Journal of Applied Analysis and Computation
Volume 11, Number 1, February 2021, 521–531

Website:http://www.jaac-online.com
DOI:10.11948/20200086

NECESSARY AND SUFFICIENT CONDITIONS AND OPTIMAL CONSTANT FACTORS FOR THE VALIDITY OF MULTIPLE INTEGRAL HALF-DISCRETE HILBERT TYPE INEQUALITIES WITH A CLASS OF QUASI-HOMOGENEOUS KERNELS*

Bing He[1], Yong Hong[2], Zhen Li[3,†] and Bicheng Yang[1]

Abstract The problem of equivalent parameters and the best constant factor for the existence of quasi-homogeneous half-discrete Hilbert type inequality

$$\int_{\mathbb{R}_+^m} \sum_{n=1}^{\infty} G\left(n^{\lambda_1}/\|x\|_{m,\rho}^{\lambda_2}\right) a_n f(x) \mathrm{d}x \leq M\|\tilde{a}\|_{p,\alpha}\|f\|_{q,\beta}$$

is discussed, and their applications in the study of operator boundedness and norm are also considered.

Keywords Quasi homogeneous kernel, half-discrete Hilbert type inequality, equivalent condition, best constant factor, boundedness of operator.

MSC(2010) 26D15, 47A07.

1. Introduction and preliminary knowledge

Suppose that $\rho > 0, m \in \mathbb{N}_+, x = (x_1, x_2, \cdots, x_m), \mathbb{R}_+^m = \{x = (x_1, x_2, \cdots, x_m) : x_i > 0, i = 1, 2, \cdots, m\}, \|x\|_{m,\rho} = (x_1^\rho + x_2^\rho + \cdots + x_m^\rho)^{1/\rho}$. Spaces l and L are defined by respectively

$$l_p^\alpha = \left\{\tilde{a} = \{a_n\} : \|\tilde{a}\|_{p,\alpha} = \left(\sum_{n=1}^{\infty} n^\alpha a_n^p\right)^{1/p} < +\infty, a_n \geq 0\right\},$$

$$L_q^\beta(\mathbb{R}_+^m) = \left\{f(x) \geq 0 : \|f\|_{q,\beta} = \left(\int_{\mathbb{R}_+^m} \|x\|_{m,\rho}^\beta f^q(x) \mathrm{d}x\right)^{1/q} < +\infty\right\}.$$

†The corresponding author. Email address: lzhhymath@163.com(Z. Li)
1Department of Mathematics, Guangdong University of Education, Guangzhou, Guangdong 510303, China
2Department of Mathematics, Guangdong Baiyun University, Guangzhou, Guangdong 510450, China
3College of Mathematics and Statistics, Guangdong University of Finance and Economics, Guangzhou, Guangdong 510320, China
*The authors were supported by NNSF of China (No. 61772140), and Innovation Team Construction Project of Guangdong Province (2018KCXTD020).

If $\lambda_1\lambda_2 > 0$, then the nonnegative measurable function $K(n, \|x\|_{m,\rho}) = G\big(n^{\lambda_1}/\|x\|_{m,\rho}^{\lambda_2}\big)$ is a kind of quasi homogeneous functions. In particular, it is homogeneous of order 0 as $\lambda_1 = \lambda_2$. If $\frac{1}{p} + \frac{1}{q} = 1(p > 1)$, what are the parameter conditions for the validity of multiple integral half-discrete Hilbert type inequality as follows?

$$\int_{\mathbb{R}_+^m} \sum_{n=1}^{\infty} G\left(n^{\lambda_1}/\|x\|_{m,\rho}^{\lambda_2}\right) a_n f(x)\mathrm{d}x \le M\|\widetilde{a}\|_{p,\alpha}\|f\|_{q,\beta}. \tag{1.1}$$

That is, what conditions do the parameters $p, q, \alpha, \beta, \lambda_1, \lambda_2$ meet if there is a constant M that makes (1.1) true? Is this condition necessary at the same time? This question is undoubtedly very important. In this paper, these issues are discussed, and the best constant factor when the inequality holds is also considered. Finally, their applications in the boundedness and norm of operators are discussed. Related literatures can be found in [1–3, 5–17]

Lemma 1.1 ([4]). *Suppose that $a_i > 0, \alpha_i > 0 (i = 1, 2, \cdots, m), \psi(u)$ is measurable, then*

$$\begin{aligned}&\int_{\sum_{i=1}^m\left(\frac{x_i}{a_i}\right)^{\alpha_i}\le 1; x_i>0} \psi\left(\sum_{i=1}^m \left(\frac{x_i}{a_i}\right)^{\alpha_i}\right)\mathrm{d}x_1\cdots\mathrm{d}x_m\\ =&\frac{a_1\cdots a_m\Gamma\left(\frac{1}{\alpha_1}\right)\cdots\Gamma\left(\frac{1}{\alpha_m}\right)}{\alpha_1\cdots\alpha_m\Gamma\left(\frac{1}{\alpha_1}+\cdots+\frac{1}{\alpha_m}\right)}\int_0^1\psi(u)u^{\frac{1}{\alpha_1}+\cdots+\frac{1}{\alpha_m}-1}\mathrm{d}u,\end{aligned}$$

where $\Gamma(t)$ is Gamma function.

