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EXISTENCE OF NON-TRIVIAL SOLUTIONS FOR THE KIRCHHOFF-TYPE EQUATIONS WITH FUČIK-TYPE RESONANCE AT INFINITY

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Abstract In this paper, we obtain the existence of nontrivial solutions for the Kirchhoff type equation with Fučik-type resonance at infinity by variational methods.

Keywords Kirchhoff type equation, Fučik spectrum, mountain pass theorem, compactness condition.

MSC(2010) 35J20, 35J25, 35J60.

1. Introduction

In this paper, we consider the following Kirchhoff type equation

$$\begin{cases} -\left(a+b\int_\Omega |\nabla u|^2dx\right)\triangle u = \alpha(u^+)^3+\beta(u^-)^3+f(x,u) & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is an open ball in $R^N(N=1,2,3)$ or $\Omega\subset R^2$ is symmetric in x and y, and convex in the x and y directions, $a>0$, $b>0$ are real constants and $\alpha,\beta\in R$, $f\in C(\bar\Omega\times R,R)$. $u^+=\max\{u,0\}$, $u^-=\min\{u,0\}$, and $u=u^++u^-$.

Let $H_0^1(\Omega)$ be the usual Hilbert space with inner product and the norm

$$(u,v)=\int_\Omega \nabla u\cdot\nabla v dx \quad \text{ and } \quad \|u\|=\left(\int_\Omega |\nabla u|^2dx\right)^{\frac12}$$

for all $u,v\in H_0^1(\Omega)$, and let $L^p(\Omega)$ $(p\in[1,\infty))$ be the usual Lebesgue space with the norm $|u|_p=(\int_\Omega |u|^pdx)^{\frac1p}$ for any $u\in L^p(\Omega)$. Since the embedding $H_0^1(\Omega)\hookrightarrow L^p(\Omega)$ is continuous and compact for any $p\in[1,2^*)$, where $2^*=\infty$ if $N=1,2$, and $2^*=6$ if $N=3$, for every $p\in[1,2^*)$, there is $S_p>0$ such that

$$|u|_p\le S_p\|u\| \quad \text{for any } u\in H_0^1(\Omega). \tag{1.2}$$

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Let μ_1 be the principal eigenvalue of $-\Delta$ in $H_0^1(\Omega)$, and for the following eigenvalue problem

$$\begin{cases} -b\left(\int_\Omega |\nabla u|^2 dx\right) \triangle u = \lambda u^3 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

Liang, Li and Shi in [6] showed that the principal eigenvalue is

$$\lambda_1 = \inf\{b\|u\|^4 : u \in H_0^1(\Omega), |u|_4^4 = 1\} > 0, \tag{1.4}$$

and the corresponding eigenfunction $\varphi_1 > 0$ in Ω with$|\varphi_1|_4^4 = 1$. Meanwhile, problem (1.3) has a sequence of eigenvalues with the variational characterization (see [12])

$$\lambda_m = \inf_{h\in\Sigma_m} \sup_{u\in h(S^{m-1})} b\|u\|^4, \tag{1.5}$$

where $\Sigma_m = \{h \in C(S^{m-1}, S) : h \text{ is old}\}(m \in N)$, and $S := \{u \in H_0^1(\Omega) : |u|_4^4 = 1\}$ and S^{m-1} is the unit sphere in R^m.

The set Σ of the points $(\alpha, \beta) \in R^2$ for which the equation

$$\begin{cases} -b\left(\int_\Omega |\nabla u|^2 dx\right) \triangle u = \alpha(u^+)^3 + \beta(u^-)^3 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

has a non-trivial solution is called Fučik spectrum for the Kirchhoff-type problem. In [7], Li, Rong and Liang have obtained two trivial curves $\{\lambda_1\} \times R$ and $R \times \{\lambda_1\}$, and a nontrivial curve ℓ of Σ. The construction of the curve ℓ is carried out as follows in [7]: for any $s \geq 0$, we define

$$J_s(u) = b\|u\|^4 - s|u^+|_4^4, \qquad c(s) = \inf_{\gamma\in\Sigma_0} \max_{t\in[0,1]} J_s(\gamma(t)),$$

where $\Sigma_0 = \{\gamma \in C([0, 1], S) : \gamma(0) = \varphi_1, \gamma(1) = -\varphi_1\}$, and it was proved that $c(s) > \lambda_1$ for every $s \geq 0$. Similarly, for every $s \geq 0$, we define

$$\tilde{J}_s(u) = b\|u\|^4 - s|u^-|_4^4, \qquad \tilde{c}(s) = \inf_{\gamma\in\Sigma_0} \max_{t\in[0,1]} \tilde{J}_s(\gamma(t)),$$

and it was also proved that $\tilde{c}(s) > \lambda_1$ for every $s \geq 0$. Then, ℓ is defined by:

$$\ell := \{(c(s) + s, c(s)) : s \geq 0\} \cup \{(\tilde{c}(s), \tilde{c}(s) + s) : s \geq 0\}.$$

