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GROUND STATES FOR A FRACTIONAL REACTION-DIFFUSION SYSTEM

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Abstract In this paper, we prove the existence of the ground state of a strongly indefinite fractional reaction-diffusion system based on the Non-Nehari method established by Tang-Chen-Lin-Yu [J. Differ. Equ., 2020(268), 4663–4690]. In particular, neither any monotonicity condition nor any Ambrosetti-Rabinowitz growth condition is required. To our knowledge, this is the first result about the ground states with the strongly indefinite case for fractional reaction-diffusion system.

Keywords Reaction-diffusion system, ground states, strongly indefinite functional.

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1. Introduction

The present paper addresses the ground state of the following fractional reaction-diffusion system

$$\begin{cases} \partial_t u + (-\Delta)^s u + V(x)u = F_v(t,x,u,v), \\ -\partial_t v + (-\Delta)^s v + V(x)v = F_u(t,x,u,v), \end{cases} \tag{1.1}$$

where $0 < s < 1, z = (u,v) : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^M \times \mathbb{R}^M$ and the fractional Laplacian operator $(-\Delta)^s$ is defined by

$$(-\Delta)^s w = C(N,s) P.V. \int_{\mathbb{R}^N} \frac{w(t,x) - w(t,y)}{|x-y|^{N+2s}} dy,$$

here $P.V.$ means the Cauchy principle value on the integral and $C(N,s)$ represents a normalizing constant which depends upon N and s, precisely given by

$$C(n,s) = \left(\int_{\mathbb{R}^N} \frac{1 - cos(\zeta_1)}{|\zeta|^{n+2s}} d\zeta \right)^{-1} = 2^{2s} \pi^{-\frac{n}{2}} \frac{\Gamma(\frac{n}{2} + s)}{|\Gamma(-s)|}.$$

Fractional diffusion problem arises in optimal control of systems governed by fractional partial differential equations [16]. One of the most interesting and physically important features of space fractional diffusion is well known as anomalous diffusion [19]. In particular, there has been tremendous interest in developing the

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fractional Laplace problem in various fields, for instance, phase transitions, stratified materials, thin obstacle problems, anomalous diffusion, crystal dislocation, conservation laws etc. Quite different from the classical Laplace operator, the usual analytical methods for elliptic PDEs cannot be directly applied to (1.1) since the operator $(-\Delta)^s$ is nonlocal. In the past two decades, a great deal of mathematical effort has been devoted to the study of solutions of reaction-diffusion system. There have been extensive work in the case of bounded domain, systems like or analogous to (1.1) with $s = 1$ were studied by a number of authors, see [2, 5–8, 10, 11, 15, 17, 20, 22, 31]. Especially, variational methods and other topological methods have attracted considerable attention in the existence of solutions for reaction-diffusion system, see [1–11, 14, 18, 24–30].

Recently, Ding and Guo [12] considered (1.1) and obtained the homoclinic solution by using strongly indefinite theory which is established by Ding [9]. However, another question arises: whether the result [12] on the existence of ground state solution, i.e., a nontrivial solution for (1.1) with the minimal energy for (1.1) can be obtained? Answering this question constitutes the goal of this paper.

As a motivation, we recall a notable work of Szulkin and Weth [20], they developed a powerful approach to find ground state solutions for strongly indefinite periodic Schrödinger equation under a strict monotone condition, which plays an important role in generalized Nehari manifold, but this method is invalid without monotone assumption. Recently, completely different from the one of Szulkin and Weth [20], Tang [21–23] developed a new approach which is called Non-Nehari method to find ground state solution of Nehari-Pankov type for Schrödinger equation, it is a powerful tool to resolve the ground state solution of strongly indefinite problem. However, to the best of our knowledge, there is no result about ground states for fractional reaction-diffusion system (1.1), motivated by the papers [4,12,21–23,29], we will continue to study the existence of ground state solutions of problem (1.1) under some weaker conditions by means of Non-Nehari method. Now, we are in a position to state our assumptions for problem (1.1):

Assume that V and F satisfy the following conditions:

(V) $V \in C(\mathbb{R}^N, \mathbb{R})$ is T_i-periodic in x_i for $i = 1, ..., N$ and $a := \min_{x\in\mathbb{R}^N} V(x) > 0$;

(F1) $F \in C^1(\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^{2M}, \mathbb{R}^+)$ is T_0-periodic in t, T_i-periodic in x_i for $i = 1, ..., N$ and there exists a constant $C > 0$ such that

$$|F_z(t, x, z)| \le C(1 + |z|^{p-1}), \quad \forall\ (t, x, z) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^{2M},$$

where $p \in (2, N^*)$, and

$$N^* = \begin{cases} \dfrac{4Ns + 2N}{2Ns + N - 2s}, & \text{if } N > \max\{2, 4s\} \\ \dfrac{2N + 8s}{N + 2s}, & \text{if } 4s < N \le 2\ (0 < s < \dfrac{1}{2}) \\ \dfrac{2(N + 2)}{N + 1}, & \text{if } 2 < N \le 4s\ (\dfrac{1}{2} < s < 1) \\ \dfrac{8}{3}, & \text{if } N \le \min\{2, 4s\} \end{cases} \tag{1.2}$$

