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SPREADING SPEEDS OF MONOSTABLE EQUATIONS IN LOCALLY SPATIALLY VARIATIONAL HABITAT WITH HYBRID DISPERSAL

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Abstract In this paper, we consider a monostable model with hybrid dispersal. This model characterizes the time evolution of a population which disperses both locally and nonlocally in locally spatially inhomogeneous media. It is shown that such equation has a spreading speed in every direction.

Keywords Monostable equations, random dispersal, nonlocal dispersal, localized spatial inhomogeneity, principal eigenvalue, sub-solution, super-solution, comparison principle, spreading speeds.

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1. Introduction

Population dynamics models are commonly used to describe the dispersal of species. The current paper is concerned with spreading properties of species in locally spatially variational environments or habitat with multiple dispersal strategies. The following model characterizes the species adopting both random and nonlocal dispersal,

$$\frac{\partial u(t,x)}{\partial t} = d[\tau\Delta u(t,x) + (1-\tau)\mathcal{K}u(t,x)] + u(t,x)f(x,u(t,x)), \quad x \in \mathbb{R}^N \tag{1.1}$$

where $u(t,x)$ denotes the density of species at location x and time t. The expression $\Delta = \Sigma_{i=1}^N \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator in $\mathbb{R}^N$ accounting for random dispersal of species.

The nonlocal operator $\mathcal{K}$ is defined by

$$(\mathcal{K}u)(t,x) = \int_{\mathbb{R}^N} \kappa(|y-x|)u(t,y)dy - u(t,x) \tag{1.2}$$

where $\kappa = \kappa(r)$ is a smooth, decreasing function with compact support such that

$$\upsilon_N \int_0^\infty \kappa(r) r^{N-1} dr = 1 \tag{1.3}$$

where υ_N denotes the area of the surface of the N-dimensional unit ball. Moreover, d is a positive constant which measures the total number of dispersal individuals

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per unit time, the constant τ, $(0 < \tau \leq 1)$, measures the total number of dispersal individuals adopting random dispersal.

If $\tau = 1$, then (1.1) is the classical reaction diffusion equation, so called random dispersal equation,

$$u_t(t,x) = d\Delta u(t,x) + u(t,x)f(x,u(t,x)), \quad x \in \mathbb{R}^N \tag{1.4}$$

which is broadly used to model the population dynamics of many species in unbounded environments, where $f(x,u)$ represents the growth rate of the population, which satisfies that $f(x,u) < 0$ for $u \gg 1$ and $\partial_u f(x,u) < 0$ for $u \geq 0$ (see [1, 2, 9, 17, 19, 20, 31, 40, 52, 53, 55, 56, 59], etc.).

If $\tau = 0$, then (1.1) is so called nonlocal dispersal equation,

$$u_t(t,x) = d[\int_{\mathbb{R}^N} \kappa(y-x)u(t,y)dy - u(t,x)] + u(t,x)f(x,u(t,x)), \quad x \in \mathbb{R}^N, \tag{1.5}$$

where $f(\cdot,\cdot)$ is of the same property as f in (1.4) (see [3, 10–12, 18, 22, 27, 35, 36], etc.).

When using (1.4) to model the population dynamics of a species, it is assumed that the underlying environment is continuous and the dispersal of cells or organisms are based on the hypothesis that the movement of the dispersing species can be described as a random walk in which there is no correlation between steps. However, dispersal of large organisms often involves mechanisms that may introduce correlations in movements. To model the population dynamics of such species in the case that the underlying environment is continuous, the nonlocal dispersal equation (1.5) is often used. This paper propose to study a mixed dispersal strategy, that is, a hybrid of random and non-local dispersal. We assume that a fraction of individuals in the population adopt random dispersal, while the rest fraction assumes non-local dispersal. Some research has been done on the hybrid dispersal in the spatially periodic habitat (see [14, 29, 30, 32], and [57]). Our main goal is to study how the hybrid dispersal affects the spreading properties of a single species and how the hybrid dispersal strategies will evolve in spatially locally inhomogeneous environment (see H1 and H2).

Since the seminal works by Fisher [20] and Kolmogorov, Petrowsky, Piscunov [31] on the following special case of (1.4),

$$\frac{\partial}{\partial t}u(t,x) = \frac{\partial^2}{\partial x^2}u(t,x) + u(t,x)(1-u(t,x)), \qquad x \in \mathbb{R}. \tag{1.6}$$

A vast number of research has been carried out toward the spatial spreading dynamics of (1.4) and (1.5) with $f(\cdot,\cdot)$ being independent of the space variable or periodic in the space variable, which reflects the spatial periodicity of the media. We refer to [1, 2, 5, 28, 38, 39, 47, 54, 55], etc. for the study of (1.4) in the case that $f(x,u)$ is independent of x and refer to [4, 6, 21, 23, 26, 41, 43, 44, 56], etc. for the study of (1.4) in the case that $f(x,u)$ is periodic in x; refer to [15, 16, 37], etc. for the study of (1.5) in the case that $f(x,u)$ is independent of x and refer to [25, 49–51], etc. for the study of (1.5) in the case that $f(x,u)$ is periodic in x and refer to [7, 8, 33, 34, 45, 48], etc. for the study of (1.4) and/or (1.5) in the case that $f(t,x,u)$ is temporally and/or spatially heterogeneous.

For instance, consider (1.4) and assume that $f(x + p_i\mathbf{e}_i, u) = f(x,u)$ for $i = 1, 2, \cdots, N$, where $p_i > 0$ $(i = 1, 2, \cdots, N)$ and

$$\mathbf{e}_i = (\delta_{i1}, \delta_{i2}, \cdots, \delta_{iN}),\ \delta_{ij} = 1 \text{ if } i = j \text{ and } 0 \text{ if } i \neq j.$$

If the principal eigenvalue of the following eigenvalue problem associated to the linearized equation of (1.4) at $u = 0$,

$$\begin{cases} \Delta u(x) + f(x,0)u(x) = \lambda u(x), & x \in \mathbb{R}^N \\ u(x + p_i \mathbf{e}_i) = u(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.7}$$

is positive, then (1.4) has a unique positive stationary solution $u^*(\cdot)$ with $u^*(\cdot + p_i \mathbf{e}_i) = u^*(\cdot)$.

In this paper, we consider (1.1) in the case that the growth rates depend on the space variable, but only when it is in some bounded subset of the underlying habitat, which reflects the localized spatial inhomogeneity of the media. More precisely, we assume

(H1) *$f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a C^2 function, $f(x,u) < 0$ for all $(x,u) \in \mathbb{R}^N \times \mathbb{R}^+$ with $u \ge \beta_0$ for some $\beta_0 > 0$, and $\partial_u f(x,u) < 0$ for all $(x,u) \in \mathbb{R}^N \times \mathbb{R}^+$.*

(H2) *$f(x,u) = f^0(u)$ for some C^2 function $f^0 : \mathbb{R} \to \mathbb{R}$ and all $(x,u) \in \mathbb{R}^N \times \mathbb{R}$ with $\|x\| \ge L_0$ for some $L_0 > 0$, and $f^0(0) > 0$.*

Assume (H1) and (H2). Then (1.1) has the following limit equations as $\|x\| \to \infty$,

$$u_t(t,x) = d[\tau \Delta u(t,x) + (1-\tau)\mathcal{K}u(t,x)] + u(t,x) f^0(u(t,x)), \quad x \in \mathbb{R}^N. \tag{1.8}$$

Equations (1.8) will play an important role in the study of (1.1). Equations (1.8) has a unique positive constant stationary solution u^0. We introduce some standing notations and then state the main results of the paper.

