Journal of Applied Analysis and Computation
Volume 11, Number 1, February 2021, 546–555
Website:http://www.jaac-online.com
DOI:10.11948/20200147

UNILATERAL PROBLEM FOR A NONLINEAR WAVE EQUATION WITH *P*-LAPLACIAN OPERATOR

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Abstract This work deals with the unilateral problem for a nonlinear wave equation with p-Laplacian operator and source term. Using an appropriate penalization, we obtain a variational inequality for the equation perturbed and then the existence of solutions is analyzed.

Keywords Unilateral problem, p-Laplacian operator, global solution.

MSC(2010) 35L85, 35A01, 35L80.

1. Introduction

In this work we prove the existence of global weak solutions to the following mixed unilateral problem

$$u'' - \Delta_p u + |u|^{r-1}u - \Delta u' \geq 0, \quad \text{in } Q, \tag{1.1}$$
$$u'(x,t) = 0, \quad \text{in } \Sigma, \tag{1.2}$$
$$u(x,t) = 0, \quad \text{in } \Sigma, \tag{1.3}$$
$$u(x,0) = u_0(x), \ u'(x,0) = u_1(x), \quad \text{in } \Omega, \tag{1.4}$$

that is related, from the physical point of view, to the motion of waves vibrating against an obstacle. More precisely, here we consider a unilateral problem, i.e. a variational inequality, see Lions [13], for the operator

$$\mathbb{L} = u'' - \Delta_p u + |u|^{r-1}u - \Delta u'$$

taking into account the source term $|u|^{r-1}u$, where $1 < r < \infty$ if $n \leq p$ and $1 < r < pn/(n-p)$ if $n > p$. Making use of the penalty method and Galerkin's approximations, we prove the existence and uniqueness of solutions.

Throughout this paper we omit the space variable x of $u(x,t)$ and simply denote $u(x,t)$ by $u(t)$ when no confusion arises. C denotes various positive constants depending on the known constants and may be different at each appearance. Let $T > 0$ be a real number, $\Omega \subset \mathbb{R}^n$ be a bounded open set with sufficiently smooth boundary Γ. We denote by $Q = \Omega \times (0,T)$ the cylinder with lateral boundary $\Sigma = \Gamma \times (0,T)$. Here we consider $2 \leq p < \infty$ and q such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

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The duality pairing between the space $W_0^{1,p}(\Omega)$ and its dual $W^{-1,q}(\Omega)$ will be denoted using the form $\langle\cdot\,,\,\cdot\rangle$. According to Poincaré's inequality, the standard norm $\|\cdot\|_{W_0^{1,p}(\Omega)}$ is equivalent to the norm $\|\nabla\cdot\|_p$ on $W_0^{1,p}(\Omega)$. Henceforth, we put $\|\cdot\|_{W_0^{1,p}(\Omega)}=\|\nabla\cdot\|_p$. We denote $\|\cdot\|_{L^2(\Omega)}=|\cdot|_2$ and the usual inner product by $(\cdot\,,\,\cdot)$. $\Delta_p u = div\left(|\nabla u|^{p-2}\nabla u\right)$ denotes the p-Laplacian operator which can be extended to a monotone, bounded, hemicontinuous and coercive operator between the spaces $W_0^{1,p}(\Omega)$ and its dual by

$$-\Delta_p\colon W_0^{1,p}(\Omega)\to W^{-1,q}(\Omega),\qquad \langle -\Delta_p u, v\rangle_p=\int_\Omega |\nabla u|^{p-2}\nabla u\nabla v\,\mathrm{d}x.$$

Now, we present a small literature overview including some new contributions on the problems with p-Laplacian operator. The existence of a global solution for wave equation of p-Laplacian type

$$u''-\Delta_p u=0 \tag{1.5}$$

without an additional dissipation term is an open problem. For $n=1$, Derher [8] proved the local in time existence of solution and showed by a generic counter-example that the global in time solution can not use expected. Adding a strong damping $-\Delta u'$ in (1.5) the well-posedness and asymptotic behavior was studied by Greenberg [10]. Weak solutions and blow-up for wave equations of the p-Laplacian type with supercritical sources was considered in [16]. Ma and Soriano [14] gave the weak solution for the problem with a dissipative source term $g(u)$ where $g(u)u\geq 0$ has a growth bound. Nevertheless, if the strong damping is replaced by a weaker damping u', then global existence and uniqueness are only know for $n=1,2$, see the works of Chueshov and Lasiecka [5] and Zhijian [22]. The damping term $-\Delta u'$ played an essential role in order to obtain global solutions. In [9] Gao and Ma analyzed existence of solution with the damping $(-\Delta)^\alpha u'$ with $0<\alpha\leq 1$

$$u''-\Delta_p u+(-\Delta)^\alpha u'+g(u)=f, \tag{1.6}$$

and extended the result of [14] for $g(u)$ without the sign condition $g(u)u\geq 0$.