According to Lemma 1.1, it is not difficult to get

$$\int_{\|x\|_{m,\rho}\le r}\psi\left(\|x\|_{m,\rho}\right)\mathrm{d}x = \frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\int_0^r \psi(u)u^{m-1}\mathrm{d}u,$$

$$\int_{\|x\|_{m,\rho}\ge r}\psi\left(\|x\|_{m,\rho}\right)\mathrm{d}x = \frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\int_r^{+\infty} \psi(u)u^{m-1}\mathrm{d}u.$$

Lemma 1.2. *Assume that $\frac{1}{p}+\frac{1}{q} = 1(p > 1), \rho > 0, \lambda_1\lambda_2 > 0, m \in \mathbb{N}_+, \frac{1}{\lambda_2}\big(\frac{\alpha\lambda_2-m\lambda_1}{p}+\frac{\beta\lambda_1-\lambda_2}{q}\big) = c$, $K(n, \|x\|_{m,\rho}) = G\left(n^{\lambda_1}/\|x\|_{m,\rho}^{\lambda_2}\right)$ is nonnegative and measurable, $K(t, 1)t^{-\frac{\alpha+1}{p}+c}$ is monotonically decreasing on $(0, +\infty)$. Denote that*

$$W_1 = \int_0^{+\infty} K(1, t)t^{-\frac{\beta+m}{q}+m-1}\mathrm{d}t, W_2 = \int_0^{+\infty} K(t, 1)t^{-\frac{\alpha+1}{p}+c}\mathrm{d}t.$$

Then

$$\lambda_2 W_1 = \lambda_1 W_2$$

and

$$\omega_1(n) = \int_{\mathbb{R}_+^m} K\left(n, \|x\|_{m,\rho}\right)\|x\|_{m,\rho}^{-\frac{\beta+m}{q}}\mathrm{d}x = \frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}n^{-\frac{\lambda_1}{\lambda_2}\left(\frac{\beta}{q}-\frac{m}{p}\right)}W_1,$$

$$\omega_2(x) = \sum_{n=1}^{\infty} K\left(n, \|x\|_{m,\rho}\right)n^{-\frac{\alpha+1}{p}+c} \le \|x\|_{m,\rho}^{-\frac{\lambda_2}{\lambda_1}\left(\frac{\alpha}{p}-\frac{1}{q}-c\right)}W_2.$$

Proof. According to $\frac{1}{\lambda_2}\left(\frac{\alpha\lambda_2-m\lambda_1}{p}+\frac{\beta\lambda_1-\lambda_2}{q}\right)=c$, let $t^{-\lambda_2/\lambda_1}=u$, we have

$$\begin{aligned}W_1&=\int_0^{+\infty}K(t^{-\frac{\lambda_2}{\lambda_1}},1)t^{-\frac{\beta+m}{q}+m-1}\mathrm{d}t\\&=\frac{\lambda_1}{\lambda_2}\int_0^{+\infty}K(u,1)\,u^{\frac{\lambda_1}{\lambda_2}(\frac{\beta+m}{q}-m+1)-\frac{\lambda_1}{\lambda_2}-1}\mathrm{d}u\\&=\frac{\lambda_1}{\lambda_2}\int_0^{+\infty}K(u,1)u^{-\frac{\alpha+1}{p}+c}\mathrm{d}u=\frac{\lambda_1}{\lambda_2}W_2.\end{aligned}$$

Hence $\lambda_2 W_1=\lambda_1 W_2$.

By Lemma 1.1, one gets

$$\begin{aligned}\omega_1(n)&=\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\int_0^{+\infty}K(n,t)t^{-\frac{\beta+m}{q}+m-1}\mathrm{d}t\\&=\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\int_0^{+\infty}K\left(1,t\cdot n^{-\frac{\lambda_1}{\lambda_2}}\right)t^{-\frac{\beta+m}{q}+m-1}\mathrm{d}t\\&=\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}n^{-\frac{\lambda_1}{\lambda_2}(\frac{\beta+m}{q}-m+1)+\frac{\lambda_1}{\lambda_2}}\int_0^{+\infty}K(1,u)u^{-\frac{\beta+m}{q}+m-1}\mathrm{d}u\\&=\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}n^{-\frac{\lambda_1}{\lambda_2}(\frac{\beta}{q}-\frac{m}{p})}W_1.\end{aligned}$$

Notice that $K(t,1)t^{-\frac{\alpha+1}{p}+c}$ is monotonically decreasing on $(0,+\infty)$, thus

$$\begin{aligned}\omega_2(x)&=\|x\|_{m,\rho}^{-\frac{\lambda_2}{\lambda_1}\left(\frac{\alpha+1}{p}-c\right)}\sum_{n=1}^{\infty}K\left(\|x\|_{m,\rho}^{-\lambda_2/\lambda_1}n,1\right)\left(\|x\|_{m,\rho}^{-\lambda_2/\lambda_1}n\right)^{-\frac{\alpha+1}{p}+c}\\&\le\|x\|_{m,\rho}^{-\frac{\lambda_2}{\lambda_1}\left(\frac{\alpha+1}{p}-c\right)}\int_0^{+\infty}K\left(\|x\|_{m,\rho}^{-\lambda_2/\lambda_1}u,1\right)\left(\|x\|_{m,\rho}^{-\lambda_2/\lambda_1}u\right)^{-\frac{\alpha+1}{p}+c}\mathrm{d}u\\&=\|x\|_{m,\rho}^{-\frac{\lambda_2}{\lambda_1}\left(\frac{\alpha+1}{p}-c\right)+\frac{\lambda_2}{\lambda_1}}\int_0^{+\infty}K(t,1)\,t^{-\frac{\alpha+1}{p}+c}\mathrm{d}t=\|x\|_{m,\rho}^{-\frac{\lambda_2}{\lambda_1}\left(\frac{\alpha}{p}-\frac{1}{q}-c\right)}W_2.\end{aligned}$$

□

2. Main results

Theorem 2.1. *Suppose that* $\frac{1}{p}+\frac{1}{q}=1(p>1),\rho>0,\lambda_1\lambda_2>0,m\in\mathbb{N}_+$, $\frac{1}{\lambda_2}\left(\frac{\alpha\lambda_2-m\lambda_1}{p}+\frac{\beta\lambda_1-\lambda_2}{q}\right)=c, K(n,\|x\|_{m,\rho})=G\left(n^{\lambda_1}/\|x\|_{m,\rho}^{\lambda_2}\right)$ *is nonnegative and measurable,* $K(t,1)t^{-\frac{\alpha+1}{p}}$ *and* $K(t,1)t^{-\frac{\alpha+1}{p}+c}$ *are monotonically decreasing on* $(0,+\infty)$, *and*

$$W_0=|\lambda_2|\int_0^{+\infty}K(1,t)t^{-\frac{\beta+m}{q}+m-1}\mathrm{d}t$$

is convergent, then

(i) There are constant $M>0$, *for* $\forall\widetilde{a}=\{a_n\}\in l_p^{\alpha}, f(x)\in L_q^{\beta}\left(\mathbb{R}_+^m\right)$, *the necessary and sufficient conditions for the validity of inequality*

$$\int_{\mathbb{R}_+^m}\sum_{n=1}^{\infty}K\left(n,\|x\|_{m,\rho}\right)a_nf(x)\mathrm{d}x\le M\|\tilde{a}\|_{p,\alpha}\|f\|_{q,\beta}. \tag{2.1}$$

is $c \geq 0$.

