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THE EXISTENCE OF NONTRIVIAL SOLUTION FOR BOUNDARY VALUE PROBLEM OF NONLINEAR FRACTIONAL DIFFERENTIAL EQUATION*

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Abstract By using the theory of degree, the existence of nontrivial solution for boundary value problem of nonlinear fractional differential equation is investigated. In order to get this conclusion, we make use of Laplace transform to obtain the conditions that the eigenvalues satisfy. Then, for three different specific problems, we use Matlab software to calculate the eigenvalues. This is the fundamental skill that Leray-Schauder degree theorem can be used.

Keywords Fractional differential equation, eigenvalue, Green's function, Leray-Schauder degree.

MSC(2010) 26A33, 34A08, 34B15.

1. Introduction

In recent years, fractional differential equations has attracted more and more attention [5–8, 10, 12], since it can describe many problems in reality more accurately. Moreover, the research results of fractional differential equations are also very fruitful [1–4, 9, 11, 13–17]. There are many different forms of the definition of fractional calculus, such as Riemann-Liuville fractional calculus, Caputo fractional derivative and conformable fractional calculus, etc.

Ma [11]combines the eigenvalue and Leray-Schauder degree to study the boundary value problem of nonlinear fractional differential equation:

$$\begin{cases} {}_0^cD_t^\alpha u(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \\ u(0) = au(1), \quad u'(0) = bu'(1). \end{cases}$$

This is a new method. But in [11], the existence range of the eigenvalue is not given, which makes the final result ambiguous.

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In this paper, we will study the boundary value problem of nonlinear fractional differential equation with Caputo derivative:

$$\begin{cases} {}_0^cD_t^\alpha u(t) + f(u(t)) = 0, \quad t \in (0,1), \\ u(0) = bu'(1), \quad u'(0) = cu(1), \end{cases} \tag{1.1}$$

where $1 < \alpha < 2$, b and c are real numbers with $b > \dfrac{1}{c} - 1, 0 < c < 1$.

This article is organized as follows. In Section 2, we give some definitions and lemmas. In Section 3, we are going to get the equations that the eigenvalue of (1.2) satisfies.

$$\begin{cases} {}_0^cD_t^\alpha u(t) + \lambda u(t) = 0, \quad t \in (0,1), \\ u(0) = bu'(1), \quad u'(0) = cu(1). \end{cases} \tag{1.2}$$

Then, we use these results and Leray-Schauder degree theorem to prove that (1.1) has solution. Finally, taking three different sets of data α, b and c, we use Matlab software to estimate the existence range of eigenvalue of (1.2). These results are in examples 4.1, 4.2 and 4.3. And, in these three cases, we study the existence of the solution of (1.1).

2. Preliminaries

Definition 2.1. The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $y : (0, \infty) \to \mathbb{R}$ is given by

$${}_0I_t^\alpha y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds,$$

provided that the right-hand side is pointwise defined on $(0, \infty)$.

Definition 2.2. The Caputo fractional derivative of order $\alpha > 0$ of a function $y : (0, \infty) \to \mathbb{R}$ is given by

$${}_0^cD_t^\alpha y(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{y^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds,$$

provided that the right-hand side is pointwise defined on $(0, \infty)$, where $n = [\alpha] + 1$.

Definition 2.3. The Mittag-Leffler type function is defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)},$$

where $\alpha > 0, \beta > 0$.

Lemma 2.1. *For $\alpha > 0$, the general solution of the fractional differential equation ${}_0^cD_t^\alpha u(t) = 0$ is given by*

$$u(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + \cdots + c_{n-1} t^{n-1},$$

where $c_i \in \mathbb{R}, i = 1, 2, \cdots, n-1, n = [\alpha] + 1$.

Lemma 2.2. *If $\alpha > 0, \beta > m$, for $m \in \mathbb{N}$, the following equation hold:*

$$\left(\frac{d}{dt}\right)^m \left[t^{\beta-1}E_{\alpha,\beta}(t^\alpha)\right] = t^{\beta-m-1}E_{\alpha,\beta-m}(t^\alpha).$$

Lemma 2.3. *The following formula holds for the Laplace transforms of the function $t^{\beta-1}E^{\gamma}_{\alpha,\beta}(\lambda t^\alpha)$.*

$$L\left\{t^{\beta-1}E^{\gamma}_{\alpha,\beta}(\lambda t^\alpha); s\right\} = \frac{s^{\alpha\gamma-\beta}}{(s^\alpha-\lambda)^\gamma},$$

where $Re(s) > 0$, $\beta > 0, \lambda \in \mathbb{C}$ and $|\lambda s^{-\alpha}| < 1$.

