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SEMICLASSICAL SOLUTIONS OF THE CHOQUARD EQUATIONS IN $\mathbb{R}^3$

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Abstract We study the nonlocal equation:

$$-\varepsilon^2\Delta u + \lambda u + V(x)u = \varepsilon^{-2}(|x|^{-1} * |u|^p)|u|^{p-2}u \qquad \text{in } \mathbb{R}^3,$$

where $\varepsilon > 0$ is a small parameter, $\lambda > 0$, $0 < p < \infty$ are positive constants and u is a real-valued measurable function. By Lyapunov–Schmidt reduction, we will prove the existence of multiple semiclassical solutions.

Keywords Choquard equations, semiclassical states, Lyapunov–Schmidt reduction.

MSC(2010) 35J20, 35J60.

1. Introduction and main results

The Choquard equation

$$-\Delta u + u = (I_2 * u^2)u \qquad \text{in } \mathbb{R}^n \tag{1.1}$$

is also called the nonlinear Hartree or Schrödinger–Newton equation, where I_2 is the Newton potential. (1.1) comes from several physical models. For example, it is used to describe the quantum mechanics of a polaron at rest by Pekar [19] and model an electron trapped in its own hole, in a certain approximation to Hartree–Fock theory of a plasma [12]. Later, Penrose proposed (1.1) as a model for the self-gravitational collapse of a quantum mechanical wave function [20]. Wei and Winter [24], Secchi [22] and Chen [4] constructed multi-peak solution by perturbation method. The more results about Hartree equations can be seen [11, 16, 28] and the references therein.

Equation (1.1) can be considered as a special case of the generalized Choquard equation

$$-\Delta u + \lambda u + V(x)u = (I_\alpha * |u|^p)|u|^{p-2}u \qquad \text{in } \mathbb{R}^n, \tag{1.2}$$

where $p > 1$ and I_α denotes the Riesz potential [21] defined by

$$I_\alpha(x) = \frac{\Gamma\left(\frac{n-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)\pi^{n/2}2^\alpha|x|^{n-\alpha}}$$

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and Γ is the Gamma function. When $V(x) = 0$ and $\lambda = 1$, Moroz and Van Schaftingen [17] stated two conclusions: one is the existence of a positive groundstate solution which is radially symmetric and monotone decaying about some points for $\frac{n-2}{n+\alpha} < \frac{1}{p} < \frac{n}{n+\alpha}$; the other is regularity, positivity and asymptotic properties at infinity of the groundstates. They also showed in [18] that (1.2) has a family of solutions concentrating to the local minimum of V under some conditions by variational methods and a novel nonlocal penalization technique. When $\lambda = 1$ and $V(x) = -\frac{\mu}{\nu^2+|x|^2}$ for $\mu,\ \nu > 0$, Cassani et al. [3] proved that there exist two thresholds μ^ν and μ_ν such that if $\mu < \mu_\nu$, then (1.2) has no ground state solution; if $\mu_\nu < \mu < \mu^\nu$, then equation admits a positive ground state solution; if $\mu > \max\{\mu^\nu, N^2(N-2)/4(N+1)\}$, a sign changing ground state solution exists with $n \geq 4$ or non-resonant case. In the case of $\lambda = V(x) = 0$ and $p = 2^*_\mu$, where $2^*_\mu = \frac{2n-\mu}{n-2}$ is the upper critical exponent on account of the Hardy-Littlewood-Sobolev inequality, under the condition that using moving plane method, Du and Yang [8] got the symmetry and uniqueness of positive solutions, the nondegeneracy of the unique solutions for (1.2) are proved in [8] and [9] when $\mu \to n$ and $\mu \to 0$ respectively. We refer reader to [5–7, 10, 13, 26, 27] and the references therein for more details.

Motivated by [4, 24] , we are interested in the existence of multiple semiclassical solutions for the following Choquard equation with potential V:

$$-\varepsilon^2\Delta u + \lambda u + V(x)u = \varepsilon^{-2}(|x|^{-1} * |u|^p)|u|^{p-2}u \qquad \text{in } \mathbb{R}^3, \tag{1.3}$$

where $\varepsilon > 0$ is a small parameter, $\Delta = \sum_{i=1}^3 \partial_{x_i x_i}$ denotes the Laplace operator, $p > 2$ and the potential V satisfies:

(V_1) $V \in C_b^3(\mathbb{R}^3)$, $\inf_{x\in\mathbb{R}^3}(\lambda + V(x)) > 0$, where

$$C_b^3(\mathbb{R}^3) := \{V \in C^3(\mathbb{R}^3) | D^J V \text{ is bounded in } \mathbb{R}^3 \text{ for all multi-index } J \text{ such that } |J| \leq 3\}.$$

Our main result in this paper can be stated as follows.

Theorem 1.1. *Suppose that V satisfies (V_1) and has a nondegenerate smooth compact critical manifold M. Then (1.3) has at least $l(M)$ solutions concentrating near points of M for $2 < p < 2 + \delta$ and $\varepsilon > 0$ sufficiently small, where $0 < \delta \leq \frac{1}{3}$ and $l(M)$ denotes the cup length of M.*

The main idea in proving Theorems 1.1 is by Lyapunov–Schmidt reduction, which has been widely used to deal with the problem of interior peak solutions for Schrödinger equations. See for example [2, 14, 15, 23] and the references therein. However, the nonlocal term in the equation (1.3) causes some technique difficulties to us. Thus we extend the classical Lyapunov–Schmidt reduction and make careful estimates to overcome some problems.

The paper is organized as follows. In Section 2, we state some useful lemmas and give the proof of Theorem 1.1 in Section 3 by Lyapunov–Schmidt reduction method.

2. Notations and preliminaries

In this section, we present some useful notations and lemmas which will be of importance later.

First we need to introduce some notations. Let $x = \varepsilon x$, (1.3) becomes

$$-\Delta u + \lambda u + V(\varepsilon x)u = (|x|^{-1} * |u|^p)|u|^{p-2}u \qquad \text{in } \mathbb{R}^3, \tag{2.1}$$

which will be considered as a perturbation of

$$-\Delta u + \lambda u = (|x|^{-1} * |u|^p)|u|^{p-2}u \qquad \text{in } \mathbb{R}^3. \tag{2.2}$$

Let $H^1(\mathbb{R}^3)$ be a Hilbert space with the inner produce

$$\langle u, v\rangle = \int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + \lambda uv)dx$$

and the corresponding norm

$$\|u\|^2 = \int_{\mathbb{R}^3} \left(|\nabla u|^2 + \lambda |u|^2\right) dx.$$

The energy functional of (2.1) is defined by

$$\begin{aligned} f_\varepsilon(u) &= \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla u|^2 + \lambda u^2)dx + \frac{1}{2}\int_{\mathbb{R}^3} V(\varepsilon x)u^2 dx - \frac{1}{2p}\int_{\mathbb{R}^3} (|x|^{-1} * |u|^p)|u|^p dx \\ &:= f_0(u) + \frac{1}{2}\int_{\mathbb{R}^3} V(\varepsilon x)u^2 dx \end{aligned}$$

for any $u \in H^1(\mathbb{R}^3)$. $f_\varepsilon(u)$ is well defined by the Hardy–Littlewood–Sobolev inequality. We shall verify that there exist solutions of (2.1) near a solution of

$$-\Delta u + \lambda u + V(\varepsilon \xi)u = (|x|^{-1} * |u|^p)|u|^{p-2}u \qquad \text{in } \mathbb{R}^3 \tag{2.3}$$

with some appropriate ξ in $\mathbb{R}^3$. Clearly, the solutions of (2.3) are critical points of

$$F_{\varepsilon,\xi}(u) = f_0(u) + \frac{1}{2}V(\varepsilon\xi)\int_{\mathbb{R}^3} u^2 dx$$

for any $u \in H^1(\mathbb{R}^3)$. Then we have

$$f_\varepsilon(u) = F_{\varepsilon,\xi}(u) + \frac{1}{2}\int_{\mathbb{R}^3} \left(V(\varepsilon x) - V(\varepsilon\xi)\right) u^2 dx. \tag{2.4}$$

We state the following lemma to describe the properties of solutions of (2.2) for $2 < p < 2 + \delta$, where $\delta \in (0, \delta^*)$, $\delta^* \in (0, \frac{1}{3})$.

Lemma 2.1. *(i). The problem (2.2) has a unique ground state solution, denoted by U;*

(ii). U is radially symmetric and strictly decreasing: $U(y) = U(|y|)$ and $U' < 0$ for $r > 0$, $r = |y|$;

(iii). The asymptotic behavior of U:

$$U(r) = A_1(1 + O(1/r))r^{-1}e^{-r}, \quad U'(r) = -A_1(1 + O(1/r))r^{-1}e^{-r} \quad \text{as } r \to \infty,$$

where $A_1 > 0$ is a constant;

(iv). U is nondegenerate, to be more precise,

$$ker\, L = span\left\{\frac{\partial U}{\partial x_1}, \frac{\partial U}{\partial x_2}, \frac{\partial U}{\partial x_3}\right\},$$

where L is the linearized operator at U given by

$$L\varphi := -\Delta\varphi + \lambda\varphi - (p-1)(|x|^{-1} * U^p)U^{p-2}\varphi - p(|x|^{-1} * (U^{p-1}\varphi))U^{p-1}$$

for any $\varphi \in H^2(\mathbb{R}^3)$.