(ii) When $c = 0$*, the best constant factor of (2.1) is*

$$\inf M = \frac{W_0}{|\lambda_1|^{1/q}|\lambda_2|^{1/p}} \left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)} \right)^{1/p}.$$

Proof. (i) Suppose that there are constant $M > 0$, such that (2.1) holds. If $c < 0$, for $0 < \varepsilon < -c/|\lambda_1|$, take

$$a_n = n^{(-\alpha-1-|\lambda_1|\varepsilon)/p}, n = 1, 2, \cdots$$

and

$$f(x) = \begin{cases} \|x\|_{m,\rho}^{(-\beta-m-|\lambda_2|\varepsilon)/q}, & \|x\|_{m,\rho} \geq 1, \\ 0, & 0 < \|x\|_{m,\rho} < 1. \end{cases}$$

Then the right side of (2.1) satisfies

$$\begin{aligned} M\|\tilde{a}\|_{p,\alpha}\|f\|_{q,\beta} &= M\left(\sum_{n=1}^{\infty} n^{-1-|\lambda_1|\varepsilon}\right)^{1/p} \left(\int_{\|x\|_{m,\rho}\geq 1} \|x\|_{m,\rho}^{-m-|\lambda_2|\varepsilon} \mathrm{d}x\right)^{1/q} \\ =&M\left(1+\sum_{n=2}^{\infty} n^{-1-|\lambda_1|\varepsilon}\right)^{1/p} \left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)} \int_1^{+\infty} t^{-1-|\lambda_2|\varepsilon}\mathrm{d}t\right)^{1/q} \\ \leq&M\left(1+\int_1^{+\infty} t^{-1-|\lambda_1|\varepsilon}\mathrm{d}t\right)^{1/p} \left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\frac{1}{|\lambda_2|\varepsilon}\right)^{1/q} \\ =&\frac{M}{\varepsilon|\lambda_1|^{1/p}|\lambda_1|^{1/q}}(1+|\lambda_1|\varepsilon)^{1/p}\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/q}. \end{aligned}$$

Since $K(t,1)t^{-\frac{\alpha+1}{p}}$ is monotonically decreasing on $(0, +\infty)$, the left side of (2.1) satisfies

$$\begin{aligned} &\int_{\mathbb{R}_+^m} \sum_{n=1}^{\infty} K\left(n, \|x\|_{m,\rho}\right) a_n f(x)\mathrm{d}x \\ =& \int_{\|x\|_{m,\rho}\geq 1} \|x\|_{m,\rho}^{-(\beta+m+|\lambda_2|\varepsilon)/q} \left(\sum_{n=1}^{\infty} K\left(n, \|x\|_{m,\rho}\right) n^{-(\alpha+1+|\lambda_1|\varepsilon)/p}\right)\mathrm{d}x \\ =& \int_{\|x\|_{m,\rho}\geq 1} \|x\|_{m,\rho}^{-\frac{\beta+m+|\lambda_2|\varepsilon}{q}-\frac{\lambda_2}{\lambda_1}\frac{\alpha+1+|\lambda_1|\varepsilon}{p}} \left[\sum_{n=1}^{\infty} K\left(n\cdot\|x\|_{m,\rho}^{-\frac{\lambda_2}{\lambda_1}}, 1\right)\left(n\cdot\|x\|_{m,\rho}^{-\frac{\lambda_2}{\lambda_1}}\right)^{-\frac{\alpha+1+|\lambda_1|\varepsilon}{p}}\right]\mathrm{d}x \\ \geq& \int_{\|x\|_{m,\rho}\geq 1} \|x\|_{m,\rho}^{-\frac{\beta+m+|\lambda_2|\varepsilon}{q}-\frac{\lambda_2}{\lambda_1}\frac{\alpha+1+|\lambda_1|\varepsilon}{p}} \left[\int_1^{+\infty} K\left(u\cdot\|x\|_{m,\rho}^{-\frac{\lambda_2}{\lambda_1}}, 1\right)\left(u\cdot\|x\|_{m,\rho}^{-\frac{\lambda_2}{\lambda_1}}\right)^{-\frac{\alpha+1+|\lambda_1|\varepsilon}{p}}\mathrm{d}u\right]\mathrm{d}x \\ =& \int_{\|x\|_{m,\rho}\geq 1} \|x\|_{m,\rho}^{-\frac{\beta+m+|\lambda_2|\varepsilon}{q}-\frac{\lambda_2}{\lambda_1}\frac{\alpha+1+|\lambda_1|\varepsilon}{p}+\frac{\lambda_2}{\lambda_1}} \left[\int_{\|x\|_{m,\rho}^{-\frac{\lambda_2}{\lambda_1}}}^{+\infty} K\left(t,1\right)t^{-\frac{\alpha+1+|\lambda_1|\varepsilon}{p}}\mathrm{d}t\right]\mathrm{d}x \\ \geq& \int_{\|x\|_{m,\rho}\geq 1} \|x\|_{m,\rho}^{-m-\frac{\lambda_2}{\lambda_1}c-|\lambda_2|\varepsilon} \left[\int_1^{+\infty} K(t,1)t^{-\frac{\alpha+1+|\lambda_1|\varepsilon}{p}}\mathrm{d}t\right]\mathrm{d}x \\ =& \frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\int_1^{+\infty} t^{-1-\frac{\lambda_2}{\lambda_1}c-|\lambda_2|\varepsilon}\mathrm{d}t\int_1^{+\infty} K(t,1)t^{-\frac{\alpha+1+|\lambda_1|\varepsilon}{p}}\mathrm{d}t. \end{aligned}$$