Lemma 2.4 (Lemma 3.4, [11])**.** *Let Ω be a bounded open set in infinite dimensional real Banach space E. $\theta \notin \partial\Omega$ and $A : \overline{\Omega} \to E$ be completely continuous. Suppose that $\| Ax \| > \| x \|, Ax \neq x, \forall x \in \partial\Omega$. Then $deg(I - A, \Omega, \theta) = 0$.*

Lemma 2.5 (Lemma 3.5, [11])**.** *Let A be a completely continuous operator which is defined on a Banach space E. Assume that 1 is not an eigenvalue of the asymptotic derivative. The completely continuous vector field $I - A$ is then nonsingular on spheres $S_\rho = \{x \mid \| x \| = \rho\}$ of sufficiently large radius ρ and $deg(I-A, B(\theta,\rho), \theta) = (-1)^k$, where k is the sum of the algebraic multiplicities of the real eigenvalues of $A^{'}(\infty)$ in $(1, \infty)$.*

Lemma 2.6. *Let $n - 1 < \alpha \leq n$, $h(t) \in C^n(\mathbb{R}, \mathbb{R}^+)$, for any $d > 0$, $h^{(n)}(t) \in L_1(0, d)$, then the following formula holds for the Laplace transform of the Caputo fractional derivative:*

$$L\left\{{}^c_0D^\alpha_t h(t); s\right\} = s^\alpha H(s) - \sum_{l=0}^{n-1} s^{\alpha-l-1}h^{(l)}(0),$$

where $H(s) = L\{h(t); s\}$.

Proof. According to Definition 2.1 and 2.2, we have

$$\begin{aligned}{}^c_0D^\alpha_t h(t) =& \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{h^{(n)}(\iota)}{(t-\iota)^{\alpha-n+1}}d\iota \\ =& {}_0I^{n-\alpha}_t h^{(n)}(t).\end{aligned}$$

Let $h^{(n)}(t) = g(t)$, then, we get

$$\begin{aligned}L\left\{{}^c_0D^\alpha_t h(t); s\right\} =& L\left\{{}_0I^{n-\alpha}_t h^{(n)}(t); s\right\} \\ =& L\left\{{}_0I^{n-\alpha}_t g(t); s\right\} \\ =& L\left\{\frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{g(s)}{(t-s)^{\alpha-n+1}}ds; s\right\} \\ =& \frac{1}{\Gamma(n-\alpha)}L\{t^{n-\alpha-1}; s\} \times L\left\{g(t); s\right\} \\ =& \frac{1}{\Gamma(n-\alpha)} \times \frac{\Gamma(n-\alpha)}{s^{n-\alpha}} \times L\{g(t); s\} \\ =& \frac{1}{s^{n-\alpha}}L\{g(t); s\}.\end{aligned}$$

Since

$$\begin{aligned}
L\{g(t);s\} =&L\left\{h^{(n)}(t);s\right\}\\
=&\int_0^\infty e^{-st}h^{(n)}(t)dt\\
=&\int_0^\infty e^{-st}dh^{(n-1)}(t)\\
=&e^{-st}h^{(n-1)}(t)\Big|_0^\infty + s\int_0^\infty h^{(n-1)}(t)e^{-st}dt\\
=&-h^{(n-1)}(0)+s\int_0^\infty e^{-st}dh^{(n-2)}(t)\\
=&-h^{(n-1)}(0)+se^{-st}h^{(n-2)}(t)\Big|_0^\infty + s^2\int_0^\infty e^{-st}h^{(n-2)}(t)dt\\
=&-h^{(n-1)}(0)-sh^{(n-2)}(0)+s^2\int_0^\infty e^{-st}h^{(n-2)}(t)dt\\
=&\cdots\cdots\cdots\\
=&-\sum_{k=0}^{n-1}s^k h^{(n-k-1)}(0)+s^nH(s),
\end{aligned}$$

we conclude that

$$\begin{aligned}
L\left\{{}_0^cD_t^\alpha h(t);s\right\} =&s^{-n+\alpha}\left(s^nH(s)-\sum_{k=0}^{n-1}s^k h^{(n-k-1)}(0)\right)\\
=&s^\alpha H(s)-\sum_{k=0}^{n-1}s^{-n+k+\alpha}h^{(n-k-1)}(0)\\
=&s^\alpha H(s)-\sum_{l=0}^{n-1}s^{\alpha-l-1}h^{(l)}(0).
\end{aligned}$$

□

Let's make the following assumptions:

(G_1) $f\in C(\mathbb{R},\mathbb{R})$;

(G_2) λ satisfies $cE_{\alpha,1}(-\lambda)+[cE_{\alpha,2}(-\lambda)-1]\,\frac{1-bE^{(1)}_{\alpha,1}(-\lambda)}{bE_{(\alpha,1)}(-\lambda)}=0$, and $\beta_\infty\neq\lambda$, where $\beta_\infty=\lim_{u\to\infty}\frac{f(u)}{u}$;

(G_3) There exists constant $r>0$ such that $f(u)>Q$, for any $|u|\leq r$, where $Q=\frac{(bc+c-1)\Gamma(\alpha+1)}{bc}r$.