We define X by

$$X = \{u \in C(\mathbb{R}^N, \mathbb{R}) \,|\, u \text{ is uniformly continuous and bounded}\} \tag{1.9}$$

with norm $\|u\|_X = \sup_{x \in \mathbb{R}^N} |u(x)|$,

Let

$$X^+ = \{u \in X \,|\, u(x) \ge 0 \ \forall x \in \mathbb{R}^N\} \tag{1.10}$$

and

$$X^{++} = \{u \in X^+ \,|\, \inf_{x \in \mathbb{R}^N} u(x) > 0\}. \tag{1.11}$$

Without occurring confusion, we may write $\|\cdot\|_X$ as $\|\cdot\|$. Assume (H1). By general semigroup theory (see [24], [46]), for any $u_0 \in X$, (1.4) has a unique local solution $u(t,\cdot;u_0)$ with $u(0,\cdot;u_0) = u_0(\cdot)$. Moreover, if $u_0 \in X^+$, then $u(t,\cdot;u_0)$ exist and $u(t,\cdot;u_0) \in X^+$ for all $t \ge 0$ (see Proposition 2.2).

Let

$$S^{N-1} = \{\xi \in \mathbb{R}^N \,|\, \|\xi\| = 1\}. \tag{1.12}$$

For given $\xi \in S^{N-1}$ and $u \in X^+$, we define

$$\liminf_{x\cdot\xi \to -\infty} u(x) = \liminf_{r \to -\infty} \inf_{x \in \mathbb{R}^N, x\cdot\xi \le r} u(x).$$

For given $u : [0,\infty) \times \mathbb{R}^N \to \mathbb{R}$ and $c > 0$, we define

$$\liminf_{x\cdot\xi \le ct, t \to \infty} u(t,x) = \liminf_{t \to \infty} \inf_{x \in \mathbb{R}^N, x\cdot\xi \le ct} u(t,x),$$

$$\limsup_{x\cdot\xi\geq ct,t\to\infty} u(t,x)=\limsup_{t\to\infty}\sup_{x\in\mathbb{R}^N,x\cdot\xi\geq ct} u(t,x).$$

The notions $\limsup_{|x\cdot\xi|\leq ct,t\to\infty} u(t,x)$, $\limsup_{|x\cdot\xi|\geq ct,t\to\infty} u(t,x)$, $\limsup_{\|x\|\leq ct,t\to\infty} u(t,x)$, and $\limsup_{\|x\|\geq ct,t\to\infty} u(t,x)$ are defined similarly. We define $X^+(\xi)$ by

$$X^+(\xi)=\{u\in X^+ \mid \liminf_{x\cdot\xi\to-\infty} u(x)>0, \quad u(x)=0 \text{ for } x\cdot\xi\gg 1\}. \tag{1.13}$$

Definition 1.1 (Spatial spreading speed)**.** For given $\xi\in S^{N-1}$, a real number $c^*(\xi)$ is called the spatial spreading speed of (1.1) in the direction of ξ if for any $u_0\in X^+(\xi)$,

$$\liminf_{x\cdot\xi\leq ct,t\to\infty} u(t,x;u_0)>0 \quad \forall c<c^*(\xi)$$

and

$$\limsup_{x\cdot\xi\geq ct,t\to\infty} u(t,x;u_0)=0 \quad \forall c>c^*(\xi).$$

Our objective is to explore the spatial spreading dynamics of (1.1) with localized spatial inhomogeneity. The main results of this paper are stated in the following two theorems.

Theorem 1.1 (Existence and characterization of spreading speeds)**.** *Assume* (H1) *and* (H2)*. Then for any given $\xi\in S^{N-1}$,* (1.1) *has a positive spreading speed $c^*(\xi)$ in the direction of ξ. Moreover, for any $u_0\in X^+(\xi)$,*

$$\liminf_{x\cdot\xi\leq ct,t\to\infty} |u(t,x;u_0)-u^*(x)|=0 \quad \forall c<c^*(\xi), \tag{1.14}$$

and

$$c^*(\xi)=c^0(\xi)$$

where

$$c^0(\xi)=\inf_{\mu>0}\frac{f^0(0)+\mu^2}{\mu}=2\sqrt{f^0(0)}, \quad \text{if} \quad \tau=1 \tag{1.15}$$

$$c^0(\xi)=\inf_{\mu>0}\frac{\int_{\mathbb{R}^N} e^{-\mu z\cdot\xi}\kappa(z)dz-1+f^0(0)}{\mu}, \quad \text{if} \quad \tau=0 \tag{1.16}$$

and

are the spatial spreading speeds of (1.8) *in the direction of ξ.*

Theorem 1.2 (Spreading features of spreading speeds)**.** *Assume* (H1) *and* (H2)*. Then for any given $\xi\in S^{N-1}$, the following hold.*

(1) For each $u_0\in X^+$ satisfying that $u_0(x)=0$ for $x\in\mathbb{R}^N$ with $|x\cdot\xi|\gg 1$,

$$\limsup_{|x\cdot\xi|\geq ct,t\to\infty} u(t,x;u_0)=0 \quad \forall c>\max\{c^*(\xi),c^*(-\xi)\}.$$

(2) For each $\sigma>0$, $r>0$, and $u_0\in X^+$ satisfying that $u_0(x)\geq\sigma$ for $x\in\mathbb{R}^N$ with $|x\cdot\xi|\leq r$,

$$\limsup_{|x\cdot\xi|\leq ct,t\to\infty} |u(t,x;u_0)-u^*(x)|=0 \quad \forall 0<c<\min\{c^*(\xi),c^*(-\xi)\}.$$

(3) For each $u_0 \in X^+$ satisfying that $u_0(x) = 0$ for $x \in \mathbb{R}^N$ with $\|x\| \gg 1$,

$$\limsup_{\|x\| \geq ct, t\to\infty} u(t, x; u_0) = 0 \quad \forall c > \sup_{\xi \in S^{N-1}} c^*(\xi).$$

(4) For each $\sigma > 0$, $r > 0$, and $u_0 \in X^+$ satisfying that $u_0(x) \geq \sigma$ for $\|x\| \leq r$,

$$\limsup_{\|x\| \leq ct, t\to\infty} |u(t, x; u_0) - u^*(x)| = 0 \quad \forall 0 < c < \inf_{\xi \in S^{N-1}} c^*(\xi).$$

To indicate the dependence of $u^*(\cdot)$ and $c^*(\xi)$ on f, we may sometime write $u^*(\cdot)$ and $c^*(\xi)$ as $u^*(\cdot; f(\cdot, \cdot))$ and $c^*(\xi; f(\cdot, \cdot))$, respectively.

The rest of the paper is organized as follows. In section 2, we present some preliminary materials to be used in later sections. Section 3 is devoted to the study of spreading speeds of (1.1). Theorem 1.1 and Theorem 1.2 are proved in this section.

2. Preliminary

In this section, we present some preliminary materials to be used in later sections, including some basic properties of solutions of (1.1); principal eigenvalue theories for spatially periodic dispersal operators with random, and nonlocal; and spatial spreading dynamics of KPP equations in spatially periodic media.

2.1. Classic properties of Monostable equations

In this subsection, we present some basic properties of solutions of (1.1), including comparison principle, global existence, convergence in open compact topology, and decreasing of the so called part metric along the solutions. Throughout this subsection, we assume (H1).

Let X be as in (1.9). For given $u_0 \in X$, let $u(t, \cdot; u_0)$ be the (local) solution of (1.4) with $u(0, \cdot; u_0) = u_0(\cdot)$.