Proof. See [25, Theorem 1.2] and [25, Theorem 1.3]. $\square$

Let $\beta = \beta(\varepsilon\xi) = (1 + V(\varepsilon\xi)/\lambda)^{1/2}$ and $\alpha = \alpha(\varepsilon\xi) = (1 + V(\varepsilon\xi)/\lambda)^{1/(p-1)}$, then $\alpha U(\beta x)$ is a solution of (2.3). Let

$$z_{\varepsilon,\xi} := \alpha(\varepsilon\xi)U(\beta(\varepsilon\xi)x) \tag{2.5}$$

and $Z_\varepsilon := \{z_{\varepsilon,\xi}(x-\xi) | \xi \in \mathbb{R}^3\}$. For convenience, we write $z_\xi = z_{\varepsilon,\xi}(x-\xi)$.

Lemma 2.2 (see [4, Lemma 5.1])**.** *For all $\xi \in \mathbb{R}^3$, it holds that:*

$$\partial_{\xi_i}[z_{\varepsilon,\xi}(x-\xi)] = -\partial_{x_i}[z_{\varepsilon,\xi}(x-\xi)] + O(\varepsilon). \tag{2.6}$$

Lemma 2.3 (see [4, Lemma 5.2])**.** *For all $\xi \in \mathbb{R}^3$ and $\varepsilon > 0$ sufficiently small, the following holds:*

$$\|Df_\varepsilon(z_\xi)\| \le C\left(\varepsilon|\nabla V(\varepsilon\xi)| + \varepsilon^2\right),$$

where $C > 0$ is a constant independent on ξ and ε.

Lemma 2.4. *For all $\varphi_j \in H^1(\mathbb{R}^3)$ $(j = 1, 2, 3, 4)$, the following hold:*

$$\left|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\varphi_1^{p-1}(x)\varphi_2(x)\varphi_3^{p-1}(y)\varphi_4(y)}{|x-y|}dxdy\right| \le C\|\varphi_1\|^{p-1}\|\varphi_2\|\|\varphi_3\|^{p-1}\|\varphi_4\| \tag{2.7}$$

and

$$\left|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\varphi_1^{p}(x)\varphi_2^{p-2}(y)\varphi_3(y)\varphi_4(y)}{|x-y|}dxdy\right| \le C\|\varphi_1\|^{p}\|\varphi_2\|^{p-2}\|\varphi_3\|\|\varphi_4\|, \tag{2.8}$$

where $C > 0$ is a constant independent on φ_j.

Proof. By the Hardy–Littlewood–Sobolev inequality and Hölder inequality, we have

$$\begin{aligned}
&\left|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\varphi_1^{p-1}(x)\varphi_2(x)\varphi_3^{p-1}(y)\varphi_4(y)}{|x-y|}dxdy\right| \\
&\le C\|\varphi_1^{p-1}\varphi_2\|_{L^{\frac{6}{5}}(\mathbb{R}^3)}\|\varphi_3^{p-1}\varphi_4\|_{L^{\frac{6}{5}}(\mathbb{R}^3)} \\
&\le C\|\varphi_1\|^{p-1}_{L^{\frac{12}{5}}(\mathbb{R}^3)}\|\varphi_2\|_{L^{\frac{12}{5(3-p)}}(\mathbb{R}^3)}\|\varphi_3\|^{p-1}_{L^{\frac{12}{5}}(\mathbb{R}^3)}\|\varphi_4\|_{L^{\frac{12}{5(3-p)}}(\mathbb{R}^3)}, \\
&\left|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\varphi_1^{p}(x)\varphi_2^{p-2}(y)\varphi_3(y)\varphi_4(y)}{|x-y|}dxdy\right| \\
&\le C\|\varphi_1\|^{p}_{L^{\frac{12}{5}}(\mathbb{R}^3)}\|\varphi_2^{p-2}\varphi_3\varphi_4\|_{L^{\frac{12}{20-5p}}(\mathbb{R}^3)} \\
&\le C\|\varphi_1\|^{p}_{L^{\frac{12}{5}}(\mathbb{R}^3)}\|\varphi_2\|^{p-2}_{L^{\frac{12}{5}}(\mathbb{R}^3)}\|\varphi_3\|_{L^{\frac{12}{5(3-p)}}(\mathbb{R}^3)}\|\varphi_4\|_{L^{\frac{12}{5(3-p)}}(\mathbb{R}^3)}.
\end{aligned}$$

Since $2 < p < 2 + \delta$, then the Sobolev imbedding gives that

$$\|\varphi_j\|_{L^{\frac{12}{5}}(\mathbb{R}^3)} \leq C\|\varphi_j\|, \ \ \|\varphi_j\|_{L^{\frac{12}{5(3-p)}}(\mathbb{R}^3)} \leq C\|\varphi_j\|, \quad j = 1, 2, 3, 4.$$

Thus we finish the proof. □

3. The proof of Theorem 1.1

In this section, we construct multiple semi-classical solutions to (1.3) and prove Theorem 1.1.

3.1. Invertibility.

Let $D^2 f_\varepsilon(z_\xi)$ be the Hessian of f_ε at z_ξ and $T_{z_\xi}(Z_\varepsilon)$ be the tangent space of Z_ε at z_ξ. Define $L_{\varepsilon,\xi} : (T_{z_\xi}(Z_\varepsilon))^\perp \to H^1(\mathbb{R}^3)$ by

$$\langle L_{\varepsilon,\xi} u, w\rangle = D^2 f_\varepsilon(z_\xi)(u, w), \ \ u \in (T_{z_\xi}(Z_\varepsilon))^\perp, w \in H^1(\mathbb{R}^3).$$

Let $P_{\varepsilon,\xi} : H^1(\mathbb{R}^3) \to (T_{z_\xi}(Z_\varepsilon))^\perp$ be the orthogonal projection. We define

$$\mathcal{L}_{\varepsilon,\xi} = P_{\varepsilon,\xi} L_{\varepsilon,\xi} : (T_{z_\xi}(Z_\varepsilon))^\perp \to (T_{z_\xi}(Z_\varepsilon))^\perp.$$

As a preparation, the following proposition gives the invertibility of $\mathcal{L}_{\varepsilon,\xi}$.

Proposition 3.1. *For any fixed $\bar{\delta} > 0$, there exist $C > 0$ and $\varepsilon_0 > 0$ sufficiently small such that for any $\varepsilon \in (0, \varepsilon_0)$ and $|\xi| \leq \bar{\delta}$, the map $\mathcal{L}_{\varepsilon,\xi}$ is both injective and surjective. Moreover,*

$$|\langle \mathcal{L}_{\varepsilon,\xi} w, w\rangle| \geq C\|w\|^2$$

for all $w \in (T_{z_\xi}(Z_\varepsilon))^\perp$.

Proof. z_ξ is a mountain pass critical point of $F_{\varepsilon,\xi}$, then for any fixed $\varepsilon_1 > 0$ small, there exists a constant $c_1 > 0$ such that for any $\varepsilon \in (0, \varepsilon_1)$ and $|\xi| \leq \bar{\delta}$,

$$D^2 F_{\varepsilon,\xi}(z_\xi)(z_\xi, z_\xi) < -c_1 < 0.$$

By (2.4) and Lemma 2.3, we have

$$\begin{aligned}\langle L_{\varepsilon,\xi} z_\xi, z_\xi\rangle &= D^2 F_{\varepsilon,\xi}(z_\xi)(z_\xi, z_\xi) + \int_{\mathbb{R}^3} (V(\varepsilon x) - V(\varepsilon\xi)) z_\xi^2 dx \\ &\leq -c_1 + c_2\left(\varepsilon|\nabla V(\varepsilon\xi)| + \varepsilon^2\right),\end{aligned}$$

which implies that

$$\langle L_{\varepsilon,\xi} z_\xi, z_\xi\rangle \leq -C_1 < 0, \tag{3.1}$$

where $C_1 > 0$ is a constant depending only on $\bar{\delta}$ and ε_0.

Let

$$K_{\varepsilon,\xi} = \operatorname{span}\{z_\xi, \partial_{x_1} z_\xi, \partial_{x_2} z_\xi, \partial_{x_3} z_\xi\}.$$

Next we show that for any $\phi \in K_{\varepsilon,\xi}^\perp$, the following holds:

$$\langle L_{\varepsilon,\xi}\phi, \phi\rangle \geq C_2\|\phi\|^2, \tag{3.2}$$

where $C_2 > 0$ is a constant depending only on $\bar{\delta}$ and ε_0.

