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AN EFFICIENT NUMERICAL METHOD BASED ON LEGENDRE-GALERKIN APPROXIMATION FOR THE STEKLOV EIGENVALUE PROBLEM IN SPHERICAL DOMAIN

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Abstract We present in this paper an efficient numerical method based on Legendre-Galerkin approximation for the Steklov eigenvalue problem in spherical domain. Firstly, by means of spherical coordinate transformation and spherical harmonic expansion, the original problem is reduced to a sequence of equivalent one-dimensional eigenvalue problems that can be solved individually in parallel. Through the introduction of the appropriate weighted Sobolev spaces, the weak form and corresponding discrete scheme are established for each one-dimensional eigenvalue problem. Then from the approximate property of orthogonal polynomials in the weighted Sobolev spaces, we prove the error estimates of approximate eigenvalues for each one dimensional eigenvalue problem. Finally, some numerical examples are provided to illustrate the validity of our algorithms.

Keywords Steklov eigenvalue problem, Legendre-Galerkin approximation, weighted Soblev space, error estimation, spherical domain.

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1. Introduction

The Steklov eigenvalue problems on the bounded domains in the plane was introduced by Steklov in 1902 [13]. The eigenfunction represents the steady state temperature on domain Ω such that the flux on the boundary is proportional to the temperature. It is also important in conductivity and harmonic analysis as it was initially studied by Calderón [10]. This connection arises because the set of eigenvalues for the Steklov problem is the same as the set of eigenvalues of the well-known Dirichlet-Neumann map. Thus, the Steklov eigenvalue problems have important physical background and applications. For more details we can refer to [4–6, 9, 11, 15]. In this paper, we consider the following model problem

$$\begin{cases} -\Delta\psi + \psi = 0, & \text{in } \Omega, \\ \frac{\partial \psi}{\partial \mathbf{n}} = \lambda\psi, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

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where Ω is a sphere of radius R and $\frac{\partial \psi}{\partial \mathbf{n}}$ is the outward normal derivative on $\partial\Omega$.

In recent years, many numerical methods have been proposed and received more and more attention for the Steklov eigenvalue problems. Andreev and Todorov [1] discussed the isoparametric finite element method. Armentano and Padra [2] introduced and analyzed the conforming linear finite element approximation in a bounded polygonal domain. Alonso and Russo [3], Yang et al. [27] and Li et al. [20] studied nonconforming finite elements approximation. Li and Yang [19], Bi and Yang [7] discussed a two-grid method of the conforming and non-conforming finite element method, respectively. Li et al. [18] studied the extrapolation and superconvergence. Tang et al. [25] studied the boundary element approximation. Bi and Yang [8] discussed the multi-scale discretization scheme based on the Rayleigh quotient iterative method. Garau and Morin [14] analyzed the convergence and quasi-optimality of adaptive FEM. Cao et al. [12] discussed multiscale asymptotic method in composite medias. Xie et al. [16, 17, 26] studied a type of multilevel method. Zhang et al. [28] discussed the spectral method with the tensor-product nodal basis in rectangular domain.

In theoretical research and practical applications, we often need to solve the Steklov eigenvalue problem in some special domains, such as circular domain, spherical domain and so on. Then an essential problem is how to efficiently solve the Steklov eigenvalue problem in these special domains, especially for three-dimensional spherical domains. To the best of our knowledge, there has few reports until recently. Thus, the aim of this paper is to present an efficient numerical method based on Legendre-Galerkin approximation for the Steklov eigenvalue problem in spherical domain. Firstly, by means of spherical coordinate transformation and spherical harmonic expansion, the original problem is reduced to a sequence of equivalent one-dimensional eigenvalue problems that can be solved individually in parallel. Through the introduction of the appropriate weighted Sobolev spaces, the weak form and corresponding discrete scheme are established for each one-dimensional eigenvalue problem. Then from the approximate property of orthogonal polynomials in the weighted Sobolev spaces, we prove the error estimates of approximate eigenvalues for each one dimensional eigenvalue problem. Finally, some numerical examples are provided to illustrate the validity of our algorithms.

The remainder of this paper is organized as follows. In §2, we derive dimension reduction scheme of the Steclov eigenvalue problem. In §3, we formulate a weak form and derive an error estimate of approximate eigenvalues. In §4, we deduce in detail an efficient implementation of the spectral Galerkin approximation. We present several numerical results in §5 to demonstrate the accuracy and efficiency of our algorithm. Finally, in §6, we give some concluding remarks.

For brief, we use the symbol $a \lesssim b$ to mean that $a \leq Cb$ hereafter, where C is a positive constant independent of N.

2. Dimension reduction scheme

In this section, we shall reduce the problem (1.1) into a series of equivalent one-dimensional eigenvalue problems. At first, applying the spherical coordinate transformation

$$x = r \sin\theta \cos\phi, y = r \sin\theta \sin\phi, z = r \cos\theta \tag{2.1}$$

to (1.1), we obtain

$$-\frac{1}{r^2}\partial_r(r^2\partial_r u) - \frac{1}{r^2\sin\theta}\partial_\theta(\sin\theta\partial_\theta u) - \frac{1}{r^2\sin^2\theta}\partial_{\phi\phi}u + u = 0,$$
$$(r,\theta,\phi) \in [0,R)\times[0,\pi]\times[0,2\pi] \tag{2.2}$$
$$\partial_r u(R,\theta,\phi) = \lambda u(R,\theta,\phi), (\theta,\phi)\in[0,\pi]\times[0,2\pi], \tag{2.3}$$

where $u(r,\theta,\phi) = \psi(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)$.