Let X^+ and X^{++} be as in (1.10) and (1.11). For given $u, v \in X$, we define

$$u \leq v \ (u \geq v) \quad \text{if } v - u \in X^+ \ (u - v \in X^+) \tag{2.1}$$

and

$$u \ll v \ (u \gg v) \quad \text{if } v - u \in X^{++} \ (u - v \in X^{++}). \tag{2.2}$$

For given continuous and bounded function $u : [0, T) \times \mathbb{R}^N \to \mathbb{R}$, it is called a *super-solution* (*sub-solution*) of (1.1) on $[0, T)$ if

$$u_t(t, x) \geq (\leq) d[\tau\Delta u(t, x) + (1-\tau)\mathcal{K}u(t, x)] + u(t, x)f(x, u(t, x)) \quad \forall (t, x) \in (0, T)\times\mathbb{R}^N.$$

Proposition 2.1 (Comparison principle)**.** *Assume* (H1).

(1) Suppose that $u^1(t, x)$ and $u^2(t, x)$ are sub- and super-solutions of (1.1) *on $[0, T)$ with $u^1(0, \cdot) \leq u^2(0, \cdot)$. Then $u^1(t, \cdot) \leq u^2(t, \cdot)$ for $t \in (0, T)$. Moreover, if $u^1(0, \cdot) \neq u^2(0, \cdot)$, then $u^1(t, x) < u^2(t, x)$ for $x \in \mathbb{R}^N$, and $t \in (0, T)$.*

(2) If $u_{01}, u_{02} \in X$ and $u_{01} \leq u_{02}$, $u_{01} \neq u_{02}$, then $u(t, x; u_{01}) < u(t, x; u_{02})$ for all $x \in \mathbb{R}^N$ and $t > 0$ at which both $u(t, \cdot; u_{01})$ and $u(t, \cdot; u_{02})$ exist.

(3) *If* $u_{01}, u_{02} \in X$ *and* $u_{01} \ll u_{02}$*, then* $u(t,\cdot;u_{01}) \ll u(t,\cdot;u_{02})$ *for* $t > 0$ *at which both* $u(t,\cdot;u_{01})$ *and* $u(t,\cdot;u_{02})$ *exist.*

Proof. (1) The case $\tau = 1$ follows from comparison principle for parabolic equations. The case $\tau = 0$ follows from [49, Propositions 2.1 and 2.2].

(2) follows from (1).

(3) We provide a proof for the case $\tau = 0$. Other cases can be proved similarly. Take any $T > 0$ such that both $u(t,\cdot;u_{01})$ and $u(t,\cdot;u_{02})$ exist on $[0,T]$. It suffices to prove that $u(t,\cdot;u_{02}) \gg u(t,\cdot;u_{01})$ for $t \in [0,T]$. To this end, let $w(t,x) = u(t,x;u_{02}) - u(t,x;u_{01})$. Then $w(t,x)$ satisfies the following equation,

$$w_t(t,x) = \int_{\mathbb{R}^N} \kappa(y-x)w(t,y)dy - w(t,x) + a(t,x)w(t,x),$$

where

$$\begin{aligned} a(t,x) =& f(x,u(t,x;u_{02})) \\ &+ u(t,x;u_{01})\int_0^1 \partial_u f(x, su(t,x;u_{02}) + (1-s)u(t,x;u_{01}))ds. \end{aligned}$$

Let $M > 0$ be such that $M \geq \sup_{x\in\mathbb{R}^N, t\in[0,T]}(1 - a(t,x))$ and $\tilde{w}(t,x) = e^{Mt}w(t,x)$. Then $\tilde{w}(t,x)$ satisfies

$$\tilde{w}_t(t,x) = \int_{\mathbb{R}^N} \kappa(y-x)\tilde{w}(t,y)dy + [M - 1 + a(t,x)]\tilde{w}(t,x).$$

Let $\mathcal{M} : X \to X$ be defined by

$$(\mathcal{M}u)(x) = \int_{\mathbb{R}^N} \kappa(y-x)u(y)dy \quad \text{for} \quad u \in X. \tag{2.3}$$

Then $\mathcal{M}$ generates an analytic semigroup on X and

$$\tilde{w}(t,\cdot) = e^{\mathcal{K}t}(u_{02} - u_{01}) + \int_0^t e^{\mathcal{K}(t-\tau)}(M - 1 + a(\tau,\cdot))\tilde{w}(\tau,\cdot)d\tau.$$

Observe that $e^{\mathcal{K}t}u_0 \geq 0$ for any $u_0 \in X^+$ and $t \geq 0$ and $e^{\mathcal{K}t}u_0 \gg 0$ for any $u_0 \in X^{++}$ and $t \geq 0$. Observe also that $u_{02} - u_{01} \in X^{++}$. By (2), $\tilde{w}(\tau,\cdot) \geq 0$ and hence $(M - 1 + a(\tau,\cdot))\tilde{w}(\tau,\cdot) \geq 0$ for $\tau \in [0,T]$. It then follows that $\tilde{w}(t,\cdot) \gg 0$ and then $w(t,\cdot) \gg 0$ (i.e. $u(t,\cdot;u_{02}) \gg u(t,\cdot;u_{01})$) for $t \in [0,T]$. □

Proposition 2.2 (Global existence)**.** *Assume* (H1). *For any given* $u(t,\cdot;u_0)$ *exists for all* $t \geq 0$.

Proof. Let $u_0 \in X^+$ be given. There is $M \gg 1$ such that $0 \leq u_0(x) \leq M$ and $f(x,M) < 0$ for all $x \in \mathbb{R}^N$. Then by Proposition 2.1,

$$0 \leq u(t,\cdot;u_0) \leq M$$

for any $t > 0$ at which $u(t,\cdot;u_0)$ exists. It is then not difficult to prove that for any $T > 0$ such that $u(t,\cdot;u_0)$ exists on $(0,T)$, $\lim_{t\to T} u(t,\cdot;u_0)$ exists in X. This implies that $u(t,\cdot;u_0)$ exists and $u(t,\cdot;u_0) \geq 0$ for all $t \geq 0$. □

For given $u, v \in X^{++}$, define

$$\rho(u,v) = \inf\{\ln\alpha \,|\, \frac{1}{\alpha}u \le v \le \alpha u,\ \alpha \ge 1\}.$$

Observe that $\rho(u,v)$ is well defined and there is $\alpha \ge 1$ such that $\rho(u,v) = \ln\alpha$. Moreover, $\rho(u,v) = \rho(v,u)$ and $\rho(u,v) = 0$ iff $u \equiv v$. In literature, $\rho(u,v)$ is called the *part metric* between u and v.

Proposition 2.3. *For given $u_0, v_0 \in X^{++}$ with $u_0 \ne v_0$, $\rho(u(t,\cdot;u_0), u(t,\cdot;v_0))$ is non-increasing in $t \in (0,\infty)$.*

Proof. It can be proved by similar argument in [33, Proposition 3.3]. For completeness, we provide a proof here.

First, note that there is $\alpha^* > 1$ such that $\rho(u_0,v_0) = \ln\alpha^*$ and $\frac{1}{\alpha^*}u_0 \le v_0 \le \alpha^* u_0$. By Proposition 2.1,

$$u(t,\cdot;v_0) \le u(t,\cdot;\alpha^* u_0) \quad \text{for} \quad t > 0.$$

Let $v(t,x) = \alpha^* u(t,x;u_0)$. Then

$$\begin{aligned}
v_t(t,x) &= d[\tau\Delta v(t,x) + (1-\tau)\mathcal{K}v(t,x)] + v(t,x)f(x,u(t,x;u_0)) \\
&= d[\tau\Delta v(t,x) + (1-\tau)\mathcal{K}v(t,x)] \\
&\quad + v(t,x)f(x,v(t,x)) + v(t,x)f(x,u(t,x;u_0)) - v(t,x)f(x,v(t,x)) \\
&> d[\tau\Delta v(t,x) + (1-\tau)\mathcal{K}v(t,x)] + v(t,x)f(x,v(t,x)).
\end{aligned}$$