So we get

$$\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)\int_1^{+\infty} t^{-1-\frac{\lambda_2}{\lambda_1}c-|\lambda_2|\varepsilon}\mathrm{d}t\int_1^{+\infty} K(t,1)t^{-\frac{\alpha+1+|\lambda_1|\varepsilon}{p}}\mathrm{d}t$$
$$\leq\frac{M}{\varepsilon|\lambda_1|^{1/p}|\lambda_2|^{1/q}}(1+|\lambda_1|\varepsilon)^{1/p}\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/q}<+\infty. \tag{2.2}$$

Notice that $0<\varepsilon<-\frac{c}{|\lambda_1|}$, then $\frac{\lambda_2}{\lambda_1}c+|\lambda_2|\varepsilon<0$ and $\int_1^{+\infty} t^{-1-\frac{\lambda_2}{\lambda_1}c-|\lambda_2|\varepsilon}\mathrm{d}t=+\infty$, which contradicts (2.2). Therefore, $c\geq 0$.

Conversely, suppose that $c\geq 0$. It follows from the mixed Hölder's inequality and Lemma 1.2 that

$$\int_{\mathbb{R}_+^m}\sum_{n=1}^{\infty}K\left(n,\|x\|_{m,\rho}\right)a_nf(x)\mathrm{d}x$$
$$=\int_{\mathbb{R}_+^m}\sum_{n=1}^{\infty}K\left(n,\|x\|_{m,\rho}\right)\left(\frac{n^{\frac{\alpha+1-cp}{pq}}}{\|x\|_{m,\rho}^{\frac{\beta+m}{pq}}}a_n\right)\left(\frac{\|x\|_{m,\rho}^{\frac{\beta+m}{pq}}}{n^{\frac{\alpha+1-cp}{pq}}}f(x)\right)\mathrm{d}x$$
$$\leq\left(\int_{\mathbb{R}_+^m}\sum_{n=1}^{\infty}K\left(n,\|x\|_{m,\rho}\right)\frac{n^{\frac{\alpha+1-cp}{q}}}{\|x\|_{m,\rho}^{\frac{\beta+m}{q}}}a_n^p\mathrm{d}x\right)^{1/p}$$
$$\times\left(\int_{\mathbb{R}_+^m}\sum_{n=1}^{\infty}K\left(n,\|x\|_{m,\rho}\right)\frac{\|x\|_{m,\rho}^{\frac{\beta+m}{p}}}{n^{\frac{\alpha+1-cp}{p}}}f^q(x)\mathrm{d}x\right)^{1/q}$$
$$=\left(\sum_{n=1}^{\infty}n^{\frac{\alpha+1-cp}{q}}a_n^p\omega_1(n)\right)^{1/p}\left(\int_{\mathbb{R}_+^m}\|x\|_{m,\rho}^{\frac{\beta+m}{p}}f^q(x)\omega_2(x)\mathrm{d}x\right)^{1/q}$$
$$\leq W_1^{1/p}W_2^{1/q}\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p}\left(\sum_{n=1}^{\infty}n^{\frac{\alpha+1-cp}{q}-\frac{\lambda_1}{\lambda_2}\left(\frac{\beta}{q}-\frac{m}{p}\right)}a_n^p\right)^{1/p}$$
$$\times\left(\int_{\mathbb{R}_+^m}\|x\|_{m,\rho}^{\frac{\beta+m}{p}-\frac{\lambda_2}{\lambda_1}\left(\frac{\alpha}{p}-\frac{1}{q}-c\right)}f^q(x)\mathrm{d}x\right)^{1/q}$$
$$=\frac{W_0}{|\lambda_1|^{1/q}|\lambda_2|^{1/p}}\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p}\left(\sum_{n=1}^{\infty}n^{\alpha-pc}a_n^p\right)^{1/p}\left(\int_{\mathbb{R}_+^m}\|x\|_{m,\rho}^{\beta}f^q(x)\mathrm{d}x\right)^{1/q}$$
$$\leq\frac{W_0}{|\lambda_1|^{1/q}|\lambda_2|^{1/p}}\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p}\left(\sum_{n=1}^{\infty}n^{\alpha}a_n^p\right)^{1/p}\left(\int_{\mathbb{R}_+^m}\|x\|_{m,\rho}^{\beta}f^q(x)\mathrm{d}x\right)^{1/q}$$
$$=\frac{W_0}{|\lambda_1|^{1/q}|\lambda_2|^{1/p}}\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p}\|\tilde{a}\|_{p,\alpha}\|f\|_{q,\beta}.$$

Take $M\geq\frac{W_0}{|\lambda_1|^{1/q}|\lambda_2|^{1/p}}\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p}$ arbitrarily, one can get (2.1).

(ii) When $c=0$, assuming the best constant factor of (2.1) is M_0, then we can see from the previous proof

$$M_0\leq\frac{W_0}{|\lambda_1|^{1/q}|\lambda_2|^{1/p}}\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p},$$

$$\int_{\mathbb{R}_+^m}\sum_{n=1}^{\infty}K\left(n,\|x\|_{m,\rho}\right)a_nf(x)\mathrm{d}x\le M_0\|\tilde{a}\|_{p,\alpha}\|f\|_{q,\beta}.$$

For $\varepsilon>0$ and $\delta>0$ small enough, let

$$a_n=n^{(-\alpha-1-|\lambda_1|\varepsilon)/p},n=1,2,\cdots$$

$$f(x)=\begin{cases}\|x\|_{m,\rho}^{(-\beta-m-|\lambda_2|\varepsilon)/q}, & \|x\|_{m,\rho}\ge\delta,\\ 0, & 0<\|x\|_{m,\rho}<\delta.\end{cases}$$