(F2) $F_z(t, x, z) = o(|z|)$ as $|z| \to 0$ uniformly in (t, x);

(F3) $\lim_{|z|\to\infty} \frac{|F(t,x,z)|}{|z|^2} = \infty$ uniformly in (t, x);

(F4) For all $\kappa \geq 0, z, \zeta \in \mathbb{R}^{2M}$,

$$F(t,x,\kappa z+\zeta)-F(t,x,z)+\frac{1-\kappa^2}{2}F_z(t,x,z)\cdot z-\kappa F_z(t,x,z)\cdot\zeta \geq 0.$$

The present paper is organized as follows. The variational form for $\mathcal{I}$ and some preliminaries are introduced in Section 2. Furthermore, our main results are obtained. The proofs of our theorem are given in Section 3.

2. The variational setting and main results

Before giving our main result, we shall introduce some notation and definitions.

Throughout this paper, we always assume that hypotheses of (V), (F1)-(F4) hold. $|\cdot|_q$ stands for the usual L^q-norm, $(\cdot,\cdot)_2$ denotes the usual L^2 inner product, c, c_i or C_i stand for different positive constants. Setting

$$\mathcal{J}=\begin{pmatrix}0 & -I\\ I & 0\end{pmatrix},\quad \mathcal{J}_0=\begin{pmatrix}0 & I\\ I & 0\end{pmatrix},\quad \mathcal{S}=(-\Delta)^s+V,\ A_0:=\mathcal{J}_0\mathcal{S}.$$

Denote $A=\mathcal{J}\partial_t z+A_0$, then (1.1) can be rewritten as follows

$$Az=\mathcal{J}\partial_t z+A_0 z=H_z(t,x,z),\quad z=(u,v).$$

Note that the time-dependent Besov space is defined as follows

$$B_r^s \doteq B_r(\mathbb{R}\times\mathbb{R}^N,\mathbb{R}^{2M}) \doteq W^{1,r}(\mathbb{R},L^r(\mathbb{R}^N,\mathbb{R}^{2M}))\cap L^r(\mathbb{R},W^{2s,r}(\mathbb{R}^N,\mathbb{R}^{2M})).$$

When $0<s<1/2$, we have the equivalent norm as

$$\|z\|_{B_r^s}=\left(\int_{\mathbb{R}}\left(\int_{\mathbb{R}^N}|z|^r+|\partial_t z|^r+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|z(t,x)-z(t,y)|^r}{|x-y|^{n+2sr}}dxdy\right)dt\right)^{\frac{1}{r}}.$$

When $1/2<s<1$, we have the equivalent norm as

$$\|z\|_{B_r^s}=\left(|z|_r^r+|\partial_t z|_r^r+|Dz|_r^r+\int_{\mathbb{R}}\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|Dz(t,x)-Dz(t,y)|^r}{|x-y|^{n+\{2s\}r}}dxdydt\right)^{\frac{1}{r}},$$

where Dz is the derivative of z with respect to x, $2s=[2s]+\{2s\}$ with $[2s]$ integer and $0<\{2s\}=2s-1<1$.

For the case of $s=\frac{1}{2}$, it can be shown that $B_r^{1/2}=W^{1,r}(\mathbb{R}\times\mathbb{R}^N,\mathbb{R}^{2M})$ with the norm

$$\|z\|_{B_r^{1/2}}=(|z|_r^r+|\partial_t z|_r^r+|Dz|_r^r)=\|z\|_{W^{1,r}(\mathbb{R}\times\mathbb{R}^N,\mathbb{R}^{2M})}.$$

B_r^s is the completion space of $C_0^\infty(\mathbb{R}\times\mathbb{R}^N,\mathbb{R}^{2M}))$ with the norm $\|\cdot\|_{B_r^s}$. In particular, B_2^s is a Hilbert space.

Let $|A|$ be the absolute of A and $|A|^{\frac{1}{2}}$ be the square root of $|A|$. Denote $E=\mathcal{D}(|A|^{1/2})$, it is easy to verify that E is a Hilbert space equipped with the inner product $\langle z,w\rangle_E=\langle |A|^{1/2}z,|A|^{1/2}w\rangle_{L^2}$ and the norm $\|z\|_E=\langle z,z\rangle_E^{1/2}$. It is well-known that $E=\mathcal{D}(|A|^{1/2})\cong(\mathcal{D}(A),L^2)_{[1/2]}\cong(B_2^s,L^2)_{[1/2]}$, where $(\cdot,\cdot)_{[\frac{1}{2}]}$ is the complex interpolation space of exponent $1/2$. The following lemmas comes from [12].