This together with Proposition 2.1 implies that

$$u(t,\cdot;\alpha^* u_0) \le \alpha^* u(t,\cdot;u_0) \quad \text{for} \quad t > 0$$

and then

$$u(t,\cdot;v_0) \le \alpha^* u(t,\cdot;u_0) \quad \text{for} \quad t > 0.$$

Similarly, it can be proved that

$$\frac{1}{\alpha^*}u(t,\cdot;u_0) \le u(t,\cdot;v_0) \quad \text{for} \quad t > 0.$$

It then follows that

$$\rho(u(t,\cdot;u_0), u(t,\cdot;v_0)) \le \rho(u_0,v_0) \quad \forall t > 0$$

and hence

$$\rho(u(t_2,\cdot;u_0), u(t_2,\cdot;v_0)) \le \rho(u(t_1,\cdot;u_0), u(t_1,\cdot;v_0)) \quad \forall 0 \le t_1 < t_2.$$

□

Proposition 2.4 (Convergence on compact subsets)**.** *Suppose that $u_{0n}, u_0 \in X^+$ $(n = 1,2,\cdots)$, $\{\|u_{0n}\|\}$ is bounded, and $u_{0n}(x) \to u_0(x)$ as $n \to \infty$ uniformly in x on bounded sets.*

(1) If $z_n, z^* \in \mathbb{R}^N$ $(n = 1,2,\cdots)$ are such that $f(x+z_n,u) \to f(x+z^*,u)$ as $n \to \infty$ uniformly in (x,u) on bounded sets, then for each $t > 0$, $u(t,x;u_{0n}, f(\cdot+z_n,\cdot)) \to u(t,x;u_0,f(\cdot+z^*,\cdot))$ as $n \to \infty$ uniformly in x on bounded sets.

(2) If $z_n \in \mathbb{R}^N$ $(n = 1,2,\cdots)$ are such that $f(x+z_n,u) \to f^0(u)$ as $n \to \infty$ uniformly in (x,u) on bounded sets, then for each $t > 0$, $u(t,x;u_{0n}, f(\cdot+z_n,\cdot)) \to u(t,x;u_0,f^0(\cdot))$ as $n \to \infty$ uniformly in x on bounded sets.

Proof. It can be proved by similar argument in [33, Proposition 3.4]. □

2.2. Principal eigenvalues of spatially periodic dispersal operators

In this subsection, we present some principal eigenvalue theories for spatially periodic dispersal operators with hybrid dispersals.

Let $p = (p_1, p_2, \cdots, p_N)$ with $p > 0$ for $i = 1, 2, \cdots, N$. We define the Banach spaces X_p by

$$X_p = \{u \in C(\mathbb{R}^N, \mathbb{R}) \,|\, u(\cdot + p_i \mathbf{e_i}) = u(\cdot), \quad i = 1, ..., N\} \tag{2.4}$$

with norm $\|u\|_{X_p} = \max_{x\in\mathbb{R}^N} |u(x)|$.

Let

$$X_p^+ = \{u \in X_p \,|\, u(x) \geq 0 \;\forall x \in \mathbb{R}^N\} \tag{2.5}$$

and

$$X_p^{++} = \{u \in X_p \,|\, u(x) > 0 \;\forall x \in \mathbb{R}^N\}. \tag{2.6}$$

We will denote $\mathcal{I}$ as an identity map on the Banach space under consideration. For given $\xi \in S^{N-1}$, $\mu \in \mathbb{R}$, $a \in X_p$, consider the following eigenvalue problems,

$$\begin{cases} \mathcal{O}u(x) = \lambda u(x), \quad x \in \mathbb{R}^N \\ u(x + p_i \mathbf{e}_i) = u(x), \quad x \in \mathbb{R}^N, \end{cases} \tag{2.7}$$

where

$$\mathcal{O}u(x) := \tau\Delta u(x) + \\ (1-\tau)[\int_{\mathbb{R}^N} e^{-\mu(y-x)\cdot\xi}\kappa(y-x)u(y)dy - u(x)] - 2\tau\mu\xi\cdot\nabla u(x) + (a(x) + \tau\mu^2)u(x), \tag{2.8}$$

and $\mathcal{O} : \mathcal{D}(\mathcal{O}) \subset X_p \to X_p$.

Observe that if $\tau = 1$,

$$(\mathcal{O}u)(x) = \Delta u(x) - 2\mu\xi\cdot\nabla u(x) + (a(x) + \mu^2)u(x) \quad \forall\, u \in \mathcal{D}(\mathcal{O}) \subset X_p. \tag{2.9}$$

If $\tau = 0$,

$$(\mathcal{O}u)(x) = \int_{\mathbb{R}^N} e^{-\mu(y-x)\cdot\xi}\kappa(y-x)u(y)dy - u(x) + a(x)u(x) \quad \forall\, u \in \mathcal{D}(\mathcal{O}) \subset X_p. \tag{2.10}$$

Let $\sigma(\mathcal{O})$ be the spectrum of $\mathcal{O}$.

Definition 2.1. Let $\mu \in \mathbb{R}$, and $\xi \in S^{N-1}$ be given. A real number $\lambda(\mu, \xi, a) \in \mathbb{R}$ is called the principal eigenvalue of $\mathcal{O}$ if it is an isolated algebraic simple eigenvalue of $\mathcal{O}$ with a positive eigenfunction and for any $\lambda \in \sigma(\mathcal{O})\backslash\{\lambda(\mu, \xi, a)\}$, $\mathrm{Re}\lambda < \lambda(\mu, \xi, a)$.

For given $\mu \in \mathbb{R}$, and $\xi \in S^{N-1}$, let

$$\lambda^0(\mu, \xi, a) = \sup\{\mathrm{Re}\mu \,|\, \mu \in \sigma(\mathcal{O})\}. \tag{2.11}$$

Observe that for any $\mu \in \mathbb{R}$ and $\xi \in S^{N-1}$, $\mathcal{O}$ generates an analytic semigroup $\{T(t)\}_{t\geq 0}$ in X_p and moreover, $T(t)$ is strongly positive (that is, $T(t)u_0 \geq 0$ for any $t \geq 0$ and $u_0 \in X_p^+$ and $T(t)u_0 \gg 0$ for any $t > 0$ and $u_0 \in X_p^+ \setminus \{0\}$). Then by [42, Proposition 4.1.1], $r(T(t)) \in \sigma(T(t))$ for any $t > 0$, where $r(T(t))$ is the spectral radius of $T(t)$. Hence by the spectral mapping theorem (see [13, Theorem

2.7]), $\lambda^0(\mu,\xi,a) \in \sigma(\mathcal{O})$. Observe also that $\lambda^0(0,\xi,a)$ are independent of $\xi \in S^{N-1}$. We may then put

$$\lambda^0(a) = \lambda^0(0,\xi,a).$$

It is well known that the principal eigenvalue $\lambda(\mu,\xi,a)$ in (2.9) exist for all $\mu \in \mathbb{R}$ and $\xi \in S^{N-1}$ and

$$\lambda(\mu,\xi,a) = \lambda^0(\mu,\xi,a).$$

The principal eigenvalue of $\mathcal{O}$ in (2.10) may not exist (see [49] for examples). If the principal eigenvalue $\lambda(\mu,\xi,a)$ exists in (2.10), then

$$\lambda(\mu,\xi,a) = \lambda^0(\mu,\xi,a).$$

Regarding the existence of principal eigenvalue of $\mathcal{O}$ in (2.10), the following proposition is proved in [49, 50].