Then

$$M_0\|\tilde{a}\|_{p,\alpha}\|f\|_{q,\beta}\le\frac{M_0}{|\lambda_1|^{1/p}|\lambda_2|^{1/q}\varepsilon}\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/q}\left(|\lambda_1|\varepsilon+1\right)^{1/p}\delta^{-\frac{|\lambda_2|\varepsilon}{q}},$$

$$\begin{aligned}
&\int_{\mathbb{R}_+^m}\sum_{n=1}^{\infty}K\left(n,\|x\|_{m,\rho}\right)a_nf(x)\mathrm{d}x\\
=&\sum_{n=1}^{\infty}n^{(-\alpha-1-|\lambda_1|\varepsilon)/p}\left(\int_{\|x\|_{m,\rho}\ge\delta}\|x\|_{m,\rho}^{(-\beta-m-|\lambda_2|\varepsilon)/q}K\left(n,\|x\|_{m,\rho}\right)\mathrm{d}x\right)\\
=&\sum_{n=1}^{\infty}n^{(-\alpha-1-|\lambda_1|\varepsilon)/p}\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\int_{\delta}^{+\infty}K(n,u)u^{-\frac{\beta+m+|\lambda_2|\varepsilon}{q}+m-1}\mathrm{d}u\right)\\
=&\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\sum_{n=1}^{\infty}n^{-\frac{\alpha+1+|\lambda_1|\varepsilon}{p}-\frac{\lambda_1}{\lambda_2}\left(\frac{\beta+m+|\lambda_2|\varepsilon}{q}-m+1\right)}\\
&\times\left(\int_{\delta}^{+\infty}K(1,n^{-\frac{\lambda_1}{\lambda_2}}u)\left(n^{-\frac{\lambda_1}{\lambda_2}}u\right)^{-\frac{\beta+m+|\lambda_2|\varepsilon}{q}+m-1}\mathrm{d}u\right)\\
=&\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\sum_{n=1}^{\infty}n^{-1-|\lambda_1|\varepsilon}\int_{\delta\cdot n^{-\frac{\lambda_1}{\lambda_2}}}^{+\infty}K(1,t)t^{-\frac{\beta+m+|\lambda_2|\varepsilon}{q}+m-1}\mathrm{d}t\\
\ge&\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\int_1^{+\infty}t^{-1-|\lambda_1|\varepsilon}\mathrm{d}t\int_{\delta}^{+\infty}K(1,t)t^{-\frac{\beta+m+|\lambda_2|\varepsilon}{q}+m-1}\mathrm{d}t\\
=&\frac{1}{|\lambda_1|\varepsilon}\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\int_{\delta}^{+\infty}K(1,t)t^{-\frac{\beta+m+|\lambda_2|\varepsilon}{q}+m-1}\mathrm{d}t.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\frac{\Gamma^m(1/\rho)}{|\lambda_1|\rho^{m-1}\Gamma(m/\rho)}\int_{\delta}^{+\infty}K(1,t)t^{-\frac{\beta+m+|\lambda_2|\varepsilon}{q}+m-1}\mathrm{d}t\\
\le&\frac{M_0}{|\lambda_1|^{1/p}|\lambda_2|^{1/q}}\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/q}\left(|\lambda_1|\varepsilon+1\right)^{1/p}\delta^{-\frac{|\lambda_2|\varepsilon}{q}}.
\end{aligned}$$

Let $\varepsilon\to0^+$, then

$$\frac{\Gamma^m(1/\rho)}{|\lambda_1|\rho^{m-1}\Gamma(m/\rho)}\int_{\delta}^{+\infty}K(1,t)t^{-\frac{\beta+m}{q}+m-1}\mathrm{d}t\le\frac{M_0}{|\lambda_1|^{1/p}|\lambda_2|^{1/q}}\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/q},$$

i.e.

$$\frac{1}{|\lambda_1|^{1/q}|\lambda_2|^{1/p}}\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p}|\lambda_2|\int_{\delta}^{+\infty}K(1,t)t^{-\frac{\beta+m}{q}+m-1}\mathrm{d}t\le M_0.$$

In addition, let $\delta \to 0^+$, we have

$$\frac{W_0}{|\lambda_1|^{1/q}|\lambda_2|^{1/p}}\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p} \le M_0.$$

Hence the best constant factor of (2.1) is

$$M_0 = \frac{W_0}{|\lambda_1|^{1/q}|\lambda_2|^{1/p}}\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p}.$$

□

3. Applications

According to the basic theory of Hilbert type inequality, (2.1) is equivalent to the following two forms

$$\int_{\mathbb{R}_+^m} \|x\|_{m,\rho}^{\beta(1-p)}\left(\sum_{n=1}^{\infty} K\left(n,\|x\|_{m,\rho}\right)a_n\right)^p \mathrm{d}x \le M^p\|\tilde{a}\|_{p,\alpha}^p, \tag{3.1}$$

$$\sum_{n=1}^{\infty} n^{\alpha(1-q)}\left(\int_{\mathbb{R}_+^m} K\left(n,\|x\|_{m,\rho}\right)f(x)\mathrm{d}x\right)^q \le M^q\|f\|_{q,\beta}^q. \tag{3.2}$$

If the series operator T_1 and singular integral operator T_2 are defined by

$$T_1(\tilde{a})(x) = \sum_{n=1}^{\infty} K\left(n,\|x\|_{m,\rho}\right)a_n, T_2(f)_n = \int_{\mathbb{R}_+^m} K\left(n,\|x\|_{m,\rho}\right)f(x)\mathrm{d}x, \tag{3.3}$$

then (3.1) and (3.2) can be written as

$$\|T_1(\tilde{a})\|_{p,\beta(1-p)} \le M\|\tilde{a}\|_{p,\alpha} \text{and} \|T_2(f)\|_{q,\alpha(1-q)} \le M\|f\|_{q,\beta}.$$

Thus by Theorem 2.1, one has

Theorem 3.1. *Assume that* $\frac{1}{p}+\frac{1}{q}=1(p>1), \rho>0, \lambda_1\lambda_2>0, m\in\mathbb{N}_+$, $\frac{1}{\lambda_2}\left(\frac{\alpha\lambda_2-m\lambda_1}{p}+\frac{\beta\lambda_1-\lambda_2}{q}\right)=c, K(n,\|x\|_{m,\rho})=G\left(n^{\lambda_1}/\|x\|_{m,\rho}^{\lambda_2}\right)$ *is nonnegative and measurable,* $K(t,1)t^{-\frac{\alpha+1}{p}}$ *and* $K(t,1)t^{-\frac{\alpha+1}{p}+c}$ *are monotonically decreasing on* $(0,+\infty)$, *operators* T_1 *and* T_2 *are defined as (3.3), and*

$$W_0 = |\lambda_2|\int_0^{+\infty} K(1,t)t^{-\frac{\beta+m}{q}+m-1}\mathrm{d}t < +\infty.$$