Let S be the unit spherical surface, and denote by Δ_S the Laplace-Beltrami operator on S, namely,

$$\triangle_S u = \frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta u) + \frac{1}{\sin^2\theta}\partial_\phi^2 u. \tag{2.4}$$

The spherical harmonics $\{Y_l^m\}$ (as normalized in [21]) are eigenfunctions of Δ_S, i.e.,

$$\Delta_S Y_l^m = -l(l+1)Y_l^m, m, l\in\mathbb{Z}, l\geq 0, |m|\leq l. \tag{2.5}$$

By spherical harmonic expansion we have

$$u(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{|m|=0}^{l} u_l^m Y_l^m. \tag{2.6}$$

From (2.5), the (2.2)-(2.3) can be reduced to

$$-\partial_r(r^2\partial_r u_l^m) + l(l+1)u_l^m + r^2 u_l^m = 0, r\in[0,R), \tag{2.7}$$
$$\partial_r u_l^m(R) = \lambda_l u_l^m(R). \tag{2.8}$$

Let $r = \frac{t+1}{2}R, \psi_l(t) = u_l^m(\frac{t+1}{2}R)$. Then (2.7) - (2.8) is equivalent to

$$-\partial_t((t+1)^2\partial_t\psi_l) + l(l+1)\psi_l + \frac{R^2}{4}(t+1)^2\psi_l = 0, t\in[-1,1), \tag{2.9}$$
$$\partial_t\psi_l(1) = \lambda_l\frac{R}{2}\psi_l(1). \tag{2.10}$$

3. Weak form and Error Estimation

3.1. Weak form and its discrete scheme

We denote by $\omega = 1+t$ a weight function and introduce the usual weighted Sobolev space

$$L_\omega^2(I) := \{\psi : \int_I \omega\psi^2 dt < \infty\}$$

equipped with the inner product and norm

$$(\psi,\phi)_\omega = \int_I \omega\psi\phi dt, \|\psi\|_w = (\int_I \omega\psi^2 dt)^{\frac{1}{2}},$$

where $I = (-1,1)$. Next, we introduce the following non-uniformly weighted Sobolev space $H^1_{\omega,l}(I)$. For $l=0$, we define

$$H^1_{\omega,0}(I) := \{\psi : \partial_t^k \psi \in L^2_{\omega^2}(I), k=0,1\}$$

equipped with the corresponding inner product and norm

$$(\psi,\phi)_{1,\omega,0} = \sum_{k=0}^{1} (\partial_t^k \psi, \partial_t^k \phi)_{\omega^2}, \|\psi\|_{1,\omega,0} = (\psi,\psi)^{\frac{1}{2}}_{1,\omega,0}.$$

For $l \geq 1$, we define

$$H^1_{\omega,l}(I) := \{\psi : \partial_t^k \psi \in L^2_{\omega^{2k}}(I), k=0,1\}$$

equipped with the corresponding inner product and norm

$$(\psi,\phi)_{1,\omega,l} = \sum_{k=0}^{1} (\partial_t^k \psi, \partial_t^k \phi)_{\omega^{2k}}, \|u\|_{1,\omega,l} = (\psi,\psi)^{\frac{1}{2}}_{1,\omega,l}.$$

Then the weak form of (2.9) -(2.10) is: Find $(\lambda_l, \psi_l) \in \mathbb{R} \times H^1_{\omega,l}(I)$, such that

$$\mathcal{A}_l(\psi_l,\phi) = \lambda_l \mathcal{B}_l(\psi_l,\phi), \forall \phi \in H^1_{\omega,l}(I), \tag{3.1}$$

where

$$\begin{aligned}
&\mathcal{A}_l(\psi_l,\phi) = \int_I ((t+1)^2 \psi_l' \phi' + l(l+1)\psi_l \phi + \frac{R^2}{4}(t+1)^2 \psi_l \phi) dt, \\
&\mathcal{B}_l(\psi_l,\phi) = 2R\psi_l(1)\phi(1).
\end{aligned}$$

Let P_N be the space of polynomials of degree less than or equal to N, and set $X_N(l) = P_N \cap H^1_{\omega,l}(I)$. Then the discrete form of (3.1) is: Find $(\lambda_{lN}, \psi_{lN}) \in \mathbb{R} \times X_N(l)$, such that

$$\mathcal{A}_l(\psi_{lN},\phi_N) = \lambda_{lN} \mathcal{B}_l(\psi_{lN},\phi_N), \forall \phi_N \in X_N(l). \tag{3.2}$$

3.2. Error estimation of approximation eigenvalues

Theorem 3.1. *$\mathcal{A}_l(\psi,\phi)$ is a continuous and coercive bilinear form on $H^1_{\omega,l}(I) \times H^1_{\omega,l}(I)$, i.e.,*

$$\begin{aligned}
&|\mathcal{A}_l(\psi,\phi)| \lesssim \|\psi\|_{1,\omega,l} \|\phi\|_{1,\omega,l}, \\
&\mathcal{A}_l(\psi,\psi) \gtrsim \|\psi\|^2_{1,\omega,l}.
\end{aligned}$$