The global existence of solution and asymptotic behaviour for wave equation with source term and p-Laplacian operator

$$u''-\Delta_p u+|u|^{r-1}u-\Delta u'=0,$$

can be obtained from (1.6) putting $\alpha=1, f=0$, and source term $g(u)=|u|^{r-1}u$.

It is well known that the energy of a PDE system with source term $|u|^{r-1}u$ is, in some sense, split into kinetic and potential energy. Following the idea of Y. Ye [21] can be built a set of stability for (1.6) with weaker damping ($\alpha=0$) and $f=0$. In fact there is a valley or a "well" of depth d created by the potential energy. If this height d is strictly positive, for the initial data in the "good part" of the well, the potential energy of the solution can never escape of the well. As a result, the total energy of the solution remains finite on any time interval $[0,T)$, which provides the global existence of the solution. In this way see for instance [17, 19] and references therein.

Unilateral problems for Klein-Gordon operator of Kirchhoff-Carrier type was studied by Raposo et al in [18]. Unilateral mixed problem with thermal effect was studied by Clark [6]. For unilateral problems related to the wave model subject to degenerate and localized nonlinear damping on a compact Riemannian manifold we cite the recent work of Cavalcanti et al [4]. For variational problems see the works of Medeiros and Milla [15], Lar'kin and Medeiros [12] and references therein.

Unilateral problem is very interesting because in general, dynamic contact problems are characterized by nonlinear hyperbolic variational inequalities. For contact problem in elasticity and finite element method see Kikuchi-Oden [11] and reference therein. For Contact Problem Viscoelastic Materials see Rivera and Oquendo [20]. For dynamic contact problems with friction, for example problems involving unilateral contact with dry friction of Coulomb, see Ballard and Basseville [1].

2. The Galerkin basis

We will show that there exists a Hilbert space $H_0^s(\Omega)$ with $0 < s$ such that $H_0^s(\Omega) \hookrightarrow W_0^p(\Omega)$ is continuous and $H_0^s(\Omega) \hookrightarrow L^2(\Omega)$ is continuous and compact.

For $v \in H^1(\mathbb{R}^n)$ consider

$$\hat{v}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-(\xi.x)i} v(x)\, dx$$

the Fourier transform of v and

$$H^s(\mathbb{R}^n) = \{v \in L^2(\mathbb{R}^n) \,:\, (1 + ||\xi||^{s/2} \hat{v}(\xi)) \in L^2(\mathbb{R}^n)\}.$$

Since that Ω is a bounded open set with sufficiently smooth boundary, we have $H^s(\Omega)$ is the set of restrictions on Ω of the functions $v \in H^s(\mathbb{R}^n)$, then

$$||v||_{H^s(\Omega)} = inf\{||V||_{H^s(\mathbb{R}^n)} \,:\, V = v \text{ a.e. in } \Omega\}$$

and

$$H_0^s(\Omega) = \overline{C_0^\infty(\Omega)}^{H^s(\Omega)}.$$

We need

$$W_0^{m,q}(\Omega) \hookrightarrow W_0^{m-k,q_k}(\Omega),\ \frac{1}{q_k} = \frac{1}{q} - \frac{k}{n}.$$

Choosing $q_k = p$, $m - k = 1$ and $q = 2$ we get

$$m = 1 + \frac{n}{2} - \frac{n}{p}.$$

For $s > m$ we have

$$H_0^s(\Omega) \hookrightarrow W_0^{1,p}(\Omega) \hookrightarrow H_0^1(\Omega) \hookrightarrow L^2(\Omega)$$

from where follows our goal. Now, from spectral theory the problem

$$((v_j, v))_{H_0^s(\Omega)} = \lambda_j(v_j, v),\ \text{for all } v \in H_0^s(\Omega)$$

has solution and moreover $\{v_j\}_{j\in\mathbb{N}}$ precisely, is a Schauder basis for $H_0^s(\Omega) \cap L^{r+1}(\Omega)$ with elements that are orthogonal in $L^2(\Omega)$.

3. Penalty method

The method consists in to consider a perturbation of the inequality (1.1) with adding singular term, called penalization, depending on a parameter $\varepsilon > 0$. We solve the mixed problem for that penalized equation that allow to pass to the limit when $\varepsilon \to 0$, in order to obtain a function which is the solution of our problem.

