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QUADRATIC APPROXIMATION OF SOLUTIONS FOR SET-VALUED FUNCTIONAL DIFFERENTIAL EQUATIONS*

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Abstract This paper investigates nonlinear set-valued functional differential equations with initial value conditions. By introducing the notion of Hukuhara partial derivative of set-valued function, using the comparison principle and the method of quasilinearization, we obtain monotone iterative sequences of approximate solutions which converge uniformly and quadratically to the solutions of such problems.

Keywords Set-valued functional differential equations, coupled lower and upper solutions, quasilinearization, convergence.

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1. Introduction

Recently, many researchers have shown great interest in set-valued differential equations in a semilinear metric space due to its applicability to multivalued differential inclusions and fuzzy differential equations and its inclusion of ordinary differential systems as a special case. For some interest results on its basic theory and applications, we can refer the reader to the results of the local existence and uniqueness of solutions, the continuous dependence of solutions, the existence of extremum solutions and global existence [4, 9, 17, 20, 25, 27]; the comparison principle, stability and instabilty of solutions for set-valued differential equations [5–8, 10, 16, 19, 22–24, 26, 29]. For a complete framework of set-valued differential equations, we can see the monograph of Lakshmikantham et al. [18], and the references cited therein. Meanwhile, the results of various kinds of set-valued differential equations have appeared successively. For example, Ahmad and Sivasundaram [2] investigated the monotone iterative technique for impulsive hybrid set integro-differential equations; Blasi, Lakshmikantham, and Bhaskar [11,12] gave the result of existence of solution for set-valued differential inclusions in a semilinear metric space; Drice and Mcrae [14, 15] obtained some basic results on existence, uniqueness, and continuous dependence of solutions with respect to initial values for set-valued differential equations with causal operators; Lupulescu [20] studied the successive approximations of solutions for set-valued differential equations and the differential inclusion and control in compact but non-convex spaces;

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Appala Naidu, Dhaigude and Devi [3], Bashir and Sivasundaram [5, 6], Hong [16] and Slyn¡ko [28] obtained some basic results and stability criteria in terms of two measures for set-valued differential equations involving causal operators, set-valued perturbed hybrid integro-differential equations with impulse, set-valued differential equations on time scales and set difference equations in space conv $\mathbb{R}^n$ respectively. We can also find some interesting results for set-valued functional differential equations in [1, 13, 21, 30]. However, we noticed that the previous studies mainly focused on the existence of solutions and the stability of solutions. There are few results of convergence for set-valued differential equations and fewer results of convergence for set-valued functional differential equations. In this paper, by introducing the notion of Hukuhara partial derivative of set-valued function, using the comparison principle and the method of quasilinearization, we consider quadratic approximation of solutions for the following set-valued functional differential equations

$$\begin{cases} D_H U(t) = F(t, U(t), U(\delta(t))), & t \in J, \\ U(t) = \Psi(t) \in K_c(\mathbb{R}^n), & t \in J_0, \end{cases} \tag{1.1}$$

where $U \in C = C[J, K_c(\mathbb{R}^n)]$, $F : J \times C \times C \to K_c(\mathbb{R}^n)$, $\Psi : J_0 \to K_c(\mathbb{R}^n)$ are continuous set-valued mapping; $J = [0, T]$, $J_0 = [-r, 0]$, $t - r \leq \delta(t) \leq t$, $r > 0$ is a constant; $K_c(\mathbb{R}^n)$ is a family of all nonempty compact and convex subsets of $\mathbb{R}^n$.

2. Preliminaries

We first give the notations and concepts for set-valued differential equations which can be found in [18].

Let A and B be the nonempty closed subsets of the space $K_c(\mathbb{R}^n)$, the Hausdorff metric between A and B is determined by the formula

$$D[A, B] = \max \Big[\sup_{x \in B} d(x, A), \sup_{y \in A} d(y, B) \Big],$$

where $d(x, A) = \inf[d(x, y) : y \in A]$.

It is known that $(K_c(\mathbb{R}^n), D)$ is a complete metric space. For any nonempty subsets A, B, C, A' and B' of the space $K_c(\mathbb{R}^n)$, $\lambda \in \mathbb{R}_+$, we have

$$\begin{aligned} & D[A + C, B + C] = D[A, B], \ D[A, B] = D[B, A], \\ & D[\lambda A, \lambda B] = \lambda D[A, B], \\ & D[A, B] \leq D[A, C] + D[C, B], \\ & D[A + A', B + B'] \leq D[A, B] + D[A', B']. \end{aligned}$$

Definition 2.1 ([18])**.** Given any $A, B \in K_c(\mathbb{R}^n)$, if there exists an element $C \in K_c(\mathbb{R}^n)$ such that $A = B + C$, then we define the $A - B$ as geometric difference of A and B. if there exists a $C \in K_c(\mathbb{R}^n)$ such that $A = B + C$, and for any $c \in C$ is a nonnegative(positive) vector of n components satisfying $c_i \geq 0$ for $i = 1, 2, \cdots, n$, then we define the $A \geq B$. Similarly, one can define $A \leq B$.