Proof. For the case of $l=0$, from Schwartz inequality we can derive that

$$\begin{aligned}
|\mathcal{A}_0(\psi,\phi)| &= |\int_I ((t+1)^2 \psi' \phi' + \frac{R^2}{4}(t+1)^2 \psi \phi) dt| \\
&\lesssim \int_I ((t+1)^2 |\psi' \phi'| + (t+1)^2 |\psi \phi|) dt \\
&\lesssim (\int_I ((t+1)^2 (\psi')^2 + (t+1)^2 \psi^2 dt)^{\frac{1}{2}} (\int_I ((t+1)^2 (\phi')^2 + (t+1)^2 \phi^2) dt)^{\frac{1}{2}}
\end{aligned}$$

$$= \|\psi\|_{1,\omega,0}\|\phi\|_{1,\omega,0},$$

$$\mathcal{A}_0(\psi,\psi) = \int_I ((t+1)^2(\psi')^2 + \frac{R^2}{4}(t+1)^2\psi^2)dt$$

$$\gtrsim \int_I ((t+1)^2(\psi')^2 + (t+1)^2\psi^2)dt = \|\psi\|^2_{1,\omega,0}.$$

For the case of $l \geq 1$, again by using Schwartz inequality we can derive that

$$|\mathcal{A}_l(\psi,\phi)| = |\int_I ((t+1)^2\psi'\phi' + l(l+1)\psi\phi + \frac{R^2}{4}(t+1)^2\psi\phi)dt|$$

$$\lesssim \int_I (t+1)^2|\psi'\phi'| + |\psi\phi|dt$$

$$\lesssim (\int_I ((t+1)^2(\psi')^2 + \psi^2)dt)^{\frac{1}{2}}(\int_I ((t+1)^2(\phi')^2 + \phi^2)dt)^{\frac{1}{2}}$$

$$= \|\psi\|_{1,\omega,l}\|\phi\|_{1,\omega,l},$$

$$\mathcal{A}_l(\psi,\psi) = \int_I ((t+1)^2(\psi')^2 + l(l+1)\psi^2 + \frac{R^2}{4}(t+1)^2\psi^2)dt$$

$$\gtrsim \int_I ((t+1)^2(\psi')^2 + \psi^2)dt = \|\psi\|^2_{1,\omega,l}.$$

□

Lemma 3.1. *Let λ_l^k be the eigenvalues of (3.1) and V_k be any k-dimensional subspace of $H^1_{\omega,l}(I)$. Then, for $\lambda_l^1 \leq \lambda_l^2 \leq \cdots \leq \lambda_l^k \leq \cdots$, it holds*

$$\lambda_l^k = \min_{V_k \subset H^1_{\omega,l}(I)} \max_{\psi \in V_k} \frac{\mathcal{A}_l(\psi,\psi)}{\mathcal{B}_l(\psi,\psi)}. \tag{3.3}$$

Proof. See Theorem 3.1 in [22]. □

Lemma 3.2. *Let λ_l^i be the eigenvalues of (3.1) and be arranged in an ascending order, and define*

$$E_{i,j} = \text{span}\left\{\psi_l^i, \cdots, \psi_l^j\right\},$$

where ψ_l^i is the eigenfunction corresponding to the eigenvalue λ_l^i. Then we have

$$\lambda_l^k = \max_{\psi \in E_{m,k}} \frac{\mathcal{A}_l(\psi,\psi)}{\mathcal{B}_l(\psi,\psi)} \quad m \leq k, \tag{3.4}$$

$$\lambda_l^k = \min_{\psi \in E_{k,n}} \frac{\mathcal{A}_l(\psi,\psi)}{\mathcal{B}_l(\psi,\psi)} \quad k \leq n. \tag{3.5}$$

Proof. See Lemma 3.2 in [22]. □

It is true that the minimax principle is also valid for the discrete formulation (3.2) (see [22]).

Lemma 3.3. *Let λ_{lN}^k be the eigenvalues of (3.2), and V_k be any k-dimensional subspace of $X_N(l)$. Then, for $\lambda_{lN}^1 \leq \lambda_{lN}^2 \leq \cdots \leq \lambda_{lN}^k \leq \cdots$, there holds*

$$\lambda_{lN}^k = \min_{V_k \subset X_N(l)} \max_{\psi \in V_k} \frac{\mathcal{A}_l(\psi,\psi)}{\mathcal{B}_l(\psi,\psi)}. \tag{3.6}$$

Let $\Pi_N^{1,l}: H^1_{\omega,l}(I) \to X_N(l)$ be an orthogonal projection, defined by

$$\mathcal{A}_l(\psi_l - \Pi_N^{1,l}\psi_l, \phi) = 0, \forall \phi \in X_N(l).$$

Theorem 3.2. *Let λ_{lN}^k be k-th eigenvalue of (3.2) and be an approximation of λ_l^k, k-th eigenvalue of (3.1). Then, we have*

$$\lambda_l^k \le \lambda_{lN}^k \le \lambda_l^k \max_{\psi \in E_{1,k}} \frac{\mathcal{B}_l(\psi,\psi)}{\mathcal{B}_l(\Pi_N^{1,l}\psi, \Pi_N^{1,l}\psi)}. \tag{3.7}$$