First of all, let us consider the penalty operators $\beta\colon L^2(\Omega) \to L^2(\Omega)$ associated to the closed convex sets K, see Lions ([13], p. 370). The operator β is monotonous, hemicontinuous, takes bounded sets of $L^2(\Omega)$ into bounded sets of $L^2(\Omega)$ and its kernel is

$$K = \{v \in L^2(\Omega); v \geq 0 \text{ a.e. in } \Omega\}$$

be a closed and convex subset of $L^2(\Omega)$ with $0 \in K$, and

$$\beta\colon L^2(0,T;L^2(\Omega)) \to L^2(0,T;L^2(\Omega))$$

is monotone and hemicontinuous.

The penalized problem associated with the variational inequality (1.1)-(1.4) consists in given $0 < \varepsilon < 1$ find u_ε solution in Q of the mixed problem

$$u_\varepsilon'' - \Delta_p u_\varepsilon + |u_\varepsilon|^{r-1}u_\varepsilon - \Delta u_\varepsilon' + \frac{1}{\varepsilon}\beta(u_\varepsilon') = 0, \qquad \text{in } Q, \tag{3.1}$$

$$u_\varepsilon(x,t) = 0, \qquad \text{in } \Sigma, \tag{3.2}$$

$$u_\varepsilon'(x,t) = 0, \qquad \text{in } \Sigma, \tag{3.3}$$

$$u_\varepsilon(x,0) = u_{\varepsilon_0}(x), \quad u_\varepsilon'(x,0) = u_{\varepsilon_1}(x), \qquad \text{in } \Omega. \tag{3.4}$$

Definition 3.1. We suppose $u_{\varepsilon_0} \in W_0^{1,p}(\Omega) \cap L^{r+1}(\Omega)$, $u_{\varepsilon_1} \in H_0^1(\Omega)$. A weak solution to the problem (3.1)-(3.4) is a function

$$u_\varepsilon \in L^\infty(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(0,T;L^{r+1}(\Omega)),$$

where

$$u_\varepsilon' \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega)),$$

and $u_\varepsilon'' \in L^q(0,T;W^{-1,q}(\Omega))$ such that for all $\varphi \in W_0^{1,p}(\Omega)$ in $\mathcal{D}'(0,T)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}(u_\varepsilon'(t),\varphi) - \langle \Delta_p u_\varepsilon(t),\varphi\rangle + (|u_\varepsilon(t)|^{r-1}u_\varepsilon(t),\varphi) + (\nabla u_\varepsilon'(t),\nabla\varphi) + \frac{1}{\varepsilon}(\beta(u_\varepsilon'(t)),\varphi) = 0,$$

with $u_\varepsilon(0) = u_{\varepsilon_0}, u_\varepsilon'(0) = u_{\varepsilon_1}$.

The solution of (3.1)-(3.4) is given by the following theorem.

Theorem 3.1. *Assume $u_{\varepsilon_0} \in W_0^{1,p}(\Omega) \cap L^{r+1}(\Omega)$, $u_{\varepsilon_1} \in H_0^1(\Omega)$. Then for each $0 < \varepsilon < 1$ there exists a weak solution of* (3.1)-(3.4).

Proof. Let s be an integer for which $H_0^s(\Omega) \hookrightarrow W_0^{1,p}(\Omega)$ is continuous. Then the eigenfunctions $-\Delta v_j = \lambda_j v_j$ in $H_0^s(\Omega)$ yields a Galerkin basis $(v_j)_{j=1}^\infty$ for both $W_0^{1,p}(\Omega)$ and $L^2(\Omega)$ with $(v_j)_{j=1}^\infty$ orthogonal in $L^2(\Omega)$.

Let $V_m = \operatorname{span}\{v_1,\dots,v_m\}$. Let us consider

$$u_{\varepsilon_m}(t) = \sum_{j=1}^m \xi_{\varepsilon_{jm}}(t) v_j$$

solution of the approximate problem

$$(u''_{\varepsilon_m}(t), w) + \langle -\Delta_p u_{\varepsilon_m}(t), w\rangle + (|u_{\varepsilon_m}(t)|^{r-1} u_{\varepsilon_m}(t), w)$$
$$+ (\nabla u'_{\varepsilon_m}(t), \nabla w) + \frac{1}{\varepsilon}(\beta(u'_{\varepsilon_m}(t)), w) = 0, \ \forall\, w \in V_m, \tag{3.5}$$
$$u_{\varepsilon_m}(0) = u_{\varepsilon 0m} \to u_{\varepsilon_0} \text{ strongly in } W_0^{1,p}(\Omega) \cap L^{r+1}(\Omega), \tag{3.6}$$
$$u'_{\varepsilon_m}(0) = u_{\varepsilon 1m} \to u_{\varepsilon_1} \text{ strongly in } H_0^1(\Omega). \tag{3.7}$$