For any compact set $J \subseteq \mathbb{R}_+$, we give the Hukuhara integral of F by

$$U(t) = U(t_0) + \int_{t_0}^{t} D_H U(s)\, ds, \quad t \in J,$$

that is

$$U(t) = U(t_0) + \int_{t_0}^{t} F(s, U(s), U(\delta(s)))\, ds, \quad t \in J,$$

where the Hukuhara integral as follow

$$\int_J F(s)\, ds = \Big[\int_J f(s)\, ds : f \text{ is a continuous selector of } F \Big].$$

Corollary 2.1. *If $F : J \to K_c(\mathbb{R}^n)$ is integrable, then*

$$\int_{t_0}^{t_2} F(s)\, ds = \int_{t_0}^{t_1} F(s)\, ds + \int_{t_1}^{t_2} F(s)\, ds, \quad t_0 \le t_1 \le t_2,$$
$$\int_{t_0}^{t} \lambda F(s)\, ds = \lambda \int_{t_0}^{t} F(s)\, ds, \quad \lambda \in \mathbb{R}.$$

Corollary 2.2. *If $F, G : J \to K_c(\mathbb{R}^n)$ is integrable, then $D[F(\cdot), G(\cdot)] : J \to \mathbb{R}$ is integrable and*

$$D\left[\int_{t_0}^{t} F(s)\, ds, \int_{t_0}^{t} G(s)\, ds \right] \le \int_{t_0}^{t} D\big[F(s), G(s)\big]\, ds.$$

Definition 2.2 ([18])**.** The set-valued mapping $F : J \times K_c(\mathbb{R}^n) \times K_c(\mathbb{R}^n) \to K_c(\mathbb{R}^n)$ is Hukuhara differentiable at a point $t_0 \in J$, if the limits

$$\lim_{h \to 0^+} \frac{F(t_0 + h, U, U(\delta)) - F(t_0, U, U(\delta))}{h}$$

and

$$\lim_{h \to 0^+} \frac{F(t_0, U, U(\delta)) - F(t_0 - h, U, U(\delta))}{h}$$

exist in $K_c(\mathbb{R}^n)$ and equal to $D_H F(t_0)$.

Next, we give the concept of Hukuhara partial derivatives of set-valued function which are useful in proving the main results.

Definition 2.3. The set-valued mapping $F : J \times K_c(\mathbb{R}^n) \times K_c(\mathbb{R}^n) \to K_c(\mathbb{R}^n)$ is Hukuhara partial differentiable at U_0, if there exists $D_{H_{U_0}} F(t, U_0, U(\delta)) \in 2^{K_c(\mathbb{R}^n)}$ such that the limits

$$\lim_{h \to 0^+} \frac{F(t, U_0 + he_k, U(\delta)) - F(t, U_0, U(\delta))}{h}$$

and

$$\lim_{h \to 0^+} \frac{F(t, U_0, U(\delta)) - F(t, U_0 - he_k, U(\delta))}{h}$$

exist in the topology of $(K_c(\mathbb{R}^n), D)$ and are equal to $D_{H_{U_0}} F(t, U_0, U(\delta))$, where $e_k = (e_k^1, \cdots, e_k^n)$ is the vector that $e_k^j = 0$, $k \ne j$ and $e_k^k = 1$.

Similarly, we can define the second partial derivative of F with respect to U_0.

Definition 2.4. The set-valued mapping $D_{H_{U_0}}F : J \times K_c(\mathbb{R}^n) \to 2^{K_c(\mathbb{R}^n)}$ is Hukuhara partial differentiable at U_0, if there exists $D_{H^2_{U_0}}F(t, U_0, U(\delta))$ such that the limits

$$\lim_{h\to 0^+} \frac{D_{H_{U_0}}F(t, U_0 + he_k, U(\delta)) - D_{H_{U_0}}F(t, U_0, U(\delta))}{h}$$

and

$$\lim_{h\to 0^+} \frac{D_{H_{U_0}}(t, U_0, U(\delta)) - D_{H_{U_0}}F(t, U_0 - he_k, U(\delta))}{h}$$

exist and are equal to $D_{H^2_{U_0}}F(t, U_0, U(\delta))$, where $e_k = (e_k^1, \cdots, e_k^n)$ is the vector that $e_k^j = 0$, $k \neq j$ and $e_k^k = 1$.

In addition, we give some suitable forms for the Hukuhara partial derivatives. We first identify $D_{H_{U_0}}F(t, U_0, U(\delta))$ in the following form:

$$D_{H_{U_0}}F(t, U_0, U(\delta)) = \Big[\frac{\partial F(t, U_0, U(\delta))}{\partial u_0} : u_0 \in U_0 \in K_c(\mathbb{R}^n)\Big] \in 2^{K_c(\mathbb{R}^n)},$$

where

$$\frac{\partial F(t, U_0, U(\delta))}{\partial u_0} = \Big(\frac{\partial F(t, U_0, U(\delta))}{\partial u_{01}}, \cdots, \frac{\partial F(t, U_0, U(\delta))}{\partial u_{0n}}\Big),$$

such that $\frac{\partial F(t,U_0,U(\delta))}{\partial u_{0i}} \in K_c(\mathbb{R}^n)$ for each i. In the special case, when U_0, F are single-valued mapping, $D_{H_{U_0}}F(t, U_0, U(\delta))$ reduces to n vectors

$$\frac{\partial F(t, U_0, U(\delta))}{\partial u_0} = \Big(\frac{\partial F(t, U_0, U(\delta))}{\partial u_{01}}, \cdots, \frac{\partial F(t, U_0, U(\delta))}{\partial u_{0n}}\Big),$$

which is usually written as an $n \times n$ matrix so that one can treat it as a linear operator mapping any vector into another vector or any matrix into another matrix. Similarly, we can identify $D_{H^2_{U_0}}F(t, U_0, U(\delta))$ in the following suitable form:

$$D_{H^2_{U_0}}F(t, U_0, U(\delta)) = \Big[\frac{\partial F^2(t, U_0, U(\delta))}{\partial u_0^2} : u_0 \in U_0 \in K_c(\mathbb{R}^n)\Big],$$