Proof. Owing to $X_N(l) \subset H^1_{\omega,l}(I)$, then from (3.3) and (3.6) we have $\lambda_l^k \le \lambda_{lN}^k$. Let $\Pi_N^{1,l}E_{1,k}$ denote the space spanned by $\Pi_N^{1,l}\psi_l^1, \Pi_N^{1,l}\psi_l^2, \cdots, \Pi_N^{1,l}\psi_l^k$. It's obvious that $\Pi_N^{1,l}E_{1,k}$ is a k-dimensional subspace of $X_N(l)$. From the minimax principle, we can derive that

$$\lambda_{lN}^k \le \max_{\psi \in \Pi_N^{1,l}E_{1,k}} \frac{\mathcal{A}_l(\psi,\psi)}{\mathcal{B}_l(\psi,\psi)} = \max_{\psi \in E_{1,k}} \frac{\mathcal{A}_l(\Pi_N^{1,l}\psi, \Pi_N^{1,l}\psi)}{\mathcal{B}_l(\Pi_N^{1,l}\psi, \Pi_N^{1,l}\psi)}.$$

From the symmetry of $\mathcal{A}_l(\cdot,\cdot)$, we have

$$\mathcal{A}_l(\psi,\psi) = \mathcal{A}_l(\Pi_N^{1,l}\psi, \Pi_N^{1,l}\psi) + 2\mathcal{A}_l(\psi - \Pi_N^{1,l}\psi, \Pi_N^{1,l}\psi) + \mathcal{A}_l(\psi - \Pi_N^{1,l}\psi, \psi - \Pi_N^{1,l}\psi).$$

From $\mathcal{A}_l(\psi - \Pi_N^{1,l}\psi, \Pi_N^{1,l}\psi) = 0$ and the non-negativity of $\mathcal{A}_l(\psi - \Pi_N^{1,l}\psi, \psi - \Pi_N^{1,l}\psi)$, we obtain

$$\mathcal{A}_l(\Pi_N^{1,l}\psi, \Pi_N^{1,l}\psi) \le \mathcal{A}_l(\psi,\psi).$$

Thus, we have

$$\begin{aligned}
\lambda_{lN}^k &\le \max_{\psi \in E_{1,k}} \frac{\mathcal{A}_l(\psi,\psi)}{\mathcal{B}_l(\Pi_N^{1,l}\psi, \Pi_N^{1,l}\psi)} \\
&= \max_{\psi \in E_{1,k}} \frac{\mathcal{A}_l(\psi,\psi)}{\mathcal{B}_l(\psi,\psi)} \frac{\mathcal{B}_l(\psi,\psi)}{\mathcal{B}_l(\Pi_N^{1,l}\psi, \Pi_N^{1,l}\psi)} \\
&\le \lambda_l^k \max_{\psi \in E_{1,k}} \frac{\mathcal{B}_l(\psi,\psi)}{\mathcal{B}_l(\Pi_N^{1,l}\psi, \Pi_N^{1,l}\psi)}.
\end{aligned}$$

This finishes our proof. □

We introduce the following non-uniformly weighted Sobole spaces:

$$H^s_{\omega^{\alpha,\beta},*}(I) := \{p : \partial_t^k p \in L^2_{\omega^{\alpha+k,\beta+k}}(I), 0 \le k \le s\}$$

equipped with the inner product and norm

$$(p,q)_{s,\omega^{\alpha,\beta},*} = \sum_{k=0}^{s} (\partial_t^k p, \partial_t^k q)_{\omega^{\alpha+k,\beta+k}}, \|p\|_{s,\omega^{\alpha,\beta},*} = (p,p)^{\frac{1}{2}}_{s,\omega^{\alpha,\beta},*},$$

and

$$\mathcal{H}^s_{\omega,l}(I) := \{\psi \in H^1_{\omega,l}(I) \cap H^1(I) : \partial_t^k \psi \in L^2_{\omega^{-1+k,-1+k}}, 2 \le k \le s\},$$

equipped with the inner product and norm

$$[\psi,\phi]_{s,\omega,l}=(\psi,\phi)_{1,\omega,l}+\sum_{k=2}^{s}(\partial_t^k\psi,\partial_t^k\phi)_{\omega^{-1+k,-1+k}},$$
$$|||\psi|||_{s,\omega,l}=[\psi,\psi]_{s,\omega,l}^{\frac{1}{2}},$$

where $\omega^{\alpha,\beta}(t)=(1-t)^{\alpha}(1+t)^{\beta}$ denotes the Jacobi weight function with index (α,β). Define orthogonal projection $\pi_{N,\omega^{-1,-1}}:L^2_{\omega^{-1,-1}}(I)\to Q_N^{-1,-1}$ by

$$(p-\pi_{N,\omega^{-1,-1}}p,q_N)_{\omega^{-1,-1}}=0,\forall q_N\in Q_N^{-1,-1},$$

where $Q_N^{-1,-1}=\{q\in P_N:q(\pm1)=0\}$. From the Theorem 1.8.2 in [23] we have the following Lemma:

Lemma 3.4. *For $\forall p\in H^s_{\omega^{-1,-1},*}(I)$, the following inequality holds:*

$$\|\partial_t^k(\pi_{N,\omega^{-1,-1}}p-p)\|_{\omega^{-1+k,-1+k}}\lesssim N^{k-s}\|\partial_t^s p\|_{\omega^{-1+s,-1+s}},0\le k\le s.$$

Theorem 3.3. *There exists an operator $\pi_N^{1,l}:H^1_{\omega,l}(I)\to X_N(l)$ such that $\pi_N^{1,l}u(\pm1)=u(\pm1)$ and for $u\in\mathcal{H}^s_{\omega,l}(I)$ with $s\ge1$, there holds*

$$\|\partial_t^k(\pi_N^{1,l}u-u)\|_{\omega^{-1+k,-1+k}}\lesssim N^{k-s}\|\partial_t^s u\|_{\omega^{-1+s,-1+s}},0\le k\le s.$$

Proof. Let $u_*(t)=\frac{1-t}{2}u(-1)+\frac{1+t}{2}u(1)$ for $\forall u\in H^1_{\omega,l}(I)$, then we have $(u-u_*)(\pm1)=0$. For $\forall u\in\mathcal{H}^s_{\omega,m}(I)$, we have $u-u_*\in H^s_{\omega^{-1,-1},*}(I)$. In fact, from Hardy inequality (cf. B8.8 in [24]) we can derive that

$$\int_I\omega^{-1,-1}(u-u_*)^2dt\lesssim\int_I\omega^{-2,-2}(u-u_*)^2dt\lesssim\int_I(\partial_t(u-u_*))^2dt. \tag{3.8}$$

Since

$$\int_I(\partial_t u_*)^2dt=\int_I\frac{1}{4}(u(1)-u(-1))^2dt=\frac{1}{2}(u(1)-u(-1))^2$$
$$=\frac{1}{2}(\int_I\partial_t udt)^2\le\int_I(\partial_t u)^2dt,$$

then we have

$$\int_I(\partial_t(u-u_*))^2dt\lesssim\int_I(\partial_t u)^2dt+\int_I(\partial_t u_*)^2dt\lesssim\int_I(\partial_t u)^2dt. \tag{3.9}$$

For $k\ge2$, we have

$$\int_I\omega^{-1+k,-1+k}(\partial_t^k(u-u_*))^2dt=\int_I\omega^{-1+k,-1+k}(\partial_t^k u)^2dt. \tag{3.10}$$

Thus, $u-u_*\in H^s_{\omega^{-1,-1},*}(I)$. Define

$$\pi_N^{1,l}u=\pi_{N,\omega^{-1,-1}}(u-u_*)+u_*\in X_N(l),\forall u\in\mathcal{H}^s_{\omega,l}(I).$$

Then from Lemma 3.4 we derive that

$$\|\partial_t^k(\pi_N^{1,l}u-u)\|_{\omega^{-1+k,-1+k}}=\|\partial_t^k(\pi_{N,\omega^{-1,-1}}(u-u_*)-(u-u_*))\|_{\omega^{-1+k,-1+k}}$$
$$\lesssim N^{k-s}\|\partial_t^s(u-u_*)\|_{\omega^{-1+s,-1+s}}.$$

Together with (3.8), (3.9) and (3.10) we can obtain desired result. □

Theorem 3.4. *Let λ_{lN}^k be the k-th approximate eigenvalue of λ_l^k. Then for all $\{\psi_l^i\}_{i=1}^k \subset \mathcal{H}_{\omega,l}^s(I)$ with $\mathcal{B}_l(\psi_l^i,\psi_l^i)=1$ and $s \geq k$, we have*

$$|\lambda_{lN}^k - \lambda_l^k| \lesssim c(k)N^{2(1-s)} \max_{i,j=1,\cdots,k} \frac{1}{\lambda_l^j}\|\partial_t^s\psi_l^i\|_{\omega^{-1+s,-1+s}}\|\partial_t^s\psi_l^j\|_{\omega^{-1+s,-1+s}},$$

where $c(k)$ is a constant independent of N.

Proof. For $\forall \psi \in E_{1,k}$, we have $\psi = \sum_{i=1}^k \mu_i\psi_l^i$. Then we can derive that

$$\begin{aligned}
\frac{\mathcal{B}_l(\psi,\psi)-\mathcal{B}_l(\Pi_N^{1,l}\psi,\Pi_N^{1,l}\psi)}{\mathcal{B}_l(\psi,\psi)} &\leq \frac{2|\mathcal{B}_l(\psi,\psi-\Pi_N^{1,l}\psi)|}{\mathcal{B}_l(\psi,\psi)}\\
&\leq \frac{2\sum_{i,j=1}^k |\mu_i||\mu_j||\mathcal{B}_l(\psi_l^i-\Pi_N^{1,l}\psi_l^i,\psi_l^j)|}{\sum_{i=1}^k|\mu_i|^2}\\
&\leq 2k\max_{i,j=1,\cdots,k}|\mathcal{B}_l(\psi_l^i-\Pi_N^{1,l}\psi_l^i,\psi_l^j)|.
\end{aligned}$$