Problem (1.1) is analogous to the stationary case of equations that arise in the study of string or membrane vibrations, namely,

$$u_{tt} - \left(a + b\int_\Omega |\nabla u|^2 dx\right)\Delta u = f(x, u), \tag{1.7}$$

where the parameters have practical physical meaning: u denotes the displacement, f is the external force, b represents the initial tension, and a is related to the intrinsic properties of the string. Problem (1.7) was first proposed by Kirchhoff (see [4]) in 1883 and it is an extension of the classical D'Alembert's wave equation by considering the effects of the changes in the length of the string during the vibrations. Some early works related to problem (1.7) are seen in [9, 10].

For the following semilinear elliptic equation:

$$\begin{cases} -\Delta u = \alpha u^+ + \beta u^- + f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.8}$$

and under different assumptions on f, many people have studied the existence and multiple of weak solutions for problem (1.8) with Fučik-type resonance(see [2, 5] and the references therein). Moreover, there are also many results on the existence of non-trivial solution for p-Laplacian problem with the Fučik resonance(see [15, 16, 19] and the references therein). In recent years, many people have studied the existence and multiplicity of weak solutions of Kirchhoff type problems ($\alpha = \beta = 0$ in problem (1.1)) by variational methods (see [1, 7, 8, 11–13, 18] and the references therein). Especially, if the nonlinearity f satisfies the certain Landesman-Lazer-type conditions, Sun and Tang in [12] obtained the existence of weak solutions for Kirchhoff type problems with resonance at higher eigenvalues, that is, $\alpha = \beta$ in problem (1.1) is the higher eigenvalue of problem (1.3). In 2019, Li, Rong and Liang in [7] considered the existence of at least two positive solutions for problem (1.1), where $a = b = 1$, $\alpha = \beta = 0$ and $f(x,t) = f_\infty t^3 + g(x,t)$ and $g(x,t) = o(t^3)$ as $t \to +\infty$ and $f_\infty > \lambda_1$ by using Mountain Pass Theorem. Rong, Li and Liang in [11] studied the existence of nontrivial solutions for problem (1.1) with jumping nonlinearities at infinity and $a = 0$ and $b = 1$.

Inspired by [7, 11, 12], we will investigate the existence of non-trivial solutions for problem (1.1) with Fučik spectrum type resonance at infinity. Let $F(x,t) = \int_0^t f(x,s)ds$ for all $(x,t) \in \Omega \times R$, assume that the nonlinearity f (or F) satisfies:

(f_1) $f(x,t) = o(|t|^3)$ as $|t| \to \infty$ uniformly for $x \in \Omega$;

(f_2) $f(x,t)t > 0$ for any $t \neq 0$ and there exist $\delta > 0$, $\theta > 2$ and $C_0 > 0$ such that

$$F(x,t) \geq C_0|t|^\theta \quad \text{for any} \quad |t| \leq \delta \text{ and } x \in \Omega;$$

$(\tilde{f_2})$ $F(x,t) > 0$ and there exist $\delta > 0$, $C_1 > 0$ and $r \in (1,2)$ such that

$$F(x,t) \geq C_1|t|^r \quad \text{for any} \quad |t| \leq \delta \text{ and } x \in \Omega;$$

(f_3) $\lim\limits_{|t|\to\infty} (f(x,t)t - 4F(x,t) + a\mu_1 t^2) = +\infty$ uniformly for $x \in \Omega$;

(f_4) $\lim\limits_{|t|\to\infty} \frac{F(x,t)}{t^2} = +\infty$ uniformly for $x \in \Omega$.

Now we are ready to state our first theorem:

Theorem 1.1. *Assume that f satisfies (f_1)–(f_4) and let $(\alpha,\beta) \in \{\lambda_1\} \times [\lambda_1, +\infty) \cup [\lambda_1, +\infty) \times \{\lambda_1\}$, then problem (1.1) has at least one non-trivial solution.*

Let $Q_m = \{(\alpha,\beta) \in R^2 : \alpha, \beta \in (\lambda_m, \lambda_{m+1}]\} (m \in N)$, the second main result is the following theorem:

Theorem 1.2. *Suppose that f satisfies $(f_1), (\tilde{f_2}), (f_4)$, then problem (1.1) has a non-trivial solution if one of the following conditions holds:*

(i) $(\alpha,\beta) \in intQ_m \cap (R^2 \setminus \Sigma)$,

(ii) $(\alpha,\beta) \in intQ_m$ and (f_3) holds.