where

$$\frac{\partial F^2(t, U_0, U(\delta))}{\partial u_0^2} = \Big(\frac{\partial \frac{\partial F(t,U_0,U(\delta))}{\partial u_0}}{\partial u_{01}}, \cdots, \frac{\partial \frac{\partial F(t,U_0,U(\delta))}{\partial u_0}}{\partial u_{0n}}\Big),$$

such that $\frac{\partial \frac{\partial F(t,U_0,U(\delta))}{\partial u_0}}{\partial u_{0i}} \in 2^{K_c(\mathbb{R}^n)}$ for each i. In the special case when U_0, F are single-valued $D_{H^2_{U_0}}F(t, U_0, U(\delta))$ reduces to n vectors or $n \times n$ matrix

$$\begin{aligned}\frac{\partial F^2(t, U_0, U(\delta))}{\partial u_0^2} &= \Big(\frac{\partial \frac{\partial F(t,U_0,U(\delta))}{\partial u_0}}{\partial u_{01}}, \cdots, \frac{\partial \frac{\partial F(t,U_0,U(\delta))}{\partial u_0}}{\partial u_{0n}}\Big) \\ &= \begin{pmatrix} \frac{\partial^2 F(t,U_0,U(\delta))}{\partial u_{01}^2} & \frac{\partial^2 F(t,U_0,U(\delta))}{\partial u_{01}\partial u_{02}} & \cdots & \frac{\partial^2 F(t,U_0,U(\delta))}{\partial u_{01}\partial u_{0n}} \\ \frac{\partial^2 F(t,U_0,U(\delta))}{\partial u_{02}\partial u_{01}} & \frac{\partial^2 F(t,U_0,U(\delta))}{\partial u_{02}^2} & \cdots & \frac{\partial^2 F(t,U_0,U(\delta))}{\partial u_{02}\partial u_{0n}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 F(t,U_0,U(\delta))}{\partial u_{0n}\partial u_{01}} & \frac{\partial^2 F(t,U_0,U(\delta))}{\partial u_{0n}\partial u_{02}} & \cdots & \frac{\partial^2 F(t,U_0,U(\delta))}{\partial u_{0n}^2} \end{pmatrix}\end{aligned}$$

which is usually written as an $n \times n^2$ matrix so that one can treat it as a linear operator mapping.

Definition 2.5. Let $A = \{a_{ij}\}^N$ be a matrix, $i, j = 1$, $N \in \mathbb{N}^+$. We call that $A > \theta$ if $a_{ij} > \theta$ for $i, j = 1, 2, ..., N$, where θ is the element in $\mathbb{R}^n$, which is regarded as a point set.

3. Quadratic Convergence

In this section, the quadratic convergence of approximate iterative sequences are proved by the quasilinearization method.

Let θ be a null set of $K_c(\mathbb{R}^n)$, $P \in K_c(\mathbb{R}^n)$, we denote $||P|| = \{(|P^1|, \cdots, |P^n|)\}$, $||P||^2 = \{(|P^1|^2, \cdots, |P^n|^2)\}$.

Definition 3.1. There exist $V, W \in C(J_0, K_c(\mathbb{R}^n)) \cup C^1(J, K_c(\mathbb{R}^n))$, and V, W are said to be

(I) natural lower and upper solutions of the problem (1.1) if

$$\begin{cases} D_H V(t) \leq F(t, V(t), V(\delta(t))), & t \in J, \\ V(t) \leq \Psi(t), & t \in J_0, \end{cases}$$

and

$$\begin{cases} D_H W(t) \geq F(t, W(t), W(\delta(t))), & t \in J, \\ W(t) \geq \Psi(t), & t \in J_0, \end{cases}$$

(II) coupled lower solution and upper solutions of the problem (1.1) if

$$\begin{cases} D_H V(t) \leq F(t, V(t), W(\delta(t))), & t \in J, \\ V(t) \leq \Psi(t), & t \in J_0, \end{cases}$$

and

$$\begin{cases} D_H W(t) \geq F(t, W(t), V(\delta(t))), & t \in J, \\ W(t) \geq \Psi(t), & t \in J_0. \end{cases}$$

Firstly, we give some lemmas which plays an important role in the proof of our results.

Lemma 3.1. *Assume that the following conditions hold:*

$(A_{3.1})$ $V, W \in C(J_0, K_c(\mathbb{R}^n)) \bigcup C^1(J, K_c(\mathbb{R}^n))$ *are natural lower and upper solutions of the problem* (1.1);

$(A_{3.2})$ $F \in C[J \times K_c(\mathbb{R}^n) \times K_c(\mathbb{R}^n), K_c(\mathbb{R}^n)]$, $F(t, X, Y)$ *is nondecreasing in both* X *and* Y *for* $t \in J$*; and for any* $X_1, X_2, Y_1, Y_2 \in K_c(\mathbb{R}^n)$, $X_1 \geq X_2$, $Y_1 \geq Y_2$, *satisfy*

$$F_i(t, X_1, Y_1) \leq F_i(t, X_2, Y_2) + L_1 \sum_{j=1}^{n} \Big[(X_{1j} - X_{2j}) + (Y_{1j} - Y_{2j})\Big],$$

where $L_1 > 0$ *is a constant, and* F_i *is a submap of the ith component of the mapping* F, $i = 1, \cdots, n$.

Then $V(t) \leq W(t)$ *for* $t \in [-r, T]$.