From Cauchy-Schwarz inequality and the property of orthogonal projection we can derive that

$$\begin{aligned}
&|\mathcal{B}_l(\psi_l^i-\Pi_N^{1,l}\psi_l^i,\psi_l^j)|\\
=&\frac{1}{\lambda_l^j}|\lambda_l^j\mathcal{B}_l(\psi_l^j,\psi_l^i-\Pi_N^{1,l}\psi_l^i)|\\
=&\frac{1}{\lambda_l^j}|\mathcal{A}_l(\psi_l^j,\psi_l^i-\Pi_N^{1,l}\psi_l^i)| = \frac{1}{\lambda_l^j}|\mathcal{A}_l(\psi_l^j-\Pi_N^{1,l}\psi_l^j,\psi_l^i-\Pi_N^{1,l}\psi_l^i)|\\
\leq&\frac{1}{\lambda_l^j}(\mathcal{A}_l(\psi_l^j-\Pi_N^{1,l}\psi_l^j,\psi_l^j-\Pi_N^{1,l}\psi_l^j))^{\frac{1}{2}}(\mathcal{A}_l(\psi_l^i-\Pi_N^{1,l}\psi_l^i,\psi_l^i-\Pi_N^{1,l}\psi_l^i))^{\frac{1}{2}}\\
\lesssim&\frac{1}{\lambda_l^j}(\mathcal{A}_l(\psi_l^j-\pi_N^{1,l}\psi_l^j,\psi_l^j-\pi_N^{1,l}\psi_l^j))^{\frac{1}{2}}(\mathcal{A}_l(\psi_l^i-\pi_N^{1,l}\psi_l^i,\psi_l^i-\pi_N^{1,l}\psi_l^i))^{\frac{1}{2}}.
\end{aligned}$$

Since

$$\begin{aligned}
\mathcal{A}_l(\psi,\psi) &= \int_I((t+1)^2(\psi')^2+l(l+1)\psi^2+\frac{R^2}{4}(t+1)^2\psi^2)dt\\
&\lesssim \int_I((t+1)^2(\psi')^2+\psi^2)dt \lesssim \int_I((\psi')^2+\psi^2)dt\\
&\lesssim \int_I((\psi')^2+\omega^{-1,-1}\psi^2)dt
\end{aligned}$$

then from Theorem 3.3 we obtain

$$|\mathcal{B}_l(\psi_l^i-\Pi_N^{1,l}\psi_l^i,\psi_l^j)| \lesssim N^{2(1-s)}\frac{1}{\lambda_l^j}\|\partial_t^s\psi_l^i\|_{\omega^{-1+s,-1+s}}\|\partial_t^s\psi_l^j\|_{\omega^{-1+s,-1+s}}.$$

Since

$$\frac{\mathcal{B}_l(\psi,\psi)}{\mathcal{B}_l(\Pi_N^{1,l}\psi,\Pi_N^{1,l}\psi)} \leq \frac{1}{1-2k\max_{i,j=1,\cdots,k}|\mathcal{B}_l(\psi_l^i-\Pi_N^{1,l}\psi_l^i,\psi_l^j)|},$$

then from Theorem 3.2 we can get desired results. □

Remark 3.1. For the error estimation of eigenfunctions, it can be treated similarly. For the sake of brevity, we omit the detail.

4. Efficient implementation of the algorithm

We propose in this section an efficient numerical algorithm to solve the problems (3.2). Let

$$\varphi_i(t) = L_i(t) - L_{i+2}(t), i = 0 \cdots N-2, \varphi_{N-1} = \frac{t+1}{2}, \varphi_N = 1, \tag{4.1}$$

where L_k is the Legendre polynomial of degree k. It is obvious that

$$X_N(l) = \text{span}\{\varphi_0(t), \cdots, \varphi_N(t)\}.$$

Set

$$\begin{aligned} a_{ij} &= \int_I (t+1)^2 \varphi_j' \varphi_i' dt, b_{ij} = \int_I \varphi_j \varphi_i dt, \\ c_{ij} &= \int_I (t+1)^2 \varphi_j \varphi_i dt, d_{ij} = \varphi_j(1)\varphi_i(1). \end{aligned}$$

Then we shall look for

$$\psi_{lN} = \sum_{i=0}^{N} \psi_i^l \varphi_i(t). \tag{4.2}$$

Now, plugging the expression (4.2) in (3.2), and taking ϕ_N through all the basis functions in $X_N(l)$, we will arrive at the following linear eigen-system:

$$(A + l(l+1)B + \frac{R^2}{4}C)\Psi^l = \lambda_{lN} 2RD\Psi^l, \tag{4.3}$$

where

$$\begin{aligned} &A = (a_{ij}), B = (b_{ij}), C = (c_{ij}), D = (d_{ij}), \\ &\Psi^l = (\psi_0^l, \cdots, \psi_N^l)^T. \end{aligned}$$

In order to illustrate the effectiveness of our algorithm, we will prove that the stiff matrix and mass matrix are all sparse matrices.