Proposition 2.5 (Existence of principal eigenvalue). *(1) If $a \in C^N(\mathbb{R}^N,\mathbb{R}) \cap X_p$ and the partial derivatives of $a(x)$ up to order $N-1$ are zero at some x_0 satisfying that $a(x_0) = \max_{x\in\mathbb{R}^N} a(x)$, then the principal eigenvalue $\lambda(\mu,\xi,a)$ of $\mathcal{O}$ exists for all $\mu \in \mathbb{R}$ and $\xi \in S^{N-1}$.*

(2) If $a(x)$ satisfies that $\max_{x\in\mathbb{R}^N} a(x) - \min_{x\in\mathbb{R}^N} a(x) < \inf_{\xi\in S^{N-1}} \int_{z\cdot\xi\le 0} k(z)dz$, then the principal eigenvalue $\lambda(\mu,\xi,a)$ of $\mathcal{O}$ exists for all $\mu \in \mathbb{R}$ and $\xi \in S^{N-1}$.

Proof. (1) It follows from [49, Theorem B].

(2) It follows from [50, Theorem B$'$]. □

Let $\hat{a}$ be the average of $a(\cdot)$, that is,

$$\hat{a} = \frac{1}{|D|}\int_D a(x)dx \quad \text{for} \tag{2.12}$$

where

$$D = [0,p_1] \times [0,p_2] \times \cdots \times [0,p_N] \cap \mathbb{R}^N \tag{2.13}$$

and

$$|D| = p_1 \times p_2 \times \cdots \times p_N \text{ for} \tag{2.14}$$

By Proposition 2.5 (2), $\lambda(\mu,\xi,\hat{a})$ exists for all $\mu \in \mathbb{R}$ and $\xi \in S^{N-1}$. The following proposition shows a relation between $\lambda^0(\mu,\xi)$ and $\lambda^0(\mu,\xi,\hat{a})$.

Proposition 2.6 (Influence of spatial variation). *For given $\mu \in \mathbb{R}$, and $\xi \in S^{N-1}$, there holds*

$$\lambda^0(\mu,\xi) \ge \lambda^0(\mu,\xi,\hat{a}).$$

Proof. It follow from [25, Theorem 2.1]. □

We remark that $\lambda(\mu,\xi,\hat{a}) (= \lambda^0(\mu,\xi,\hat{a}))$ have the following explicit expressions,

$$\lambda(\mu,\xi,\hat{a}) = \tau\mu^2 + (1-\tau)\Big(\int_{\mathbb{R}^N} e^{-\mu z\cdot\xi}\kappa(z)dz - 1\Big) + \hat{a}. \tag{2.15}$$

2.3. Monostable equations in spatially periodic media

In this subsection, we recall some spatial spreading dynamics of KPP equations in spatially periodic media.

Consider

$$u_t(t,x) = d[\tau\Delta u(t,x) + (1-\tau)\mathcal{K}u(t,x)] + u(t,x)g(x,u(t,x)), \quad x \in \mathbb{R}^N, \tag{2.16}$$

where $g(\cdot,\cdot)$ are periodic in the first variable and monostable in the second variable. More precisely, we assume

(P1) *$g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a C^2 function, $g(x + p_l\mathbf{e}_l, u) = g(x,u)$, where $p_l > 0$ and $g(x,u) < 0$ for all $(x,u) \in \mathbb{R}^N \times \mathbb{R}^+$ with $u \ge \alpha_0$ for some $\alpha_0 > 0$ and $\partial_u g(x,u) < 0$ for all $(x,u) \in \mathbb{R}^N \times \mathbb{R}^+$.*

(P2) *$\lambda^0(g(\cdot,0)) > 0$.*

Assume (P1). Similarly, by general semigroup theory, for any $u_0 \in X$, (2.16) has a unique (local) solution $u(t,\cdot;u_0,g(\cdot,\cdot))(\in X)$ with initial data $u_0(\cdot)$. Moreover, if $u_0 \in X_p$, then $u(t,\cdot;u_0,g(\cdot,\cdot)) \in X_p$ for any $t > 0$ at which $u(t,\cdot;u_0,g(\cdot,\cdot))$ exists. By Proposition 2.1, if $u_0 \in X^+$, then $u(t,\cdot;u_0,g(\cdot,\cdot))$ exists and $u(t,\cdot;u_0,g(\cdot,\cdot)) \in X^+$ for all $t > 0$.

Proposition 2.7 (Spatially periodic positive stationary solution)**.** *Assume (P1) and (P2). Then* (2.16) *has a unique spatially periodic stationary solution $u^*(\cdot;g(\cdot,\cdot)) \in X_p^{++}$ which is globally asymptotically stable with respect to perturbations in $X_p^+ \setminus \{0\}$.*

Proof. It follows from [58, Theorem 2.3] and [50, Theorem C]. □

Proposition 2.8 (Spreading speeds)**.** *Assume (P1) and (P2). Then for any $\xi \in S^{N-1}$,* (2.16) *has a positive spreading speed $c^*(\xi;g_1(\cdot,\cdot))$ in the direction of ξ. Moreover,*

$$c^*(\xi;g(\cdot,\cdot)) = \inf_{\mu>0} \frac{\lambda^0(\mu,\xi,g(\cdot,0))}{\mu}$$

and the following hold.

(1) For each $u_0 \in X^+$ satisfying that $u_0(x) = 0$ for $x \in \mathbb{R}^N$ with $|x\cdot\xi| \gg 1$,

$$\limsup_{|x\cdot\xi|\ge ct,t\to\infty} u(t,x;u_0,g(\cdot,\cdot)) = 0 \quad \forall c > \max\{c^*(\xi;g(\cdot,\cdot)), c^*(-\xi;g(\cdot,\cdot))\}.$$

(2) For each $\sigma > 0$, $r > 0$, and $u_0 \in X^+$ satisfying that $u_0(x) \ge \sigma$ for $x \in \mathbb{R}^N$ with $|x\cdot\xi| \le r$,

$$\limsup_{|x\cdot\xi|\le ct,t\to\infty} |u(t,x;u_0,g(\cdot,\cdot)) - u^*(x;g(\cdot,\cdot))| = 0$$

for all $0 < c < \min\{c^*(\xi;g(\cdot,\cdot)), c^*(-\xi;g(\cdot,\cdot))\}$.

(3) For each $u_0 \in X^+$ satisfying that $u_0(x) = 0$ for $x \in \mathbb{R}^N$ with $\|x\| \gg 1$,

$$\limsup_{\|x\|\ge ct,t\to\infty} u(t,x;u_0,g(\cdot,\cdot)) = 0 \quad \forall c > \sup_{\xi\in S^{N-1}} c^*(\xi;g(\cdot,\cdot)).$$

(4) For each $\sigma > 0$, $r > 0$, and $u_0 \in X^+$ satisfying that $u_0(x) \ge \sigma$ for $x \in \mathbb{R}^N$ with $\|x\| \le r$,

$$\limsup_{\|x\| \le ct, t \to \infty} |u(t, x; u_0, g(\cdot,\cdot)) - u^*(x; g(\cdot,\cdot))| = 0 \quad \forall 0 < c < \inf_{\xi \in S^{N-1}} c^*(\xi; g(\cdot,\cdot)).$$

Proof. It follows from [50, Theorems D and E]. □

Let $\hat{g}(u)$ be the spatial average of $g(x, u)$, that is,

$$\hat{g}(u) = \frac{1}{|D|} \int_D g(x, u) dx \quad \text{for} \tag{2.17}$$

where D, $|D|$ is as in (2.13) and (2.14).

Assume

(P3) $\hat{g}(0) > 0$.

Observe that $\lambda(\hat{g}(0)) = \hat{g}(0)$. Then by Proposition 2.6, (P3) implies (P2).