2. Proof of the main results

The energy functional corresponding to problem (1.1) is defined by

$$I(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{\alpha}{4}|u^+|_4^4 - \frac{\beta}{4}|u^-|_4^4 - \int_\Omega F(x,u)dx \tag{2.1}$$

for any $u \in H_0^1(\Omega)$. From (f_1), we can obtain $I \in C^1(H_0^1(\Omega), R)$, and it is well known that a critical point of the functional I corresponds to a weak solution of problem (1.1). We will prove Theorem 1.1 by using Mountain Pass Theorem with the (Ce) condition (see [17]), and Theorem 1.2 by Γ-linking Theorem (see [14]). Let W be a real Banach space, the functional I satisfies the $(Ce)_c$ condition at the level $c \in R$, if any sequence $\{u_n\} \subset W$ such that $I(u_n) \to c$, $(1+\|u_n\|)\|I'(u_n)\|_{W^*} \to 0$ as $n \to \infty$, has a convergent subsequence. The functional I satisfies the (Ce) condition if I satisfies the $(Ce)_c$ condition at any $c \in R$. We note that the (Ce) condition is weaker than the usual (PS) condition. If I satisfies (PS) or (Ce) condition, then I satisfies the deformation lemma.

Lemma 2.1 (Lemma 11, [16])**.** *Let I be a C^1 functional on a Banach space E and suppose that I satisfies the (Ce) condition at any level $c \in [a,b]$ and I has no critical value in (a,b). Assume that $K_a := \{u \in E : I'(u) = 0, I(u) = a\}$ consists only of isolated points ($K_a = \emptyset$). Moreover, the set $\{u \in E : I(u) \le c\}$ is denoted by I^c for every $c \in R$. Then, there exists a $\eta \in C([0,1] \times E, E)$ satisfying the following conditions:*

(i) $\eta(\cdot,\cdot)$ is non-increasing in t for every $u \in E$;

(ii) $\eta(t,u) = u$ for any $u \in I^a, t \in [0,1]$;

(iii) $\eta(0,u) = u$ and $\eta(1,u) \in I^a$ for any $I^b \backslash K_b$; that is, I^a is a strong deformation retract of $I^b \backslash K^b$.

Lemma 2.2. *Let f satisfy (f_1). Then the following assertions hold:*

(i) if $(\alpha,\beta) \notin \Sigma$, then I satisfies the (PS) condition.

(ii) if $(\alpha,\beta) \in \Sigma$ and (f_3), then I satisfies the (Ce) condition.

Proof. (i) Let $(\alpha,\beta) \notin \Sigma$ and $\{u_n\} \subset H_0^1(\Omega)$ be a (PS) sequence of the functional I, namely

$$I(u_n) \to c \quad \text{and} \quad I'(u_n) \to 0 \ \text{ as } \ n \to \infty. \tag{2.2}$$

We first claim that $\{u_n\}$ is bounded. If not, without loss of generality, we assume $\|u_n\| \to \infty$ as $n \to \infty$. Define $v_n = \frac{u_n}{\|u_n\|}$, then $\{v_n\}$ is bounded and $\|v_n\| = 1$. Therefore there is a subsequence of $\{v_n\}$, still denoted by $\{v_n\}$, and $v \in H_0^1(\Omega)$ such that

$$v_n \rightharpoonup v \text{ in } H_0^1(\Omega), \quad \text{and} \quad v_n \to v \text{ in } L^p(\Omega) \text{ for any } p \in [1, 2^*), \tag{2.3}$$

as $n \to \infty$. Estimating the following equality

$$\begin{aligned}\frac{\langle I'(u_n), v_n - v\rangle}{\|u_n\|^3} =& \frac{a}{\|u_n\|^2}\int_\Omega \nabla v_n \cdot \nabla(v_n - v)dx + b\int_\Omega \nabla v_n \cdot \nabla(v_n - v)dx \\ &- \int_\Omega \left(\alpha(v_n^+)^3 + \beta(v_n^-)^3 + \frac{f(x,u_n)}{\|u_n\|^3}\right)(v_n - v)dx. \end{aligned}\tag{2.4}$$

From (f_1), for any $\varepsilon > 0$, there exists a $C_2 > 0$ such that

$$|f(x,t)| \le \varepsilon|t|^3 + C_2 \ \text{ for any } (x,t) \in \Omega \times R. \tag{2.5}$$

From the boundedness of Ω and (1.2), there exists a constant $C_3 > 0$ such that

$$|f(x,u_n)|_{\frac{4}{3}} \leq 2\left(\int_\Omega (\varepsilon^{\frac{4}{3}}|u_n|^4 + C_2^{\frac{4}{3}})dx\right)^{\frac{3}{4}} \leq \varepsilon\|u_n\|^3 + C_3.$$