Lemma 2.1. *If* $0 < s < 1$ *and (V) is satisfied, then for any* $z \in B_2^s$ *we have* $c_1\|z\|^2_{B_2^s} \leq |Az|_2^2 \leq c_2\|z\|^2_{B_2^s}$.

Lemma 2.2. E *is continuously embedded in* $L^q(\mathbb{R}\times\mathbb{R}^N, \mathbb{R}^{2M})$ *and compactly embedded in* $L^{q'}_{loc}(\mathbb{R}\times\mathbb{R}^N, \mathbb{R}^{2M})$, *where* $2 \leq q \leq N^*, 2 \leq q' < N^*$, N^* *is defined in (1.2).*

Lemma 2.3. *Let (V) be satisfied, then*

(1) $\sigma(A) = \sigma_e(A)$, *i.e., A has only essential spectrum;*

(2) $\sigma(A) \subset \mathbb{R}\setminus(-a, a)$, $\sigma(A)$ *is symmetric with respect to* 0, *that is,* $\sigma(A)\bigcap(-\infty, 0) = -\sigma(A)\bigcap(0, \infty)$.

Lemma 2.3 yields that L^2 possesses the orthogonal decomposition $L^2 = L^- \oplus L^+$, $z = z^- + z^+$, $z^\mp \in L^\mp$ such that A is negative definite in L^- (resp. L^+).

Based on the above agrument, we have $E = E^+ \oplus E^-$, where $E^\pm = E \cap L^\pm$. We define the following energy functional of (1.1) on E

$$\mathcal{I}(z) = \frac{1}{2}(\|z^+\|^2 - \|z^-\|^2) - \mathcal{F}(z), \tag{2.1}$$

where $\mathcal{F}(z) = \int_{\mathbb{R}\times\mathbb{R}^N} F(t, x, z)$. Clearly, $\mathcal{I}$ is strongly indefinite, our hypotheses imply that $\mathcal{I} \in C^1(E, \mathbb{R})$. Moreover,

$$\langle \mathcal{I}'(z), z\rangle = (\|z^+\|^2 - \|z^-\|^2) - \langle \mathcal{F}'(z), z\rangle, \quad z = z^- + z^+ \in E = E^- \oplus E^+, \tag{2.2}$$

standard argument shows that critical points of $\mathcal{I}$ are solutions of (1.1).

If $z_0 = (u_0, v_0) \in E$ is a nontrivial solution of problem (1.1), then $z_0 \in \mathcal{N}^-$, where

$$\mathcal{N}^- := \{z \in E\setminus E^- : \langle \mathcal{I}'(z), z\rangle = \langle \mathcal{I}'(z), w\rangle = 0, \ \forall\, w \in E^-\}, \tag{2.3}$$

the set $\mathcal{N}^-$ of (2.3) was first introduced by Pankov [17], which is a natural constraint and contains nontrivial critical points of $\mathcal{I}$, The purpose of the present paper is to seek a solution z_0 for (1.1) that satisfies $\mathcal{I}(z_0) = \inf_{\mathcal{N}^-}\mathcal{I}(z)$ under some suitable assumptions.

Our main result are the following:

Theorem 2.1. *Assume that* $V(x)$ *and* $F(t, x, z)$ *satisfy the basic assumptions (V) and (F1)-(F4) respectively, then system (1.1) has at least a solution* $z_0 \in E$ *such that* $\mathcal{I}(z_0) = \inf_{\mathcal{N}^-}\mathcal{I} > 0$.

Remark 2.2. It is easy to verify that the following functions

$$F(t, x, u, v) = (|u|^2 + u\cdot v + |v|^2)\ln(1 + |u|^2 + u\cdot v + |v|^2)$$
$$F(t, x, u, v) = |u + 2v|^{\sigma_1} + |3u + 2v|^{\sigma_2}, \quad \sigma_1, \sigma_2 \in (2, N^*)$$

satisfy (F1)-(F4).

Remark 2.3. Theorem 2.1 can be thought as an extension of the results in [12], we remark that in our assumptions neither any monotonicity condition nor any Ambrosetti-Rabinowitz growth condition is required, and we need a new method different from those used in [20] to overcome the difficulty in studying the least energy solution.

3. Proof of the main result

Let W be a real Hilbert space with $W = W^- \oplus W^+$ and $W^- \perp W^+$. For a functional $\psi \in C^1(W,\mathbb{R})$, ψ is said to be weakly sequentially lower semi-continuous if for any $u_n \rightharpoonup u$ in W one has $\psi(u) \le \liminf_{n\to\infty} \psi(u_n)$, and ψ' is said to be weakly sequentially continuous if $\lim_{n\to\infty}\langle \psi'(u_n), v\rangle = \langle \psi'(u_n), v\rangle$ for each $v \in W$.