Putting $w = v_i$, $i = 1, \ldots, m$, we observe that (3.5)-(3.7) is a system of ODEs in the variable t and has a local solution $u_{\varepsilon_m}(t)$ defined in $[0, t_m)$, $0 < t_m \leq T$. In the next step we obtain the a priori estimates for the solution $u_{\varepsilon_m}(t)$ so that it can be extended to the whole interval $[0, T]$.

3.1. A priori estimates

We consider $w = u'_{\varepsilon_m}(t)$ in (3.5) to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}|u'_{\varepsilon_m}(t)|_2^2 + \frac{1}{p}\|\nabla u_{\varepsilon_m}(t)\|_p^p + \frac{1}{r+1}|u_{\varepsilon_m}(t)|_{r+1}^{r+1}\right]$$
$$+ |\nabla u'_{\varepsilon_m}(t)|_2^2 + \ \frac{1}{\varepsilon}\left(\beta(u'_{\varepsilon_m}(t)), u'_{\varepsilon_m}(t)\right) = 0. \tag{3.8}$$

We have, $(\beta(u'_{\varepsilon_m}(t)), u'_{\varepsilon_m}(t)) \geq 0$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}|u'_{\varepsilon_m}(t)|_2^2 + \frac{1}{p}\|\nabla u_{\varepsilon_m}(t)\|_p^p + \frac{1}{r+1}|u_{\varepsilon_m}(t)|_{r+1}^{r+1}\right] + |\nabla u'_{\varepsilon_m}(t)|_2^2 \leq 0.$$

3.2. The approximate energy

$$E_{\varepsilon_m}(t) = \frac{1}{2}|u'_{\varepsilon_m}(t)|_2^2 + \frac{1}{p}\|\nabla u_{\varepsilon_m}(t)\|_p^p + \frac{1}{r+1}|u_{\varepsilon_m}(t)|_{r+1}^{r+1}$$

satisfies $\frac{\mathrm{d}}{\mathrm{d}t}E_{\varepsilon_m}(t) \leq -|\nabla u'_{\varepsilon_m}(t)|_2^2$.

Integrating from 0 to t, $t \leq t_m$, and using (3.6)-(3.7) we obtain

$$E_{\varepsilon_m}(t) + \int_0^t |\nabla u'_{\varepsilon_m}(s)|_2^2 \,\mathrm{d}s \leq E_{\varepsilon_m}(0) \leq C, \tag{3.9}$$

being C positive constant independent of m and t. Therefore,

$$u_{\varepsilon_m} \text{ is bounded in } L^\infty(0, T; W_0^{1,p}(\Omega)) \cap L^\infty(0, T; L^{r+1}(\Omega)), \tag{3.10}$$
$$u'_{\varepsilon_m} \text{ is bounded in } L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega)), \tag{3.11}$$
$$-\Delta_p u_{\varepsilon_m} \text{ is bounded in } L^\infty(0, T; W^{-1,q}(\Omega)), \tag{3.12}$$
$$|u_{\varepsilon_m}|^{r-1} u_{\varepsilon_m} \text{ is bounded in } L^{\frac{r+1}{r}}(0, T; L^{\frac{r+1}{r}}(\Omega)). \tag{3.13}$$

3.3. Passage to the limit

By the estimates (3.10)-(3.13) implies that the existence of subsequence of (u_{ε_m}), still denoted by (u_{ε_m}) such that

$$u_{\varepsilon_m} \overset{*}{\rightharpoonup} u_\varepsilon \text{ in } L^\infty(0, T; W_0^{1,p}(\Omega)) \cap L^\infty(0, T; L^{r+1}(\Omega)), \tag{3.14}$$

$$u'_{\varepsilon_m} \overset{*}{\rightharpoonup} u'_\varepsilon \text{ in } L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega)), \tag{3.15}$$

$$-\Delta_p u_{\varepsilon_m} \overset{*}{\rightharpoonup} \mathcal{X} \text{ in } L^\infty(0,T;W^{-1,q}(\Omega)), \tag{3.16}$$

$$|u_{\varepsilon_m}|^{r-1} u_{\varepsilon_m} \rightharpoonup \psi_2 \text{ weakly in } L^{\frac{r+1}{r}}(0,T;L^{\frac{r+1}{r}}(\Omega)). \tag{3.17}$$