Proof. For $t \in J_0$, by the condition $(A_{3.1})$, it's easy to see that the inequality holds. When $t \in J$, putting $\epsilon = (\epsilon, \epsilon, \cdots, \epsilon) > 0$ and defining $\overline{W}(t) = W(t) + \epsilon e^{2(n+1)L_1 t}$. Noting that $V(0) \leq W(0) < \overline{W}(0)$, we can only prove that $V(t) < \overline{W}(t)$ to arrive at the conclusion, due to the fact ϵ is arbitrary. Suppose that $t_1 > 0$ is the supremum of all positive numbers ν, we have $V(t) < \overline{W}(t)$ on $[0, \nu]$ by $V(0) < \overline{W}(0)$, it implies that $V(\delta(t_1)) < \overline{W}(\delta(t_1))$. Now using the nondecreasing of $F(t, U, U(\delta))$ in both U and $U(\delta)$ and the assumption $(A_{3.2})$, we have

$$\begin{aligned} D_H V_i(t_1) \leq & F_i(t_1, V(t_1), V(\delta(t_1))) \\ \leq & F_i(t_1, W(t_1), W(\delta(t_1))) \\ & + L_1 \sum_{j=1}^{n} \Big[(\overline{W}_j(t_1) - W_j(t_1)) + (\overline{W}_j(\delta(t_1)) - W_j(\delta(t_1))) \Big] \\ < & D_H W_i(t_1) + 2(n+1) L_1 \epsilon e^{2(n+1)L_1 t_1} \\ = & D_H \overline{W}_i(t_1). \end{aligned}$$

Therefore there exists an $\eta > 0$ satisfying

$$V_i(t_1) - \overline{W}_i(t_1) \leq V_i(t) - \overline{W}_i(t), \quad t_1 - \eta < t < t_1.$$

This contradicts that $t_1 > 0$ is the supremum due to the continuity of the functions involved and consequently $V(t) \leq W(t)$ is true for $t \in [-r, T]$. □

Remark 3.1. Assume that $(A_{3.2})$ holds and there exists $P \in C(J_0, K_c(\mathbb{R}^n)) \bigcup C^1(J, K_c(\mathbb{R}^n))$ satisfying

$$\begin{cases} D_H P(t) \leq F(t, P(t), P(\delta(t))), & t \in J, \\ P(t) \leq \theta, & t \in J_0. \end{cases}$$

Then $P(t) \leq \theta$ for $t \in [-r, T]$.

Lemma 3.2. *Assume that the following conditions hold:*

$(A_{3.3})$ $V, W \in C(J_0, K_c(\mathbb{R}^n)) \bigcup C^1(J, K_c(\mathbb{R}^n))$ *are coupled lower and upper solutions of the problem* (1.1)*;*

$(A_{3.4})$ $F \in C[J \times K_c(\mathbb{R}^n) \times K_c(\mathbb{R}^n), K_c(\mathbb{R}^n)]$*,* $F(t, X, Y)$ *is nondecreasing in* X *for each* (t, Y) *and nonincreasing in* Y *for each* (t, X)*; and for any* $X_1, X_2, Y_1, Y_2 \in K_c(\mathbb{R}^n)$*,* $X_1 \geq X_2$*,* $Y_1 \geq Y_2$*,* $t \in J$*, satisfy*

$$F_i(t, X_1, Y) \leq F_i(t, X_2, Y) + L_2 \sum_{j=1}^{n} (X_{1j} - X_{2j}),$$

$$F_i(t, X, Y_1) \geq F_i(t, X, Y_2) - L_2 \sum_{j=1}^{n} (Y_{1j} - Y_{2j}),$$

where $L_2 > 0$ *is a constant, and* F_i *is a submap of the ith component of the mapping* F*,* $i = 1, \cdots, n$*.*

Then $V(t) \leq W(t)$ *for* $t \in [-r, T]$.

Proof. For $t \in J_0$, by the condition $(A_{3.3})$, it's easy to see that the inequality holds. When $t \in J$, putting $\epsilon = (\epsilon, \epsilon, \cdots, \epsilon) > 0$ and defining $\overline{W}(t) = W(t) + \epsilon e^{3(n+1)L_2 t}$, $\overline{V}(t) = V(t) - \epsilon e^{3(n+1)L_2 t}$. Noting that $\overline{V}(0) < V(0) \leq W(0) < \overline{W}(0)$, we can prove that $\overline{V}(t) < \overline{W}(t)$ to arrive at the conclusion $V(t) \leq W(t)$ for $t \in J$. In order to prove that conclusion, suppose that $t_1 > 0$ is the supremum of all positive mumbers ν, we have $\overline{V}(t) < \overline{W}(t)$ on $[0, \nu]$ by $\overline{V}(0) < \overline{W}(0)$, it implies that $\overline{V}(\delta(t_1)) < \overline{W}(\delta(t_1))$. Now using the assumption $(A_{3.4})$, we have

$$\begin{aligned}
D_H \overline{V}_i(t_1) &\leq F_i(t_1, V(t_1), W(\delta(t_1))) - 3(n+1)L_2 \epsilon e^{3(n+1)L_2 t} \\
&\leq F_i(t_1, \overline{W}(t_1), \overline{W}(\delta(t_1))) - (n+3)L_2 \epsilon e^{3(n+1)L_2 t} \\
&\leq F_i(t_1, W(t_1), V(\delta(t_1))) - (n+3)L_2 \epsilon e^{3(n+1)L_2 t} \\
&\quad + L_2 \sum_{j=1}^{n} (\overline{W}_j(t_1) - W_j(t_1)) + L_2 \sum_{j=1}^{n} (V_j(\delta(t_1)) - \overline{V_j}(\delta(t_1)) \\
&< D_H W_i(t_1) + 3(n+1)L_2 \epsilon e^{3(n+1)L_2 t_1} \\
&= D_H \overline{W}_i(t_1).
\end{aligned}$$