Combining this inequality with $\|u_n\| \to \infty$ as $n \to \infty$, we have

$$\lim_{n\to\infty} |f(x,u_n)|_{\frac{4}{3}}/\|u_n\|^3 = 0. \tag{2.6}$$

Hence, from (2.2), (2.4), (2.6) and the Hölder's inequality, we have $\int_\Omega \nabla v_n \cdot \nabla(v_n - v)dx \to 0$. Moreover, from (2.3), it follows that $\int_\Omega \nabla v \cdot \nabla(v_n - v)dx \to 0$. Therefore, we have $v_n \to v$ in $H_0^1(\Omega)$ and $\|v\| = 1$. By a similar way and from $\frac{\langle I'(u_n), w\rangle}{\|u_n\|^3} \to 0$ as $n \to \infty$ for any $w \in H_0^1(\Omega)$, we have

$$b\int_\Omega \nabla v \cdot \nabla w dx - \alpha\int_\Omega (v^+)^3 w dx - \beta\int_\Omega (v^-)^3 w dx = 0 \text{ for any } w \in H_0^1(\Omega).$$

From $\|v\| = 1$, the above equality implies that v is an eigenfunction of problem (1.6) related with (α,β), which is a contradiction to $(\alpha,\beta) \notin \Sigma$. Hence, $\{u_n\}$ is bounded. And then, there is a $u \in H_0^1(\Omega)$ such that

$$u_n \rightharpoonup u \quad \text{in } H_0^1(\Omega), \quad u_n \to u \quad \text{in } L^p(\Omega) \text{ for any } p \in [1, 2^*). \tag{2.7}$$

From (2.3), (2.5) and the Hölder's inequality, there exists a $C_4 > 0$ such that

$$\begin{aligned}&\left|\int_\Omega \left(\alpha(u_n^+)^3 + \beta(u_n^-)^3 + f(x,u_n)\right)(u_n - u)dx\right| \\ \leq\, & C_4(|u_n^+|_4^3 + |u_n^-|_4^3 + |f(x,u_n)|_{\frac{4}{3}})|u_n - u|_4 \to 0 \text{ as } n \to \infty.\end{aligned}$$

So, we have

$$\begin{aligned}&(a + b\|u_n\|^2)\int_\Omega \nabla u_n \cdot \nabla(u_n - u)dx \\ =&\langle I'(u_n), u_n - u\rangle + \int_\Omega (\alpha(u_n^+)^3 + \beta(u_n^-)^3 + f(x,u_n))(u_n - u)dx \\ \to& 0 \quad \text{as} \quad n \to \infty.\end{aligned}$$

From the boundedness of $\{u_n\}$, we have $\int_\Omega \nabla u_n \cdot \nabla(u_n - u)dx \to 0$ *as* $n \to \infty$. In addition, we have $\int_\Omega \nabla u \cdot \nabla(u_n - u)dx \to 0$ *as* $n \to \infty$ by (2.7), which implies $\|u_n - u\|^2 \to 0$ *as* $n \to \infty$. Hence, $u_n \to u$ in $H_0^1(\Omega)$.

(ii) Let $(\alpha,\beta) \in \Sigma$ and $\{u_n\} \subset H_0^1(\Omega)$ be a (Ce) sequence of the functional I, namely

$$I(u_n) \to c \quad \text{and} \quad (1 + \|u_n\|)\|I'(u_n)\|_{H^*} \to 0 \text{ as } n \to \infty.$$

We first show that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. For this we suppose by contradiction that $\|u_n\| \to \infty$ as $n \to \infty$. Letting $v_n = \frac{u_n}{\|u_n\|}$, there is a $v \in H_0^1(\Omega)$ such that

$$v_n \rightharpoonup v \quad \text{in } H_0^1(\Omega), \quad v_n \to v \quad \text{in } L^p(\Omega) \text{ for any } p \in [1, 2^*),$$

as $n \to \infty$. By the same argument as case (i), we have $v_n \to v$ in $H_0^1(\Omega)$ and $\|v\| = 1$. Hence, let $\Omega_0 = \{x \in \Omega : |v(x)| > 0\}$, we have the Lebesgue measure of

Ω_0 is not zero. Then $|u_n(x)| \to \infty$ for a.e $x \in \Omega_0$. By (f_3) and Fatou's Lemma, we see

$$\liminf_{n\to\infty} \int_{\Omega_0} (f(x,u_n)u_n - 4F(x,u_n) + a\mu_1|u_n|^2)dx = +\infty,$$

and it follows that

$$\begin{aligned}
4c + o(1) =& 4I(u_n) - \langle I'(u_n), u_n\rangle \\
=& a\|u_n\|^2 - a\mu_1|u_n|_2^2 + \int_\Omega (f(x,u_n)u_n - 4F(x,u_n) + a\mu_1|u_n|^2)dx \\
\geq& \int_{\Omega_0} (f(x,u_n)u_n - 4F(x,u_n) + a\mu_1|u_n|^2)dx \to +\infty \text{ as } n \to \infty,
\end{aligned}$$

which is a contradiction. Hence $\{u_n\}$ is bounded. Similar with the proof of case (i), we have $u_n \to u$ in $H_0^1(\Omega)$. $\square$

We define two C^1 functionals on $H_0^1(\Omega)$ as follows:

$$\begin{aligned}
I_\beta^-(u) &= \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{\beta}{4}|u^-|_4^4 - \int_\Omega F_-(x,u)dx, \\
I_\alpha^+(u) &= \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{\alpha}{4}|u^+|_4^4 - \int_\Omega F_+(x,u)dx,
\end{aligned}$$

where $f_\pm(x,t) := 0$ if $\pm t \leq 0$ and $f_\pm(x,t) = f(x,t)$ if $\pm t > 0$, and $F_\pm(x,t) := \int_0^t f_\pm(x,s)ds$.