For $x, y \in \mathbb{R}$ and $p \geq 2$, consider the elementary inequalities

$$\left||x|^{\frac{p-2}{2}} x - |y|^{\frac{p-2}{2}} y\right| \leq C\left(|x|^{\frac{p-2}{2}} + |y|^{\frac{p-2}{2}}\right)|x-y|, \tag{3.18}$$

$$\left||x|^{p-2} x - |y|^{p-2} y\right| \leq C\left(|x|^{\frac{p-2}{2}} + |y|^{\frac{p-2}{2}}\right)\left||x|^{\frac{p-2}{2}} x - |y|^{\frac{p-2}{2}} y\right|. \tag{3.19}$$

The inequality (3.18) is a consequence of the mean value theorem and (3.19) can be found in [7]. As in [19] applying (3.18), (3.19) and Hölder generalized inequality with

$$\frac{p-2}{4p} + \frac{p-2}{4p} + \frac{1}{2} + \frac{1}{p} = 1$$

we deduce for all $v \in W_0^{1,p}(\Omega)$

$$\left|\int_0^T \langle -\Delta_p u_m(t), v\rangle_p - \langle -\Delta_p u_\varepsilon(t), v\rangle_p \,\mathrm{d}t\right| \leq C\int_0^T |\nabla u_m(t) - \nabla u_\varepsilon(t)|_2 \,\mathrm{d}t. \tag{3.20}$$

Now we are going to obtain an estimate for $u''_{\varepsilon_m}(t)$. Since our Galerkin basis was taken in the Hilbert space $L^2(\Omega)$ we can use the standard projection arguments as described in Lions [13]. Then from the approximate equation and the estimates (3.10)-(3.13) we get

$$u''_{\varepsilon_m} \text{ is bounded in } L^\infty(0,T;W^{-1,q}(\Omega)). \tag{3.21}$$

Applying Aubin-Lions compactness lemma, see [13], we get respectively by (3.14), (3.15) and (3.21),

$$u_{\varepsilon_m} \to u_\varepsilon \text{ strongly in } L^2(0,T;L^2(\Omega)) \text{ and a.e. in } Q, \tag{3.22}$$

$$u'_{\varepsilon_m} \to u'_\varepsilon \text{ strongly in } L^2(0,T;L^2(\Omega)) \text{ and a.e. in } Q. \tag{3.23}$$

Using (3.22) we get that $u_m \to u$ almost everewhere in $\Omega \times (0,T)$ and by from (3.20) we have that,

$$-\Delta_p u_{\varepsilon_m} \overset{*}{\rightharpoonup} -\Delta_p u_\varepsilon \text{ in } L^\infty(0,T;W^{-1,q}(\Omega)). \tag{3.24}$$

From (3.16), (3.24) and uniqueness of the limit we conclude that $\mathcal{X} = -\Delta_p u_\varepsilon$. By statements (3.23), the continuity of β imply

$$\beta(u'_{\varepsilon_m}) \longrightarrow \beta(u'_\varepsilon) \text{ a.e. in } Q. \tag{3.25}$$

To prove that $\psi_2 = |u_\varepsilon|^{r-1} u_\varepsilon$ note that

$$\int_Q \left||u_{\varepsilon_m}(t)|^{r-1} u_{\varepsilon_m}(t)\right|^{\frac{r+1}{r}} \mathrm{d}x\,\mathrm{d}t = \int_Q |u_{\varepsilon_m}(t)|^{r+1} \mathrm{d}x\,\mathrm{d}t \leq C.$$

By (3.22) we have

$$|u_{\varepsilon_m}|^{r-1} u_{\varepsilon_m} \longrightarrow |u_\varepsilon|^{r-1} u_\varepsilon \text{ a.e. in } Q.$$

Therefore from [13] lemma 1.3, we infer that

$$|u_{\varepsilon_m}|^{r-1}u_{\varepsilon_m} \rightharpoonup |u_\varepsilon|^{r-1}u_\varepsilon \ \text{ weak in } \ L^{\frac{r+1}{r}}(0,T;L^{\frac{r+1}{r}}(\Omega)) \tag{3.26}$$

so have from (3.17) and (3.26) that $\psi_2 = |u_\varepsilon|^{r-1}u_\varepsilon$.