Therefore there exists an $\eta > 0$ satisfying $V_i(t_1) - \overline{W}_i(t_1) \leq V_i(t) - \overline{W}_i(t), \ \ t_1 - \eta < t < t_1$. This contradicts that $t_1 > 0$ is the supremum due to the continuity of the functions involved and consequently $V(t) \leq W(t)$ is true for $t \in [-r, T]$. □

Remark 3.2. Assume that $(A_{3.4})$ holds and if $P, Q \in C(J_0, K_c(R^n)) \bigcup C^1(J, K_c(R^n))$ are satisfying

$$\begin{cases} D_H P(t) \leq F(t, P(t), Q(\delta(t))), & t \in J, \\ P(t) \leq \theta, & t \in J_0, \end{cases}$$

$$\begin{cases} D_H Q(t) \geq F(t, Q(t), P(\delta(t))), & t \in J, \\ Q(t) \geq \theta, & t \in J_0. \end{cases}$$

Then $P(t) \leq \theta \leq Q(t)$ for $t \in [-r, T]$.

Lemma 3.3 ([18]). *Let $U \in C^1(J, K_c(\mathbb{R}^n))$. If $D_H U \leq \hat{A}U + \alpha$. Then the following inequality*

$$U(t) \leq U(t_0) e^{\hat{A}t} + \int_{t_0}^{t} e^{\hat{A}(t-s)} \alpha(s) ds, \quad t \in J$$

holds, where $\hat{A} = (a_{ij})$ is a $n \times n$ matrix satisfying $a_{ij} \geq \theta$, $i \neq j$, $\alpha \in C(J, K_c(\mathbb{R}^n))$.

In order to prove the convergence of the approximate solutions, we will apply the method of lower and upper solutions coupled with the method of quasilinearization, and will give the following two theorems finally.

Theorem 3.1. *Assuming that the following conditions are satisfied:*

($C_{3.1}$) $V_0, W_0 \in C(J_0, K_c(\mathbb{R}^n)) \cup C^1(J, K_c(\mathbb{R}^n))$ are coupled lower and upper solutions of the problem (1.1)*;*

($C_{3.2}$) there exist bound matric mappings $D_{H_1}F \geq \theta$, $D_{H_2}F \leq \theta$, and $D_{H_i H_j}F$, $i, j = 1, 2$. Moreover, the quadratic form $K(F(t, X, Y))$ satisfying $K(F) \geq \theta$

on $J \times K_c(\mathbb{R}^n) \times K_c(\mathbb{R}^n)$ *is given by*

$$\begin{aligned} K(F) =&(X-\alpha)^T D_{H_1H_1}F(t,X_1,Y_1)(X-\alpha) \\ &+ 2(X-\alpha)^T D_{H_1H_2}F(t,X_1,Y_1)(\beta - Y) \\ &+ (\beta - Y)^T D_{H_2H_2}F(t,X_1,Y_1)(\beta - Y), \end{aligned}$$

where $V_0 \leq \alpha \leq X_1 \leq X \leq W_0$, $V_0 \leq \beta \leq Y_1 \leq Y \leq W_0$.

Then there exist monotone sequences $\{V_n(t)\}$, $\{W_n(t)\}$ *converging uniformly to the solution* $U(t)$ *of the problem* (1.1) *and the convergence are quadratic.*

Proof. Let $D_{H_1}F(t,X,Y) = \overline{A}(t,X,Y)$, $D_{H_2}F(t,X,Y) = \overline{B}(t,X,Y)$. Consider the sequences $\{V_n(t)\}$, $\{W_n(t)\}$ as follows

$$\begin{cases} D_H V_{n+1}(t) = F(t,V_n(t),W_n(\delta)) + \overline{A}(t,V_n(t),W_n(\delta(t))\Big[V_{n+1}(t) - V_n(t)\Big] \\ \qquad\qquad + \overline{B}(t,V_n(t),W_n(\delta))\Big[W_{n+1}(\delta(t)) - W_n(\delta(t))\Big], \quad t \in J, \\ V_{n+1}(t) = \Psi(t), \quad t \in J_0, \end{cases} \tag{3.1}$$

and

$$\begin{cases} D_H W_{n+1}(t) = F(t,W_n(t),V_n(\delta)) + \overline{A}(t,V_n(t),W_n(\delta))\Big[W_{n+1}(t) - W_n(t)\Big] \\ \qquad\qquad + \overline{B}(t,V_n(t),W_n(\delta))\Big[V_{n+1}(\delta(t)) - V_n(\delta(t))\Big], \quad t \in J, \\ W_{n+1}(t) = \Psi(t), \quad t \in J_0. \end{cases} \tag{3.2}$$

Firstly, we show that $V_0(t) \leq V_1(t) \leq W_1(t) \leq W_0(t)$ for $t \in [-r,T]$. Putting $P(t) = V_0(t) - V_1(t)$, $Q(t) = W_0(t) - W_1(t)$.

Case 1. For $t \in J_0$, by $V_1(t) = \Psi(t) = W_1(t)$ and the condition $(C_{3.1})$, we can see that

$$P(t) = V_0(t) - V_1(t) \leq \theta \ \text{ and } \ Q(t) = W_0(t) - W_1(t) \geq \theta.$$

Case 2. For $t \in J$, by (3.1) and (3.2), we have

$$\begin{aligned} D_H P(t) \leq& \overline{A}(t,V_0(t),W_0(\delta))P(t) + \overline{B}(t,V_0(t),W_0(\delta))Q(\delta(t)), \\ D_H Q(t) \geq& \overline{A}(t,V_0(t),W_0(\delta))Q(t) + \overline{B}(t,V_0(t),W_0(\delta))P(\delta(t)). \end{aligned}$$

In view of Remark 3.2, we obtain that $P(t) \leq \theta \leq Q(t)$ on $[-r,T]$ and we have $V_0(t) \leq V_1(t)$, $W_1(t) \leq W_0(t)$ for $t \in [-r,T]$.