From the definition of $L_{\varepsilon,\xi}$, we obtain that

$$\langle L_{\varepsilon,\xi}\phi, \phi\rangle = D^2 F_{\varepsilon,\xi}(z_\xi)(\phi,\phi) + \int_{\mathbb{R}^3} (V(\varepsilon x) - V(\varepsilon\xi))\phi^2 dx$$

for any $\phi \in K^{\perp}_{\varepsilon,\xi}$. Since z_ξ is a mountain pass critical point of $F_{\varepsilon,\xi}$, then

$$D^2 F_{\varepsilon,\xi}(z_\xi)(\phi,\phi) \geq C_3\|\phi\|^2, \ \ \forall\phi \in K^{\perp}_{\varepsilon,\xi}. \tag{3.3}$$

Let $\eta_1 : \mathbb{R}^3 \to \mathbb{R}$ be a radial smooth cut-off function such that for any $R > 0$ sufficiently large,

$$\eta_1(x) = 1 \text{ for } |x| \leq R, \ \ \eta_1(x) = 0 \text{ for } |x| \geq 2R \text{ and } |\nabla\eta_1(x)| \leq \frac{2}{R} \text{ for } R \leq |x| \leq 2R.$$

Let $\eta_2 = 1 - \eta_1$ and $\phi_i = \eta_i\phi$ $(i = 1, 2)$. Then

$$\begin{aligned} \|\phi\|^2 &= \|\phi_1\|^2 + \|\phi_2\|^2 + 2\int_{\mathbb{R}^3} (\nabla\phi_1 \cdot \nabla\phi_2 + \lambda\phi_1\phi_2)\, dx \\ &= \|\phi_1\|^2 + \|\phi_2\|^2 + 2\int_{\mathbb{R}^3} \eta_1\eta_2 \left(|\nabla\phi|^2 + \lambda\phi^2\right) dx + o_R(1)\|\phi\|^2 \end{aligned} \tag{3.4}$$

and

$$\langle L_{\varepsilon,\xi}\phi, \phi\rangle = \langle L_{\varepsilon,\xi}\phi_1, \phi_1\rangle + \langle L_{\varepsilon,\xi}\phi_2, \phi_2\rangle + 2\langle L_{\varepsilon,\xi}\phi_1, \phi_2\rangle := T_1 + T_2 + T_3. \tag{3.5}$$

We estimate (3.5) term by term. For T_1, using the definition of $L_{\varepsilon,\xi}$ again, we have

$$T_1 = \langle L_{\varepsilon,\xi}\phi_1, \phi_1\rangle = D^2 F_{\varepsilon,\xi}(z_\xi)(\phi_1,\phi_1) + \int_{\mathbb{R}^3} (V(\varepsilon x) - V(\varepsilon\xi))\phi_1^2 dx.$$

Splitting $\phi_1 = \bar{\phi}_1 + \psi$, where $\bar{\phi}_1 \in K^{\perp}_{\varepsilon,\xi}$ and $\psi \in K_{\varepsilon,\xi}$. Hence

$$\psi = \langle\phi_1, z_\xi\rangle\|z_\xi\|^{-2} z_\xi + \sum_{i=1}^{3} \langle\phi_1, \partial_{x_i} z_\xi\rangle\|\partial_{x_i} z_\xi\|^{-2}\partial_{x_i} z_\xi$$

and

$$D^2 F_{\varepsilon,\xi}(z_\xi)(\phi_1,\phi_1) = D^2 F_{\varepsilon,\xi}(z_\xi)(\bar{\phi}_1,\bar{\phi}_1) + D^2 F_{\varepsilon,\xi}(z_\xi)(\psi,\psi) + 2D^2 F_{\varepsilon,\xi}(z_\xi)(\bar{\phi}_1,\psi).$$

(3.3) gives that

$$D^2 F_{\varepsilon,\xi}(z_\xi)(\bar{\phi}_1,\bar{\phi}_1) \geq C_4\|\bar{\phi}_1\|^2. \tag{3.6}$$

Since $\phi \in K^{\perp}_{\varepsilon,\xi}$, the following holds:

$$\begin{aligned} \langle\phi_1, z_\xi\rangle &= \langle(1-\eta_2)\phi, z_\xi\rangle = -\langle\eta_2\phi, z_\xi\rangle \\ &= -\lambda\int_{\mathbb{R}^3} \eta_2(x)\phi(x)z_\xi(x)dx - \int_{\mathbb{R}^3} \nabla(\eta_2\phi)\cdot\nabla z_\xi dx \\ &= -\lambda\int_{\mathbb{R}^3\setminus B_R(0)} \eta_2(x)\phi(x)z_\xi(x)dx - \int_{\mathbb{R}^3\setminus B_R(0)} \eta_2\nabla\phi\cdot\nabla z_\xi dx \\ &\quad - \int_{B_{2R}(0)\setminus B_R(0)} \phi\nabla\eta_2\cdot\nabla z_\xi dx \\ &\leq \frac{C}{R}e^{-R}\|\phi\|, \end{aligned}$$

here we use the Hölder inequality and Lemma 2.1. This implies that as $R \to \infty$,

$$\langle \phi_1, z_\xi \rangle = o_R(1)\|\phi\|.$$

Similarly, it is easy to see that

$$\langle \phi_1, \partial_{x_i} z_\xi \rangle = o_R(1)\|\phi\|.$$

The two formulas in the above show that

$$\|\psi\| = o_R(1)\|\phi\|. \tag{3.7}$$

By simple calculation, we deduce that

$$\begin{aligned} D^2F_{\varepsilon,\xi}(z_\xi)(\psi,\psi) =& \|\psi\|^2 + V(\varepsilon\xi)\int_{\mathbb{R}^3}\psi^2 dx - p\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{z_\xi^{p-1}(x)\psi(x)z_\xi^{p-1}(y)\psi(y)}{|x-y|}dxdy \\ & - (p-1)\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{z_\xi^{p}(x)z_\xi^{p-2}(y)\psi^2(y)}{|x-y|}dxdy. \end{aligned} \tag{3.8}$$

Using (2.7) and (2.8), we have

$$\left|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{z_\xi^{p-1}(x)\psi(x)z_\xi^{p-1}(y)\psi(y)}{|x-y|}dxdy\right| \le C\|\psi\|^2 \tag{3.9}$$

and

$$\left|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{z_\xi^{p}(x)z_\xi^{p-2}(y)\psi^2(y)}{|x-y|}dxdy\right| \le C\|\psi\|^2. \tag{3.10}$$

Combinging (3.7)-(3.10), we get

$$D^2F_{\varepsilon,\xi}(z_\xi)(\psi,\psi) = o_R(1)\|\phi\|^2. \tag{3.11}$$

A same estimate shows that

$$\begin{aligned} & D^2F_{\varepsilon,\xi}(z_\xi)(\bar\phi_1,\psi) \\ =& \langle\bar\phi_1,\psi\rangle + V(\varepsilon\xi)\int_{\mathbb{R}^3}\bar\phi_1\psi dx - p\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{z_\xi^{p-1}(x)\bar\phi_1(x)z_\xi^{p-1}(y)\psi(y)}{|x-y|}dxdy \\ & - (p-1)\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{z_\xi^{p}(x)z_\xi^{p-2}(y)\psi(y)\bar\phi_1(y)}{|x-y|}dxdy \\ =& o_R(1)\|\bar\phi_1\|\|\phi\| = o_R(1)\|\phi\|^2. \end{aligned} \tag{3.12}$$

From (3.6), (3.11) and (3.12), it holds that

$$D^2F_{\varepsilon,\xi}(z_\xi)(\phi_1,\phi_1) \ge C\|\phi_1\|^2 + o_R(1)\|\phi\|^2. \tag{3.13}$$

Recalling that D^JV is bounded and $\eta_1 = 0$ for $|x| \ge 2R$, by Taylor expansion, we have that for $|\xi| \le \bar\delta$,

$$\begin{aligned} \int_{\mathbb{R}^3}|V(\varepsilon x) - V(\varepsilon\xi)|\phi_1^2 dx \le& C_5\varepsilon\int_{B_{2R}(0)}|x-\xi|\eta_1^2(x)\phi^2(x)dx \\ \le& C_6\varepsilon R\int_{B_{2R}(0)}\eta_1^2(x)\phi^2(x)dx \le C_6\varepsilon R\|\phi\|^2. \end{aligned} \tag{3.14}$$

Choosing $R = \varepsilon^{-1/2}$, then (3.13) and (3.14) imply that there exists $\varepsilon_0 > 0$ sufficiently small such that for any $\varepsilon \le \varepsilon_0$,

$$T_1 \ge C\|\phi_1\|^2 + o_R(1)\|\phi\|^2. \tag{3.15}$$

Now, we estimate T_2. According to the definition of T_2, we have

$$\begin{aligned} T_2 = \langle L_{\varepsilon,\xi}\phi_2, \phi_2\rangle = & \int_{\mathbb{R}^3} \left(|\nabla \phi_2|^2 + (\lambda + V(\varepsilon x))\phi_2^2\right) dx \\ & - p\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{z_\xi^{p-1}(x)\phi_2(x) z_\xi^{p-1}(y)\phi_2(y)}{|x-y|} dxdy \\ & - (p-1)\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{z_\xi^{p}(x) z_\xi^{p-2}(y)\phi_2^2(y)}{|x-y|} dxdy. \end{aligned} \tag{3.16}$$

Since $\inf_{x\in\mathbb{R}^3}(\lambda + V(x)) > 0$, one finds that for $|\xi| \le \bar{\delta}$ and $\varepsilon > 0$ sufficiently small,

$$\int_{\mathbb{R}^3} \left(|\nabla \phi_2|^2 + (\lambda + V(\varepsilon x))\phi_2^2\right) dx \ge C_7\|\phi_2\|^2.$$

By Lemma 2.4, Hölder inequality and $\eta_2 = 0$ in $B_R(0)$, we obtain

$$\begin{aligned} & \left|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{z_\xi^{p-1}(x)\phi_2(x) z_\xi^{p-1}(y)\phi_2(y)}{|x-y|} dxdy\right| \\ \le & C_8\|z_\xi^{p-1}\phi_2\|^2_{L^{6/5}(\mathbb{R}^3)} \le C_8\|z_\xi^{p-1}\eta_2\|^2_{L^3(\mathbb{R}^3\setminus B_R(0))}\|\phi\|^2 \\ \le & C_8 R^{-1}e^{-R}\|\phi\|^2 \le o_R(1)\|\phi\|^2. \end{aligned}$$