3. Main results

Lemma 3.1. *Let $1<\alpha<2$, $y(t)\in C[0,1]$. The unique solution of*

$$\begin{cases}{}_0^cD_t^\alpha u(t)+y(t)=0, \quad 0<t<1,\\ u(0)=bu'(1), \quad u'(0)=cu(1),\end{cases} \tag{3.1}$$

is $u(t)=\int_0^1 G(t,s)y(s)ds$, *where*

$$G(t,s)=\begin{cases} -\frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1}+\frac{c(b+t)}{(bc+c-1)\Gamma(\alpha)}(1-s)^{\alpha-1} & \\ +\frac{b(ct+1-c)}{(bc+c-1)\Gamma(\alpha-1)}(1-s)^{\alpha-2}, & 0\le s<t<1,\\ \frac{c(b+t)}{(bc+c-1)\Gamma(\alpha)}(1-s)^{\alpha-1}+\frac{b(ct+1-c)}{(bc+c-1)\Gamma(\alpha-1)}(1-s)^{\alpha-2}, & 0<t\le s<1. \end{cases} \tag{3.2}$$

Proof. It follows from Lemma 2.1 that (3.1) is equivalent to integral equation

$$u(t)=-\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds+c_0+c_1t, \tag{3.3}$$

where $c_0, c_1\in\mathbb{R}$. So, we get

$$u'(t)=-\frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2}y(s)ds+c_1. \tag{3.4}$$

Applying the boundary conditions $u(0)=bu'(1)$ and $u'(0)=cu(1)$ to (3.3) and (3.4), we get

$$c_0=-\frac{b\int_0^1(1-s)^{\alpha-2}y(s)ds}{\Gamma(\alpha-1)}+\frac{bc\int_0^1(1-s)^{\alpha-1}y(s)ds}{(bc+c-1)\Gamma(\alpha)}+\frac{b^2c\int_0^1(1-s)^{\alpha-2}y(s)ds}{(bc+c-1)\Gamma(\alpha-1)},$$

$$c_1=\frac{c\int_0^1(1-s)^{\alpha-1}y(s)ds}{(bc+c-1)\Gamma(\alpha)}+\frac{bc\int_0^1(1-s)^{\alpha-2}y(s)ds}{(bc+c-1)\Gamma(\alpha-1)}.$$

Thus, the unique solution of (3.1) is

$$\begin{aligned} u(t)=&-\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}y(s)ds-\frac{b\int_0^1(1-s)^{\alpha-2}y(s)ds}{\Gamma(\alpha-1)}+\frac{bc\int_0^1(1-s)^{\alpha-1}y(s)ds}{(bc+c-1)\Gamma(\alpha)}\\ &+\frac{b^2c\int_0^1(1-s)^{\alpha-2}y(s)ds}{(bc+c-1)\Gamma(\alpha-1)}+\frac{ct\int_0^1(1-s)^{\alpha-1}y(s)ds}{(bc+c-1)\Gamma(\alpha)}+\frac{bct\int_0^1(1-s)^{\alpha-2}y(s)ds}{(bc+c-1)\Gamma(\alpha-1)}\\ =&\int_0^t\left[-\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{c(b+t)(1-s)^{\alpha-1}}{(bc+c-1)\Gamma(\alpha)}+\frac{b(ct+1-c)(1-s)^{\alpha-2}}{(bc+c-1)\Gamma(\alpha-1)}\right]y(s)ds\\ &+\int_t^1\left[\frac{c(b+t)(1-s)^{\alpha-1}}{(bc+c-1)\Gamma(\alpha)}+\frac{b(ct+1-c)(1-s)^{\alpha-2}}{(bc+c-1)\Gamma(\alpha-1)}\right]y(s)ds\\ =&\int_0^1 G(t,s)y(s)ds. \end{aligned}$$

Since $b>\frac{1}{c}-1$, $0<c<1$, $G(t,s)>0$. Taking $N_1=\frac{bc+c}{(bc+c-1)\Gamma(\alpha+1)}+\frac{b}{(bc+c-1)\Gamma(\alpha)}$, we have $\int_0^1 G(t,s)ds\le\int_0^1\frac{c(b+t)(1-s)^{\alpha-1}}{(bc+c-1)\Gamma(\alpha)}+\frac{b(ct+1-c)(1-s)^{\alpha-2}}{(bc+c-1)\Gamma(\alpha-1)}ds\le N_1$. □

