1-s)^{\frac{8}{3}}]}{\Gamma(\frac{19}{6})(\Gamma(\frac{11}{3})-2\Gamma(2))} ds \approx 0.0822,\\
\gamma_0 &= \left(\frac{1}{2}\right)^{\frac{11}{3}-1} = 0.1575.
\end{aligned}
$$

Choosing $a = \frac{1}{2},\ b = 2$ and $c = 35$, we have $a < b < \frac{\gamma_0}{l}c$ and

$$
\begin{aligned}
& f(t,u,v) < \frac{c}{\max\{M,N\}} \approx 200.1570,\\
& \text{for } \ (t,u,v) \in [0,1]\times[0,440.4098]\times[-440.4098,440.4098]\,;\\
& f(t,u,v) > \frac{b}{\gamma_0 \max\{S,T\}} \approx 154.5319,\\
& \text{for } \ (t,u,v) \in \left[\frac{1}{2},1\right]\times[0,25.1663]\times[-25.1663,25.1663]\,;\\
& f(t,u,v) < \frac{a}{M+N} \approx 1.7373, \quad \text{for } \ (t,u,v) \in [0,1]\times\left[0,\frac{1}{2}\right]\times\left[-\frac{1}{2},\frac{1}{2}\right];
\end{aligned}
$$

It follows from the Theorem 6.1 that this problem has at least three distinct positive solutions such that

$$
\begin{aligned}
& 0 < \|u_1\|_1 + \|u_1\|_2 < \frac{1}{2} < \|u_2\|_1 + \|u_2\|_2,\\
& \max\left\{\min_{t\in\left[\frac{1}{2},1\right]} |D_{0+}^{\frac{1}{2}}u_2|,\ \min_{t\in\left[\frac{1}{2},1\right]} u_2\right\} < 2,\\
& 2 < \max\left\{\min_{t\in\left[\frac{1}{2},1\right]} |D_{0+}^{\frac{1}{2}}u_3|,\ \min_{t\in\left[\frac{1}{2},1\right]} u_3\right\} < 35.
\end{aligned}
$$

By the iterative method of Example 5.1, the numerical simulation of existence to the solutions is obtained. With the same scale set in the graph, the solution u_3 is very small and not easy to observe, and that is the reason why another graph is established with larger scale. The graph of the approximation of the solutions is given in Figure 2.

Theorem 6.2. *Assume that there exist real numbers a,b,c such that $0 < a < b < \frac{\gamma_0}{l}c$, here $l = \frac{\sqrt{\Gamma(\alpha-\beta)^2+\Gamma(\alpha)^2}}{\Gamma(\alpha-\beta)}$. In addition, if $f \in C([0,1]\times[0,+\infty)\times\mathbb{R},\ [0,+\infty))$ and $f(t,u,v)$ satisfies the following conditions:*

(H1) $f(t,u,v) > \frac{c}{\gamma_0\max\{S,T\}}$ *for* $(t,u,v) \in \left[\frac{1}{2},1\right]\times\left[0,\frac{l}{\gamma_0}c\right]\times\left[-\frac{l}{\gamma_0}c,\frac{l}{\gamma_0}c\right]$*;*

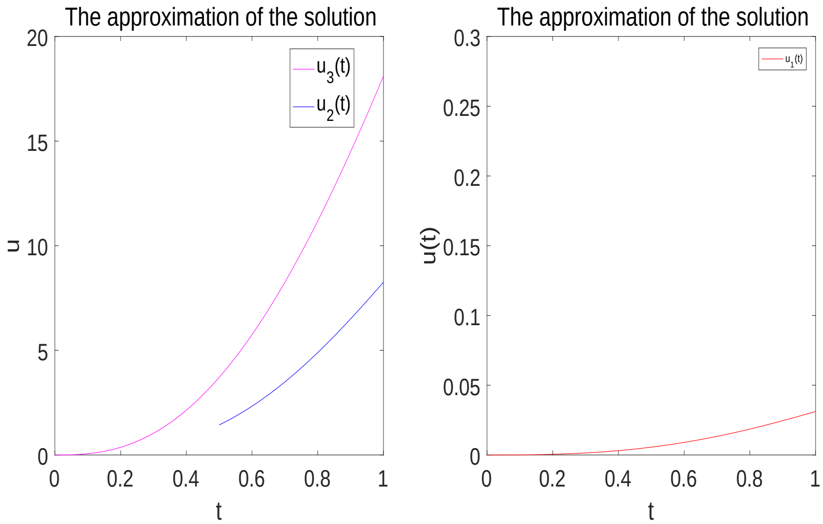

Figure 2. the approximation of the solutions to problem (6.3)