Lemma 3.1([15]). *Let W be a a real Hilbert space, $W = W^- \oplus W^+$ and $W^- \perp W^+$, and $\psi \in C^1(X,\mathbb{R})$ of the form*

$$\psi(u) = \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2) - \psi(u), \quad u = u^- + u^+ \in W^- \oplus W^+.$$

Suppose that the following assumptions hold:

(A1) $\psi \in C^1(W,\mathbb{R})$ is bounded from below and weakly sequentially lower semi-continuous;

(A2) ψ' is weakly sequentially continuous;

(A3) there exist $r > \rho > 0$, $e \in W^+$ with $\|e\| = 1$ such that

$$\kappa := \inf \psi(S_\rho^+) > \sup \varphi(\partial Q),$$

where

$$S_\rho^+ = \{u \in X^+ : \|u\| = \rho\}, \quad Q = \{v + se : v \in X^-, s \ge 0, \|v + se\| \le r\}.$$

Then for some $c \in [\kappa, \sup \varphi(Q)]$, there exists a sequence $\{u_n\} \subset W$ satisfying

$$\psi(u_n) \to c, \quad \|\psi'(u_n)\|(1 + \|u_n\|) \to 0.$$

Employing a standard argument, one can easily derive the following lemma.

Lemma 3.2. *Assume that (V), (F1)-(F4) are satisfied. Then $\mathcal{F}$ is nonnegative, weakly sequentially lower semicontinuous, and $\mathcal{F}'$ is weakly sequentially continuous.*

Lemma 3.3. *Assume that (V), (F1)-(F4) are satisfied. Then for all $\kappa \ge 0, z \in E, \zeta \in E^-$,*

$$\mathcal{I}(z) \ge \mathcal{I}(\kappa z + \zeta) + \frac{1}{2}\|\zeta\|^2 + \frac{1-\kappa^2}{2}\langle \mathcal{I}'(z), z\rangle - \kappa\langle \mathcal{I}'(z), \zeta\rangle. \tag{3.1}$$

Proof. From (2.1), (2.2) and (F4) we have

$$\begin{aligned}
\mathcal{I}(z) - \mathcal{I}(\kappa z + \zeta) &= \frac{1}{2}\|\zeta\|^2 + \frac{1-\kappa^2}{2}(\|z^+\|^2 - \|z^-\|^2) + \kappa(z,\zeta) \\
&\quad - \int_{\mathbb{R}\times\mathbb{R}^N} [F(t,x,z) - F(t,x,\kappa z + \zeta)] \\
&= \frac{1}{2}\|\zeta\|^2 + \frac{1-\kappa^2}{2}\langle \mathcal{I}'(z), z\rangle - \kappa\langle \mathcal{I}'(z), \zeta\rangle \\
&\quad + \int_{\mathbb{R}\times\mathbb{R}^N} \frac{1-\kappa^2}{2} F_z(t,x,z)\cdot z - \kappa F_z(t,x,z)\cdot\zeta \\
&\quad + \int_{\mathbb{R}\times\mathbb{R}^N} F(t,x,\kappa z + \zeta) - F(t,x,z)
\end{aligned}$$

$$\geq \frac{1}{2}\|\zeta\|^2+\frac{1-\kappa^2}{2}\langle \mathcal{I}'(z),z\rangle-\kappa\langle \mathcal{I}'(z),\zeta\rangle,\ \forall\,\kappa\geq 0, z\in E,\ \zeta\in E^-.$$

□

Using Lemma 3.3, some important corollaries are given as follows, the proof will be omitted.

Corollary 3.4. *Assume that (V), (F1)-(F4) are satisfied. Then for $z\in\mathcal{N}^-$, we have*

$$\mathcal{I}(z)\geq \mathcal{I}(\kappa z+\zeta),\quad \forall\ \kappa\geq 0,\ \zeta\in E^-.$$

Corollary 3.5. *Assume that (V), (F1)-(F4) are satisfied. Then for all $z\in E,\theta\geq 0$,*

$$\mathcal{I}(z)\geq \frac{\kappa^2}{2}\|z\|^2+\frac{1-\kappa^2}{2}\langle \mathcal{I}'(z),z\rangle+\kappa^2\langle \mathcal{I}'(z),z^-\rangle-\int_{\mathbb{R}\times\mathbb{R}^N}F(t,x,\kappa z^+). \quad (3.2)$$

Lemma 3.6. *Assume that (V), (F1)-(F4) are satisfied. Then*

(i) there exists $\rho>0$ such that

$$m:=\inf_{\mathcal{N}^-}\mathcal{I}\geq \Lambda:=\inf\{\mathcal{I}(z):z\in E^+,\ \|z\|=\rho\}>0;$$

(ii) $\|z^+\|\geq \max\{\|z^-\|,\sqrt{2m}\}$ for all $z\in\mathcal{N}^-$.