Theorem 3.1. *The eigenfunction of* (1.2) *satisfies*

$$u(t)=u(0)\left[E_{\alpha,1}(-\lambda t^\alpha)+\frac{1-bE^{(1)}_{\alpha,1}(-\lambda)}{bE_{\alpha,1}(-\lambda)}tE_{\alpha,2}(-\lambda t^\alpha)\right],$$

where $u(0)$ is a constant, $E^{(1)}_{\alpha,1}(-\lambda)=\sum_{k=1}^{\infty}\frac{(-\lambda)^k}{\Gamma(\alpha k)}$, and the eigenvalue λ satisfies equation

$$cE_{\alpha,1}(-\lambda)+\Big[cE_{\alpha,2}(-\lambda)-1\Big]\frac{1-bE^{(1)}_{\alpha,1}(-\lambda)}{bE_{\alpha,1}(-\lambda)}=0.$$

Proof. For linear problem (1.2), by Lemma 2.6, we get

$$L\Big\{\,{}^{c}_{0}D^{\alpha}_{t}u(t)+\lambda u(t);s\Big\}=s^{\alpha}L\Big\{u(t);s\Big\}-s^{\alpha-1}u(0)-s^{\alpha-2}u'(0)+\lambda L\Big\{u(t);s\Big\}=0,$$

i.e.

$$L\left\{u(t);s\right\}=\frac{s^{\alpha-1}}{s^{\alpha}+\lambda}u(0)+\frac{s^{\alpha-2}}{s^{\alpha}+\lambda}u^{'}(0).$$

By Lemma 2.3, we have

$$u(t)=u(0)E_{\alpha,1}(-\lambda t^{\alpha})+u'(0)tE_{\alpha,2}(-\lambda t^{\alpha}),\tag{3.5}$$

$$u'(t)=u(0)\sum_{k=1}^{\infty}\frac{(-\lambda)^k t^{\alpha k-1}}{\Gamma(\alpha k)}+u'(0)E_{\alpha,1}(-\lambda t^{\alpha}).\tag{3.6}$$

Then, for $u(0)=bu'(1)$, we get eigenfunction of (1.2) satisfies following equation

$$u(t)=u(0)\left[E_{\alpha,1}(-\lambda t^{\alpha})+\frac{1-bE^{(1)}_{\alpha,1}(-\lambda)}{bE_{\alpha,1}(-\lambda)}tE_{\alpha,2}(-\lambda t^{\alpha})\right],$$

where $u(0)$ is a constant and $E^{(1)}_{\alpha,1}(-\lambda)=\sum_{k=1}^{\infty}\frac{(-\lambda)^k}{\Gamma(\alpha k)}$. According to $u'(0)=cu(1)$, we get the eigenvalue λ satisfies following equation

$$cE_{\alpha,1}(-\lambda)+\Big[cE_{\alpha,2}(-\lambda)-1\Big]\frac{1-bE^{(1)}_{\alpha,1}(-\lambda)}{bE_{\alpha,1}(-\lambda)}=0.\tag{3.7}$$

□

By Lemma 3.1, the problem (1.2) is equivalent to $u(t)=\lambda\int_0^1 G(t,s)u(s)ds$. We define $(Tu)(t)=\int_0^1 G(t,s)u(s)ds$. Then, the eigenvalue of operator T is $\frac{1}{\lambda}$ $(\lambda\neq 0)$. Further, we define $(Au)(t)=\int_0^1 G(t,s)f(u(s))ds$, $E=C[0,1]$ and $\parallel u\parallel= max_{0\leq t\leq 1}\mid u(t)\mid$.

Lemma 3.2. *If (G_1) and (G_2) holds, the operator $A: E\to E$ is completely continuous.*

Proof. Obviously, the operator A is continuous in consideration of continuity of $f(u)$. Let $\Omega\subset E$ be bounded, there exists constants R, $M>0$ such that $\parallel u\parallel\leq R$, $u\in\Omega$ and $\mid f(u(t))\mid< M$, $t\in[0,1]$. Taking $H=\frac{1}{\Gamma(\alpha+1)}+\frac{c(b+1)}{(bc+c-1)\Gamma(\alpha+1)}+\frac{b}{(bc+c-1)\Gamma(\alpha)}$, then, for $u\in\Omega$, we have

$$\begin{aligned}\Big|Au(t)\Big|=&\Big|\int_0^1 G(t,s)f(u(s))ds\Big|\\ =&\Big|\frac{1}{(bc+c-1)\Gamma(\alpha)}\int_0^1 c(b+t)(1-s)^{\alpha-1}+b(ct+1-c)(\alpha-1)(1-s)^{\alpha-2}\\ &\times f(u(s))ds-\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(u(s))ds\Big|\end{aligned}$$

$$
\begin{aligned}
\leq & \frac{M}{(bc+c-1)\Gamma(\alpha)} \int_0^1 c(b+1)(1-s)^{\alpha-1} + b(\alpha-1)(1-s)^{\alpha-2} ds \\
& + \frac{M}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds \\
\leq & \left[\frac{1}{\Gamma(\alpha+1)} + \frac{c(b+1)}{(bc+c-1)\Gamma(\alpha+1)} + \frac{b}{(bc+c-1)\Gamma(\alpha)} \right] M \\
= & HM.
\end{aligned}
$$