(H2) $f(t,u,v) < \frac{b}{\max\{M,N\}}$ *for* $(t,u,v) \in [0,1] \times \left[0, \frac{l}{\gamma_0} b\right] \times \left[-\frac{l}{\gamma_0} b, \frac{l}{\gamma_0} b\right]$;

(H3) $f(t,u,v) > \frac{a}{\gamma_0(S+T)}$ *for* $(t,u,v) \in [0,1] \times [0,a] \times [-a,a]$.

Then problem (1.1) has at least three distinct positive solutions $u_1, u_2, u_3 \in \overline{P}(\nu, c)$ *such that*

$$
\begin{aligned}
&0 < \|u_1\|_1 + \|u_1\|_2 < a < \|u_2\|_1 + \|u_2\|_2, \\
&\max\left\{ \min_{t\in\left[\frac{1}{2},1\right]} |D^{\beta}_{0+} u_2|, \min_{t\in\left[\frac{1}{2},1\right]} u_2 \right\} < b, \\
&b < \max\left\{ \min_{t\in\left[\frac{1}{2},1\right]} |D^{\beta}_{0+} u_3|, \min_{t\in\left[\frac{1}{2},1\right]} u_3 \right\} < c.
\end{aligned}
$$

Proof. According to the Lemma 2.5, and using a way similar to the proof of Theorem 6.1, we can obtain the desired result. □

Now, for $u \in \overline{P}$, we define the following functionals, by

$$
\begin{aligned}
\varphi(u) &= \max_{t\in[0,1]} |D^{\beta}_{0+} u(t)| + \max_{t\in[0,1]} |u(t)|, \\
\theta(u) &= \eta(u) = \max_{t\in[0,1]} |u(t)|, \\
\delta(u) &= \min_{t\in\left[\frac{1}{2},1\right]} u(t).
\end{aligned}
$$

It is obvious that φ and η are nonnegative continuous convex functionals on P, δ is a nonnegative continuous concave functional on P and θ is a nonnegative continuous functional on P. Moreover, for any $u \in P$, we have

$$\varphi(u) \le 2\|u\| \text{ and } \gamma_0 \eta(u) \le \delta(u) \le \eta(u) = \theta(u) \le \|u\|.$$

Theorem 6.3. *Suppose that $f \in C([0,1]\times[0,+\infty)\times\mathbb{R},\ [0,+\infty))$ and there exist constants a,b,d such that $0<a<b<\gamma_0 d$. In addition, $f(t,u,v)$ satisfies the following conditions:*

(R1) $f(t,u,v) \le \frac{d}{M+N}$ for $(t,u,v)\in[0,1]\times[0,\frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)+\Gamma(\alpha-\beta)}d]\times[-d,d]$;

(R2) $f(t,u,v) > \frac{b}{\gamma_0 S}$ for $(t,u,v)\in\left[\frac{1}{2},1\right]\times[b,\frac{b}{\gamma_0}]\times[-d,d]$;

(R3) $f(t,u,v) < \frac{a}{M}$ for $(t,u,v)\in[0,1]\times[0,a]\times[-d,d]$.

Then problem (1.1) has at least triple positive solutions $u_1,u_2,u_3\in P$ such that

$$\max_{t\in[0,1]}|D^{\beta}_{0+}u_i| + \max_{t\in[0,1]}|u(t)| \le d,\ \ i=1,2,3,$$
$$b < \min_{t\in\left[\frac{1}{2},1\right]}|u_1|,\ \ a<\max_{t\in[0,1]}|u_2|,\ \min_{t\in\left[\frac{1}{2},1\right]}|u_2|<b,\ \ \max_{t\in[0,1]}|u_3|<a.$$

Proof. For arbitrary $u\in P$, we have $\delta(u)\le\eta(u)$ and

$$\|u\| \le \|u\|_1+\|u\|_2 = \max_{t\in[0,1]}|D^{\beta}_{0+}u(t)| + \max_{t\in[0,1]}|u(t)| = \varphi(u).$$

This implies that the inequality (2.1) of Lemma 2.6 is satisfied.