Then

(i) $T_1 : l_p^\alpha \to L_p^{\beta(1-p)}\left(\mathbb{R}_+^m\right)$ *and* $T_2 : L_q^\beta\left(\mathbb{R}_+^m\right) \to l_q^{\alpha(1-q)}$ *are bounded operators if and only if* $c \ge 0$;

(ii) When $c=0$, *the operator norms of* T_1 *and* T_2 *are*

$$\|T_1\| = \|T_2\| = \frac{W_0}{|\lambda_1|^{1/q}|\lambda_2|^{1/p}}\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p}.$$

Corollary 3.1. *Suppose that* $\frac{1}{p}+\frac{1}{q}=1(p>1), \rho>0, m\in\mathbb{N}_+, a\geq 0, b\geq 0\ (a^2+b^2\neq 0), \lambda_1>0, \lambda_2>0, \frac{1}{\lambda_2}\left(\frac{\alpha\lambda_2-m\lambda_1}{p}+\frac{\beta\lambda_1-\lambda_2}{q}\right)=c$, $\alpha\geq\max\{-1,pc-1\}$, $m(q-1)<\beta<m(q-1)+q\lambda_2(a+b)$, *and*

$$W_0=\int_0^1\frac{1}{(1+t)^a}\left[t^{\frac{1}{\lambda_2}(\frac{\beta+m}{q}-m)-1}+t^{a+b-\frac{1}{\lambda_2}(\frac{\beta+m}{q}-m)-1}\right]\mathrm{d}t.$$

The operators T_1 *and* T_2 *are defined by*

$$T_1(\tilde{a})(x)=\sum_{n=1}^{\infty}\frac{a_n}{\left(1+n^{\lambda_1}/\|x\|_{m,\rho}^{\lambda_2}\right)^a\left(\max\{1,n^{\lambda_1}/\|x\|_{m,\rho}^{\lambda_2}\}\right)^b},\tilde{a}=\{a_n\}\in l_p^{\alpha},$$

$$T_2(f)_n=\int_{\mathbb{R}_+^m}\frac{f(x)\mathrm{d}x}{\left(1+n^{\lambda_1}/\|x\|_{m,\rho}^{\lambda_2}\right)^a\left(\max\{1,n^{\lambda_1}/\|x\|_{m,\rho}^{\lambda_2}\}\right)^b},f(x)\in L_q^{\beta}\left(\mathbb{R}_+^m\right),$$

then

(i) T_1 *is a bounded operator from* l_p^{α} *to* $L_p^{\beta(1-p)}\left(\mathbb{R}_+^m\right)$ *and* T_2 *is a bounded operator from* $L_q^{\beta}\left(\mathbb{R}_+^m\right)$ *to* $l_q^{\alpha(1-q)}$ *if and only if* $c\geq 0$;

(ii) When $c=0$, *the operator norms of* T_1 *and* T_2 *are*

$$\|T_1\|=\|T_2\|=\frac{W_0}{\lambda_1^{1/q}\lambda_2^{1/p}}\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p}.$$

Proof. Notice that $\alpha\geq pc-1, \alpha\geq -1$, then $-\frac{\alpha+1}{p}\leq 0, -\frac{\alpha+1}{p}+c\leq 0$,

$$K(t,1)t^{-\frac{\alpha+1}{p}}=\frac{1}{(1+t^{\lambda_1})^a\left(\max\{1,t^{\lambda_1}\}\right)^b}t^{-\frac{\alpha+1}{p}}$$

and

$$K(t,1)t^{-\frac{\alpha+1}{p}+c}=\frac{1}{(1+t^{\lambda_1})^a\left(\max\{1,t^{\lambda_1}\}\right)^b}t^{-\frac{\alpha+1}{p}+c}$$

are monotonically decreasing on $(0,+\infty)$.

According to $m(q-1)<\beta<m(q-1)+q\lambda_2(a+b)$, one has $\frac{1}{\lambda_2}\left(\frac{\beta+m}{q}-m\right)>0, a+b-\frac{1}{\lambda_2}\left(\frac{\beta+m}{q}-m\right)>0$, it can be seen that the integral in W_0 is convergent. And since

$$\begin{aligned}&|\lambda_2|\int_0^{+\infty}K(1,t)t^{-\frac{\beta+m}{q}+m-1}\mathrm{d}t\\&=\lambda_2\int_0^{+\infty}\frac{t^{-\frac{\beta+m}{q}+m-1}}{(1+t^{-\lambda_2})^a\left(\max\{1,t^{-\lambda_2}\}\right)^b}\mathrm{d}t\\&=\int_0^{+\infty}\frac{u^{\frac{1}{\lambda_2}(\frac{\beta+m}{q}-m)-1}}{(1+u)^a\left(\max\{1,u\}\right)^b}\mathrm{d}u\\&=\int_0^1\frac{1}{(1+t)^a}\left[t^{\frac{1}{\lambda_2}(\frac{\beta+m}{q}-m)-1}+t^{a+b-\frac{1}{\lambda_2}(\frac{\beta+m}{q}-m)-1}\right]\mathrm{d}t,\end{aligned}$$

it follows by Theorem 3.1 that Corollary 3.1 holds. □

Take $b=0$ in Corollary 3.1, then

$$\begin{aligned}W_0 &= \int_0^1 \frac{1}{(1+t)^a}\left[t^{\frac{1}{\lambda_2}(\frac{\beta+m}{q}-m)-1}+t^{a-\frac{1}{\lambda_2}(\frac{\beta+m}{q}-m)-1}\right]\mathrm{d}t\\ &= B\Big(\frac{1}{\lambda_2}(\frac{\beta+m}{q}-m), a-\frac{1}{\lambda_2}(\frac{\beta+m}{q}-m)\Big).\end{aligned}$$