The last term in (3.16) can be estimated as follows:

$$\begin{aligned} & \left|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{z_\xi^{p}(x) z_\xi^{p-2}(y)\phi_2^2(y)}{|x-y|} dxdy\right| \le \int_{B_R^c(0)}\int_{\mathbb{R}^3} \frac{z_\xi^{p}(x) z_\xi^{p-2}(y)\phi^2(y)}{|x-y|} dxdy \\ = & \int_{B_R^c(0)}\int_{B_{R/2}(0)} \frac{z_\xi^{p}(x) z_\xi^{p-2}(y)\phi^2(y)}{|x-y|} dxdy + \int_{B_R^c(0)}\int_{B_{R/2}^c(0)} \frac{z_\xi^{p}(x) z_\xi^{p-2}(y)\phi^2(y)}{|x-y|} dxdy \\ \le & C_9 R^{-1}\|\phi\|^2 + \int_{B_R^c(0)} z_\xi^{p-2}(y)\phi^2(y)\left(\int_{B_{R/2}^c(0)\cap B_1(y)} \frac{z_\xi^p(x)}{|x-y|}dx\right) dy \\ & + \int_{B_R^c(0)} z_\xi^{p-2}(y)\phi^2(y)\left(\int_{B_{R/2}^c(0)\cap B_1^c(y)} \frac{z_\xi^p(x)}{|x-y|}dx\right) dy \\ \le & C_9 R^{-1}\|\phi\|^2 + C_{10}e^{-R}\|\phi\|^2 = o_R(1)\|\phi\|^2. \end{aligned}$$

Hence choosing R large enough, the following holds:

$$T_2 \ge C_{11}\|\phi\|^2 + o_R(1)\|\phi\|^2. \tag{3.17}$$

Similarly, we continue in the same way as for T_3,

$$\begin{aligned}T_3 =&2\langle L_{\varepsilon,\xi}\phi_1,\phi_2\rangle = 2\int_{\mathbb{R}^3}\left(\nabla\phi_1\cdot\nabla\phi_2+(\lambda+V(\varepsilon x))\phi_1\cdot\phi_2\right)dx\\&-2p\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{z_\xi^{p-1}(x)\phi_1(x)z_\xi^{p-1}(y)\phi_2(y)}{|x-y|}dxdy\\&-2(p-1)\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{z_\xi^{p}(x)z_\xi^{p-2}(y)\phi_1(y)\phi_2(y)}{|x-y|}dxdy\\ \geq& C_{12}\int_{\mathbb{R}^3}\eta_1\eta_2\left(|\nabla\phi|^2+\lambda\phi^2\right)dx+o_R(1)\|\phi\|^2,\end{aligned}\tag{3.18}$$

here (3.4) is used in the last inequality.

Combining (3.15), (3.17) and (3.18), (3.2) holds. By Lemma 2.2, we have

$$|\langle L_{\varepsilon,\xi}w,w\rangle|\geq C\|w\|^2$$

for all $w\in(T_{z_\xi}(Z_\varepsilon))^\perp$. This completes the proof. □

3.2. Lyapunov–Schmidt reduction.

In this subsection, we prove the following equation

$$P_{\varepsilon,\xi}Df_\varepsilon(z_\xi+w)=0\tag{3.19}$$

has a unique solution $w=w_{\varepsilon,\xi}\in(T_{z_\xi}(Z_\varepsilon))^\perp$ and $w_{\varepsilon,\xi}$ is C^1 in ξ.

Expand $Df_\varepsilon(z_\xi+w)$ as follows:

$$Df_\varepsilon(z_\xi+w)=Df_\varepsilon(z_\xi)+D^2f_\varepsilon(z_\xi)(w)+\mathcal{R}(z_\xi,w),$$

where

$$\begin{aligned}\mathcal{R}(z_\xi,w)\quad:\quad H^1(\mathbb{R}^3)\quad&\to\quad\mathbb{R}\\ \varphi\quad&\to\quad\int_{\mathbb{R}^3}R(z_\xi,w)\varphi dx.\end{aligned}$$

Here $R(z_\xi,w)$ is a high order nonlocal term given by

$$\begin{aligned}R(z_\xi,w)=&-(|x|^{-1}*|z_\xi+w|^p)|z_\xi+w|^{p-2}(z_\xi+w)+(|x|^{-1}*z_\xi^p)z_\xi^{p-1}\\&+(p-1)(|x|^{-1}*z_\xi^p)z_\xi^{p-2}w+p(|x|^{-1}*(z_\xi^{p-1}w))z_\xi^{p-1}.\end{aligned}$$

Thus (3.19) becomes

$$\mathcal{L}_{\varepsilon,\xi}w+P_{\varepsilon,\xi}Df_\varepsilon(z_\xi)+P_{\varepsilon,\xi}\mathcal{R}(z_\xi,w)=0,\quad w\in(T_{z_\xi}(Z_\varepsilon))^\perp.\tag{3.20}$$

Since $\mathcal{L}_{\varepsilon,\xi}$ is invertible, we can rewrite (3.20) as

$$w=-\mathcal{L}_{\varepsilon,\xi}^{-1}\left(P_{\varepsilon,\xi}Df_\varepsilon(z_\xi)+P_{\varepsilon,\xi}\mathcal{R}(z_\xi,w)\right):=N_{\varepsilon,\xi}(w).\tag{3.21}$$

We will show that the operator $N_{\varepsilon,\xi}$ is a contraction on $B_{\delta_0}(0)$.

Lemma 3.1. *For any w_1, $w_2\in B_1(0)\subset H^1(\mathbb{R}^3)$,*

$$\|\mathcal{R}(z_\xi,w_2)-\mathcal{R}(z_\xi,w_1)\|\leq C\left(\|w_1\|^{2p-2}+\|w_2\|^{2p-2}\right)\|w_2-w_1\|,\tag{3.22}$$

where $C>0$ is a constant independent on w_1 and w_2.

Proof. A simple computation shows that for any $\varphi \in H^1(\mathbb{R}^3)$,

$$
\begin{aligned}
&|\mathcal{R}(z_\xi, w_2)(\varphi) - \mathcal{R}(z_\xi, w_1)(\varphi)| \\
\le &\int_{\mathbb{R}^3} \big|(|x|^{-1} * |z_\xi + w_2|^p)|z_\xi + w_2|^{p-2}(z_\xi + w_2) - (|x|^{-1} * |z_\xi + w_1|^p)|z_\xi \\
&+ w_1|^{p-2}(z_\xi + w_1)\big||\varphi|dx \\
&+ (p-1)\int_{\mathbb{R}^3} (|x|^{-1} * z_\xi^p) z_\xi^{p-2}|w_2 - w_1||\varphi|dx \\
&+ p\int_{\mathbb{R}^3} (|x|^{-1} * (z_\xi^{p-1}|w_2 - w_1|))|z_\xi^{p-1}\varphi|dx \\
\le &\int_{\mathbb{R}^3} \left[|x|^{-1} * \big||z_\xi + w_2|^p - |z_\xi + w_1|^p\big|\right] |z_\xi + w_2|^{p-2}(z_\xi + w_2)|\varphi|dx \\
&+ \int_{\mathbb{R}^3} (|x|^{-1} * |z_\xi + w_1|^p) \left||z_\xi + w_2|^{p-2}(z_\xi + w_2) - |z_\xi + w_1|^{p-2}(z_\xi + w_1)\right| |\varphi|dx \\
&+(p-1)\int_{\mathbb{R}^3}(|x|^{-1} * z_\xi^p) z_\xi^{p-2}|w_2 - w_1||\varphi|dx + p\int_{\mathbb{R}^3}(|x|^{-1} * (z_\xi^{p-1}|w_2 - w_1|))|z_\xi^{p-1}\varphi|dx \\
\le &C\left(\|w_1\|^{2p-2} + \|w_2\|^{2p-2}\right)\|w_2 - w_1\|\|\varphi\|.
\end{aligned}
$$

Here we use the Lemma 2.4 and Mean Value Theorem. Thus we finish the proof of (3.22). □

Lemma 3.2. *There exists a small ball $B_{\delta_0}(0) \subset (T_{z_\xi}(Z_\varepsilon))^\perp$ such that $N_{\varepsilon,\xi}$ maps $B_{\delta_0}(0)$ into itself for $0 < \varepsilon \le \varepsilon_0$ and $|\xi| \le \bar{\delta}$. Moreover, for all w_1, $w_2 \in B_{\delta_0}(0)$,*

$$\|N_{\varepsilon,\xi}(w_2) - N_{\varepsilon,\xi}(w_1)\| \le C\left(\|w_1\|^{2p-2} + \|w_2\|^{2p-2}\right)\|w_2 - w_1\|,$$

where $C > 0$ is a constant independent on w_1 and w_2. In particular, $N_{\varepsilon,\xi}$ is a contraction map on $B_{\delta_0}(0)$.

Proof. We have $\|\mathcal{R}(z_\xi, w)\| = O(\|w\|^{2p-1})$ by Lemma 3.1. Lemma 2.3 and (3.21) yield that:

$$\|N_{\varepsilon,\xi}(w)\| \le C\|Df_\varepsilon(z_\xi)\| + O\left(\|w\|^{2p-1}\right) \le C\left(\varepsilon|\nabla V(\varepsilon\xi)| + O(\varepsilon^2)\right) + O\left(\|w\|^{2p-1}\right), \tag{3.23}$$

which implies that $N_{\varepsilon,\xi}$ is a map from $B_{\delta_0}(0)$ to $B_{\delta_0}(0)$ for $0 < \varepsilon \le \varepsilon_0$, $|\xi| \le \bar{\delta}$ and $\delta_0 > 0$ sufficiently small.