Similarly, we can show that $V_1(t) \leq W_1(t)$ for $t \in [-r,T]$. Putting $P(t) = V_1(t) - W_1(t)$.

Case 1. For $t \in J_0$, since $V_1(t) = \Psi(t) = W_1(t)$ and the condition $(C_{3.1})$, we can see that $P(t) = V_1(t) - W_1(t) = \theta$.

Case 2. For $t \in J$, using the condition $(C_{3.2})$ with

$$\begin{aligned} &K(F(t,W_0(t),V_0(\delta))) \\ =&\Big[W_0(t) - V_0(t)\Big]^T D_{H_1H_1}F(t,X_2,Y_2)\Big[W_0(t) - V_0(t)\Big] \\ &+ 2\Big[W_0(t) - V_0(t)\Big]^T D_{H_1H_2}F(t,X_2,Y_2)\Big[V_0(\delta(t)) - W_0(\delta(t))\Big] \end{aligned}$$

$$+\Big[V_0(\delta(t))-W_0(\delta(t))\Big]^T D_{H_2H_2}F(t,X_2,Y_2)\Big[V_0(\delta(t))-W_0(\delta(t))\Big],$$

where $V_0 \le X_2 \le W_0$, $V_0 \le Y_2 \le W_0$. Then we have

$$\begin{aligned} D_H P(t) =&F(t,V_0(t),W_0(\delta)) + \overline{A}(t,V_0(t),W_0(\delta))\Big[V_1(t)-V_0(t)\Big] \\ &+ \overline{B}(t,V_0(t),W_0(\delta))\Big[W_1(\delta(t))-W_0(\delta(t))\Big] \\ &- F(t,W_0(t),V_0(\delta)) - \overline{A}(t,V_0(t),W_0(\delta))\Big[W_1(t)-W_0(t)\Big] \\ &- \overline{B}(t,V_0(t),W_0(\delta))\Big[V_1(\delta(t))-V_0(\delta(t))\Big] \\ \le&\overline{A}(t,V_0(t),W_0(\delta))P(t) - \overline{B}(t,V_0(t),W_0(\delta))P(\delta(t)). \end{aligned}$$

In view of Remark 3.1, we obtain that $P(t) \le \theta$ on $[-r,T]$, which means $V_1(t) \le W_1(t)$ on $[-r,T]$. Thus, it proves that

$$V_0(t) \le V_1(t) \le W_1(t) \le W_0(t), \quad t \in [-r,T].$$

Next, we will show that $V_1(t)$, $W_1(t)$ are coupled lower solution and upper solutions of the problem (1.1).

For $t \in J_0$, by (3.1) and (3.2), we know that $V_1(t) = \Psi(t) = W_1(t)$. For $t \in J$, using (3.1) and the condition $(C_{3.2})$ with

$$\begin{aligned} &K(F(t,V_1(t),W_1(\delta))) \\ =&\Big[V_1(t)-V_0(t)\Big]^T D_{H_1H_1}F(t,X_3,Y_3)\Big[V_1(t)-V_0(t)\Big] \\ &+ 2\Big[V_1(t)-V_0(t)\Big]^T D_{H_1H_2}F(t,X_3,Y_3)\Big[W_1(\delta(t))-W_0(\delta(t))\Big] \\ &+ \Big[W_1(\delta(t))-W_0(\delta(t))\Big]^T D_{H_2H_2}F(t,X_3,Y_3)\Big[W_1(\delta(t))-W_0(\delta(t))\Big], \end{aligned}$$

where $V_0 \le X_3 \le V_1$, $W_1 \le Y_3 \le W_0$. Then, we have

$$\begin{aligned} D_H V_1(t) =&F(t,V_0(t),W_0(\delta)) + \overline{A}(t,V_0,W_0(\delta))\Big[V_1(t)-V_0(t)\Big] \\ &+ \overline{B}(t,V_0,W_0(\delta))\Big[W_1(\delta(t))-W_0(\delta(t))\Big] \\ \le&F(t,V_1(t),W_1(\delta)). \end{aligned}$$

Similarly, we have

$$D_H W_1(t) \ge F(t,W_1(t),V_1(\delta(t))),$$

which means that $V_1(t)$ and $W_1(t)$ are coupled lower solution and upper solutions of the problem (1.1). Therefore, by induction we can show that

$$V_0(t) \le V_1(t) \le \ldots \le V_n(t) \le W_n(t) \le \ldots \le W_1(t) \le W_0(t), \quad t \in [-r,T]$$

and $V_n(t)$, $W_n(t)$ are coupled lower solution and upper solutions of the problem (1.1).