Firstly, we show that $T:\overline{P(\varphi,d)}\to\overline{P(\varphi,d)}$. For any $u\in\overline{P(\varphi,d)}$, we have

$$\varphi(u) = \max_{t\in[0,1]}|D^{\beta}_{0+}u(t)| + \max_{t\in[0,1]}|u(t)| \le d,$$

By (6.1) and the above inequality, we have

$$\max_{t\in[0,1]}|u(t)| \le \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)+\Gamma(\alpha-\beta)}d.$$

Using the condition $(R1)$ that is $f(t,u,v)\le\frac{d}{M+N}$, we get

$$\begin{aligned}
\varphi(Tu) &= \max_{t\in[0,1]}|D^{\beta}_{0+}Tu(t)| + \max_{t\in[0,1]}|Tu(t)| \\
&= \max_{t\in[0,1]}\int_0^1 D^{\beta}_t G(t,s) f\left(s,u,D^{\beta}_{0+}u\right)ds + \max_{t\in[0,1]}\int_0^1 G(t,s) f\left(s,u,D^{\beta}_{0+}u\right)ds \\
&= \int_0^1 \max_{t\in[0,1]} D^{\beta}_t G(t,s) f\left(s,u,D^{\beta}_{0+}u\right)ds + \int_0^1 \max_{t\in[0,1]} G(t,s) f\left(s,u,D^{\beta}_{0+}u\right)ds \\
&\le \frac{d}{M+N}\left(\int_0^1 D^{\beta}_t G(1,s)ds + \int_0^1 G(1,s)ds\right) = d.
\end{aligned}$$

Hence, $T:\overline{P(\varphi,d)}\to\overline{P(\varphi,d)}$.

Secondly, taking $u(t)=\frac{b}{\gamma_0}$, since $0<\gamma_0<1$, then we have $\delta(u)=\delta(\frac{b}{\gamma_0})=\frac{b}{\gamma_0}>b,\ \eta(u)=\eta(\frac{b}{\gamma_0})=\frac{b}{\gamma_0},\ \varphi(\frac{b}{\gamma_0})=\frac{b}{\gamma_0}<d$. This implies that

$$\frac{b}{\gamma_0}\in\{u\in P(\varphi,\eta,\delta,b,\frac{b}{\gamma_0},d):\delta(u)>b\}.$$

Moreover, for $u\in P(\varphi,\eta,\delta,b,\frac{b}{\gamma_0},d)$, we have $b\le u(t)\le\frac{b}{\gamma_0},\ |D^{\beta}_{0+}u(t)|\le d,\ t\in\left[\frac{1}{2},1\right]$. By the condition $(R2)$, we obtain

$$\delta(Tu)=\min_{t\in\left[\frac{1}{2},1\right]}Tu=\min_{t\in\left[\frac{1}{2},1\right]}\int_0^1 G(t,s)f\left(s,u(s),D^{\beta}_{0+}u(s)\right)ds$$

$$
\begin{aligned}
&= \int_0^1 \min_{t\in\left[\frac{1}{2},1\right]} G(t,s) f\left(s, u(s), D_{0+}^{\beta} u(s)\right) ds \\
&\geq \int_{\frac{1}{2}}^1 \min_{t\in\left[\frac{1}{2},1\right]} G(t,s) f\left(s, u(s), D_{0+}^{\beta} u(s)\right) ds \\
&\geq \gamma_0 \int_{\frac{1}{2}}^1 G(1,s) f\left(s, u(s), D_{0+}^{\beta} u(s)\right) ds \\
&> \gamma_0 \int_{\frac{1}{2}}^1 G(1,s) \frac{b}{\gamma_0 S} ds = b,
\end{aligned}
$$

which means condition (i) of Lemma 2.6 is true.

Thirdly, when $u \in P(\varphi, \delta, b, d)$ with $\eta(Tu) > \frac{b}{\gamma_0}$, we have

$$
\delta(Tu) > \gamma_0 \eta(Tu) > \gamma_0 \frac{b}{\gamma_0} = b.
$$

Thus, condition (ii) of Lemma 2.6 is satisfied.