Theorem 4.1. *For the basis functions* (4.1)*, the matrices A, B, C, D are symmetric banded matrices such that*

$$\begin{aligned} &a_{ij} = 0 \quad \textit{for} \quad 0 \le i, j \le N-2 \quad \textit{and} \quad |i-j| > 2, \\ &a_{N-1,k} = a_{k,N-1} = 0 \quad \textit{for} \quad 2 \le k \le N-2, \\ &a_{N,k} = a_{k,N} = 0 \quad \textit{for} \quad 0 \le k \le N; \\ &b_{ij} = 0 \quad \textit{for} \quad 0 \le i, j \le N-2 \quad \textit{and} \quad |i-j| > 2, \\ &b_{N-1,k} = b_{k,N-1} = 0 \quad \textit{for} \quad 2 \le k \le N-2, \\ &b_{N,k} = b_{k,N} = 0 \quad \textit{for} \quad 1 \le k \le N-2; \end{aligned}$$

$$
\begin{aligned}
&c_{ij} = 0 \quad for \quad 0 \le i,j \le N-2 \quad and \quad |i-j| > 4, \\
&c_{N-1,k} = c_{k,N-1} = 0 \quad for \quad 4 \le k \le N-2, \\
&c_{N,k} = c_{k,N} = 0 \quad for \quad 3 \le k \le N-2; \\
&d_{ij} = 0 \quad for \quad 0 \le i,j \le N-2, \\
&d_{N-1,k} = d_{k,N-1} = d_{N,k} = d_{k,N} = 0 \quad for \quad 0 \le k \le N-2.
\end{aligned}
$$

Proof. For $0 \le i, j \le N-2$, we derive that

$$
\begin{aligned}
a_{ij} &= \int_I (t+1)^2 \varphi_i' \varphi_j' dt = -\int_I \varphi_i ((t+1)^2 \varphi_j')' dt \\
&= \int_I (L_i - L_{i+2})((t+1)^2 \varphi_j')' dt.
\end{aligned}
$$

Thus, $a_{ij} = 0$ for $|i-j| > 2$ since $((t+1)^2\varphi_j')'$ is a j-degree polynomial. For $2 \le k \le N-2$, from the orthogonal property of Legendre polynomial we have

$$
\begin{aligned}
a_{N-1,k} = a_{k,N-1} &= \int_I (t+1)^2 \varphi_k' \varphi_{N-1}' dt = -\int_I \varphi_k ((t+1)^2 \varphi_{N-1}')' dt \\
&= -\int_I (t+1)(L_k - L_{k+2}) dt = 0.
\end{aligned}
$$

For $0 \le k \le N$, from $\varphi_N' = 0$ we have

$$
a_{N,k} = a_{k,N} = \int_I (t+1)^2 \varphi_k' \varphi_N' dt = 0.
$$

For $0 \le i, j \le N-2$, we have

$$
b_{ij} = \int_I \varphi_i \varphi_j dt = \int_I (L_i - L_{i+2})(L_j - L_{j+2}) dt,
$$

i.e., $b_{ij} = 0$ for $|i-j| > 2$. For $2 \le k \le N-2$, we have

$$
\begin{aligned}
b_{N-1,k} = b_{k,N-1} &= \int_I \varphi_k \varphi_{N-1} dt \\
&= \int_I \frac{t+1}{2} (L_k - L_{k+2}) dt = 0.
\end{aligned}
$$

For $1 \le k \le N-2$, we have

$$
\begin{aligned}
b_{N,k} = b_{k,N} &= \int_I \varphi_k \varphi_N dt \\
&= \int_I (L_k - L_{k+2}) dt = 0.
\end{aligned}
$$

Similarly, we can prove that

$$
\begin{aligned}
&c_{ij} = 0 \quad \text{for} \quad 0 \le i,j \le N-2 \quad \text{and} \quad |i-j| > 4, \\
&c_{N-1,k} = c_{k,N-1} = 0 \quad \text{for} \quad 4 \le k \le N-2, \\
&c_{N,k} = c_{k,N} = 0 \quad \text{for} \quad 3 \le k \le N-2.
\end{aligned}
$$

In addition, from $\varphi_i(1) = 0 (i = 0, 1, \cdots, N-2)$, we obtain

$$
\begin{aligned}
&d_{ij} = \varphi_j(1)\varphi_i(1) = 0 \quad \text{for} \quad 0 \le i,j \le N-2, \\
&d_{N-1,k} = d_{k,N-1} = d_{N,k} = d_{k,N} = 0 \quad \text{for} \quad 0 \le k \le N-2.
\end{aligned}
$$

□

5. Numerical experiments

We now perform a sequence of numerical tests to study the convergence behavior and show the effectiveness of our algorithm. We perform our programs in MATLAB 2015b. We take $R = 1$ and $l = 0, 1, 2, 3$ as our examples. The numerical results of first meaningful physical eigenvalue for different l and N are listed in Table 1.

Table 1. The first meaningful physical eigenvalue for different l and N in a unit ball.

N	λ_{0N}	λ_{1N}	λ_{2N}	λ_{3N}
6	0.313035285499535	1.194528049467097	2.140646825765955	3.110007601420826
8	0.313035285499331	1.194528049465325	2.140646825733227	3.110007600859386
10	0.313035285499332	1.194528049465325	2.140646825733230	3.110007600859385
12	0.313035285499331	1.194528049465325	2.140646825733231	3.110007600859386
14	0.313035285499332	1.194528049465327	2.140646825733231	3.110007600859385
16	0.313035285499332	1.194528049465326	2.140646825733227	3.110007600859385
18	0.313035285499332	1.194528049465325	2.140646825733231	3.110007600859389
20	0.313035285499331	1.194528049465325	2.140646825733229	3.110007600859382

We know from Table 1 that the eigenvalues achieve at least fourteen-digit accuracy with $N \geq 8$. In order to further show the efficiency of our algorithm, we choose the solutions of $N = 60$ as reference solutions, the error figures of the approximate eigenvalue $\lambda_{lN}(l = 0, 1, 2, 3)$ with different N are listed in Figure 1.