Next, we can show that the sequences $\{V_n(t)\}$, $\{W_n(t)\}$ are uniformly bounded and equicontinuous. Obviously, the sequences $\{W_n(t)\}$ are uniformly bounded, we only prove that sequences $\{W_n(t)\}$ are equicontinuous on $[-r, T]$. For $s, t \in [-r, T]$, when $s < t$, we have

$$\begin{aligned}
D[W_n(t), W_n(s)] \leq & D\bigg[W_{n-1}(0) + \int_0^t \Big\{F(s, W_{n-1}(s), V_{n-1}(\delta(s))) \\
& + \overline{A}(s, V_{n-1}(s), W_{n-1}(\delta))\Big[W_n(s) - W_{n-1}(s)\Big] \\
& + \overline{B}(s, V_{n-1}(s), W_{n-1}(\delta))\Big[V_n(\delta(s)) - V_{n-1}(\delta(s))\Big]\Big\}ds, \\
& W_{n-1}(0) + \int_0^s \Big\{F(t, W_{n-1}(t), V_{n-1}(\delta(t))) \\
& + \overline{A}(t, V_{n-1}(t), W_{n-1}(\delta))\Big[W_n(t) - W_{n-1}(t)\Big] \\
& + \overline{B}(t, V_{n-1}(t), W_{n-1}(\delta))\Big[V_n(\delta(t)) - V_{n-1}(\delta(t))\Big]\Big\}dt\bigg] \\
= & D\bigg[\int_s^t \Big\{F(\zeta, W_{n-1}(\zeta), V_{n-1}(\delta(\zeta))) \\
& + \overline{A}(\zeta, V_{n-1}(\zeta), W_{n-1}(\delta))\Big[W_n(\zeta) - W_{n-1}(\zeta)\Big] \\
& + \overline{B}(\zeta, V_{n-1}(\zeta), W_{n-1}(\delta))\Big[V_n(\delta(\zeta)) - V_{n-1}(\delta(\zeta))\Big]\Big\}ds, \theta\bigg] \\
\leq & \int_s^t D\bigg[F(\zeta, W_{n-1}(\zeta), V_{n-1}(\delta(\zeta))) \\
& + \overline{A}(\zeta, V_{n-1}(\zeta), W_{n-1}(\delta))\Big[W_n(\zeta) - W_{n-1}(\zeta)\Big] \\
& + \overline{B}(\zeta, V_{n-1}(\zeta), W_{n-1}(\delta))\Big[V_n(\delta(\zeta)) - V_{n-1}(\delta(\zeta))\Big], \theta\bigg]d\zeta \\
\leq & M|t - s|.
\end{aligned}$$

Analogically we can show that the sequences $\{V_n(t)\}$ are equicontinuous on $[-r, T]$. In view of **Ascoli-Arzela** theorem, there exist the subsequences $\{V_{n_k}\}$ and $\{W_{n_k}\}$ converging uniformly on J to continuous functions V and W respectively.

When there exists a unique solution of the problem (1.1), then, $V = W$ for $t \in [-r, T]$; When the solution of problem (1.1) is not unique, let $U(t)$ be one solution of the problem (1.1), it is easily to obtain that $V \leq U \leq W$, that is V, W are the minimal and maximal solutions of (1.1), respectively.

Finally, we show quadratic convergence of the approximate solution.

Let $U(t)$ be the solution of the problems (1.1), and putting

$$P_{n+1}(t) = U(t) - V_{n+1}(t) \geq \theta \quad \text{and} \quad Q_{n+1}(t) = W_{n+1}(t) - U(t) \geq \theta.$$

Case 1. For $t \in J_0$, since $V_{n+1}(t) = \Psi(t) = W_{n+1}(t)$, we have

$$\begin{aligned}
P_{n+1}(t) &= U(t) - V_{n+1}(t) = \theta \leq \max P_n^2(t) + \max Q_n^2(t), \\
Q_{n+1}(t) &= W_{n+1}(t) - U(t) = \theta \leq \max P_n^2(t) + \max Q_n^2(t).
\end{aligned}$$

Case 2. For $t \in J$, by (3.1) and (3.2), we have

$$\begin{aligned} D_H P_{n+1}(t) \leq & \overline{A}(t,U,U(\delta))P_n(t) - \overline{A}(t,V_n,W_n(\delta))\Big[P_n(t) - P_{n+1}(t)\Big] \\ & \overline{B}(t,V_n,W_n(\delta))\Big[Q_{n+1}(\delta(t)) - Q_n(\delta(t))\Big] \\ \leq & \overline{A}(t,V_n,W_n(\delta))P_{n+1}(t) - \overline{B}(t,V_n,W_n(\delta))Q_{n+1}(\delta(t)) \\ & + P_n^T(t)\left[\int_0^1 D_{H_1^2}F(t,sU(t)+(1-s)V_n(t),U(\delta))ds\right]P_n(t) \\ & + Q_n^T(t)\left[\int_0^1 D_{H_2^2}F(t,V_n(t),sW_n(\delta)+(1-s)U(\delta))ds\right]Q_n(\delta(t)) \\ & - Q_n^T(\delta(t))\left[\int_0^1 D_{H_1H_2}F(t,V_n(t),sU(\delta)+(1-s)V_n(\delta))ds\right]P_n(t) \\ \leq & A_1P_{n+1}(t) + A_2Q_{n+1}(\delta(t)) + C_1P_n^2(t) + D_1Q_n^2(\delta(t)), \end{aligned}$$

where $D_{H_1}F \leq A_1$, $-A_2 \leq D_{H_2}F \leq A_2$, $D_{H_1^2}F \leq B_1$, $-B_2 \leq D_{H_1H_2}F \leq B_2$, $-B_3 \leq D_{H_2H_1}F \leq B_3$, $D_{H_2^2}F \leq B_4$, $C_1 = B_1 + \frac{1}{2}B_2$, $D_1 = B_4 + \frac{1}{2}B_2$ and A_i, B_j are nonnegative matrices, $i = 1,2$, $j = 1,2,3,4$.