Finally, we prove that condition (iii) of Lemma 2.6 holds. By $\theta(0) = 0 < a$, we know $0 \notin R(\varphi, \theta, a, d)$. When $u \in R(\varphi, \theta, a, d)$ with $\theta(u) = a$, using condition $(R3)$, we obtain

$$
\begin{aligned}
\theta(Tu) &= \max_{t\in[0,1]} |Tu(t)| \\
&= \max_{t\in[0,1]} \int_0^1 G(t,s) f\left(s, u(s), D_{0+}^{\beta} u(s)\right) ds \\
&= \int_0^1 \max_{t\in[0,1]} G(t,s) f\left(s, u(s), D_{0+}^{\beta} u(s)\right) ds \\
&< \int_0^1 G(1,s) \frac{a}{M} ds = a.
\end{aligned}
$$

This gives the desired result.

Consequently, by Lemma 2.6, problem (1.1) has at least triple positive solutions $u_1, u_2, u_3 \in P$ such that

$$
\begin{aligned}
&\max_{t\in[0,1]} |D_{0+}^{\beta} u_i| + \max_{t\in[0,1]} |u_i| \leq d, \quad i = 1,2,3, \\
&b < \min_{t\in\left[\frac{1}{2},1\right]} |u_1|, \; a < \max_{t\in[0,1]} |u_2|, \; \min_{t\in\left[\frac{1}{2},1\right]} |u_2| < b, \; \max_{t\in[0,1]} |u_3| < a.
\end{aligned}
$$

□

Example 6.2. Let the fractional equation be given as (6.3) and

$$
f(t,u,v) = \begin{cases} 8, & u \in [0,1], \\ 327u - 319, & u \in [1,2], \\ 335. & u \in [2, +\infty). \end{cases} \tag{6.5}
$$

By means of (6.4), choosing $a = 1,\ b = 2$ and $d = 100$, we have $a < b < \gamma_0 d$ and

$$
f(t,u,v) \leq \frac{d}{M+N} \approx 347.4647, \quad \text{for } (t,u,v) \in [0,1], \times [0, 36.8880] \times [-100, 100];
$$

$$f(t,u,v) > \frac{b}{\gamma_0 S} \approx 246.8517, \quad \text{for } (t,u,v) \in \left[\frac{1}{2},1\right], \times [2, 12.6992] \times [-100,100];$$
$$f(t,u,v) < \frac{a}{M} \approx 8.8546, \quad \text{for } (t,u,v) \in [0,1] \times [0,1] \times [-100,100];$$

Using the Theorem 6.3, this problem has at least three distinct positive solutions such that

$$\max_{t\in[0,1]} |D^{\beta}_{0+} u_i| + \max_{t\in[0,1]} |u_i| \le 100, \quad i = 1,2,3,$$
$$2 < \min_{t\in\left[\frac{1}{2},1\right]} |u_1|, \ 1 < \max_{t\in[0,1]} |u_2|, \quad \min_{t\in\left[\frac{1}{2},1\right]} |u_2| < 2, \ \max_{t\in[0,1]} |u_3| < 1.$$

By the use of the iterative method of Example 5.1, the numerical simulation of existence to the solutions is obtained. The graph of the approximation of the solutions is given in Figure 3.

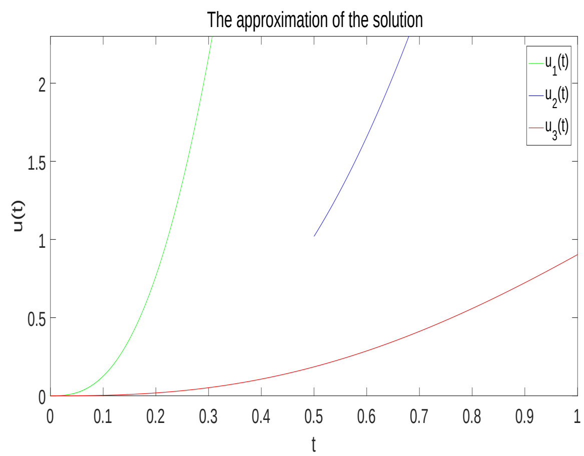

Figure 3. the approximation of the solutions to problem (6.5)

6.2. Arbitrary n Solutions

In this subsection, we will obtain that the existence for multiple positive solutions to the problem (1.1) by using the generalized Avery and Henderson fixed point Theorem.