Hence, the operator A is uniformly bounded on E. From

$$
\begin{aligned}
\left| (Au)'(t) \right| = & \Big| \frac{c}{(bc+c-1)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(u(s)) ds + \frac{bc(\alpha-1)}{(bc+c-1)\Gamma(\alpha)} \\
& \times \int_0^1 (1-s)^{\alpha-2} f(u(s)) ds - \frac{\alpha-1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-2} f(u(s)) ds \Big| \\
< & \frac{c}{\alpha(bc+c-1)\Gamma(\alpha)} M + \frac{bcM}{(bc+c-1)\Gamma(\alpha)} + \frac{M}{\Gamma(\alpha)} t^{\alpha-1} \\
< & \left[\frac{c}{(bc+c-1)\Gamma(\alpha+1)} + \frac{bc}{(bc+c-1)\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)} \right] M,
\end{aligned}
$$

for $u \in \Omega$, $t_1, t_2 \in [0,1]$, $t_1 < t_2$, we have

$$
\begin{aligned}
\left| Au(t_2) - Au(t_1) \right| = & \Big| \int_{t_1}^{t_2} (Au)'(s) ds \Big| \\
\leq & \int_{t_1}^{t_2} \left| (Au)'(s) \right| ds \\
< & \left[\frac{c}{(bc+c-1)\Gamma(\alpha+1)} + \frac{bc}{(bc+c-1)\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)} \right] M(t_2 - t_1).
\end{aligned}
$$

Therefore, the operator A is equicontinuous on E. By Arzela-Ascoli theorem, we get that $A : E \to E$ is a completely continuous operator. □

Lemma 3.3. *The operator A is Fréchet differentiable at ∞, If (G_1) and (G_2) hold, $A'(\infty) = \beta_\infty T$.*

Proof. By Lemma 3.1, we have $\int_0^1 G(t,s) ds \leq N_1$. Since $\beta_\infty = \lim_{u \to \infty} \frac{f(u)}{u}$, there exist a constant u_ε such that $|u| > u_\varepsilon$, for any $\varepsilon > 0$, we have

$$
\Big| \frac{f(u)}{u} - \beta_\infty \Big| \leq \frac{\varepsilon}{2N_1},
$$

i.e.

$$
\left| f(u) - \beta_\infty u \right| \leq \frac{\varepsilon |u|}{2N_1}.
$$

On the one hand, when $|u| \leq u_\varepsilon$, there exists $M_1(\varepsilon) > 0$ such that $\left| f(u) - \beta_\infty u \right| \leq M_1(\varepsilon)$. So, we have $\left| f(u) - \beta_\infty u \right| \leq \frac{\varepsilon |u|}{2N_1} + M_1(\varepsilon)$, for all $u \in \mathbb{R}$. Then, for $u \in E$, we have

$$
\left| Au - \beta_\infty Tu \right| = \Big| \int_0^1 G(t,s) f(u(s)) ds - \beta_\infty \int_0^1 G(t,s) u(s) ds \Big|
$$

$$\le \int_0^1 G(t,s)\Big|f(u(s)) - \beta_\infty u(s)\Big|ds \le \int_0^1 G(t,s)\left[\frac{\varepsilon \parallel u \parallel}{2N_1} + M_1(\varepsilon)\right]ds$$
$$\le \parallel u \parallel \frac{\varepsilon}{2} + M_1(\varepsilon)N_1.$$

So, we get $\frac{\|Au-\beta_\infty Tu\|}{\|u\|} \le \left(\frac{\varepsilon}{2N_1} + \frac{M_1(\varepsilon)}{\|u\|}\right)N_1$. Let $U_0 = max\left\{u_\varepsilon, \frac{2M_1(\varepsilon)N_1}{\varepsilon}\right\}$. When $\parallel u \parallel > U_0$, we obtain $\frac{\|Au-\beta_\infty Tu\|}{\|u\|} \le \varepsilon$. Therefore, $A'(\infty) = \beta_\infty T$. □

Theorem 3.2. *If $(G_1)-(G_3)$ hold, There exists at least one nontrivial solution of* (1.1).