Proof. It follows from (F1) and (F2) that there exists a constant C_ε such that

$$|F(t,x,z)|\leq \varepsilon|z|^2+C_\varepsilon|z|^p,\quad p\in(2,N^*),\ \forall\ (t,x,z)\in\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^{2M},$$

From Corollary 3.4, we have for $z\in\mathcal{N}^-$

$$\begin{aligned}\mathcal{I}(z)&\geq \mathcal{I}(\kappa z^+)\\&\geq \frac{\kappa^2}{2}\|z^+\|^2-\int_{\mathbb{R}\times\mathbb{R}^N}F(t,x,\kappa z^+)\\&\geq \frac{\kappa^2}{2}\|z^+\|^2-\kappa^2\varepsilon\|z^+\|^2-\kappa^pC_\varepsilon\|z^+\|_p^p\\&\geq \frac{\kappa^2}{2}(1-2\varepsilon)\|z^+\|^2-\kappa^p\gamma_p^pC_\varepsilon\|z^+\|^p>0,\quad \text{for small } \kappa>0.\end{aligned}$$

This shows that there exists a $\rho>0$ such that (i) holds.

By (A1), $\mathcal{F}(z)>0$ for all $(t,x,z)\in\mathbb{Z}\times\mathbb{R}^N\times\mathbb{R}^{2M}$, so we have for $z\in\mathcal{N}^-$

$$m\leq \frac{1}{2}\|z^+\|^2-\frac{1}{2}\|z^-\|^2-\mathcal{F}(z)\leq \frac{1}{2}\|z^+\|^2-\frac{1}{2}\|z^-\|^2\leq \frac{1}{2}\|z^+\|^2,$$

which implies that $\|z^+\|\geq \max\{\|z^-\|,\sqrt{2m}\}$. □

With the help of the preceding two corollaries, an argument similar to the one used in [21] shows that we can now prove the following lemma in the same way as [21].

Lemma 3.7. *Assume that (V), (F1)-(F4) are satisfied. Then for every $e\in E^+,\sup\mathcal{I}(E^-\oplus\mathbb{R}^+e)<\infty$ and there exists $R_e>0$ such that*

$$\mathcal{I}(z)\leq 0,\quad \forall\ z\in E^-\oplus\mathbb{R}^+e,\ \|z\|\geq R_e.$$

Corollary 3.8. *Assume that (V), (F1)-(F4) are satisfied. Let* $e \in E^+$ *with* $\|e\| = 1$. *Then there exists* $r_0 > \rho$ *such that* $\sup \mathcal{I}(\partial Q) \le 0$ *as* $r \ge r_0$, *where*

$$Q = \{\zeta + se : \zeta \in E^-,\ s \ge 0,\ \|\zeta + se\| \le r\}. \tag{3.3}$$

Lemma 3.9. *Assume that (V), (F1)-(F4) are satisfied. Then there exists a constant* $c \in [\Lambda, \sup \mathcal{I}(Q)]$ *and a sequence* $\{z_n\} \subset E$ *satisfying*

$$\mathcal{I}(z_n) \to c, \quad \|\mathcal{I}'(z_n)\|(1 + \|z_n\|) \to 0,$$

where Q *is defined in (3.3).*

Proof. Combining with Lemma 3.1, Lemma 3.2, Lemma 3.6 and Corollary 3.8, it is easy to verify Lemma 3.9. The proof will be omitted. □

Lemma 3.10. *Assume that (V), (F1)-(F4) are satisfied. Then there exists a constant* $c_* \in [\kappa, m]$ *and a sequence* $\{z_n\} = \{(u_n, v_n)\} \subset E$ *satisfying*

$$\mathcal{I}(z_n) \to c_*, \quad \|\mathcal{I}'(z_n)\|(1 + \|z_n\|) \to 0. \tag{3.4}$$

Proof. This is a standard result which can be found in [21,22], for the convenience of readers, we give the detailed proof process here. Choose $\xi_k \in \mathcal{N}^-$ such that

$$m \le \mathcal{I}(\xi_k) < m + \frac{1}{k}, \quad k \in \mathbb{N}. \tag{3.5}$$

Using Lemma 3.6, we can derive $\|\xi_k^+\| \ge \sqrt{2m} > 0$. Let $e_k = \xi_k / \|\xi_k\|$, then $e_k \in E^+$ with $\|e_k\| = 1$. Applying Lemma 3.8, there exists a constant $r_k > \max\{\rho, \|\xi_k\|\}$ satisfying $\sup \mathcal{I}(\partial Q_k) \le 0$, where

$$Q_k = \{\zeta + se_k : \zeta \in E^-, s \ge 0, \|\zeta + se_k\| \le r_k\},\ k \in \mathbb{N}. \tag{3.6}$$