Now, with the convergence (3.14), (3.17), (3.25), (3.26) we can pass to the limit in the approximate equation (3.5) and we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(u'_\varepsilon(t),\varphi)+\langle-\Delta_p u_\varepsilon(t),\varphi\rangle+(|u_\varepsilon(t)|^{r-1}u_\varepsilon(t),\varphi)\\&+(\nabla u'_\varepsilon(t),\varphi)+\frac{1}{\varepsilon}(\beta(u'_\varepsilon(t)),\varphi)=0,\end{aligned} \tag{3.27}$$

for all $\varphi \in W_0^{1,p}(\Omega)$ in $\mathcal{D}'(0,T)$ at the sense of distributions. $\square$

4. Global weak solutions

Now we in position to present our principal result. The existence of global weak solutions to the mixed unilateral problem (1.1)-(1.4).

Theorem 4.1. *If $u_0 \in W_0^{1,p}(\Omega)\cap L^{r+1}(\Omega)$ and $u_1 \in H_0^1(\Omega)\cap K$ then there exists a function $u:\Omega\times(0,T)\to\mathbb{R}$ such that*

$$u \in L^\infty(0,T;W_0^{1,p}(\Omega))\cap L^\infty(0,T;L^{r+1}(\Omega)), \tag{4.1}$$

$$u' \in L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H_0^1(\Omega)), \tag{4.2}$$

$$u'' \in L^q(0,T;W^{-1,q}(\Omega)), \tag{4.3}$$

satisfying for all $v(t)\in K$ a.e. in $(0,T)$

$$\begin{aligned}&\int_0^T\langle-\Delta_p u(s),v(s)-u'(s)\rangle\,\mathrm{d}s+\int_0^T(\nabla u'(s),\nabla v(s)-\nabla u'(s))\,\mathrm{d}s\\&+\int_0^T(|u(s)|^{r-1}u(s),v(s)-u'(s))\,\mathrm{d}s+|u_1(x)|_2^2-|u'(x,T)|_2^2 \qquad (4.4)\\&+\ (u'(x,T),v(x,T))-(u_1(x),v(x,0))\geq 0,\end{aligned} \tag{4.5}$$

$$u(x,0)=u_0(x),\ \ u'(x,0)=u_1(x). \tag{4.6}$$

Proof. Let $v \in L^2(0,T;H_0^1(\Omega))$, $v(t)\in K$ a.e. for $t\in[0,T]$. From (3.27) it follows that

$$\begin{aligned}&\int_0^T\frac{\mathrm{d}}{\mathrm{d}s}(u'_\varepsilon(s),v(s)-u'_\varepsilon(s))\,\mathrm{d}s+\int_0^T\langle-\Delta_p u_\varepsilon(s),v(s)-u'_\varepsilon(s)\rangle\,\mathrm{d}s\\&+\int_0^T\left(|u_\varepsilon(s)|^{r-1}u_\varepsilon(s),v(s)-u_\varepsilon(s)\right)\mathrm{d}s+\int_0^T(\nabla u'_\varepsilon(s),\nabla v(s)-\nabla u'_\varepsilon(s))\,\mathrm{d}s\\&+\frac{1}{\varepsilon}\int_0^T(\beta(u'_\varepsilon(s)),v(s)-u'_\varepsilon(s))\,\mathrm{d}s=0,\end{aligned}$$

so,

$$
\begin{aligned}
&\int_0^T \langle -\Delta_p u_\varepsilon(s), v(s) - u'_\varepsilon(s)\rangle \,\mathrm{d}s + \int_0^T \big(|u_\varepsilon(s)|^{r-1}u_\varepsilon(s), v(s) - u_\varepsilon(s)\big)\,\mathrm{d}s \\
&+ \int_0^T (\nabla u'_\varepsilon(s), \nabla v(s) - \nabla u'_\varepsilon(s))\,\mathrm{d}s + (u'_\varepsilon(x,T), v(x,T) - u'_\varepsilon(x,T)) \\
&- (u_{\varepsilon_1}(x), v(x,0) - u_{\varepsilon_1}(x)) = \frac{1}{\varepsilon}\int_0^T (\beta(u'_\varepsilon(s)), u'_\varepsilon(s) - v(s))\,\mathrm{d}s \geq 0,
\end{aligned}
$$

because $v(t) \in K$ ($\beta(v) = 0$) and β is monotone.