Proof. Apparently, $B(\theta,r)$ is a bounded open set, $\theta \notin \partial(B(\theta,r))$. Via Lemma 3.2, we get $A : \overline{B(\theta,r)} \to E$ is completely continuous. Now, we prove $\parallel Au \parallel > \parallel u \parallel$, for $u \in \partial(B(\theta,r))$. Since

$$\begin{aligned}\Big|(Au)(t)\Big| =& \Big|\int_0^1 G(t,s)f(u(s))ds\Big| \\ =& \Big|\int_0^t \frac{-(bc+c-1)(t-s)^{\alpha-1} + c(b+t)(1-s)^{\alpha-1} + b(ct+c-1)}{(bc+c-1)\Gamma(\alpha)} \\ &\times (\alpha-1)(1-s)^{\alpha-2}f(u(s))ds + \int_t^1 \frac{c(b+t)(1-s)^{\alpha-1} + b(ct+c-1)}{(bc+c-1)\Gamma(\alpha)} \\ &\times (\alpha-1)(1-s)^{\alpha-2}f(u(s))ds\Big| \\ >& \Big|\int_t^1 \frac{c(b+t)}{(bc+c-1)\Gamma(\alpha)}(1-s)^{\alpha-1}f(u(s))ds\Big|,\end{aligned}$$

taking $t = 0$ and combining (G_3), we have

$$\begin{aligned}\Big|(Au)(0)\Big| >& \Big|\int_0^1 \frac{bc}{(bc+c-1)\Gamma(\alpha)}(1-s)^{\alpha-1}f(u(s))ds\Big| \\ >& \frac{bc}{(bc+c-1)\Gamma(\alpha+1)} \times Q = r \ge \parallel u \parallel .\end{aligned}$$

Because of $\parallel Au \parallel = max_{t\in[0,1]}\Big\{(Au)(t)\Big\}$, we have $\parallel Au \parallel > \parallel u \parallel$. Therefore, from Lemma 2.4, we get

$$deg(I - A, B(\theta,r), \theta) = 0. \tag{3.8}$$

According to (G_2), we obtain $\frac{\beta_\infty}{\lambda} \ne 1$ ($\frac{\beta_\infty}{\lambda}$ is the eigenvalue of the operator $A'(\infty)$). That is to say, 1 is not an eigenvalue of the asymptotic derivative. Consequently, by Lemma 2.5, we have

$$deg(I - A, B(\theta,\rho), \theta) = (-1)^k, k \ge 1. \tag{3.9}$$

It follows from Eq. (3.8) and Eq. (3.9) that $deg(I - A, B(\theta,\rho) \setminus B(\theta,r), \theta) = (-1)^k - 0 = (-1)^k$, where k is the sum of the algebraic multiplicities of the real eigenvalues of $A'(\infty)$ in $(1,\infty)$. If Eq. (1.2) has no eigenvalue, we have $k = 0$. So, $deg(I - A, B(\theta,\rho) \setminus B(\theta,r), \theta) = (-1)^0 - 0 = 1$.

By the theory of degree, we can obtain that the operator A has at least one fixed point in $B(\theta,\rho) \setminus B(\theta,r)$. So, (1.1) has at least one nontrivial solution. □

Remark 3.1. As long as β_∞ is not equal to everyone of eigenvalues, the boundary value problem (1.1) has at least one nontrivial solution. In the boundary value problem (1.1), for different values of α, b, and c, the distribution of eigenvalue corresponding to Eq. (1.2) are different, see Examples 4.1, 4.2 and 4.3. Thus, in [11], it is not feasible that β_∞ is not equal to one of the eigenvalues.

Remark 3.2. In order to calculate the eigenvalue of Eq. (1.2) when α, b and c are given, we should find the solutions of Eq. (3.7). In other words, we have to investigate the following equation

$$\left[\sum_{k=0}^{\infty}\frac{bc(-\lambda)^k}{\Gamma(\alpha k+1)}\right]^2+\sum_{k=0}^{\infty}\frac{c(-\lambda)^k}{\Gamma(\alpha k+2)}-\sum_{k=0}^{\infty}\frac{bc(-\lambda)^k}{\Gamma(\alpha k+2)}\sum_{k=1}^{\infty}\frac{(-\lambda)^k}{\Gamma(\alpha k)}+\sum_{k=1}^{\infty}\frac{b(-\lambda)^k}{\Gamma(\alpha k)}-1=0. \tag{3.10}$$