From Lemma 3.1, we have that

$$
\begin{aligned}
\|N_{\varepsilon,\xi}(w_2) - N_{\varepsilon,\xi}(w_1)\| &\le \|\mathcal{L}_{\varepsilon,\xi}^{-1}\left(\mathcal{R}(z_\xi, w_1) - \mathcal{R}(z_\xi, w_2)\right)\| \\
&\le C\|\mathcal{R}(z_\xi, w_1) - \mathcal{R}(z_\xi, w_2)\| \\
&\le C\left(\|w_1\|^{2p-2} + \|w_2\|^{2p-2}\right)\|w_2 - w_1\|.
\end{aligned}
$$

This completes the proof. □

Proposition 3.2. *For $0 < \varepsilon \le \varepsilon_0$ and $|\xi| \le \bar{\delta}$, there exists a unique $w = w_{\varepsilon,\xi} \in (T_{z_\xi}(Z_\varepsilon))^\perp$ of class C^1 with respect to ξ satisfying $Df_\varepsilon(z_\xi + w_{\varepsilon,\xi}) \in T_{z_\xi}(Z_\varepsilon)$. Moreover, the functional $\Phi_\varepsilon(\xi) := f_\varepsilon(z_\xi + w_{\varepsilon,\xi})$ has the same regularity as w and satisfies:*

$$\nabla\Phi_\varepsilon(\xi_0) = 0 \Rightarrow Df_\varepsilon(z_{\xi_0} + w_{\varepsilon,\xi_0}) = 0.$$

Proof. Since $N_{\varepsilon,\xi}$ is a contraction map on $B_{\delta_0}(0)$ for $0<\varepsilon\le\varepsilon_0$ and $|\xi|\le\bar{\delta}$, the existence of a fixed point $w=w_{\varepsilon,\xi}$ follows from the contraction mapping principle and hence $w_{\varepsilon,\xi}$ is a solution of (3.21). Furthermore, for fixed ε with $|\varepsilon|\le\varepsilon_0$, we apply the Implicit function theorem to

$$H_\varepsilon(\xi,w):=w-N_{\varepsilon,\xi}(w)=0,\quad w\in(T_{z_\xi}(Z_\varepsilon))^\perp.$$

Setting $w=w_{\varepsilon,\xi}$, we find

$$D_wH_\varepsilon(\xi,w)[v]=v-\mathcal{L}_{\varepsilon,\xi}^{-1}P_{\varepsilon,\xi}\left[h'(z_\xi+w)-h'(z_\xi)\right]v,$$

where $h'(z_\xi+w)v$ is defined by

$$\begin{aligned}h'(z_\xi+w)v=\int_{\mathbb{R}^3}&\big[p\left(|x|^{-1}*|z_\xi+w|^{p-1}\right)|z_\xi+w|^{p-1}\\&+(p-1)\left(|x|^{-1}*|z_\xi+w|^{p}\right)|z_\xi+w|^{p-2}\big]\varphi v dx\end{aligned}$$

for $\varphi\in H^1(\mathbb{R}^3)$. In fact, D_wH_ε is a Fredholm map of index zero.

Consider the equation

$$D_wH_\varepsilon(\xi,w)[v]=0,$$

which is equivalent to

$$v=\mathcal{L}_{\varepsilon,\xi}^{-1}P_{\varepsilon,\xi}\left[h'(z_\xi+w)-h'(z_\xi)\right]v. \tag{3.24}$$

Since $w(\varepsilon)\to0$ as $\varepsilon\to0$, it follows that for ε sufficiently small, there exists a unique trivial solution such that (3.24) holds. Thus the only fixed point $w_{\varepsilon,\xi}$ of $N_{\varepsilon,\xi}$ is smooth with respect to ξ. According to the argument in [1], then the critical points of $\Phi_\varepsilon(\xi)$ give rise to critical points of f_ε $\square$

Finally, we show that Φ_ε is a perturbation of some functions of V.

By the definition of $\Phi_\varepsilon(\xi)$, we see that

$$\begin{aligned}\Phi_\varepsilon(\xi)=&\frac12\|z_\xi+w_{\varepsilon,\xi}\|^2+\frac12\int_{\mathbb{R}^3}V(\varepsilon x)(z_\xi+w_{\varepsilon,\xi})^2dx\\&-\frac1{2p}\int_{\mathbb{R}^3}(|x|^{-1}*|z_\xi+w_{\varepsilon,\xi}|^p)|z_\xi+w_{\varepsilon,\xi}|^pdx.\end{aligned}$$

Since z_ξ is the solution of (2.3), we have that

$$\|z_\xi\|^2=-V(\varepsilon\xi)\int_{\mathbb{R}^3}z_\xi^2dx+\int_{\mathbb{R}^3}(|x|^{-1}*|z_\xi|^p)|z_\xi|^pdx$$

and

$$\langle z_\xi,w\rangle=-V(\varepsilon\xi)\int_{\mathbb{R}^3}z_\xi wdx+\int_{\mathbb{R}^3}(|x|^{-1}*|z_\xi|^p)|z_\xi|^{p-1}wdx.$$

Then it is easy to get that

$$\begin{aligned}\Phi_\varepsilon(\xi)=&\frac12\int_{\mathbb{R}^3}\left(V(\varepsilon x)-V(\varepsilon\xi)\right)z_\xi^2dx+\int_{\mathbb{R}^3}\left(V(\varepsilon x)-V(\varepsilon\xi)\right)z_\xi w_{\varepsilon,\xi}dx\\&+\frac12\|w_{\varepsilon,\xi}\|^2+\frac12\int_{\mathbb{R}^3}V(\varepsilon x)w_{\varepsilon,\xi}^2dx+\int_{\mathbb{R}^3}(|x|^{-1}*|z_\xi|^p)|z_\xi|^{p-1}w_{\varepsilon,\xi}dx\\&+\frac12\int_{\mathbb{R}^3}(|x|^{-1}*|z_\xi|^p)|z_\xi|^pdx-\frac1{2p}\int_{\mathbb{R}^3}(|x|^{-1}*|z_\xi+w_{\varepsilon,\xi}|^p)|z_\xi+w_{\varepsilon,\xi}|^pdx.\end{aligned}$$

For convenience, let

$$K_\varepsilon(\xi) = \left(\frac{1}{2} - \frac{1}{2p}\right) \int_{\mathbb{R}^3} (|x|^{-1} * |z_\xi|^p)|z_\xi|^p dx,$$
$$\Gamma_\varepsilon(\xi) = \frac{1}{2} \int_{\mathbb{R}^3} \left(V(\varepsilon x) - V(\varepsilon\xi)\right) z_\xi^2 dx + \int_{\mathbb{R}^3} \left(V(\varepsilon x) - V(\varepsilon\xi)\right) z_\xi w_{\varepsilon,\xi} dx$$

and

$$\begin{aligned}\Psi_\varepsilon(\xi) =& \frac{1}{2}\|w_{\varepsilon,\xi}\|^2 + \frac{1}{2}\int_{\mathbb{R}^3} V(\varepsilon x) w_{\varepsilon,\xi}^2 dx + \int_{\mathbb{R}^3} (|x|^{-1} * |z_\xi|^p)|z_\xi|^{p-1} w_{\varepsilon,\xi} dx \\ &+ \frac{1}{2p}\int_{\mathbb{R}^3} (|x|^{-1} * |z_\xi|^p)|z_\xi|^p dx - \frac{1}{2p}\int_{\mathbb{R}^3} (|x|^{-1} * |z_\xi + w_{\varepsilon,\xi}|^p)|z_\xi + w_{\varepsilon,\xi}|^p dx.\end{aligned}$$

Using (2.5), then we have

$$\begin{aligned}K_\varepsilon(\xi) &= \left(\frac{1}{2} - \frac{1}{2p}\right) \int_{\mathbb{R}^3} (|x|^{-1} * |z_\xi|^p)|z_\xi|^p dx \\ &= \left(\frac{1}{2} - \frac{1}{2p}\right) \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{(\alpha(\varepsilon\xi))^p U^p(\beta(\varepsilon\xi)(x-\xi))(\alpha(\varepsilon\xi))^p U^p(\beta(\varepsilon\xi)(y-\xi))}{|x-y|} dxdy \\ &= \left(\frac{1}{2} - \frac{1}{2p}\right) \int_{\mathbb{R}^3} (\alpha(\varepsilon\xi))^{2p} (\beta(\varepsilon\xi))^{-5} \int_{\mathbb{R}^3} \frac{U^p(\bar{x})U^p(\bar{y})}{|\bar{x}-\bar{y}|} d\bar{x}d\bar{y} \\ &= \left(\frac{1}{2} - \frac{1}{2p}\right) C_0 \left(1 + V(\varepsilon\xi)/\lambda\right)^{\frac{5-p}{2(p-1)}},\end{aligned}$$

where $C_0 = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{U^p(\bar{x})U^p(\bar{y})}{|\bar{x}-\bar{y}|} d\bar{x}d\bar{y}$.

Before we compute $\Phi_\varepsilon(\xi)$, we will give the estimate of $\nabla_\xi w$.