Proposition 2.9 (Influence of spatial variation)**.** *Assume (P1) and (P3). Then for any $\xi \in S^{N-1}$,*

$$c^*(\xi; g(\cdot,\cdot)) \ge c^*(\xi; \hat{g}(\cdot)).$$

Proof. Let $a(\cdot) = g(\cdot, 0)$. By Proposition 2.8,

$$c^*(\xi; g(\cdot,\cdot)) = \inf_{\mu > 0} \frac{\lambda^0(\mu, \xi, a)}{\mu} \quad \text{and} \quad c^*(\xi; \hat{g}(\cdot)) = \inf_{\mu > 0} \frac{\lambda^0(\mu, \xi, \hat{a})}{\mu}.$$

By Proposition 2.6,

$$\lambda^0(\mu, \xi, a) \ge \lambda^0(\mu, \xi, \hat{a}).$$

The proposition then follows. □

3. Spatial Spreading Speeds and Proofs of Theorems 1.1 and 1.2

In this section, we explore the spreading speeds of (1.1), and prove Theorems 1.1 and 1.2. Throughout this section, we assume (H1) and (H2).

We first prove four lemmas.

Lemma 3.1. *For any $\epsilon > 0$, there are $p = (p_1, p_2, \cdots, p_N) \in \mathbb{N}^N$ and $h \in X_p \cap C^N(\mathbb{R}^N, \mathbb{R})$ such that*

$$f(x, 0) \ge h(x) \quad \text{for} \quad x \in \mathbb{R}^N,$$

$$\hat{h} \ge f^0(0) - \epsilon \quad (\text{hence} \quad \lambda^0(h(\cdot)) \ge f^0(0) - \epsilon),$$

and the partial derivatives of $h(x)$ up to order $N-1$ are zero at some $x_0 \in \mathbb{R}^N$ with $h(x_0) = \max_{x \in \mathbb{R}^N} h(x)$, where $\hat{h}$ is the average of $h(\cdot)$ (see (2.12) for the definition).

Proof. By (H2), there is $L_0 > 0$ such that $f(x, 0) = f^0(0)$ for $x \in \mathbb{R}^N$ with $\|x\| \ge L_0$. Let $M_0 = \inf_{x \in \mathbb{R}^N} f(x, 0)$. Let $h_0 : \mathbb{R} \to [0, 1]$ be a smooth function such that $h_0(s) = 1$ for $|s| \le 1$ and $h_0(s) = 0$ for $|s| \ge 2$. For any $p = (p_1, p_2, \cdots, p_N) \in \mathbb{N}^N$ with $p_j > 4L_0$, let $h \in X_p \cap C^N(\mathbb{R}^N, \mathbb{R})$ be such that

$$h(x) = f^0(0) - h_0\left(\frac{\|x\|^2}{L_0^2}\right)(f^0(0) - M_0)$$

for

$$x \in \Big([-\frac{p_1}{2}, \frac{p_1}{2}] \times [-\frac{p_2}{2}, \frac{p_2}{2}] \times \cdots \times [-\frac{p_N}{2}, \frac{p_N}{2}]\Big) \cap \mathbb{R}^N.$$

Then

$$f(x,0) \geq h(x) \quad \forall x \in \mathbb{R}^N.$$

It is clear that the partial derivatives of $h(x)$ up to order $N-1$ are zero at some $x_0 \in \mathbb{R}^N$ with $h(x_0) = \max_{x\in\mathbb{R}^N} h(x)(= f^0(0))$. For given $\epsilon > 0$, choosing $p_j \gg 1$, we have

$$\hat{h} > f^0(0) - \epsilon.$$

By Proposition 2.6, $\lambda^0(h(\cdot)) \geq \lambda^0(\hat{h}) = \hat{h}$ and hence

$$\lambda^0(h(\cdot)) \geq f^0(0) - \epsilon.$$

The lemma is thus proved. □

Lemma 3.2. *Suppose that $u^*(\cdot) \in X^{++}$ and $u = u^*(\cdot)$ is a stationary solution of (1). Then*

$$u^*(x) \to u^0 \quad \text{as} \quad \|x\| \to \infty.$$

Proof. Assume that $u^*(x) \not\to u^0$ as $\|x\| \to \infty$. Then there are $\epsilon_0 > 0$ and $x_n \in \mathbb{R}^N$ such that $\|x_n\| \to \infty$ and

$$|u^*(x_n) - u^0| \geq \epsilon_0 \quad \text{for} \quad n = 1, 2, \cdots.$$

By the uniform continuity of $u^*(x)$ in $x \in \mathbb{R}^N$, without loss of generality, we may assume that there is a continuous function $\tilde{u}^* : \mathbb{R}^N \to [\sigma_0, M_0]$ for some $\sigma_0, M_0 > 0$ such that

$$u(x + x_n) \to \tilde{u}^*(x)$$

as $n \to \infty$ uniformly in x on bounded sets. By the Lebesgue Dominated Convergence Theorem, we have

$$\tau\Delta\tilde{u}^*(x) + (1-\tau)[\int_{\mathbb{R}^N} \kappa(y-x)\tilde{u}^*(y)dy - \tilde{u}^*(x)] + \tilde{u}^*(x)f^0(\tilde{u}^*(x)) = 0 \quad \forall x \in \mathbb{R}^N.$$

Since $\tilde{u}^* \in X^{++}$, by Proposition 2.7 again, we have $\tilde{u}^*(x) \equiv u^0$ and then $u^*(x_n) \to u^0$ as $n \to \infty$. This is a contradiction. Therefore $u^*(x) \to u^0$ as $\|x\| \to \infty$. □

Lemma 3.3. *Let $\xi \in S^{N-1}$, $c > 0$, and $u_0 \in X^+$ be given.*

(1) If $\liminf_{x\cdot\xi \leq ct, t\to\infty} u(t,x;u_0) > 0$, then for any $0 < c' < c$,

$$\limsup_{x\cdot\xi \leq c't, t\to\infty} |u(t,x;u_0) - u^*(x)| = 0.$$

(2) If $\liminf_{|x\cdot\xi| \leq ct, t\to\infty} u(t,x;u_0) > 0$, then for any $0 < c' < c$,

$$\limsup_{|x\cdot\xi| \leq c't, t\to\infty} |u(t,x;u_0) - u^*(x)| = 0.$$

(3) If $\liminf_{\|x\| \leq ct, t\to\infty} u(t,x;u_0) > 0$, then for any $0 < c' < c$,

$$\limsup_{\|x\| \leq c't, t\to\infty} |u(t,x;u_0) - u^*(x)| = 0.$$

Proof. (1) Suppose that $\liminf_{x\cdot\xi\le ct, t\to\infty} u(t,x;u_0) > 0$. Then there are δ and $T > 0$ such that

$$u(t,x;u_0) \ge \delta \quad \forall (t,x) \in \mathbb{R}^+ \times \mathbb{R}^N,\ x\cdot\xi \le ct,\ t \ge T.$$

Assume that the conclusion of (1) is not true. Then there are $0 < c^{'} < c$, $\epsilon_0 > 0$, $x_n \in \mathbb{R}^N$, and $t_n \in \mathbb{R}^+$ with $x_n \cdot \xi \le c^{'} t_n$ and $t_n \to \infty$ such that

$$|u(t_n, x_n; u_0) - u^*(x_n)| \ge \epsilon_0 \quad \forall n \ge 1. \tag{3.1}$$

Without loss of generality, we may assume that $x_n \to x^*$ as $n \to \infty$ in the case that $\{\|x_n\|\}$ is bounded (this implies that $f(x+x_n,u) \to f(x+x^*,u)$ uniformly in (x,u) in bounded sets) and $f(x+x_n,u) \to f^0(u)$ as $n \to \infty$ uniformly in (x,u) on bounded sets in the case that $\{\|x_n\|\}$ is unbounded.