Therefore

$$
\begin{aligned}
&\int_0^T \langle -\Delta_p u_\varepsilon(s), v(s) - u'_\varepsilon(s)\rangle \,\mathrm{d}s + \int_0^T (|u_\varepsilon(s)|^{r-1}u_\varepsilon(s), v(s) - u'_\varepsilon(s))\,\mathrm{d}s \\
&+ \int_0^T (\nabla u'_\varepsilon(s), \nabla v(s) - \nabla u'_\varepsilon(s))\,\mathrm{d}s + (u'_\varepsilon(x,T), v(x,T)) \\
&- |u'_\varepsilon(x,T)|_2^2 + |u_{\varepsilon_1}(x)|_2^2 - (u_{\varepsilon_1}(x), v(x,0)) \geq 0.
\end{aligned} \tag{4.7}
$$

From (3.17), (3.21), (3.22), (3.23) and Banach-Steinhauss Theorem, it follows that there exists a subsequence $(u_\varepsilon)_{0<\varepsilon<1}$ such that it converges to u as $\varepsilon \to 0$, in the sense of (3.14)-(3.23), that is,

$$u_\varepsilon \overset{*}{\rightharpoonup} u \quad \text{in} \quad L^\infty(0,T; W_0^{1,p}(\Omega)) \cap L^\infty(0,T; L^{r+1}(\Omega)), \tag{4.8}$$

$$u'_\varepsilon \overset{*}{\rightharpoonup} u' \quad \text{in} \quad L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega)), \tag{4.9}$$

$$u_\varepsilon \to u \text{ strongly in } L^2(0,T; L^2(\Omega)) \quad \text{and a.e. in} \quad Q, \tag{4.10}$$

$$u'_\varepsilon \to u' \text{ strongly in } L^2(0,T; L^2(\Omega)) \quad \text{and a.e. in} \quad Q. \tag{4.11}$$

By (4.11) we have in particular

$$u'_\varepsilon(x,T) \longrightarrow u'(x,T) \quad \text{weakly in} \quad L^2(\Omega). \tag{4.12}$$

The convergence above are sufficient to pass to the limit in (4.7) with $\varepsilon \to 0$ to conclude that (4.4) is valid.

To complete the proof of Theorem 4.1, it remains to show that $u'(t) \in K$ a.e. in $[0,T]$.

Integrating (3.8) from 0 to t we obtain

$$E_{\varepsilon_m}(t) + \int_0^t |\nabla u_{\varepsilon_m}(s)|_2^2\,\mathrm{d}s + \frac{1}{\varepsilon}\int_0^t \big(\beta(u'_{\varepsilon_m}(s)), u'_{\varepsilon_m}(s)\big)_2^2\,\mathrm{d}s = E_{\varepsilon_m}(0) \leq C,$$

$\forall\ m \geq m_0$ and $\forall\ t \in [0,T]$.

So,

$$0 \leq \frac{1}{\varepsilon}\int_0^t \big(\beta(u'_{\varepsilon_m}(s)), u'_{\varepsilon_m}(s)\big)_2^2\,\mathrm{d}s \leq C.$$

Taking $t = T$ we have

$$0 \leq \frac{1}{\varepsilon}\int_0^T \big(\beta(u'_{\varepsilon_m}(s)), u'_{\varepsilon_m}(s)\big)_2^2\,\mathrm{d}s \leq C.$$

By the properties of β according to Brezis [3] we obtain

$$\int_0^T \left|\beta(u'_{\varepsilon_m}(s))\right|_2^2 \mathrm{d}s \leq \int_0^T \left(\beta(u'_{\varepsilon_m}(s)), u'_{\varepsilon_m}(s)\right)_2^2 \mathrm{d}s \leq \varepsilon C.$$

Taking the limit when $m \to \infty$ we obtain

$$\lim_{m\to\infty} \int_0^T \left|\beta(u'_{\varepsilon_m}(s))\right|_2^2 \mathrm{d}s \leq \varepsilon C.$$

By (3.23) and continuity of β we have

$$\int_0^T \left|\beta(u'_{\varepsilon_m}(s))\right|_2^2 \mathrm{d}s \longrightarrow \int_0^T |\beta(u'_\varepsilon(s))|_2^2 \mathrm{d}s.$$

Thus,

$$\int_0^T |\beta(u'_\varepsilon(s))|_2^2 \mathrm{d}s \leq \varepsilon C. \tag{4.13}$$

Now, taking the limit when $\varepsilon \to 0$ in (4.13) we obtain,

$$\int_0^T |\beta(u'_\varepsilon(s))|_2^2 \mathrm{d}s \longrightarrow 0.$$

Thus, $\beta(u'_\varepsilon(t)) \to 0$ in $L^2(0,T;L^2(\Omega))$. By (4.11) and continuity of β we have

$$\beta(u'_\varepsilon) \longrightarrow \beta(u') \quad \text{in} \quad L^2(0,T;L^2(\Omega)). \tag{4.14}$$

By (4.13),(4.14) and the uniqueness of the limit we have $\beta(u'(t)) = 0$. So, $u'(t) \in K$ a.e. in Ω and the proof of existence of solution is complete. $\square$

Acknowledgments. The authors would like to thank the anonymous referees for the careful reading and suggestions that improved this manuscript.