Lemma 3.3. *For $0 < \varepsilon \le \varepsilon_0$ and $|\xi| \le \bar{\delta}$, the following holds:*

$$|\nabla_\xi w| \le C(\varepsilon|\nabla V(\varepsilon\xi)| + O(\varepsilon^2)),$$

where $C > 0$ is a constant depending on $\bar{\delta}$ and ε_0.

Proof. For any $\varphi \in (T_{z_\xi}(Z_\varepsilon))^\perp$, (3.20) gives that

$$\langle L_{\varepsilon,\xi} w, \varphi\rangle + \langle Df_\varepsilon(z_\xi), \varphi\rangle + \langle \mathcal{R}(z_\xi, w), \varphi\rangle = 0.$$

From the definition of $L_{\varepsilon,\xi}$ and $DF_{\varepsilon,\xi}(z_\xi) = 0$, we have

$$\begin{aligned}0 =& \langle w, \varphi\rangle + \int_{\mathbb{R}^3} V(\varepsilon x) w\varphi dx - (p-1)\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{z_\xi^p(x) z_\xi^{p-2}(y) w(y)\varphi(y)}{|x-y|} dxdy \\ &- p\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{z_\xi^{p-1}(x) w(x) z_\xi^{p-1}(y)\varphi(y)}{|x-y|} dxdy \\ &+ \int_{\mathbb{R}^3} \left(V(\varepsilon x) - V(\varepsilon\xi)\right) z_\xi \varphi dx + \int_{\mathbb{R}^3} R(z_\xi, w)\varphi dx,\end{aligned}$$

which implies that

$$\begin{aligned}
0 =& \langle \partial_{\xi_i} w, \varphi\rangle + \int_{\mathbb{R}^3} V(\varepsilon x)\partial_{\xi_i} w\varphi dx - (p-1)\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{z_\xi^p(x) z_\xi^{p-2}(y)\partial_{\xi_i} w(y)\varphi(y)}{|x-y|} dxdy \\
&- p(p-1)\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{z_\xi^{p-1}(x)\partial_{\xi_i} z_\xi(x) z_\xi^{p-2}(y) w(y)\varphi(y)}{|x-y|} dxdy \\
&- (p-1)(p-2)\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{z_\xi^p(x) z_\xi^{p-3}(y)\partial_{\xi_i} z_\xi(y) w(y)\varphi(y)}{|x-y|} dxdy \\
&- p(p-1)\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{z_\xi^{p-2}(x)\partial_{\xi_i} z_\xi(x) w(x) z_\xi^{p-1}(y)\varphi(y)}{|x-y|} dxdy \\
&- p(p-1)\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{z_\xi^{p-1}(x) w(x) z_\xi^{p-2}(y)\partial_{\xi_i} z_\xi(y)\varphi(y)}{|x-y|} dxdy \\
&- p\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{z_\xi^{p-1}(x)\partial_{\xi_i} w(x) z_\xi^{p-1}(y)\varphi(y)}{|x-y|} dxdy - \varepsilon\left(\partial_{\xi_i} V\right)(\varepsilon\xi)\int_{\mathbb{R}^3} z_\xi\varphi dx \\
&+ \int_{\mathbb{R}^3} \left(V(\varepsilon x) - V(\varepsilon\xi)\right)\partial_{\xi_i} z_\xi\varphi dx + \partial_{\xi_i}\left(\int_{\mathbb{R}^3} R(z_\xi, w)\varphi dx\right).
\end{aligned}$$

Using the definition of $L_{\varepsilon,\xi}$ again, we obtain that

$$\begin{aligned}
\langle L_{\varepsilon,\xi}\partial_{\xi_i} w, \varphi\rangle =& p(p-1)\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{z_\xi^{p-1}(x)\partial_{\xi_i} z_\xi(x) z_\xi^{p-2}(y) w(y)\varphi(y)}{|x-y|} dxdy \\
&+ (p-1)(p-2)\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{z_\xi^p(x) z_\xi^{p-3}(y)\partial_{\xi_i} z_\xi(y) w(y)\varphi(y)}{|x-y|} dxdy \\
&+ p(p-1)\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{z_\xi^{p-2}(x)\partial_{\xi_i} z_\xi(x) w(x) z_\xi^{p-1}(y)\varphi(y)}{|x-y|} dxdy \\
&+ p(p-1)\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{z_\xi^{p-1}(x) w(x) z_\xi^{p-2}(y)\partial_{\xi_i} z_\xi(y)\varphi(y)}{|x-y|} dxdy \\
&+ \varepsilon\left(\partial_{\xi_i} V\right)(\varepsilon\xi)\int_{\mathbb{R}^3} z_\xi\varphi dx - \partial_{\xi_i}\left(\int_{\mathbb{R}^3} R(z_\xi, w)\varphi dx\right) \\
&- \int_{\mathbb{R}^3} \left(V(\varepsilon x) - V(\varepsilon\xi)\right)\partial_{\xi_i} z_\xi\varphi dx.
\end{aligned} \tag{3.25}$$

We estimate the right side of (3.25) term by term.

Lemma 2.4 and Hölder inequality yield that

$$\left|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{z_\xi^p(x) z_\xi^{p-3}(y)\partial_{\xi_i} z_\xi(y) w(y)\varphi(y)}{|x-y|} dxdy\right| \leq C\|w\|\|\varphi\|, \tag{3.26}$$

$$\left|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{z_\xi^{p-1}(x)\partial_{\xi_i} z_\xi(x) z_\xi^{p-2}(y) w(y)\varphi(y)}{|x-y|} dxdy\right| \leq C\|w\|\|\varphi\| \tag{3.27}$$

and

$$\begin{aligned}
&\left|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{z_\xi^{p-2}(x)\partial_{\xi_i}z_\xi(x)w(x)z_\xi^{p-1}(y)\varphi(y)}{|x-y|}dxdy\right| \\
&+\left|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{z_\xi^{p-1}(x)w(x)z_\xi^{p-2}(y)\partial_{\xi_i}z_\xi(y)\varphi(y)}{|x-y|}dxdy\right| \le C\|w\|\|\varphi\|.
\end{aligned} \tag{3.28}$$

Similarly, we obtain that

$$\begin{aligned}
&\left|\partial_{\xi_i}\left(\int_{\mathbb{R}^3}R(z_\xi,w)\varphi dx\right)\right| \\
\le&\left|\int_{\mathbb{R}^3}\left(p\left[|x|^{-1}*\left(z_\xi^{p-1}\partial_{\xi_i}z_\xi\right)\right]z_\xi^{p-1}+(p-1)\left[|x|^{-1}*z_\xi^p\right]z_\xi^{p-2}\partial_{\xi_i}z_\xi\right)\varphi dx\right| \\
&+\left|\int_{\mathbb{R}^3}\left(p(p-1)\left[|x|^{-1}*\left(z_\xi^{p-1}\partial_{\xi_i}z_\xi\right)\right]z_\xi^{p-2}w+(p-1)\left[|x|^{-1}*z_\xi^p\right]z_\xi^{p-2}\partial_{\xi_i}w\right.\right. \\
&\left.\left.+(p-1)(p-2)\left[|x|^{-1}*z_\xi^p\right]z_\xi^{p-3}w\partial_{\xi_i}z_\xi\right)\varphi dx\right| \\
&+\left|\int_{\mathbb{R}^3}\left(p\left[|x|^{-1}*\left(z_\xi^{p-1}\partial_{\xi_i}w\right)\right]z_\xi^{p-1}+p(p-1)\left[|x|^{-1}*\left(z_\xi^{p-2}w\partial_{\xi_i}z_\xi\right)\right]z_\xi^{p-1}\right.\right. \\
&\left.\left.+p(p-1)\left[|x|^{-1}*\left(z_\xi^{p-1}w\right)\right]z_\xi^{p-2}\partial_{\xi_i}z_\xi\right)\varphi dx\right| \\
&+\left|\int_{\mathbb{R}^3}p\left[|x|^{-1}*\left(|z_\xi+w|^{p-1}\partial_{\xi_i}\left(z_\xi+w\right)\right)\right]|z_\xi+w|^{p-1}\varphi dx\right| \\
&+\left|\int_{\mathbb{R}^3}(p-1)\left[|x|^{-1}*|z_\xi+w|^p\right]|z_\xi+w|^{p-2}\partial_{\xi_i}\left(z_\xi+w\right)\varphi dx\right| \\
\le& C\|w\|^{2p-2}\|\partial_{\xi_i}w\|\|\varphi\|.
\end{aligned} \tag{3.29}$$

By Hölder inequality, we have

$$\left|\varepsilon\left(\partial_{\xi_i}V\right)(\varepsilon\xi)\int_{\mathbb{R}^3}z_\xi\varphi dx\right| \le C\varepsilon\left|\nabla V(\varepsilon\xi)\right|\|\varphi\| \tag{3.30}$$

and

$$\left|\int_{\mathbb{R}^3}\left(V(\varepsilon x)-V(\varepsilon\xi)\right)\partial_{\xi_i}z_\xi\varphi dx\right| \le C\left(\varepsilon\left|\nabla V(\varepsilon\xi)\right|+\varepsilon^2\right)\|\varphi\|. \tag{3.31}$$

Combining (3.26), (3.27), (3.28), (3.29), (3.30) and (3.31), we deduce that

$$\|\partial_{\xi_i}w\| \le C\left(\varepsilon\left|\nabla V(\varepsilon\xi)\right|+\varepsilon^2\right),$$

where $C>0$ is a constant depending on $\bar{\delta}$ and ε_0. We finish the proof. □

Now we are in the position to eatimate $\Gamma_\varepsilon(\xi)$ and $\Psi_\varepsilon(\xi)$.