Lemma 2.3. *Let f satisfy (f_1), the following assertions hold:*

(i) if $\alpha \neq \lambda_1$, I_α^+ satisfies the (PS) condition.

(ii) if $\beta \neq \lambda_1$, I_β^- satisfies the (PS) condition.

(iii) if (f_3) holds, I_α^+ or I_β^- satisfies the (Ce) condition.

Proof. (i) Because $(\alpha, 0) \notin \Sigma$ and f^+ also satisfies(f_1), I_α^+ satisfies the (PS) condition by Lemma 2.3.

(ii) We note that $(\beta, 0) \notin \Sigma$ and f^- also satisfies (f_1). Hence, I_β^- satisfies the (PS) condition by Lemma 2.2.

(iii) We just prove the case I_α^+, the other case can be proved similarly. If $\alpha \neq \lambda_1$, then $(\alpha, 0) \notin \Sigma$, and from (i), the conclusion holds. Therefore, we assume that $\alpha = \lambda_1$, and by the same argument as case (ii) of Lemma 2.2, the conclusion also holds. $\square$

Lemma 2.4. *Suppose that $(f_1), (f_2), (f_4)$ hold, then there is a $h_0 > 0$ and $v \in H_0^1(\Omega) \setminus \{0\}$ such that*

$$\max_{t\in[0,1]} I(h_0tv^+ + h_0(1-t)v^-) < 0.$$

Proof. From (f_2) and (f_4), for any $M_1 > 0$, there is a constant $M_2 > 0$ such that

$$F(x,t) \geq M_1|t|^2 + M_2|t|^\theta \text{ for any } (x,t) \in \Omega \times R.$$

Let $v \in H_0^1(\Omega)$ be a nontrivial solution of problem (1.6) with $(\alpha_0, \beta_0) \in \ell$, we have that v is sign-changing and

$$b\|v\|^2\|v^+\|^2 = \alpha_0|v^+|_4^4, \quad b\|v\|^2\|v^-\|^2 = \beta_0|v^-|_4^4. \tag{2.8}$$

Hence, for any $t \in [0,1]$, let $M_1 = \max\{\frac{a\|v^+\|^2}{|v^+|_2^2}, \frac{a\|v^-\|^2}{|v^-|_2^2}\}$, by (2.8) and $\max\limits_{t\in[0,1]}(t(1-t))^2 = \frac{1}{16}$, we have

$$
\begin{aligned}
&I(htv^+ + h(1-t)v^-)\\
\leq&\frac{a}{2}\|v^+\|^2(ht)^2 + \frac{b}{4}\|v\|^2\|v^+\|^2(ht)^4 - \frac{\alpha}{4}|v^+|_4^4(ht)^4 - M_1|v^+|_2^2(ht)^2\\
&- M_2|v^+|_\theta^\theta(ht)^\theta + \frac{a}{2}\|v^-\|^2(h(1-t))^2 + \frac{b}{4}\|v\|^2\|v^-\|^2(h(1-t))^4\\
&- \frac{\beta}{4}|v^-|_4^4(h(1-t))^4 - M_1|v^-|_2^2(h(1-t))^2 - M_2|v^-|_\theta^\theta(h(1-t))^\theta\\
&+ \frac{b}{2}\|v^+\|^2\|v^-\|^2h^4[t(1-t)]^2\\
\leq&(\frac{a}{2}\|v^+\|^2 - M_1|v^+|_2^2)h^2 + (\frac{a}{2}\|v^-\|^2 - M_1|v^-|_2^2)h^2\\
&- M_2|v^+|_\theta^\theta(ht)^\theta - M_2|v^-|_\theta^\theta(h(1-t))^\theta + \frac{1}{4}(\alpha_0-\alpha)|v^+|_4^4h^4\\
&+ \frac{1}{4}(\beta_0-\beta)|v^-|_4^4h^4 + \frac{b}{32}\|v^+\|^2\|v^-\|^2h^4.
\end{aligned}
$$

Since $\theta > 2$, there is a sufficiently small $h_0 > 0$ such that $\max\limits_{t\in[0,1]} I(h_0tv^+ + h_0(1-t)v^-) < 0$. □