Let $\tilde{u}_0 \in X^+$,

$$\tilde{u}_0(x) = \delta \quad \forall x \in \mathbb{R}^N.$$

There is $\tilde{T} > 0$ such that

$$|u(\tilde{T}, x; \tilde{u}_0) - u^*(x)| < \epsilon_0 \quad \forall x \in \mathbb{R}^N, \tag{3.2}$$

$$|u(\tilde{T}, x; \tilde{u}_0, f(\cdot + x^*, \cdot)) - u^*(x + x^*)| < \frac{\epsilon_0}{2}, \tag{3.3}$$

and

$$|u(\tilde{T}, x; \tilde{u}_0, f^0) - u^0| < \frac{\epsilon_0}{2}. \tag{3.4}$$

Without loss of generality, we may assume that $t_n - \tilde{T} \ge T$ for $n \ge 1$. Let $\tilde{u}_{0n} \in X^+$ be such that $\tilde{u}_{0n}(x) = \delta$ for $x \cdot \xi \le \frac{c^{'}+c}{2}(t_n - \tilde{T})$, $0 \le \tilde{u}_{0n}(x) \le \delta$ for $\frac{c^{'}+c}{2}(t_n - \tilde{T}) \le x \cdot \xi \le c(t_n - \tilde{T})$, and $\tilde{u}_{0n}(x) = 0$ for $x \cdot \xi \ge c(t_n - \tilde{T})$. Then

$$u(t_n - \tilde{T}, \cdot; u_0) \ge \tilde{u}_{0n}(\cdot)$$

and hence

$$\begin{aligned} u(t_n, x_n; u_0) &= u(\tilde{T}, x_n; u(t_n - \tilde{T}, \cdot; u_0)) \\ &= u(\tilde{T}, 0; u(t_n - \tilde{T}, \cdot + x_n; u_0), f(\cdot + x_n, \cdot)) \\ &\ge u(\tilde{T}, 0; \tilde{u}_{0n}(\cdot + x_n), f(\cdot + x_n, \cdot)). \end{aligned} \tag{3.5}$$

Observe that $\tilde{u}_{0n}(x + x_n) \to \tilde{u}_0$ as $n \to \infty$ uniformly in x on bounded sets. In the case that $f(x + x_n, u) \to f^0(u)$, by Proposition 2.4,

$$u(\tilde{T}, 0; \tilde{u}_{0n}(\cdot + x_n), f(\cdot + x_n, \cdot)) \to u(\tilde{T}, 0; \tilde{u}_0, f^0(\cdot))$$

as $n \to \infty$. By (3.4) and (3.5),

$$u(t_n, x_n; u_0) > u^0 - \epsilon_0/2 \quad \text{for} \quad n \gg 1. \tag{3.6}$$

By Lemma 3.2,

$$u^0 > u^*(x_n) - \epsilon_0/2 \quad \text{for} \quad n \gg 1. \tag{3.7}$$

By (3.2), (3.6), and (3.7),

$$|u(t_n, x_n; u_0) - u^*(x_n)| < \epsilon_0 \quad \text{for} \quad n \gg 1.$$

This contradicts to (3.1).

In the case that $x_n \to x^*$, by Proposition 2.4 again,

$$u(\tilde{T}, 0; \tilde{u}_{0n}(\cdot + x_n), f(\cdot + x_n, \cdot)) \to u(\tilde{T}, 0; \tilde{u}_0, f(\cdot + x^*, \cdot))$$

as $n \to \infty$. By (3.3) and (3.5),

$$u(t_n, x_n; u_0) > u^*(x^*) - \epsilon_0/2 \quad \text{for} \quad n \gg 1. \tag{3.8}$$

By the continuity of $u^*(\cdot)$,

$$u^*(x^*) > u^*(x_n) - \epsilon_0/2 \quad \text{for} \quad n \gg 1. \tag{3.9}$$

By (3.2), (3.8), and (3.9),

$$|u(t_n, x_n; u_0) - u^*(x_n)| < \epsilon_0 \quad \text{for} \quad n \gg 1.$$

This contradicts to (3.1) again.

Hence

$$\lim_{x\cdot\xi \le c' t, t\to\infty} |u(t, x; u_0) - u^*(x)| = 0$$

for all $0 < c^{'} < c$.

(2) It can be proved by the similar arguments as in (1).

(3) It can also be proved by the similar arguments as in (1). □

Lemma 3.4. *Let $M > 0$ be such that $f(x, u) < 0$ for $x \in \mathbb{R}^N$, $u \geq M$. Then for any $\epsilon > 0$, there are $p \in \mathbb{N}^N$ and $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ such that for any $u \in \mathbb{R}$, $g(\cdot, u) \in X_p$, $g(\cdot, \cdot)$ satisfies* **(P1)** *and* **(P3)**, *and*

$$f(x, u) \geq g(x, u) \quad \forall x \in \mathbb{R}^N, \ u \in [0, M],$$
$$\hat{g}(0) \geq f^0(0) - \epsilon,$$

where $\hat{g}(\cdot)$ is as in (2.17).

Proof. By Lemma 3.1, for any $\epsilon > 0$, there are $p \in \mathbb{N}^N$ and $h(\cdot) \in X_p \cap C^N(\mathbb{R}^N, \mathbb{R})$ such that

$$f(x, 0) \geq h(x) \ \forall x \in \mathbb{R}^N \quad \text{and} \quad \hat{h} \geq f^0(0) - \epsilon.$$

Choose $M > 0$ such that

$$f(x, u) \geq h(x) - Mu \quad \text{for} \quad x \in \mathbb{R}^N, \ 0 \leq u \leq M.$$

Let

$$g(x, u) = h(x) - M \quad \forall x \in \mathbb{R}^N, \ \ u \in \mathbb{R}.$$

It is not difficult to see that $g(\cdot, \cdot)$ satisfy the lemma. □

In the following, $c^0(\xi)$ as in (1.15), and (1.16). Observe that $\lambda(\mu, \xi, f^0(0))$ exist. and

$$\lambda(\mu, \xi, f^0(0)) = \tau\mu^2 + (1 - \tau)[\int_{\mathbb{R}^N} e^{-\mu z\cdot\xi}\kappa(z)dz - 1] + f^0(0).$$

If no confusion occurs, we may denote $\lambda(\mu, \xi, f^0(0))$ by $\lambda(\mu, \xi)$. Observe also that $v(t, x) = e^{-\mu(x\cdot\xi - \frac{\lambda(\mu,\xi)}{\mu}t)}$ is solution of,

$$v_t(t, x) = \tau\Delta v(t, x) + (1 - \tau)\mathcal{K}v(t, x) + f^0(0)v(t, x), \quad x \in \mathbb{R}^N. \tag{3.10}$$

Proof of Theorem 1.1. Fix $\xi \in S^{N-1}$, we first prove that for any $c^{'} > c^0(\xi)$ and $u_0 \in X^+(\xi)$,

$$\limsup_{x\cdot\xi \geq c^{'}t, t\to\infty} u(t,x;u_0) = 0. \tag{3.11}$$

To this end, take a c such that $c^{'} > c > c^*(\xi)$. Note that there is $\mu^* > 0$ such that

$$c^0(\xi) = \frac{\lambda(\xi,\mu^*)}{\mu^*}$$

and there is $\mu \in (0,\mu^*)$ such that

$$c = \frac{\lambda(\mu,\xi)}{\mu}.$$

Take $d > M > 0$ such that

$$\begin{aligned} &u_0(x) \leq M \quad \text{and} \quad u_0(x) \leq de^{-\mu x\cdot\xi} \quad \forall x \in \mathbb{R}^N, \\ &f(x,M) < 0 \quad \forall x \in \mathbb{R}^N, \end{aligned} \tag{3.12}$$

and

$$f(x,u) = f^0(u) \quad \text{for} \quad x\cdot\xi \geq -\frac{1}{\mu}\ln\frac{M}{d}(>0). \tag{3.13}$$

Observe that by (3.12) and (H1), for $(t,x) \in (0,\infty)\times\mathbb{R}^N$ with $de^{-\mu(x\cdot\xi-ct)} \geq M$, i.e., $x\cdot\xi \leq -\frac{1}{\mu}\ln\frac{M}{d} + ct$,

$$f(x, de^{-\mu(x\cdot\xi-ct)}) < 0 < f^0(0).$$

By (3.13), for $(t,x) \in (0,\infty)\times\mathbb{R}^N$ with $de^{-\mu(x\cdot\xi-ct)} \leq M$, i.e, $x\cdot\xi \geq -\frac{1}{\mu}\ln\frac{M}{d} + ct$,

$$f(x, de^{-\mu(x\cdot\xi-ct)}) = f^0(de^{-\mu(x\cdot\xi-ct)}) \leq f^0(0).$$

It then follows that $u = de^{-\mu(x\cdot\xi-ct)}$, which is a solution of (3.10), is a super-solution of (1.1) and hence by Proposition 2.1,

$$u(t,x;u_0) \leq de^{-\mu(x\cdot\xi-ct)} \quad \forall t > 0 \ x \in \mathbb{R}^N. \tag{3.14}$$

This implies that (3.11) holds.