Then, by lemma 3.9, there exist a constant $c_k \in [\kappa, \sup \mathcal{I}(Q_k)]$ and a sequence $\{z_{k,n}\}_{n \in \mathbb{N}} \subset E$

$$\mathcal{I}(z_{k,n}) \to c_k, \quad \|\mathcal{I}'(z_{k,n})\|(1 + \|z_{k,n}\|) \to 0,\ k \in \mathbb{N}. \tag{3.7}$$

In virtue of Corollary 3.4, we get

$$\mathcal{I}(\xi_k) \ge \mathcal{I}(\eta \xi_k + \zeta), \quad \forall\ \eta \ge 0,\ \zeta \in E^-. \tag{3.8}$$

Since $\xi_k \in Q_k$, then by (3.6) and (3.8) we have $\mathcal{I}(\xi_k) = \sup \mathcal{I}(Q_k)$. Furthermore, by (3.5) and (3.7), we have

$$\mathcal{I}(z_{k,n}) \to c_k < m + \frac{1}{k}, \quad \|\mathcal{I}'(z_{k,n})\|(1 + \|z_{k,n}\|) \to 0,\ k \in \mathbb{N}.$$

We can choose $\{n_k\} \subset \mathbb{N}$ such that

$$\mathcal{I}(z_{k,n_k}) < m + \frac{1}{k}, \quad \|\mathcal{I}'(z_{k,n_k})\|(1 + \|z_{k,n_k}\|) < \frac{1}{k},\ k \in \mathbb{N}.$$

Set $z_k = z_{k,n_k}, k \in \mathbb{N}$, then we have

$$\mathcal{I}(z_n) \to c_* \in [\kappa, m], \quad \|\mathcal{I}'(z_n)\|(1 + \|z_n\|) \to 0.$$ □

Lemma 3.11. *Assume that (V), (F1)-(F4) are satisfied. Then for any* $z \in E \setminus E^-$ $\mathcal{N}^- \cap (E^- \oplus \mathbb{R}^+ z) \neq \emptyset$*, there exist* $\eta(z) > 0, \zeta(z) \in E^-$ *such that* $\eta(z)z + \zeta(z) \in \mathcal{N}^-$.

Proof. Note that $E^- \oplus \mathbb{R}^+ z = E^- \oplus \mathbb{R}^+ z^+$, then we may assume that $z \in E^+$. It follows from Lemma 3.7 that there exists a constant $R > 0$ such that $\mathcal{I}(z) \leq 0$ for any $z \in (E^- \oplus \mathbb{R}^+ z) \setminus B_R(0)$. For sufficiently small $s \geq 0$, we have $\mathcal{I}(sz) > 0$. Thus, $0 < \sup \mathcal{I}(E^- \oplus \mathbb{R}^+ z) < \infty$. It is easy to show that $\mathcal{I}$ is weakly continue on $E^- \oplus \mathbb{R}^+ z$, then for some $z_0 \in E^- \oplus \mathbb{R}^+ z$ we have $\mathcal{I}(z_0) = \sup \mathcal{I}(E^- \oplus \mathbb{R}^+ z)$. So z_0 is a critical point of $\mathcal{I}|_{E^- \oplus \mathbb{R} z}$. Moreover, $\langle \mathcal{I}'(z_0, z_0) \rangle = \langle \mathcal{I}'(z_0, \zeta) \rangle$, $\forall\, \zeta \in E^- \oplus \mathbb{R} z$. From the above discussion, we can derive that $z_0 \in \mathcal{N}^- \cap (E^- \oplus \mathbb{R}^+ z)$. □

Lemma 3.12. *Assume that (V), (F1)-(F4) are satisfied. Then for any* $\{z_n\} \subset E$ *such that*

$$\mathcal{I}(z_n) \to c \geq 0, \quad \langle \mathcal{I}'(z_n), z_n \rangle \to 0, \quad \langle \mathcal{I}'(z_n), z_n^- \rangle \to 0 \tag{3.9}$$

is bounded in E.

Proof. We prove the boundedness of $\{z_n\}$ by negation, if the assertion would not hold, then $\|z_n\| \to \infty$. Denote $\omega_n = z_n / \|z_n\|$, we have $\|\omega_n\| = 1$. Taking into account Sobolev embedding theorem, there exists a constant $C_1 > 0$ such that $\|\omega_n\|_2 \leq C_1$. If

$$\delta := \limsup_{n \to \infty} \sup_{y \in \mathbb{R}^{N+1}} \int_{B_1(y)} |\omega_n^+|^2 dx = 0,$$

it is easy to verify that $\omega_n^+ \to 0$ in $L^p(p \in (2, N^*))$ by using Lions' concentration compactness principle. Fix $R > [2(1+c)]^{1/2}$, combining (F1) with (F2), we see that there exists a constant $C_\varepsilon > 0$ such that