Similarly, we can show that

$$\begin{aligned} D_H Q_{n+1}(t) \leq & \overline{A}(t,V_n,W_n(\delta))Q_{n+1}(t) - \overline{B}(t,V_n,W_n(\delta))P_{n+1}(\delta) \\ & + \left[\int_0^1 D_{H_1}F(t,SW_n(t)+(1-s)U(t),V_n(\delta))ds\right]Q_n(t) \\ & - \left[\int_0^1 D_{H_2}F(t,U,sV_n(\delta)(1-s)U(\delta))ds\right]p_n(\delta) \\ & + \overline{B}(t,V_n,W_n(\delta))p_n(\delta) - \overline{A}(t,V_n,W_n(\delta))Q_n(t) \\ \leq & A_1Q_{n+1}(t) + A_2P_{n+1}(\delta) + C_2P_n^2(t) + C_3P_n^2(\delta) + D_2Q_n^2(t) + D_3Q_n^2(\delta), \end{aligned}$$

where $C_2 = \frac{1}{2}(B_1 + B_3)$, $C_3 = \frac{1}{2}(B_2 + B_3 + 3B_4)$, $D_2 = \frac{3}{2}B_1 + B_2$, $D_3 = \frac{1}{2}(B_2 + B_4)$.

Consider the following problems

$$\begin{cases} D_H\xi_1(t) = A_1P_{n+1}(t) + A_2Q_{n+1}(\delta(t)) + C_1P_n^2(t) + D_1Q_n^2(\delta(t)), & t \in J, \\ \xi_1(t) = P_{n+1}(t), & t \in J_0, \end{cases}$$

and

$$\begin{cases} D_H\xi_2(t) = C_2P_n^2(t) + C_3P_n^2(\delta(t)) + D_2Q_n^2(t) + D_3Q_n^2(\delta(t)) \\ \qquad\qquad + A_1Q_{n+1}(t) + A_2P_{n+1}(\delta(t)), \quad t \in J, \\ \xi_2(t) = Q_{n+1}(t), \quad t \in J_0. \end{cases}$$

Then, in view of $D_H\xi_1(t) \geq \theta$, $D_H\xi_2(t) \geq \theta$ and $P_{n+1}(t) \leq \xi_1(t)$, $Q_{n+1}(t) \leq \xi_2(t)$ for $t \in J$, we have

$$\begin{cases} D_H\xi_1(t) \leq A_1\xi_1(t) + A_2\xi_2(t) + C_1P_n^2(t) + D_1Q_n^2(\delta(t)), \\ D_H\xi_2(t) \leq A_1\xi_2(t) + A_2\xi_1(t) + C_2P_n^2(t) + C_3P_n^2(\delta(t)) + D_2Q_n^2(t) + D_3Q_n^2(\delta(t)), \end{cases}$$

Furthermore, by Lemma 3.3, we have

$$R_{n+1}(t) \leq \xi(t) \leq \int_0^t e^{B(t-s)}\Big\{E_1R_n^2(s) + E_2R_n^2(\delta)\Big\}ds \leq B^{-1}e^{BT}A||R_n||^2, \quad t \in J,$$

where $R_{n+1}(t) = \begin{pmatrix} P_{n+1}(t) \\ Q_{n+1}(t) \end{pmatrix}$, $\xi(t) = \begin{pmatrix} \xi_1(t) \\ \xi_2(t) \end{pmatrix}$, $B = \begin{pmatrix} A_1 & A_2 \\ A_2 & A_1 \end{pmatrix}$, $E_1 = \begin{pmatrix} C_1 & \theta \\ C_2 & D_2 \end{pmatrix}$, $E_2 = \begin{pmatrix} \theta & D_1 \\ C_3 & D_3 \end{pmatrix}$, $A = E_1 + E_2$.

Therefore, we have

$$\max P_{n+1}(t) \le K_1 \max P_n^2(t) + K_2 \max Q_n^2(t),$$

$$\max Q_{n+1}(t) \le K_3 \max P_n^2(t) + K_4 \max Q_n^2(t),$$

where K_i are suitable positive matrices, $i = 1, 2, 3, 4$. □

Similar to the proof of Theorem 3.1, we have the following result.

Theorem 3.2. *Assuming that the conditions V_0, W_0 are natural lower and upper solutions of the problem* (1.1), *and* $(C_{3.2})$ *hold.*

Then there exist monotone sequences $\{V_n(t)\}$, $\{W_n(t)\}$ converging uniformly to the solution $U(t)$ of the problem (1.1) *and the convergence are quadratic.*

In fact, we can consider the sequences $\{V_n(t)\}$, $\{W_n(t)\}$ as follow

$$\begin{cases} D_H V_{n+1}(t) = F(t, V_n(t), V_n(\delta(t))) + \overline{A}(t, V_n(t), V_n(\delta))\Big[V_{n+1}(t) - V_n(t)\Big] \\ \qquad\qquad + \overline{B}(t, V_n(t), V_n(\delta))\Big[V_{n+1}(\delta(t)) - V_n(\delta(t))\Big], \quad t \in J, \\ V_{n+1}(t) = \Psi(t), \quad t \in J_0, \end{cases}$$

and

$$\begin{cases} D_H W_{n+1}(t) = F(t, W_n(t), W_n(\delta(t))) + \overline{A}(t, V_n(t), V_n(\delta))\Big[W_{n+1}(t) - W_n(t)\Big] \\ \qquad\qquad + \overline{B}(t, V_n(t), V_n(\delta))\Big[W_{n+1}(\delta(t)) - W_n(\delta(t))\Big], \quad t \in J, \\ W_{n+1}(t) = \Psi(t), \quad t \in J_0, \end{cases}$$

for $n = 0, 1, 2, \cdots$. Similar to the proof of Theorem 3.1, We can show that monotone sequences $\{V_n(t)\}$, $\{W_n(t)\}$ converging uniformly to the solution $U(t)$ of (1.1) and its convergence is quadratic. We omit its details here.

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