Next, we prove that for any $c^{'} < c^0(\xi)$ and any $u_0 \in X^+(\xi)$,

$$\limsup_{x\cdot\xi \leq c^{'}t, t\to\infty} |u(t,x;u_0) - u^*(x)| = 0. \tag{3.15}$$

To this end, take a $c \in \mathbb{R}$ such that $c^{'} < c < c^0(\xi)$. Let $M > 0$ be such that $u_0(x) \leq M$ and $f(x,M) < 0$ for all $x \in \mathbb{R}^N$. Then $u \equiv M$ is a super-solution of (1.1) and

$$u(t,x;u_0) \leq M \quad \forall t \geq 0, \ x \in \mathbb{R}^N.$$

For any $\epsilon > 0$, let $g(\cdot,\cdot)$ be as in Lemma 3.4. By Proposition 2.9, for $\epsilon > 0$ sufficiently small,

$$c^*(\xi, g(\cdot,\cdot)) \geq c^*(\xi,\hat{g}(\cdot)) > c.$$

By Propositions 2.1 and 2.8,

$$\liminf_{x\cdot\xi\le ct, t\to\infty} u(t,x;u_0) \ge \liminf_{x\cdot\xi\le ct, t\to\infty} u(t,x;u_0,g) > 0.$$

(3.15) then follows from Lemma 3.3.

By (3.11) and (3.15), $c^*(\xi)$ exists and $c^*(\xi) = c^0(\xi)$. Moreover, (1.14) holds □

Proof of Theorem 1.2. (1) Fix $\xi \in S^{N-1}$. Let $u_0 \in X^+$ satisfy that $u_0(x) = 0$ for $x \in \mathbb{R}^N$ with $|x\cdot\xi| \gg 1$. Then there are $u_0^+ \in X^+(\xi)$ and $u_0^- \in X^+(-\xi)$ such that

$$u_0(x) \le u_0^\pm(x) \quad \forall x \in \mathbb{R}^N.$$

By Proposition 2.1 and Theorem 1.1,

$$\limsup_{x\cdot\xi\ge c' t, t\to\infty} u(t,x;u_0) \le \limsup_{x\cdot\xi\ge c' t, t\to\infty} u(t,x;u^+) = 0 \quad \forall c' > c^*(\xi)$$

and

$$\limsup_{x\cdot(-\xi)\ge c' t, t\to\infty} u(t,x;u_0) \le \limsup_{x\cdot(-\xi)\ge c' t, t\to\infty} u(t,x;u^-) = 0 \quad \forall c' > c^*(-\xi).$$

It then follows that

$$\limsup_{|x\cdot\xi|\ge c' t, t\to\infty} u(t,x;u_0) = 0 \quad \forall c' > \max\{c^*(\xi), c^*(-\xi)\}.$$

(2) Fix $\xi \in S^{N-1}$. For given $0 < c' < \min\{c^*(\xi), c^*(-\xi)\}$, take a $c > 0$ such that $c' < c < \min\{c^*(\xi), c^*(-\xi)\}$. For given $u_0 \in X^+$ satisfying the condition in Theorem 2.3 (2), let $M > 0$ be such that $u_0(x) \le M$ and $f(x,M) < 0$ for all $x \in \mathbb{R}^N$. Then $u \equiv M$ is a super-solution of (1.1) and

$$u(t,x;u_0) \le M \quad \forall t \ge 0,\ x \in \mathbb{R}^N.$$

For any $\epsilon > 0$, let $g(\cdot,\cdot)$ be as in Lemma 3.4. By Proposition 2.9, for $\epsilon > 0$ sufficiently small,

$$c^*(\xi, g(\cdot,\cdot)) \ge c^*(\xi, \hat{g}(\cdot)) > c.$$

By Propositions 2.1 and 2.8,

$$\liminf_{|x\cdot\xi|\le ct, t\to\infty} u(t,x;u_0) \ge \liminf_{|x\cdot\xi|\le ct, t\to\infty} u(t,x;u_0,g) > 0.$$

It then follows from Lemma 3.3 that

$$\limsup_{|x\cdot\xi|\le c' t, t\to\infty} |u(t,x;u_0) - u^*(x)| = 0.$$

(3) It can be proved by similar arguments as in [49, Theorem E (1)]. For completeness again, we provide a proof in the following.

Fix $\xi \in S^{N-1}$, let $c > \sup_{\xi\in S^{N-1}} c^*(\xi)$. Let $u_0 \in X^+$ be such that $u_0(x) = 0$ for $\|x\| \gg 1$. Note that for every given $\xi \in S^{N-1}$, there is $\tilde{u}_0(\cdot;\xi) \in X^+(\xi)$ such that $u_0(\cdot) \le \tilde{u}_0(\cdot;\xi)$. By Proposition 2.1,

$$0 \le u(t,x;u_0) \le u(t,x;\tilde{u}_0(\cdot;\xi))$$

for $t > 0$ and $x \in \mathbb{R}^N$. It then follows from Theorem 1.1 that

$$0 \le \limsup_{x\cdot\xi \ge ct, t\to\infty} u(t,x;u_0) \le \limsup_{x\cdot\xi \ge ct, t\to\infty} u(t,x;\tilde{u}_0(\cdot;\xi)) = 0.$$

Take any $c^{'} > c$. Consider all $x \in \mathbb{R}^N$ with $\|x\| = c^{'}$. By the compactness of $\partial B(0,c^{'}) = \{x \in \mathbb{R}^N \,|\, \|x\| = c^{'}\}$, there are $\xi_1, \xi_2, \cdots, \xi_L \in S^{N-1}$ such that for every $x \in \partial B(0,c^{'})$, there is l $(1 \le l \le L)$ such that $x \cdot \xi_l \ge c$. Hence for every $x \in \mathbb{R}^N$ with $\|x\| \ge c^{'} t$, there is $1 \le l \le L$ such that $x \cdot \xi_l = \frac{\|x\|}{c^{'}}\left(\frac{c^{'}}{\|x\|}x\right)\cdot \xi_l \ge \frac{\|x\|}{c^{'}} c \ge ct$. By the above arguments,

$$0 \le \limsup_{x\cdot\xi_l \ge ct, t\to\infty} u(t,x;u_0) \le \limsup_{x\cdot\xi_l \ge ct, t\to\infty} u(t,x;\tilde{u}_0(\cdot;\xi_l)) = 0$$

for $l = 1, 2, \cdots L$. This implies that

$$\limsup_{\|x\| \ge c^{'} t, t\to\infty} u(t,x;u_0) = 0.$$

Since $c^{'} > c$ and $c > \sup_{\xi\in S^{N-1}} c^*(\xi)$ are arbitrary, we have that for $c > \sup_{\xi\in S^{N-1}} c^*(\xi)$,

$$\limsup_{\|x\| \ge ct, t\to\infty} u(t,x;u_0) = 0.$$

(4) It can be proved by similar arguments as in (2). □

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