Now, for $(\alpha,\beta) \in R^2$, we define

$$E(\alpha,\beta) = \{u \in H_0^1(\Omega) : b\|u\|^4 \geq \alpha|u^+|_4^4 + \beta|u^-|_4^4\},$$

$$\Gamma(\alpha,\beta) := \{\gamma \in C([0,1], H_0^1(\Omega)) : \gamma(0) = \gamma(0)^+ \notin E(\alpha,\beta), \gamma(1) = \gamma(1)^- \notin E(\alpha,\beta)\}.$$

Proof of Theorem 1.1. (i) We first consider the case $\alpha \geq \lambda_1, \beta = \lambda_1$. Because $\int_\Omega F(x,u)dx = o(|u|_4^4)$ as $|u|_4 \to \infty$ from (f_1), for any $u \in E(\alpha + c(\alpha), c(\alpha))$, we have

$$
\begin{aligned}
I(u) &= \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{\alpha}{4}|u^+|_4^4 - \frac{\beta}{4}|u^-|_4^4 - \int_\Omega F(x,u)dx\\
&\geq \frac{a}{2}\|u\|^2 + \frac{1}{4}c(\alpha)|u^+|_4^4 + \frac{1}{4}(c(\alpha)-\beta)|u^-|_4^4 - o(|u|_4^4)\\
&\geq \frac{a}{2}\|u\|^2 + \frac{1}{4}(c(\alpha)-\beta)|u|_4^4 - o(|u|_4^4).
\end{aligned}
$$

Hence by $c(\alpha) > \lambda_1 = \beta$, I is bounded from below on $E(\alpha + c(\alpha), c(\alpha))$ and

$$m = \inf\{I(u) : u \in E(\alpha + c(\alpha), c(\alpha))\} > -\infty.$$

Setting

$$c := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I(\gamma(t)),$$

where $\Gamma := \{\gamma \in \Gamma(\alpha + c(\alpha), c(\alpha)) : I(\gamma(0)), I(\gamma(1)) < m-1\}$. Next, we will show that $\Gamma \neq \emptyset$ and c is negative, namely, there is a $\gamma_0 \in \Gamma$ such that $\max_{t\in[0,1]} I(\gamma_0(t)) < 0$. From Lemma 2.4, there is $h_0 > 0$ and $v \neq 0$ such that

$$\max_{t\in[0,1]} I(h_0tv^+ + h_0(1-t)v^-) < 0.$$

By Lemma 2.3, I_β^- satisfies the (PS) or the (Ce) condition in the case $\alpha > \lambda_1$ or $\alpha = \lambda_1$ and I_α^+ satisfies the (Ce) condition. Without loss of generality, we can assume that I_β^- and I_α^+ have not non-trivial critical points. Thus, by Lemma 2.1, we can obtain a $\xi \in C([0,1], H_0^1(\Omega))$ and an $\eta \in C([0,1], H_0^1(\Omega))$ satisfying

$$\begin{cases} \xi(0) = h_0 v^+,\ \eta(0) = h_0 v^-,\ I_\alpha^+(\xi(t)) < m-1,\ I_\beta^-(\eta(t)) < m-1, \\ I_\alpha^+(\xi(t)) \le I_\alpha^+(\xi(0)) = I(h_0 v^+) < 0 \text{ for every } t \in [0,1], \\ I_\beta^-(\eta(t)) \le I_\beta^-(\eta(0)) = I(h_0 v^-) < 0 \text{ for every } t \in [0,1], \end{cases}$$

and

$$\begin{cases} \xi(0)^+ = h_0 v^+, \quad \eta(0)^- = h_0 v^-, \\ I(\xi(t)^+) = I_\alpha^+(\xi(t)^+) \le I_\alpha^+(\xi(t)) < 0 \text{ for every } t \in [0,1], \\ I(\eta(t)^-) = I_\beta^-(\eta(t)^-) \le I_\beta^-(\eta(t) < 0 \text{ for every } t \in [0,1], \end{cases}$$

which yields that

$$\xi(1)^+ \notin E(\alpha + c(\alpha), c(\alpha)) \ \text{ and } \ \eta(1)^- \notin E(\alpha + c(\alpha), c(\alpha)).$$

Define

$$\gamma_0(t) = \begin{cases} \eta(1-4t)^-, & \text{if } \ 0 \le t \le \frac{1}{4}, \\ h_0(2t - 1/2)v^+ + h_0(3/2 - 2t)v^-, & \text{if } \frac{1}{4} \le t \le \frac{3}{4}, \\ \xi(4t-3)^+, & \text{if } \frac{3}{4} \le t \le 1, \end{cases}$$

then, we have $\gamma_0 \in \Gamma$ and $\max\limits_{t \in [0,1]} I(\gamma_0(t)) < 0$.