$$F(t, x, z) \leq \varepsilon |z|^2 + C_\varepsilon |z|^p$$

for $\varepsilon = 1/4(RC_1)^2 > 0$, where $(t, x, z) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^{2M}$. Hence, we have

$$\begin{aligned}
&\limsup_{n \to \infty} \int_{\mathbb{R} \times \mathbb{R}^N} F(t, x, R z_n^+ / \|z_n\|) \\
&= \limsup_{n \to \infty} \int_{\mathbb{R} \times \mathbb{R}^N} F(t, x, R \omega_n^+) \\
&\leq \limsup_{n \to \infty} R^2 \varepsilon \int_{\mathbb{R} \times \mathbb{R}^N} |\omega_n^+|^2 + \limsup_{n \to \infty} R^p C_\varepsilon \int_{\mathbb{R} \times \mathbb{R}^N} |\omega_n^+|^2 \\
&\leq \varepsilon (RC_1)^2 = \frac{1}{4}.
\end{aligned} \tag{3.10}$$

Set

$$\eta_n = R / \|z_n\|,$$

combining Lemma 3.4 with Lemma 3.11, we have, in light of (3.10)

$$\begin{aligned}
c + o(1) &= \mathcal{I}(z_n) \\
&\geq \frac{\eta_n^2}{2} \|z_n\|^2 - \int_{\mathbb{R} \times \mathbb{R}^N} F(t, x, \eta_n z_n^+) + \frac{1 - \eta_n^2}{2} \langle \mathcal{I}'(z_n), z_n \rangle + \eta_n^2 \langle \mathcal{I}'(z_n), z_n^- \rangle \\
&= \frac{R^2}{2} - \int_{\mathbb{R} \times \mathbb{R}^N} F(t, x, R z_n^+ / \|z_n\|) + \left(\frac{1}{2} - \frac{R^2}{2 \|z_n\|^2} \right) \langle \mathcal{I}'(z_n), z_n \rangle
\end{aligned}$$

$$
\begin{aligned}
&+\frac{R^2}{\|z_n\|^2}\langle \mathcal{I}'(z_n), z_n^-\rangle \\
&= \frac{R^2}{2} - \int_{\mathbb{R}\times\mathbb{R}^N} F(t,x,Rz_n^+/\|z_n\|) + o(1) \\
&\geq \frac{R^2}{2} - \frac{1}{4} + o(1) > \frac{3}{4} + c + o(1).
\end{aligned}
$$

This leads to a contradiction, so $\delta > 0$. Without loss of generality we suppose the existence of $k_n \in \mathbb{Z}^{N+1}$ such that $\int_{B_{1+\sqrt{N+1}}(0)} |\omega_n^+|^2 > \frac{\delta}{2}$. Denote $\zeta_n(x) = \omega_n(x+k_n)$, then

$$
\int_{B_{1+\sqrt{N+1}}(0)} |\zeta_n^+|^2 > \frac{\delta}{2}. \tag{3.11}
$$

Put $\tilde{z}_n(x) = z_n(x+k_n)$, $\tilde{z}_n/\|z_n\| = \zeta_n$, then $\|\zeta_n\| = 1$. Passing to a subsequence, we may assume that $\zeta_n \rightharpoonup \zeta$ on E, and $\zeta_n \to \zeta, \zeta_n \to \zeta$ on L^2_{loc}. a.e. on $\mathbb{R}\times\mathbb{R}^N$. It is evident that (3.11) implies that $\zeta \neq 0$. Thus, by virtue of (3.1), (F3) and Fatou lemma, we see that

$$
\begin{aligned}
0 &= \lim_{n\to\infty} \frac{c+o(1)}{\|z_n\|^2} \\
&= \lim_{n\to\infty} \frac{\mathcal{I}(z_n)}{\|z_n\|^2} \\
&= \lim_{n\to\infty} \left[\frac{1}{2}(\|\omega_n^+\|^2 - \|\omega_n^-\|^2) - \int_{\mathbb{R}\times\mathbb{R}^N} \frac{F(t,x,z_n)}{\|z_n\|^2}\right] \\
&= \lim_{n\to\infty} \left[\frac{1}{2}(\|\omega_n^+\|^2 - \|\omega_n^-\|^2) - \int_{\mathbb{R}\times\mathbb{R}^N} \frac{F(t,x,z_n)}{|\tilde{z}_n|^2}|\zeta_n|^2\right] \\
&\leq \frac{1}{2} - \liminf_{n\to\infty} \int_{\mathbb{R}\times\mathbb{R}^N} \frac{F(t,x,z_n)}{|\tilde{z}_n|^2}|\zeta_n|^2 \\
&= -\infty,
\end{aligned}
$$

which is a contradiction. Hence the statement of Lemma 3.11 are proved. □

Proof of Theorem 2.1. In light of Lemma 3.12, there exists a bounded sequence $\{z_n\} \subset E$ satisfying Lemma 3.9. Hence, there exists a constant $C_2 > 0$ such that $\|z_n\|_2 \leq C_2$. If $\delta := \lim_{n\to\infty} \sup \sup_{y\in\mathbb{R}^{N+1}} \int_{B_1(y)} |z_n|^2 = 0$, then $z_n \to 0$ in L^p, where $p \in (2, N^*)$. On the other hand, by virtue of (F1) and (F2), for $\varepsilon = c_*/4C_2^2 > 0$, there exists a a constant $C_\varepsilon > 0$ such that

$$
F(t,x,z) \leq \varepsilon|z|^2 + C_\varepsilon|z|^p, \quad \forall\ (t,x,z) \in \mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^{2M}.
$$