References

[1] P. Ballard and S. Basseville, *Existence and uniqueness for dynamical unilateral contact with Coulomb friction: a model problem*, ESAIM - Math. Model. Num., 2005, 39, 59–77.

[2] L. Bociu and I. Lasiecka, *Uniqueness of weak solutions for the semilinear wave equations with supercritical boundary/interior sources and damping*, Discrete Contin. Dyn. Syst., 2008, 22, 835–860.

[3] H. Brézis, *Analyse fonctionnelle: théorie et applications*, Dunod, 1999.

[4] A. D. D. Cavalcanti, M. M. Cavalcanti, L. H. Fatori and M. A. Jorge Silva, *Unilateral problems for the wave equation with degenerate a nonlocalized non-linear damping: well-posedness and non-stability results*, Math. Nachr., 2018, 291, 1–24.

[5] I. Chueshov and I. Lasiecka, *Existence, uniqueness of weak solution and global attractors for a class of nonlinear 2D Kirchhoff-Boussinesq models*, Discrete Contin. Dyn. Syst., 2006, 15, 777–809.

[6] M. R. Clark and O. A. Lima, *Existence of solutions for a variational unilateral system*, Electron. J. Differential Equations, 2002, 22, 1–18.

[7] A. Domokos and J. J. Manfredi, *A second order differentiability technique of Bojarski-Iwaniec in the Heisenberg group*, Functiones et Approximation, 2009, 40, 69–74.

[8] M. Dreher, *The wave equation for the p-Laplacian*, Hokkaido Math. J., 2007, 36, 21–52.

[9] H. Gao and T. Ma, *Global solutions for a nonlinear wave equation with the p–Laplacian operator*, Electron. J. Qual. Theory Differ. Equ., 1999, 11, 1–13.

[10] J. M. Greenberg, R. C. MacCamy and V. J. Vizel, *On the existence, uniqueness, and stability of solution of the equation* $\sigma^{'}(u_x)u_{xx} + \lambda u_{xtx} = \rho_0 u_{tt}$, J. Math. Mech., 1968, 17, 707–728.

[11] N. Kikuchi and J. T. Oden, *Contact problems in elasticity: A study of variational inequalities and finite element methods*, SIAM - Studies in Applied and Numerical Mathematics, Philadelphia, 1988.

[12] N. A. Lar'kin and L. A. Medeiros, *On a variational inequality for the equations of the theory of the elasticity, well posed boundary value problem for non classical equation,* Novosibrisk, Moscow, 1991, 6–11.

[13] J. L. Lions, *Quelques methódes de resolution des probléms aux limites non lineáires,* Dunod, Paris, 1969.

[14] T. Ma and J. A. Soriano, *On weak solutions for an evolution equation with exponential nonlinearities*, Nonlinear Anal., 1999, 37, 1029–1038.

[15] L. A. Medeiros and M. M. Milla, *Local solutions for a nonlinear unilateral problem*, Rev. Roumaine Math. Pures Appl., 1986, XXXI, 371–382.

[16] P. Pei, M. A. Rammaha and D. Toundykov, *Weak solutions and blow-up for wave equations of p-Laplacian type with supercritical sources*, J. Math. Phys., 56, 2015, Article ID 081503.

[17] D. C. Pereira, H. Hguyen, C. A. Raposo and C. H. M. Maranhão, *On the solutions for an extensible beam equation with internal damping and source terms*, Differ. Equ. Appl., 2019, 11, 367–377.

[18] C. A. Raposo, D. C. Pereira, G. Araujo and A. Baena, *Unilateral problems for the klein-gordon operator with nonlinearity of kirchhoff-carrier type*, Electron. J. Differential Equations, 2015, 137, 1–14.

[19] C. A. Raposo, J. O. Ribeiro and A. P. Cattai, *Global solution for a thermoelastic system with p-Laplacian*, Appl. Math. Lett., 2018, 86, 119–125.

[20] J. E. M. Rivera and H. P. Oquendo, *Exponential stability to a contact problem of partially viscoelastic materials*, J. Elasticity, 2001, 63, 87–111.

[21] Y. Ye, *Global existence and asymptotic behavior of solutions for a class of nonlinear degenerate wave equations*, Int. J. Difference Equ., 2007, Article ID 19685.

[22] Y. Zhijian, *Global existence, asymptotic behavior and blowup of solutions for a class of nonlinear wave equations with dissipative term*, J. Differential Equations, 2003, 187, 520–540.