So we have

Corollary 3.2. *Assume that* $\frac{1}{p}+\frac{1}{q}=1(p>1), \rho>0, m\in\mathbb{N}_+, a>0, \lambda_1>0, \lambda_2>0, \frac{1}{\lambda_2}\left(\frac{\alpha\lambda_2-m\lambda_1}{p}+\frac{\beta\lambda_1-\lambda_2}{q}\right)=c,\ \alpha\geq\max\{-1, pc-1\},\ m(q-1)<\beta<m(q-1)+q\lambda_2 a$, *operators* T_1 *and* T_2 *are defined by respectively*

$$\begin{aligned}T_1(\tilde{a})(x) &= \sum_{n=1}^{\infty}\frac{a_n}{\left(1+n^{\lambda_1}/\|x\|_{m,\rho}^{\lambda_2}\right)^a}, \tilde{a}=\{a_n\}\in l_p^{\alpha},\\ T_2(f)_n &= \int_{\mathbb{R}_+^m}\frac{f(x)\mathrm{d}x}{\left(1+n^{\lambda_1}/\|x\|_{m,\rho}^{\lambda_2}\right)^a}, f(x)\in L_q^{\beta}\left(\mathbb{R}_+^m\right).\end{aligned}$$

Then

(i) T_1 *is a bounded operator from* l_p^{α} *to* $L_p^{\beta(1-p)}\left(\mathbb{R}_+^m\right)$ *and* T_2 *is a bounded operator from* $L_q^{\beta}\left(\mathbb{R}_+^m\right)$ *to* $l_q^{\alpha(1-q)}$ *if and only if* $c\geq 0$.

(ii) When $c=0$, *the operator norms of* T_1 *and* T_2 *are*

$$\|T_1\|=\|T_2\|=\frac{1}{\lambda_1^{1/q}{\lambda_2}^{1/p}}B\Big(\frac{1}{\lambda_2}(\frac{\beta+m}{q}-m), a-\frac{1}{\lambda_2}(\frac{\beta+m}{q}-m)\Big)\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p}.$$

Take $a=0$ in Corollary 3.1, then

$$\begin{aligned}W_0 &= \int_0^1\left[t^{\frac{1}{\lambda_2}(\frac{\beta+m}{q}-m)-1}+t^{b-\frac{1}{\lambda_2}(\frac{\beta+m}{q}-m)-1}\right]\mathrm{d}t\\ &= \frac{1}{\frac{1}{\lambda_2}(\frac{\beta+m}{q}-m)}+\frac{1}{b-\frac{1}{\lambda_2}(\frac{\beta+m}{q}-m)}.\end{aligned}$$

So we have

Corollary 3.3. *Suppose that* $\frac{1}{p}+\frac{1}{q}=1(p>1), \rho>0, m\in\mathbb{N}_+, b>0, \lambda_1>0, \lambda_2>0, \frac{1}{\lambda_2}\left(\frac{\alpha\lambda_2-m\lambda_1}{p}+\frac{\beta\lambda_1-\lambda_2}{q}\right)=c,\ \alpha\geq\max\{-1, pc-1\},\ m(q-1)<\beta<m(q-1)+q\lambda_2 b$, *operators* T_1 *and* T_2 *are defined by respectively*

$$\begin{aligned}T_1(\tilde{a})(x) &= \sum_{n=1}^{\infty}\frac{a_n}{\left(\max\{1, n^{\lambda_1}/\|x\|_{m,\rho}^{\lambda_2}\}\right)^b}, \tilde{a}=\{a_n\}\in l_p^{\alpha},\\ T_2(f)_n &= \int_{\mathbb{R}_+^m}\frac{f(x)\mathrm{d}x}{\left(\max\{1, n^{\lambda_1}/\|x\|_{m,\rho}^{\lambda_2}\}\right)^b}, f(x)\in L_q^{\beta}\left(\mathbb{R}_+^m\right).\end{aligned}$$

Then

(i) T_1 is a bounded operator from l_p^α to $L_p^{\beta(1-p)}\left(\mathbb{R}_+^m\right)$ and T_2 is a bounded operator from $L_q^\beta\left(\mathbb{R}_+^m\right)$ to $l_q^{\alpha(1-q)}$ if and only if $c \geq 0$.

(ii) When $c = 0$, the operator norms of T_1 and T_2 are

$$\|T_1\| = \|T_2\| = \frac{1}{\lambda_1^{1/q}{\lambda_2}^{1/p}}\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p}\left[\frac{1}{\frac{1}{\lambda_2}(\frac{\beta+m}{q}-m)}+\frac{1}{b-\frac{1}{\lambda_2}(\frac{\beta+m}{q}-m)}\right].$$

In Corollary 3.2, let $m = 1, \alpha = \frac{1}{\lambda_2}(\lambda_1 - p), \beta = \frac{1}{\lambda_1}(\lambda_2 + q)$, the following results can be obtained.

Corollary 3.4. *Assume that $\frac{1}{p} + \frac{1}{q} = 1(p > 1), a > 0, \lambda_1 > 0, \lambda_2 > 0, \lambda_1 + \lambda_2 > p, 0 < \frac{1}{\lambda_1\lambda_2} + \frac{1}{\lambda_1 q} - \frac{1}{\lambda_2 p} < a$, $\alpha = \frac{1}{\lambda_2}(\lambda_1 - p), \beta = \frac{1}{\lambda_1}(\lambda_2 + q)$, operators T_1 and T_2 are defined by respectively*

$$T_1(\tilde{a})(x) = \sum_{n=1}^{\infty}\frac{a_n}{(1+n^{\lambda_1}/x^{\lambda_2})^a}, \tilde{a} = \{a_n\} \in l_p^\alpha,$$

$$T_2(f)_n = \int_0^{+\infty}\frac{f(x)\mathrm{d}x}{(1+n^{\lambda_1}/x^{\lambda_2})^a}, f(x) \in L_q^\beta(0,+\infty).$$