Based on the above discussion, we have

$$
\begin{aligned}
&\lim_{n\to\infty} \sup \int_{\mathbb{R}\times\mathbb{R}^N} \left[\frac{1}{2}F_z(t,x,z_n)\cdot z_n - F(t,x,z_n)\right] \\
&\leq \frac{3\varepsilon}{2}C_2^2 + \frac{3\varepsilon}{2}C_\varepsilon \lim_{n\to\infty} |z_n|^p \\
&= \frac{3c_*}{8}.
\end{aligned}
$$

Thus,

$$\begin{aligned}c_* &= \mathcal{I}(z_n) - \frac{1}{2}\langle \mathcal{I}'(z_n), z_n\rangle + o(1)\\ &= \int_{\mathbb{R}\times\mathbb{R}^N}\left[\frac{1}{2}F_z(t,x,z_n)\cdot z_n - F(t,x,z_n)\right] + o(1)\\ &\le \frac{3c_*}{8} + o(1),\end{aligned}$$

which is a contraction. Then $\delta > 0$.

Passing to the subsequence, we may assume that there exists $k_n \in \mathbb{Z}^{N+1}$ such that $\int_{B_{1+\sqrt{N+1}}(0)} |z_n^+|^2 > \frac{\delta}{2}$. Set $\zeta_n(x) = z_n(x+k_n)$, then

$$\int_{B_{1+\sqrt{N+1}}(0)} |\zeta_n^+|^2 > \frac{\delta}{2}. \tag{3.12}$$

Due to the periodic assumption of $V(x)$ and $F(t,x,z)$, it follows that $\|\zeta_n\| = \|z_n\|$ and

$$\mathcal{I}(\zeta_n) \to c_*, \quad \|\mathcal{I}'(\zeta_n)\|(1+\|\zeta_n\|) \to 0. \tag{3.13}$$

Thus, Passing to the subsequence, suppose that $\zeta_n \rightharpoonup \zeta$ in E, $\zeta_n \to \zeta$ in L^2_{loc}, $\zeta_n(t,x) \to \zeta(t,x)$ a.e on $\mathbb{R}^{N+1}$. In light of (3.8), we see that $\zeta \neq 0$. For every $\phi = (\phi_1,\phi_2) \in C_0^\infty(\mathbb{R}^{N+1}) \times C_0^\infty(\mathbb{R}^{N+1})$, by (2.2) and (3.9), we have $\langle \mathcal{I}'(\zeta), \phi\rangle = \lim_{n\to\infty}\langle \mathcal{I}'(\zeta_n), \phi\rangle = 0$. Hence, $\mathcal{I}'(\zeta) = 0$, which implies that $\zeta \in \mathcal{N}^-$. Then, $\mathcal{I}(\zeta) \ge m$. On the other way, it follows from (F2), (F3), (F4), Lemma 3.6, Lemma 3.10 and Fatou Lemma that

$$\begin{aligned}m \ge c_* &= \lim_{n\to\infty}\left[\mathcal{I}(z_n) - \frac{1}{2}\langle \mathcal{I}'(z_n), z_n\rangle\right]\\ &= \lim_{n\to\infty}\int_{\mathbb{R}\times\mathbb{R}^N}\left[\frac{1}{2}F_z(t,x,z_n)\cdot z_n - F(t,x,z_n)\right]\\ &\ge \int_{\mathbb{R}\times\mathbb{R}^N}\lim_{n\to\infty}\left[\frac{1}{2}F_z(t,x,z_n)\cdot z_n - F(t,x,z_n)\right]\\ &= \int_{\mathbb{R}\times\mathbb{R}^N}\left[\frac{1}{2}F_\zeta(t,x,\zeta)\cdot \zeta - F(t,x,\zeta)\right]\\ &= \mathcal{I}(\zeta) - \frac{1}{2}\langle \mathcal{I}'(\zeta), \zeta\rangle = \mathcal{I}(\zeta),\end{aligned}$$

which implies $\mathcal{I}(\zeta) \le m$. So $\mathcal{I}(\zeta) = m = \inf_{\mathcal{N}^-}\mathcal{I} > 0$. The proof is completed. □

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