First, taking $\alpha=1$, we make $f_1(\lambda)=\sum_{k=10001}^{n}\frac{(-\lambda)^k}{\Gamma(k+1)}$, $n=20000,30000,40000\cdots$. Through Matlab calculation, when $\lambda\in[1,200]$, the value of $f_1(\lambda)$ is approximately equal to 0. So, $\sum_{k=10001}^{n}\frac{(-\lambda)^k}{\Gamma(\alpha k+1)}\approx 0$ for $1<\alpha<2$, $\lambda\in[1,200]$, which means $\sum_{k=10001}^{\infty}\frac{(-\lambda)^k}{\Gamma(\alpha k+1)}\approx 0$, in the same way, we can obtain $\sum_{k=10001}^{\infty}\frac{(-\lambda)^k}{\Gamma(\alpha k+2)}\approx 0, \sum_{k=10001}^{\infty}\frac{(-\lambda)^k}{\Gamma(\alpha k)}\approx 0$. Therefore, the Eq. (3.10) is equivalent to

$$\begin{aligned}&bc\left[\sum_{k=0}^{10000}\frac{(-\lambda)^k}{\Gamma(\alpha k+1)}\right]^2+c\sum_{k=0}^{10000}\frac{(-\lambda)^k}{\Gamma(\alpha k+2)}-bc\sum_{k=0}^{10000}\frac{(-\lambda)^k}{\Gamma(\alpha k+2)}\sum_{k=1}^{10000}\frac{(-\lambda)^k}{\Gamma(\alpha k)}\\&+b\sum_{k=1}^{10000}\frac{(-\lambda)^k}{\Gamma(\alpha k)}-1=0,\end{aligned} \tag{3.11}$$

for $\lambda\in[1,200]$. That is to say, when $\lambda\in[1,200]$, the eigenvalue λ of (1.2) satisfies Eq. (3.11).

4. Examples

Example 4.1. Consider the following boundary value problem

$$\begin{cases}{}_0^cD_t^{\frac{5}{3}}u(t)+f(u(t))=0, & 0<t<1,\\ u(0)=2u'(1), \quad u'(0)=\dfrac{1}{2}u(1),\end{cases} \tag{4.1}$$

where

$$f(u)=\begin{cases}\frac{8}{3}u+\frac{8}{3}, & u<-\frac{1}{2},\\ \frac{4}{3}, & -\frac{1}{2}\le u\le\frac{1}{2},\\ \frac{8}{3}u, & u>\frac{1}{2}.\end{cases}$$

Now, we show that $(G_1)-(G_3)$ hold. Obviously, $f\in C(\mathbb{R},\mathbb{R})$. Corresponding (1.1), we get $\alpha=\dfrac{5}{3}$, $b=2$, $c=\dfrac{1}{2}$. For convenience, we let

$$\begin{aligned}f_2(\lambda)=&\left[\sum_{k=0}^{10000}\frac{(-\lambda)^k}{\Gamma(\frac{5k}{3}+1)}\right]^2+\frac{1}{2}\sum_{k=0}^{10000}\frac{(-\lambda)^k}{\Gamma(\frac{5k}{3}+2)}-\sum_{k=0}^{10000}\frac{(-\lambda)^k}{\Gamma(\frac{5k}{3}+2)}\sum_{k=1}^{10000}\frac{(-\lambda)^k}{\Gamma(\frac{5k}{3})}\\&+2\sum_{k=1}^{10000}\frac{(-\lambda)^k}{\Gamma(\frac{5k}{3})}-1.\end{aligned}$$

Through Matlab calculation, we get the function image of $f_2(\lambda)$ (see Figure 1). The intersections of $f_2(\lambda)$ and $y = 0$ in Figure 1 are the eigenvalues of (1.2) corresponding to $\alpha = \frac{5}{3}$, $b = 2$ and $c = \frac{1}{2}$. As showed in Figure 1, the image fluctuates around

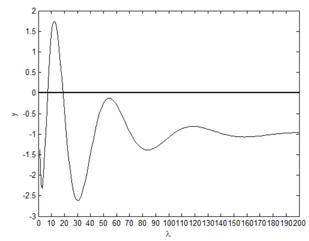

Figure 1. Graph of equation f_2

-1 and tends to be stable with the increase of λ. So, the two eigenvalues are in the intervals $(7, 8)$ and $(18, 19)$, respectively. In (4.1), $\beta_\infty = \frac{8}{3}$. Clearly, $\frac{\beta_\infty}{\lambda} \neq 1$. Choosing $r = \frac{1}{2}$, we have $Q = \frac{1}{4}\Gamma(\frac{8}{3})$. So, we get $f(u) = \frac{4}{3} > Q$, for $|u| \leq \frac{1}{2}$.

With the use of Theorem 3.2, we conclude that the problem (4.1) has at least one nontrivial solution.