Theorem 6.4. *If there exist real numbers a_i, b_i, c_i such that $0 < a_1 < b_1 < \frac{l}{\gamma_0} c_1 < \cdots < a_n < b_n < \frac{l}{\gamma_0} c_n$, where $l = \frac{\sqrt{\Gamma(\alpha-\beta)^2 + \Gamma(\alpha)^2}}{\Gamma(\alpha-\beta)}$ and $i = 1,2,\cdots,n$. In addition, if $f \in C([0,1]\times[0,+\infty)\times\mathbb{R}, \ [0,+\infty))$ and $f(t,u,v)$ satisfies the following conditions:*

(HH1) $f(t,u,v) < \frac{c_i}{\max\{M,N\}}$ *for* $(t,u,v) \in [0,1] \times \left[0, \frac{l}{\gamma_0} c_i\right] \times \left[-\frac{l}{\gamma_0} c_i, \frac{l}{\gamma_0} c_i\right]$;

(HH2) $f(t,u,v) > \frac{b_i}{\gamma_0 \max\{S,T\}}$ *for* $(t,u,v) \in \left[\frac{1}{2},1\right] \times \left[0, \frac{l}{\gamma_0} b_i\right] \times \left[-\frac{l}{\gamma_0} b_i, \frac{l}{\gamma_0} b_i\right]$;

(HH3) $f(t,u,v) < \frac{a_i}{M+N}$ *for* $(t,u,v) \in [0,1] \times [0,a_i] \times [-a_i,a_i]$.

Then problem (1.1) has at least n distinct positive solutions.

Proof. Now, we use mathematic induction to prove the result.

When $n = 1$, from the condition $(HH3)$, we have

$$T : \overline{P}_{a_1} \to P_{a_1} \subset \overline{P}_{a_1}.$$

By Schauder fixed point theorem, we can obtain that T has at least one fixed point $u_{01} \in \overline{P}_{a_1}$.

When $n = 2$, let $a = a_1,\ b = b_1,\ c = c_1$, then it implies that the conditions of Theorem 6.1 are satisfied. Thus, T has at least three distinct positive solutions u_{11}, u_{12} and u_{13} such that

$$\begin{aligned}
&0 < \|u_{11}\|_1 + \|u_{11}\|_2 < a_1 < \|u_{12}\|_1 + \|u_{12}\|_2,\\
&\max\left\{\min_{t\in\left[\frac{1}{2},1\right]} |D^{\beta}_{0+}u_{12}|,\ \min_{t\in\left[\frac{1}{2},1\right]} u_{12}\right\} < b_1,\\
&b_1 < \max\left\{\min_{t\in\left[\frac{1}{2},1\right]} |D^{\beta}_{0+}u_{13}|,\ \min_{t\in\left[\frac{1}{2},1\right]} u_{13}\right\} < c_1,
\end{aligned}$$

Thus, the statement is valid for $n = 2$.

Assume that it is true for $n = k$. Then for $n = k+1$, let $a = a_{k+1},\ b = b_{k+1},\ c = c_{k+1}$, we denote the solution by u_i again. In addition, from the solution position and local properties, we know that

$$0 < \max\left\{\min_{t\in\left[\frac{1}{2},1\right]} |D^{\beta}_{0+}u_i|, \min_{t\in\left[\frac{1}{2},1\right]} u_i\right\} < c_k, \quad i = 1, 2, \cdots, k, \tag{6.6}$$

where $c_0 = a_1$.

By Theorem 6.1, T has at least three distinct positive solutions $u_{k+1,1}$, $u_{k+1,2}$ and $u_{k+1,3}$ such that

$$\begin{aligned}
&0 < \|u_{k+1,1}\|_1 + \|u_{k+1,1}\|_2 < a_{k+1} < \|u_{k+1,2}\|_1 + \|u_{k+1,2}\|_2,\\
&\max\left\{\min_{t\in\left[\frac{1}{2},1\right]} |D^{\beta}_{0+}u_{k+1,2}|,\ \min_{t\in\left[\frac{1}{2},1\right]} u_{k+1,2}\right\} < b_1,\\
&b_1 < \max\left\{\min_{t\in\left[\frac{1}{2},1\right]} |D^{\beta}_{0+}u_{k+1,3}|,\ \min_{t\in\left[\frac{1}{2},1\right]} u_{k+1,3}\right\} < c_{k+1}.
\end{aligned} \tag{6.7}$$

By (6.6) and (6.7), we get that

$$\begin{aligned}
&\max\left\{\min_{t\in\left[\frac{1}{2},1\right]} |D^{\beta}_{0+}u_i|,\ \min_{t\in\left[\frac{1}{2},1\right]} u_i\right\} < c_k < b_{k+1},\\
&b_{k+1} < \max\left\{\min_{t\in\left[\frac{1}{2},1\right]} |D^{\beta}_{0+}u_{k+1,3}|,\ \min_{t\in\left[\frac{1}{2},1\right]} u_{k+1,3}\right\}.
\end{aligned} \tag{6.8}$$