Lemma 3.4. *For $0<\varepsilon\le\varepsilon_0$ and $|\xi|\le\bar{\delta}$, the following hold:*

$$|\Gamma_\varepsilon(\xi)|+|\Psi_\varepsilon(\xi)| \le C\left(\varepsilon\left|\nabla V(\varepsilon\xi)\right|+\varepsilon^2\right) \tag{3.32}$$

and

$$|\nabla \Gamma_\varepsilon(\xi)| + |\nabla \Psi_\varepsilon(\xi)| \leq C\varepsilon^2. \tag{3.33}$$

Proof. Firstly, we compute (3.32). By Lemma 2.3 and Lemma 2.4,

$$\begin{aligned}|\Gamma_\varepsilon(\xi)| &\leq C\left[\varepsilon\,|\nabla V(\varepsilon\xi)| \int_{\mathbb{R}^3} |x-\xi| z_\xi (z_\xi + w_{\varepsilon,\xi}) dx + \varepsilon^2 \int_{\mathbb{R}^3} |x-\xi|^2 z_\xi (z_\xi + w_{\varepsilon,\xi}) dx\right] \\ &\leq C\left(\varepsilon\,|\nabla V(\varepsilon\xi)| + \varepsilon^2\right)\end{aligned}$$

and

$$\begin{aligned}|\Psi_\varepsilon(\xi)| \leq\, & C\|w_{\varepsilon,\xi}\|^2 + C\|w_{\varepsilon,\xi}\| - \frac{1}{2p}\int_{\mathbb{R}^3} \left(|x|^{-1} * |z_\xi|^p\right)\left(|z_\xi + w_{\varepsilon,\xi}|^p - |z_\xi|^p\right) dx \\ & - \frac{1}{2p}\int_{\mathbb{R}^3} \left(|x|^{-1} * |z_\xi + w_{\varepsilon,\xi}|^p\right)\left(|z_\xi + w_{\varepsilon,\xi}|^p - |z_\xi|^p\right) dx \\ \leq\, & C\left(\varepsilon\,|\nabla V(\varepsilon\xi)| + \varepsilon^2\right).\end{aligned}$$

Thus the proof of (3.32) is complete. Next, we estimate (3.33).

By Taylor expansion of V, we have

$$\begin{aligned}&\int_{\mathbb{R}^3} \left(V(\varepsilon x) - V(\varepsilon\xi)\right) z_\xi^2 dx \\ =&\varepsilon \int_{\mathbb{R}^3} \nabla V(\varepsilon\xi)\cdot(x-\xi) z_\xi^2 dx + \varepsilon^2 \int_{\mathbb{R}^3} D^2 V(\varepsilon\xi + \theta\varepsilon(x-\xi))[x-\xi, x-\xi] z_\xi^2 dx \\ =&\varepsilon \int_{\mathbb{R}^3} \nabla V(\varepsilon\xi)\cdot y z_\xi^2 dy + \varepsilon^2 \int_{\mathbb{R}^3} D^2 V(\varepsilon\xi + \theta\varepsilon(x-\xi))[x-\xi, x-\xi] z_\xi^2 dx \\ =&\varepsilon^2 \int_{\mathbb{R}^3} D^2 V(\varepsilon\xi + \theta\varepsilon(x-\xi))[x-\xi, x-\xi] z_\xi^2 dx,\end{aligned}$$

where $\theta \in (0,1)$. Because V satisfies (V_1), then it holds that

$$\begin{aligned}&\left|\partial_{\xi_i}\left(\int_{\mathbb{R}^3} \left(V(\varepsilon x) - V(\varepsilon\xi)\right) z_\xi^2 dx\right)\right| \\ \leq&\varepsilon^2 \left|\partial_{\xi_i}\left(\int_{\mathbb{R}^3} D^2 V(\varepsilon\xi + \theta\varepsilon(x-\xi))[x-\xi, x-\xi] z_\xi^2 dx\right)\right| \leq C\varepsilon^2.\end{aligned} \tag{3.34}$$

By Hölder inequality, we easily calculate

$$\begin{aligned}&\left|\partial_{\xi_i}\left(\int_{\mathbb{R}^3} \left(V(\varepsilon x) - V(\varepsilon\xi)\right) z_\xi w_{\varepsilon,\xi} dx\right)\right| \\ \leq&\varepsilon|\nabla V(\varepsilon\xi)| \int_{\mathbb{R}^3} |z_\xi w_{\varepsilon,\xi}| dx + \int_{\mathbb{R}^3} |V(\varepsilon x) - V(\varepsilon\xi)|\, |\partial_{\xi_i} z_\xi w_{\varepsilon,\xi}| dx \\ &+ \int_{\mathbb{R}^3} |V(\varepsilon x) - V(\varepsilon\xi)|\, |z_\xi \partial_{\xi_i} w_{\varepsilon,\xi}| dx \\ \leq&C\varepsilon|\nabla V(\varepsilon\xi)|\|w_{\varepsilon,\xi}\| + \left(\int_{\mathbb{R}^3} |V(\varepsilon x) - V(\varepsilon\xi)|^2 \,|\partial_{\xi_i} z_\xi|^2\right)^{1/2} \|w_{\varepsilon,\xi}\| \\ &+ \left(\int_{\mathbb{R}^3} |V(\varepsilon x) - V(\varepsilon\xi)|^2 \,|z_\xi|^2 dx\right)^{1/2} \|\partial_{\xi_i} w_{\varepsilon,\xi}\|.\end{aligned}$$

It follows that by Lemma 2.3 and Lemma 3.3

$$\left|\nabla\left(\int_{\mathbb{R}^3}[V(\varepsilon x)-V(\varepsilon\xi)]\,z_\xi w dx\right)\right| \le C\varepsilon\left(\varepsilon+|\nabla V(\varepsilon\xi)|\right)\left(\|w_{\varepsilon,\xi}\|+\|\nabla w_{\varepsilon,\xi}\|\right)\le C\varepsilon^2. \tag{3.35}$$

(3.34) and (3.35) imply that

$$|\nabla\Gamma_\varepsilon(\xi)|\le C\varepsilon^2. \tag{3.36}$$

To finish the proof of this lemma, we are going to estimate $|\nabla\Psi_\varepsilon(\xi)|$. Compute

$$\begin{aligned}\partial_{\xi_i}(\Psi_\varepsilon(\xi)) =&\langle w_{\varepsilon,\xi},\partial_{\xi_i}w_{\varepsilon,\xi}\rangle+\int_{\mathbb{R}^3}V(\varepsilon x)w_{\varepsilon,\xi}\partial_{\xi_i}w_{\varepsilon,\xi}dx\\ &+p\int_{\mathbb{R}^3}\left[|x|^{-1}*(|z_\xi|^{p-1}\partial_{\xi_i}z_\xi)\right]|z_\xi|^{p-1}w_{\varepsilon,\xi}dx\\ &+(p-1)\int_{\mathbb{R}^3}\left(|x|^{-1}*|z_\xi|^p\right)|z_\xi|^{p-2}\partial_{\xi_i}z_\xi w_{\varepsilon,\xi}dx\\ &+\int_{\mathbb{R}^3}(|x|^{-1}*|z_\xi|^p)|z_\xi|^{p-1}\partial_{\xi_i}(z_\xi+w_{\varepsilon,\xi})dx\\ &-\int_{\mathbb{R}^3}(|x|^{-1}*|z_\xi+w_{\varepsilon,\xi}|^p)|z_\xi+w_{\varepsilon,\xi}|^{p-1}\partial_{\xi_i}(z_\xi+w_{\varepsilon,\xi})dx\\ :=&I_1+I_2+I_3.\end{aligned}$$

For I_1, we have

$$\begin{aligned}&\left|\langle w_{\varepsilon,\xi},\partial_{\xi_i}w_{\varepsilon,\xi}\rangle+\int_{\mathbb{R}^3}V(\varepsilon x)w_{\varepsilon,\xi}\partial_{\xi_i}w_{\varepsilon,\xi}dx\right|\\ \le&\int_{\mathbb{R}^3}V(\varepsilon x)|\partial_{\xi_i}w_{\varepsilon,\xi}||w_{\varepsilon,\xi}|dx+|\langle w,\partial_{\xi_i}w_{\varepsilon,\xi}\rangle|\\ \le& C\|w_{\varepsilon,\xi}\|\|\partial_{\xi_i}w_{\varepsilon,\xi}\|.\end{aligned}$$

For I_2 and I_3, Lemma 2.4 gives that

$$\begin{aligned}|I_2|\le&\left|p\int_{\mathbb{R}^3}\left[|x|^{-1}*(|z_\xi|^{p-1}\partial_{\xi_i}z_\xi)\right]|z_\xi|^{p-1}w_{\varepsilon,\xi}dx\right|\\ &+\left|(p-1)\int_{\mathbb{R}^3}\left(|x|^{-1}*|z_\xi|^p\right)|z_\xi|^{p-2}\partial_{\xi_i}z_\xi w_{\varepsilon,\xi}dx\right|\\ \le& C\|w_{\varepsilon,\xi}\|\end{aligned}$$

and

$$\begin{aligned}|I_3|\le&\int_{\mathbb{R}^3}\left[|x|^{-1}*(|z_\xi+w_{\varepsilon,\xi}|^p-|z_\xi|^p)\right]|z_\xi+w_{\varepsilon,\xi}|^{p-1}\partial_{\xi_i}(z_\xi+w_{\varepsilon,\xi})dx\\ &+\int_{\mathbb{R}^3}\left(|x|^{-1}*|z_\xi|^p\right)\left(|z_\xi+w_{\varepsilon,\xi}|^{p-1}-|z_\xi|^{p-1}\right)\partial_{\xi_i}(z_\xi+w_{\varepsilon,\xi})dx\\ \le& C\|w_{\varepsilon,\xi}\|+C\|w_{\varepsilon,\xi}\|\|\partial_{\xi_i}w_{\varepsilon,\xi}\|.\end{aligned}$$