Example 4.2. Consider the following boundary value problem

$$\begin{cases} {}_0^cD_t^{\frac{7}{4}} u(t) + f(u(t)) = 0, \quad 0 < t < 1, \\ u(0) = 4u'(1), \quad u'(0) = \dfrac{1}{4}u(1), \end{cases} \tag{4.2}$$

where

$$f(u) = \begin{cases} \frac{11}{4}u + \frac{11}{8}, & u < -\frac{1}{4}, \\ \frac{11}{16}, & -\frac{1}{4} \leq u \leq \frac{1}{4}, \\ \frac{11}{4}u, & u > \frac{1}{4}. \end{cases}$$

Now, we prove that $(G_1) - (G_3)$ hold. Clearly, $f \in C(\mathbb{R}, \mathbb{R})$. Corresponding (1.1), we get $\alpha = \frac{7}{4}$, $b = 4$, $c = \frac{1}{4}$. As a matter of convenience, we make

$$\begin{aligned} f_3(\lambda) = {} & \left[\sum_{k=0}^{10000} \frac{(-\lambda)^k}{\Gamma(\frac{7k}{4}+1)}\right]^2 + \frac{1}{4}\sum_{k=0}^{10000} \frac{(-\lambda)^k}{\Gamma(\frac{7k}{4}+2)} - \sum_{k=0}^{10000} \frac{(-\lambda)^k}{\Gamma(\frac{7k}{4}+2)} \sum_{k=1}^{10000} \frac{(-\lambda)^k}{\Gamma(\frac{7k}{4})} \\ & + 4\sum_{k=1}^{10000} \frac{(-\lambda)^k}{\Gamma(\frac{7k}{4})} - 1. \end{aligned}$$

As showed in Figure 2, when $\alpha = \frac{7}{4}$, $b = 4$, and $c = \frac{1}{4}$, the eigenvalues of (1.2) are in the intervals $(7, 8)$, $(23, 24)$, $(52, 53)$, $(81, 82)$, $(134, 135)$ and $(162, 163)$, respectively. In (4.2), $\beta_\infty = \frac{11}{4}$. So, we have $\frac{\beta_\infty}{\lambda} \neq 1$. Making $r = \frac{1}{4}$, we have $Q = \frac{1}{16}\Gamma(\frac{11}{4})$. Hence, we get $f(u) = \frac{11}{16} > Q$, for $|u| \leq \frac{1}{4}$.

With the use of Theorem 3.2, we conclude that the problem (4.2) has at least one nontrivial solution.

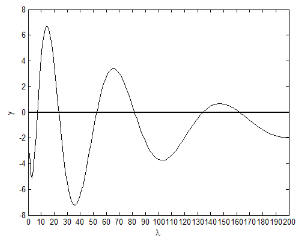

Figure 2. Graph of equation f_3

Example 4.3. Consider the following boundary value problem

$$\begin{cases} {}_0^cD_t^{\frac{4}{3}}u(t) + f(u(t)) = 0, \quad 0 < t < 1, \\ u(0) = 2u'(1), \quad u'(0) = \dfrac{1}{2}u(1), \end{cases} \tag{4.3}$$

where

$$f(u) = \begin{cases} \frac{7}{3}u + \frac{7}{3}, & u < -\frac{1}{2}, \\ \frac{7}{6}, & -\frac{1}{2} \leq u \leq \frac{1}{2}, \\ \frac{7}{3}u, & u > \frac{1}{2}. \end{cases}$$

Now, we show that $(G_1) - (G_3)$ hold. Obviously, $f \in C(\mathbb{R}, \mathbb{R})$. Corresponding (1.1), we get $\alpha = \dfrac{4}{3}$, $b = 2$, $c = \dfrac{1}{2}$. For convenience, we set

$$\begin{aligned} f_4(\lambda) = {} & \left[\sum_{k=0}^{10000} \frac{(-\lambda)^k}{\Gamma(\frac{4k}{3}+1)}\right]^2 + \frac{1}{2}\sum_{k=0}^{10000} \frac{(-\lambda)^k}{\Gamma(\frac{4k}{3}+2)} - \sum_{k=0}^{10000} \frac{(-\lambda)^k}{\Gamma(\frac{4k}{3}+2)} \sum_{k=1}^{10000} \frac{(-\lambda)^k}{\Gamma(\frac{4k}{3})} \\ & + 2\sum_{k=1}^{10000} \frac{(-\lambda)^k}{\Gamma(\frac{4k}{3})} - 1. \end{aligned}$$

As showed in Figure 3, when $\alpha = \dfrac{4}{3}$, $b = 2$ and $c = \dfrac{1}{2}$, (1.2) has no eigenvalue.

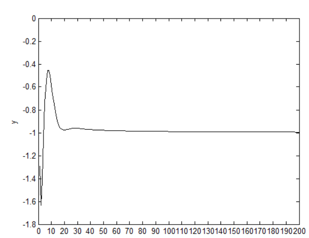

Figure 3. Graph of equation f_4

Taking $r = \dfrac{1}{2}$, we have $Q = \dfrac{1}{4}\Gamma(\dfrac{7}{3})$. Therefore, we get $f(u) = \dfrac{7}{6} > Q$, for $|u| \leq \dfrac{1}{2}$.

With the use of Theorem 3.2, we conclude that the problem (4.3) has at least one nontrivial solution.

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