Thus, we have

$$u_i \neq u_{k+1,3}, \quad i = 1, 2, \cdots, k.$$

Therefore, the statement holds for $n = k + 1$, that is, problem (1.1) has at least n distinct positive solutions. $\square$

Theorem 6.5. *If there exist real numbers a_i, b_i, c_i such that $0 < a_1 < b_1 < \frac{l}{\gamma_0}c_1 < \cdots < a_n < b_n < \frac{l}{\gamma_0}c_n$, where $l = \frac{\sqrt{\Gamma(\alpha-\beta)^2+\Gamma(\alpha)^2}}{\Gamma(\alpha-\beta)}$ and $i = 1, 2, \cdots, n$. In addition, if $f \in C([0, 1]\times[0, +\infty)\times\mathbb{R},\ [0, +\infty))$ and $f(t, u, v)$ satisfies the following conditions:*

(HH1') $f(t,u,v) > \frac{c_i}{\gamma_0 \max\{S,T\}}$ *for* $(t,u,v) \in \left[\frac{1}{2},1\right] \times \left[0, \frac{l}{\gamma_0}c_i\right] \times \left[-\frac{l}{\gamma_0}c_i, \frac{l}{\gamma_0}c_i\right]$;

(HH2') $f(t,u,v) < \frac{b_i}{\max\{M,N\}}$ *for* $(t,u,v) \in [0,1] \times \left[0, \frac{l}{\gamma_0}b_i\right] \times \left[-\frac{l}{\gamma_0}b_i, \frac{l}{\gamma_0}b_i\right]$;

(HH3') $f(t,u,v) > \frac{a_i}{\gamma_0(S+T)}$ *for* $(t,u,v) \in \left[\frac{1}{2},1\right] \times [0,a_i] \times [-a_i,a_i]$.

Then problem (1.1) has at least n *distinct positive solutions.*

6.3. Arbitrary $2n-1$ Solutions

In this subsection, we will provide the existence for arbitrary odd positive solutions to problem (1.1) by using Avery-Peterson fixed point Theorem.

Similarly, by mathematical induction, we get the following result.

Theorem 6.6. *Assume that* $f \in C([0,1] \times [0,+\infty) \times \mathbb{R},\ [0,+\infty))$ *and there exist constants* a_i, b_i, d_i *such that*

$$0 < a_1 < b_1 < \gamma_0 d_1 < a_2 < b_2 < \gamma_0 d_2 < a_3 < \cdots < a_n < b_n < \gamma_0 d_n,\ n \in \mathbb{N},$$

where $i = 1,2,\cdots,n$. *In addition,* f *satisfies the following conditions:*

(RR1) $f(t,u,v) \le \frac{d_i}{M+N}$ *for* $(t,u,v) \in [0,1] \times [0, \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)+\Gamma(\alpha-\beta)}d_i] \times [-d_i,d_i]$;

(RR2) $f(t,u,v) > \frac{b_i}{\gamma_0 S}$ *for* $(t,u,v) \in \left[\frac{1}{2},1\right] \times [b_i, \frac{b_i}{\gamma_0}] \times [-d_i,d_i]$;

(RR3) $f(t,u,v) < \frac{a_i}{M}$ *for* $(t,u,v) \in [0,1] \times [0,a_i] \times [-d_i,d_i]$.

Then problem (1.1) has at least $2n-1$ *positive solutions.*

References

[1] B. Ahmad, N. Alghamdi, A. Alsaedi et al., *On multi-term fractional differential equations with multi-point boundary conditions*, Eur. Phys. J. Spec. Top., 2017, 226, 3369–3390.

[2] R. Avery and A. Peterson, *Three positive fixed points of nonlinear operators on ordered Banach spaces*, Comput. Math. Appl., 2001, 42, 313–322.

[3] D. E. Betancur-Herrera and N. Muñoz-Galeano, *A numerical method for solving Caputo's and Riemann-Liouville's fractional differential equations which includes multi-order fractional derivatives and variable coefficients*, Commun. Nonlinear Sci. Numer. Simulat., in Press.

[4] A. H. Bhrawy, Y. A. Alhamed, D. Baleanu et al., *New spectral techniques for systems of fractional differential equations using fractional-order generalized Laguerre orthogonal functions*, Fract. Calc. Appl. Anal., 2014, 17, 1137–1157.