Putting these estimates together, we conclude that

$$|\nabla\Psi_\varepsilon(\xi)|\le C\|w_{\varepsilon,\xi}\|+C\|w_{\varepsilon,\xi}\|\|\partial_{\xi_i}w_{\varepsilon,\xi}\|\le C\varepsilon^2. \tag{3.37}$$

Combining (3.36) and (3.37), we finish the proof. □

3.3. The proof of Theorem 1.1.

Let $M \subset \mathbb{R}^3$ and M_δ be a non-empty set and its δ–neighbourhood, respectively. We denote by

$$l(M) := 1 + \sup\{k \in \mathbb{N} | \exists \Lambda_1, \Lambda_2, \cdots, \Lambda_k \in \check{H}^*(M) \backslash 1, \Lambda_1 \cup \Lambda_2 \cup \cdots \cup \Lambda_k \neq 0\}$$

the cup length of M, where $\check{H}^*(M)$ is the Alexander cohomology of M with real coefficients.

Proof of Theorem 1.1. Choose $\bar{R} > 0$ sufficiently large such that $M \subset B_{\bar{R}}(0)$, where M is a non-degenerate critical manifold of $\left(\frac{1}{2} - \frac{1}{2p}\right) C_0 \left(1 + V(\varepsilon\xi)/\lambda\right)^{\frac{5-p}{2(p-1)}}$. Let

$$f(\xi) = \left(\frac{1}{2} - \frac{1}{2p}\right) C_0 \left(1 + V(\varepsilon\xi)/\lambda\right)^{\frac{5-p}{2(p-1)}} \quad \text{and} \quad g(\xi) = \Phi_\varepsilon(\xi/\varepsilon).$$

Choose a δ-neighbourhood M_δ of M such that $M_\delta \subset B_{\bar{R}}(0)$, so the set of critical points of V in M_δ is M. Since

$$\Phi_\varepsilon(\xi) = \left(\frac{1}{2} - \frac{1}{2p}\right) C_0 \left(1 + V(\varepsilon\xi)/\lambda\right)^{\frac{5-p}{2(p-1)}} + \Gamma_\varepsilon(\xi) + \Psi_\varepsilon(\xi)$$

and Lemma 3.4, then the function $\Phi_\varepsilon(\cdot/\varepsilon)$ is converges to $f(\cdot)$ in $C^1(\overline{M_\delta})$ as $\varepsilon \to 0$. Thus there exist at least $l(M)$ critical points of g for ε sufficiently small.

Assume $\xi_k \in M_\delta$ satisfying that ξ_k/ε is a critical point of Φ_ε. Then Proposition 3.2 yields that

$$u_{\varepsilon,\xi_\varepsilon}(x) := z_{\xi_k}\left(x - \frac{\xi_k}{\varepsilon}\right) + w_{\varepsilon,\xi_k}$$

is a critical point of f_ε. Hence

$$u_{\varepsilon,\xi_\varepsilon}\left(\frac{x}{\varepsilon}\right) \simeq z_{\xi_k}\left(x - \frac{\xi_k}{\varepsilon}\right)$$

is a solution of (1.3). When $\varepsilon \to 0$, ξ_k converges to some point $\bar{\xi}_k \in M_\delta$. In conclusion, we have that $\bar{\xi}_k$ is a critical point of V by Lemma 3.4. Note that δ is arbitrary, so $\bar{\xi}_k \in M$. Therefore, $u_{\varepsilon,\xi_\varepsilon}\left(\frac{x}{\varepsilon}\right)$ concentrates to a point of M. This completes the proof. □

References

[1] A. Ambrosetti, M. Badiale, S. Cingolani, *Semiclassical states of nonlinear Schrödinger equations,* Arch. Rational Mech. Anal., 1997, 140, 285–300.

[2] W. Ao, J. Wei, J. Zeng, *An optimal bound on the number of interior peak solutions for the Lin-Ni-Takagi problem,* J. Funct. Anal., 2013, 265(7) 1324–1356.

[3] D. Cassani, J. Van Schaftingen, J. Zhang, *Groundstates for Choquard type equations with Hardy–Littlewood–Sobolev lower critical exponent,* Proceedings of the Royal Society of Edinburgh, 2017. DOI: 10.1017/prm.2018.135

[4] G. Chen, *Nondegeneracy of ground states and semiclassical solutions of the Hartree equation for general dimensions,* arXiv:1610.05503.

[5] S. Chen, X. Tang, J. Wei, *Nehari-type ground state solutions for a Choquard equation with doubly critical exponents,* Adv. Nonlinear Anal., 2021, 10(1) 152–171.

[6] Z. Cheng, M. Yang, *Stability of standing waves for a generalized Choquard equation with potential,* Acta Appl. Math., 2018, 157(1), 25–44.

[7] S. Cingolani, S. Secchi, *Semiclassical analysis for pseudo-relativistic Hartree equations,* J. Differential Equations, 2015, 258(12), 4156–4179.

[8] L. Du, M. Yang, *Uniqueness and nondegeneracy of solutions for a critical nonlocal equation,* Discrete and Continuous Dynamical Systems, 2019, 39(10), 5847–5866.

[9] J. Giacomoni, Y. Wei, M. Yang, *Nondegeneracy of solutions for a critical Hartree equation,* Nonlinear Analysis, 2020, 199, 111969.

[10] L. Guo, T. Hu, *Multi-bump solutions for nonlinear Choquard equation with potential wells and a general nonlinearity,* Acta Mathematica Scientia, 2020, 40(2), 316–340.

[11] R. Harrison, I. Moroz, K. P. Tod, *A numerical study of the Schrödinger–Newton equations,* Nonlinearity, 2003, 16(1), 101–122.

[12] E. H. Lieb, *Existence and uniqueness of the minimizing solution of Choquard's nonlinear equation,* Stud. Appl. Math., 1977 57(2), 93–105.

[13] Y. Li, G. Li, C. Tang, *Existence and concentration of ground state solutions for Choquard equations involving critical growth and steep potential well,* Nonlinear Analysis, 2020, 200, 111997.

[14] C. Lin, S. Peng, *Segregated vector solutions for linearly coupled nonlinear Schrödinger systems,* Indiana Univ. Math. J., 2014, 63(4), 939–967.

[15] F. Lin, W. Ni, J. Wei, *On the number of interior peak solutions for singularly perturbered Neumann problem,* Comm. Pure Appl. Math., 2007, 60(2), 252–281.

[16] L. Ma, L. Zhao, *Classification of positive solitary solutions of the nonlinear Choquard equation,* Arch. Rational Mech. Anal., 2010, 195(2), 455–467.

[17] V. Moroz, J. Van Schaftingen, *Groundstates of nonlinear Choquard equations: Existence, qualitative properties and decay asymptotics,* J. Funct. Anal., 2013, 265(2), 153–184.

[18] V. Moroz, J. Van Schaftingen, *Semi-classical states for the Choquard equation,* Calc. Var. Partial Differential Equations, 2015, 52(1), 199–235.

[19] S. Pekar, *Untersuchung über die Elektronentheorie der Kristalle,* Akademie Verlag, Berlin, **1**954.

[20] R. Penrose, *On gravity's role in quantum state reduction,* Gen. Relativity Gravitation, 1996, 28(5) 581–600.

[21] M. Riesz, *L'intégrale de Riemann-Liouville et le problème de Cauchy,* Acta Math., 1949 81(1) 1–223.

[22] S. Secchi, *A note on Schrödinger–Newton systems with decaying electric potential,* Nonlinear Analysis, 2010, 72(9–10), 3842–3856.

[23] L. Wang, *Number of synchronized and segregated interior spike solutions for nonlinear coupled elliptic systems with continuous potentials,* Science China-Math., 2019, 62(3), 509–534.

[24] J. Wei, M. Winter, *Strongly interacting bumps for the Schrödinger–Newton equations,* Journal of Mathematical Physics, 2009, 50(1), 012905.

[25] C. Xiang, *Uniqueness and nondegeneracy of ground states for Choquard equations in three dimensions,* Calc. Var. Partial Differential Equations, 2016, 55(6), 134–158.

[26] H. Zhang, J. Xu, F. Zhang, *Existence and multiplicity of solutions for a generalized Choquard equation,* Computers and Mathematics with Applications, 2017, 73(8), 1803–1814.

[27] H. Zhang, F. Zhang, *Infinitely many radial and nonradial solutions for a Choquard equation with general nonlinearity,* Applied Mathematics Letters, 2020, 102, 106142.

[28] J. Zhang, C. Lei, L. Guo, *Positive solutions for a nonlocal Schrödinger–Newton system involving critical nonlinearity,* Computers and Mathematics with Applications, 2018, 76(8), 1966–1974.