(ii) Case $\beta \ge \lambda_1, \alpha = \lambda_1$. We can also prove the existence of a negative critical value by using $E(\tilde{c}(\beta), \tilde{c}(\beta) + \beta)$ instead of $E(\alpha + c(\alpha), c(\alpha))$. $\square$

In order to prove Theorem 1.2, we define

$$A_m := \{u \in H_0^1(\Omega) : b\|u\|^4 \ge \lambda_{m+1}|u|_4^4\},$$

then we have the following results.

Lemma 2.5. *Assume that f satisfies (f_1), then $p := \inf_{A_m} I(u) > -\infty$ for any $(\alpha, \beta) \in intQ_m$.*

Proof. From (f_1), we have $\int_\Omega F(x,u)dx = o(\|u\|^4)$ as $\|u\| \to \infty$. For any $(\alpha, \beta) \in$ intQ_m, we have

$$\begin{aligned} I(u) &= \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{\alpha}{4}|u^+|_4^4 - \frac{\beta}{4}|u^-|_4^4 - \int_\Omega F(x,u)dx \\ &\ge \frac{a}{2}\|u\|^2 + \frac{1}{4\lambda_{m+1}}(\lambda_{m+1} - \max\{\alpha, \beta\})\|u\|^4 - o(\|u\|^4) > -\infty, \end{aligned}$$

hence, from $\lambda_{m+1} > \max\{\alpha, \beta\}$, it follows that $\inf\limits_{u \in A_m} I(u) > -\infty$. $\square$

Define

$$\Gamma_m := \{h \in C(S_+^m, H_0^1(\Omega)) : h|_{S^{m-1}} \text{ is old}, -\infty < I(h(S^{m-1})) \le p - 1\},$$

where S_+^m is the upper hemisphere of S^m with boundary S^{m-1}, we have

Lemma 2.6. *Let (f_1) hold, then $h(S_+^m) \cap A_m \neq \emptyset$ for every $h \in \Gamma_m$.*

Proof. Let $h \in \Gamma_m$. If $0 \in h(S_+^m)$, we have $h(S_+^m) \cap A_m \neq \emptyset$. Now, we assume that $0 \notin h(S_+^m)$. We define the odd extension $h^* : S^m \to H_0^1(\Omega)\backslash\{0\}$ by

$$h^*(z_1, z_2, \cdots, z_{m+1}) = \begin{cases} h(z_1, z_2, \cdots, z_{m+1}), & \text{if} \quad z_{m+1} \geq 0, \\ -h(-z_1, -z_2, \cdots, -z_{m+1}), & \text{if} \quad z_{m+1} < 0, \end{cases}$$

and let $\pi(u) = u/|u|^4$ $(u \neq 0)$, then $\pi \circ h^* \in \Sigma_{m+1}$. By the definition of λ_{m+1}, there is a $z^* \in S^m$ such that $\|(\pi \circ h^*)(z^*)\|^4 \geq \lambda_{m+1}$, namely, $\|h^*(z^*)\|^4 \geq \lambda_{m+1}|h^*(z^*)|_4^4$, thus $h^*(z^*), h^*(-z^*) \in A_m$ since h^* is odd . This shows that $h(S_+^m) \cap A_m \neq \emptyset$ because h^* is the odd extension of h. □

Proof of Theorem 1.2. By Lemma 2.1, we know that the purpose of the conditions (i) and (ii) is to guarantee the compactness condition. Now, we define a minimax value of the functional I:

$$c_m := \inf_{h\in\Gamma_m} \sup_{u \in h(S_+^m)} I(u).$$

From Lemma 2.2, Lemma 2.5, Lemma 2.6 and Γ-linking theorem, c_m is a critical value of I and

$$c_m := \inf_{h\in\Gamma_m} \sup_{u \in h(S_+^m)} I(u) \geq p := \inf_{u\in A_m} I(u) > -\infty.$$

In the following, let us show that $c_m < 0$ to prove the existence of non-trivial critical point of I. Namely, there exists a $h \in \Gamma_m$ such that $\sup_{u\in h(S_+^m)} I(u) < 0$. Now, we fix $\varepsilon > 0$ with $\min\{\alpha, \beta\} - \lambda_m > \varepsilon$. By the definition of λ_m, there is a $h_0 \in \Sigma_m$ such that

$$\sup_{z\in S^{m-1}} b\|h_0(z)\|^4 < \lambda_m + \varepsilon. \tag{2.9}$$

We shall prove the existence of a continuous extension $h_0^* \in C(S_+^m, S)$ of h_0. Define $g(z) = h_0(z)/\|h_0(z)\|$ for any $z \in S^{m-1}$, then g is a continuous from S^{m-1} to $G := \{u \in H_0^1(\Omega) : \|u\| = 1\}$. Because S_+^m is homeomorphic to the m dimensional closed unit disc, there is a continuous extension $g^* : S_+^m \to B := \{u \in H_0^1(\Omega) : \|u\| \leq 1\}$. Indeed, for every $z = (z', z_m) \in S_+^m (z' \in R^{m-1})$, we define

$$g^*(z) = \begin{cases} \sqrt{1 - z_m^2}\, g(z'/\sqrt{1 - z_m^2}), & \text{if} \quad z_m \in [0, 1), \\ 0, & \text{if} \quad z_m = 1. \end{cases}$$