[5] C. Chen, Y. Su and Z. Feng, *Positive solutions of fractional differential equations with derivative terms*, Electron. J. Differ. Equ., 2012, 2012, 1–27.

[6] C. Goodrich, *Existence of a positive solution to a class of fractional differential equations*, Appl. Math. Lett., 2010, 23, 1050–1055.

[7] D. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, San Diego, 1988.

[8] K. Jong, H. Choi and Y. Ri, *Existence of positive solutions of a class of multi-point boundary value problems for p-Laplacian fractional differential equations with singular source terms*, Commun. Nonlinear Sci. Numer. Simulat., 2019, 72, 272–281.

[9] A. Kilbas, V. Srivastava and J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, 2006.

[10] R. Luca, *On a class of nonlinear singular Riemann-Liouville fractional differential equations*, Results Math., 2018, 125, 1–15.

[11] I. Masahiro and S. Motohiro, *Sharp upper bound for lifespan of solutions to some critical semilinear parabolic, dispersive and hyperbolic equations via a test function method*, Nonlinear Anal., 2019, 182, 57–74.

[12] S. Migorski and S. Zeng, *Mixed variational inequalities driven by fractional evolutionary equations*, Acta Math. Appl. Sin., 2019, 39B(2), 461–468.

[13] K. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley and Sons, Inc, New York, 1993.

[14] U. R. Mujeeb, A. K. Rahmat and A. A. Naseer, *Three point boundary value problems for nonlinear fractional differential equations*, Acta Math. Sci., 2011, 31B(4), 1337–1346.

[15] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[16] J. Ren, W. Ge and B. Ren, *Existence of three solutions for quas-linear boundary value problems*, Acta Math. Appl. Sin., 2005, 21, 353–358.

[17] Y. Su, *Multiple positive pseudo-symmetric solutions of p-Laplacian dynamic equations on time scales*, Math. Comput. Model., 2009, 49, 1664–1681.

[18] Y. Su and Z. Feng, *Existence theory for an arbitrary order fractional differential equation with deviating argument.* Acta Appl. Math., 2012, 112, 81–105.

[19] Y. Su and Z. Feng, *Existence of pseudo-symmetric solutions to a p-Laplacian four-point BVPs involving derivatives on time scales*, Differ. Integal. Equ., 2012, 25, 441–466.

[20] Y. Su, Y. Yun and D. Wang, *Existence of solutions to nonlinear p-Laplacian fractional differential equations with higher-order derivative term*, Electron. J. Differ. Equ., 2018, 2018, 1–24.

[21] B. Vanessa, D. M. Roger and S. Gleison, *Local well-posedness for the nonlocal derivative nonlinear Schrödinger equation in Besov spaces*, Nonlinear Anal., 2019, 187, 320–338.

[22] P. Verma, I. Wong and N. A. Melosh, *Continuum model of mechanical interactions between biological cells and artificial nanostructures*, Biointerphases, 2010, 5, 37–44.

[23] Y. Wei, Q. Song and Z. Bai, *Existence and iterative method for some fourth order nonlinear boundary value problems*, Appl. Math. Lett., 2019, 87, 101–107.

[24] M. Xiang, B. Zhang and D. Yang, M*multiplicity results for variable-order fractional Laplacian equations with variable growth*, Nonlinear Anal., 2019, 178, 190–204.

[25] M. Xu and S. Sun, *Positivity for integral boundary value problems of fractional differential equations with two nonlinear terms*, Appl. Math. Comput., 2019, 59, 271–283.

[26] X. Yang, Z. Wei and W. Dong, *Existence of positive solutions for the boundary value problem of nonlinear fractional differential equations*, Commun. Nonlinear Sci. Numer. Simulat., 2012, 17, 85–92.

[27] Y. Yun, Y. Su and W. Hu, *Existence and uniqueness of solutions to a class of anti-periodic boundary value problem of fractional differential equations with p-Laplacian operator.* Acta Math. Sci., 2018, 38, 1161–1172.

[28] S. Zhang, *Positive solutions to singular boundary value problem for nonlinear fractional differential equation*, Comput. Math. Appl., 2010, 59, 1300–1309.

[29] K. V. Zhukovsky, *The operational solution of fractional-order differential equations, as well as BlackScholes and heat-conduction equations*, Moscow Univ. Phys., 2016, 71, 237–244.