Then T_1 is a bounded operator from l_p^α to $L_p^{\beta(1-p)}(0,+\infty)$, T_2 is a bounded operator from $L_q^\beta(0,+\infty)$ to $l_q^{\alpha(1-q)}$, and the operator norms of T_1 and T_2 are

$$\|T_1\| = \|T_2\| = \frac{1}{\lambda_1^{1/q}{\lambda_2}^{1/p}}B\left(\frac{1}{\lambda_1\lambda_2}+\frac{1}{\lambda_1 q}-\frac{1}{\lambda_2 p}, a-\frac{1}{\lambda_1\lambda_2}-\frac{1}{\lambda_1 q}+\frac{1}{\lambda_2 p}\right).$$

In Corollary 3.3, take $m = 1, \alpha = \frac{1}{\lambda_2}(\lambda_1 - p), \beta = \frac{1}{\lambda_1}(\lambda_2 + q)$, we have

Corollary 3.5. *Suppose that $\frac{1}{p} + \frac{1}{q} = 1(p > 1), b > 0, \lambda_1 > 0, \lambda_2 > 0, \lambda_1 + \lambda_2 > p, 0 < \frac{1}{\lambda_1\lambda_2} + \frac{1}{\lambda_1 q} - \frac{1}{\lambda_2 p} < b$, $\alpha = \frac{1}{\lambda_2}(\lambda_1 - p), \beta = \frac{1}{\lambda_1}(\lambda_2 + q)$, operators T_1 and T_2 are defined by respectively*

$$T_1(\tilde{a})(x) = \sum_{n=1}^{\infty}\frac{a_n}{(\max\{1, n^{\lambda_1}/x^{\lambda_2}\})^b}, \tilde{a} = \{a_n\} \in l_p^\alpha,$$

$$T_2(f)_n = \int_0^{+\infty}\frac{f(x)\mathrm{d}x}{(\max\{1, n^{\lambda_1}/x^{\lambda_2}\})^b}, f(x) \in L_q^\beta(0,+\infty).$$

Then T_1 is a bounded operator from l_p^α to $L_p^{\beta(1-p)}(0,+\infty)$, T_2 is a bounded operator from $L_q^\beta(0,+\infty)$ to $l_q^{\alpha(1-q)}$, and the operator norms of T_1 and T_2 are

$$\|T_1\| = \|T_2\| = \frac{1}{\lambda_1^{1/q}{\lambda_2}^{1/p}}\left(\frac{1}{\frac{1}{\lambda_1\lambda_2}+\frac{1}{\lambda_1 q}-\frac{1}{\lambda_2 p}}+\frac{1}{b-\frac{1}{\lambda_1\lambda_2}-\frac{1}{\lambda_1 q}+\frac{1}{\lambda_2 p}}\right).$$

References

[1] T. Batbold, M. Krnić and P. Vuković, *A unified approach to fractal Hilbert-type inequalities*, J. Inequal. Appl., 2019. DOI: 10.1186/s13660–019–2076–9.

[2] O. F. Brevig, *Sharp norm estimates for composition operators and Hilbert-type inequalities*, B. Lond. Math. Soc., 2017, 49(6), 965–978.

[3] A. Čižmešija, M. Krnić and J. Pečarić, *General Hilbert-Type Inequalities with Non-conjugate Exponents*, Math. Inequal. Appl., 2008, 11(2), 237–269.

[4] G. M. Fichtingoloz, *A course in differential and integral calculus*, People's Education Press, Beijing, China, 1957, 404–423.

[5] L. He, H. Liu and B. Yang, *On a more accurate reverse Mulholland-type inequality with parameters*, J. Inequal. Appl., 2019. DOI: 10.1186/s13660–019–2139–y.

[6] Y. Hong, *On the norm of a singular multiple integral operator with homogeneous kernel and its application*, Chinese Annals of Mathematics, 2011, 32(5), 599–606.

[7] Y. Hong, *A Hilbert-Type Integral Inequality with Quasi-homoeneous Kernel and Several Functions*, Acta Mathematics Sinica, Chinese Series, 2014, 57(5), 833–840.

[8] Y. Hong, B. He and B. Yang, *Necessary and sufficient conditions for the validity of Hilbert type integral inequalities with a class of quasi-homogeneous kernels and its application in operator theory*, J. Math. Inequal., 2018, 12(3), 777–788.

[9] Y. Hong, Q. Huang and B. Yang, et al., *The necessary and sufficient conditions for the existence of a kind of Hilbert-type multiple integral inequality with the non-homogeneous kernel and its applications*, J. Inequal. Appl., 2017. DOI: 10.1186/s13660-017-1592-8.

[10] M. Krnić and P. Vuković, *A class of Hilbert-type inequalities obtained via the improved Young inequality*, Results math., 2017, 71, 185–196.

[11] J. Kuang, *Applied Inequalities*, Shangdong Science and Technology Press, Jinan, China, 2004, 534–535.

[12] R. Luo and B. Yang, *Parameterized discrete Hilbert-type inequalities with intermediate variables*, J. Inequal. Appl., 2019. DOI: 10.1186/s13660–019–2095–6.

[13] A. Osekowski, *Inequalities for Hilbert operator and its extensions: The probabilistic approach*, Ann. Probab., 2017, 45(1), 535–563.

[14] M. T. Rassias, B. Yang and A. Raigorodskii, *Two Kinds of the Reverse Hardy-Type Integral Inequalities with the Equivalent Forms Related to the Extended Riemann Zeta Function*, Appl. Anal. Discrete Math., 2018, 12, 273–296.

[15] S. H. Saker, A. A. El-Deeb and H. M. Rezk, et al., *On Hilbert's inequality on time scales*, Appl. Anal. Discr. Math., 2017, 11(2), 399–423.

[16] B. Tserendorj, A. Vandanjav and L. E. Azar, *A new discrete Hilbert-type inequality involving partial sums*, J. Inequal. Appl., 2019. DOI: 10.1186/s13660–019–2087–6.

[17] B. Yang, M. Huang and Y. Zhong, *On a parametric Mulholland-type inequality and applications*, Abstr. Appl. Anal., 2019. DOI: 10.1155/2019/8317029.