Because G is a retract of the unit ball B, there is an $R \in C(B, G)$ such that $R(u) = u$ for any $u \in G$. Let $h_0^* = \pi \circ R \circ g^*(z)$ for $z \in S_+^m$, where $\pi(u) = u/|u|_4$ for any $u \in H_0^1(\Omega)\backslash\{0\}$, hence, h_0^* is the desired continuous extension of h_0.

By $(\tilde{f_2}), (f_4)$, for any $M_3 > 0$, there is a constant $M_4 > 0$ such that

$$F(x, t) \geq M_3|t|^2 + M_4|t|^r \quad \text{for any } (x, t) \in \Omega \times R. \tag{2.10}$$

Then, let $M_5 = \max\limits_{u\in h_0^*(S_+^m)} \frac{\|u\|^4}{|u|_r^4}$, we have $\|u\|^4 \leq M_5|u|_r^4$ for any $u \in h_0^*(S_+^m)$. Let $M_3 > \max\limits_{u\in h_0^*(S_+^m)} \frac{a\|u\|^2}{2|u|_2^2}$ and for any $u \in h_0^*(S_+^m)$ and $s > 0$, one gets

$$I(su) = \frac{a}{2}\|u\|^2 s^2 + \frac{b}{4}\|u\|^4 s^4 - \frac{\alpha}{4}|u^+|_4^4 s^4 - \frac{\beta}{4}|u^-|_4^4 s^4 - \int_\Omega F(x, su)dx$$

$$\leq (\frac{a}{2}\|u\|^2 - M_3|u|_2^2)s^2 + \frac{b}{4}\|u\|^4 s^4 - M_4|u|_r^r s^r$$
$$\leq (\frac{bM_5}{4}s^{4-r}\max_{u\in h_0^*(S_+^m)}|u|_r^{4-r} - M_4)|u|_r^r s^r.$$

Hence from $r \in (1,2)$ and the above inequality, there is a sufficiently small $s_0 > 0$ such that

$$\max_{u\in h_0^*(S_+^m)} I(s_0 u) < 0. \tag{2.11}$$

Let $M_3 > \max_{u\in h_0(S^{m-1})} \frac{a\|u\|^2}{2|u|_2^2}$, for any $u \in h_0(S^{m-1}) = h_0^*|_{S^{m-1}}(S^{m-1})$, from (2.10), for every $t \geq s_0$, we have

$$\begin{aligned} I(tu) &= \frac{a}{2}\|u\|^2 t^2 + \frac{b}{4}\|u\|^4 t^4 - \frac{\alpha}{4}|u^+|_4^4 t^4 - \frac{\beta}{4}|u^-|_4^4 t^4 - \int_\Omega F(x,tu)dx \\ &\leq \frac{a}{2}\|u\|^2 t^2 + \frac{b}{4}\|u\|^4 t^4 - \frac{1}{4}\min\{\alpha,\beta\}|u|_4^4 t^4 - M_3|u|_2^2 t^2 - M_4|u|_r^r t^r \\ &\leq \frac{1}{4}(\lambda_m + \varepsilon - \min\{\alpha,\beta\})t^4 - M_4|u|_r^r t^r \\ &< 0, \end{aligned} \tag{2.12}$$

and $I(tu) \to -\infty$ as $t \to \infty$ using (2.9). Therefore, there is a $t_0 > s_0$ such that

$$\sup_{u\in h_0^*(S^{m-1})} I(t_0 u) \leq p - 1. \tag{2.13}$$

Define a continuous map h_1^* from S_+^m to $H_0^1(\Omega)$ as follows:

$$h_1^*(z) = \begin{cases} ((1-2z_{m+1})t_0 + 2z_{m+1}s_0)h_0(z'/\sqrt{1-z_{m+1}^2}), & \text{if } \ 0 \leq z_{m+1} \leq \frac{1}{2}, \\ s_0 h_0^*(\frac{2}{\sqrt{3}}z', \frac{2}{\sqrt{3}}\sqrt{z_{m+1}^2 - \frac{1}{4}}), & \text{if } \ \frac{1}{2} \leq z_{m+1} \leq 1, \end{cases}$$

where $z = (z_1, \cdots, z_{m+1})$ and $z' = (z_1, \cdots, z_m)$. Therefore, $h_1^* \in \Gamma_m$ by (2.13), and $\sup_{u\in h_1^*(S_+^m)} I(u) < 0$ by (2.11) and (2.12), that is, $c_m < 0$. □

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