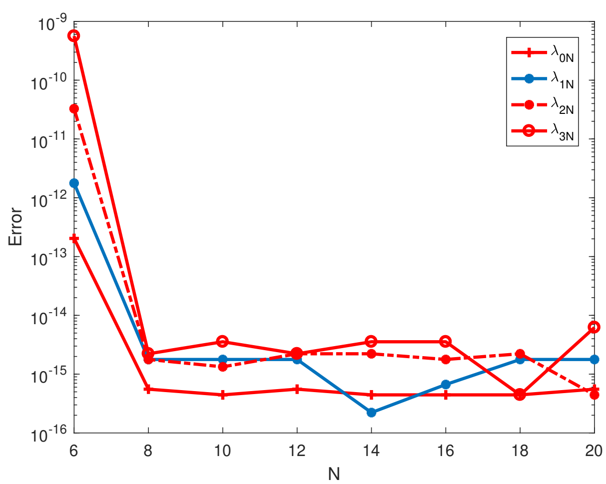

Figure 1. Errors between numerical solutions and the reference solution for $l = 0, 1, 2, 3$.

Before concluding this section, we would like to present some figures of the real part of eigenfunctions $u_l^m(r)Y_l^m(\theta, \phi)$ for different l and m with $N = 60$. Since the eigenfunctions are three dimensional functions, we only present the figures on the cutting plane along the direction of $z = 0$ in Figure 2-9.

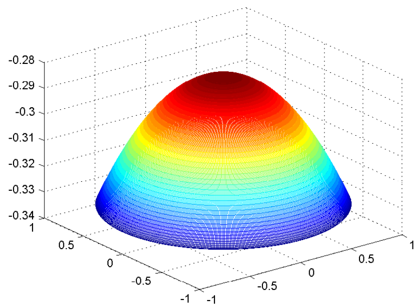

Figure 2. Mesh image of $real(u_0^0(r)Y_0^0(\pi/2, \phi))$.

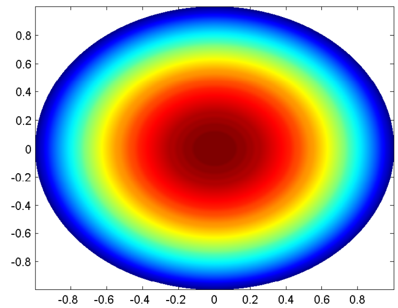

Figure 3. Contour image of $real(u_0^0(r)Y_0^0(\pi/2, \phi))$.

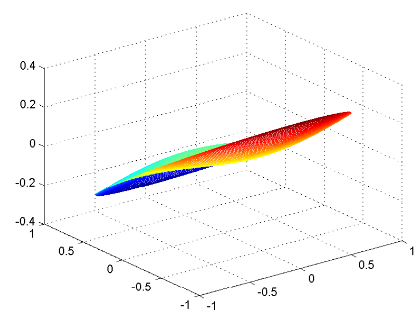

Figure 4. Mesh image of $real(u_1^0(r)Y_1^0(\pi/2, \phi))$.

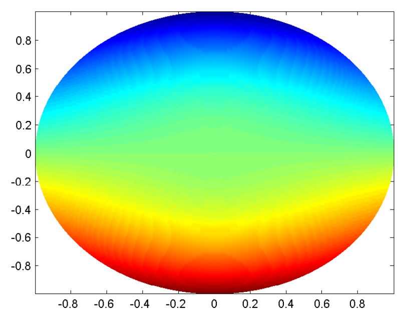

Figure 5. Contour image of $real(u_1^0(r)Y_1^0(\pi/2, \phi))$.

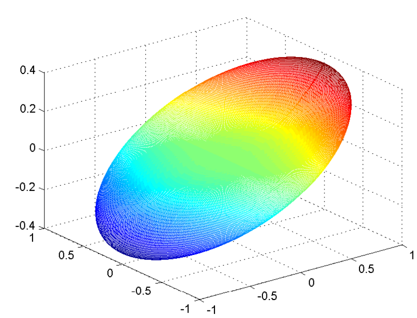

Figure 6. Mesh image of $real(u_1^1(r)Y_1^1(\pi/2, \phi))$.

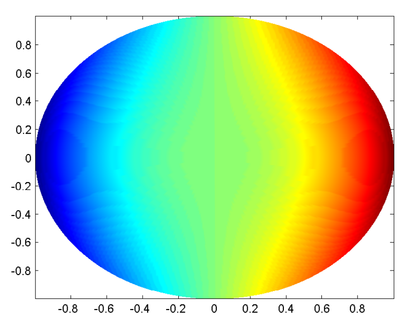

Figure 7. Contour image of $real(u_1^1(r)Y_1^1(\pi/2, \phi))$.

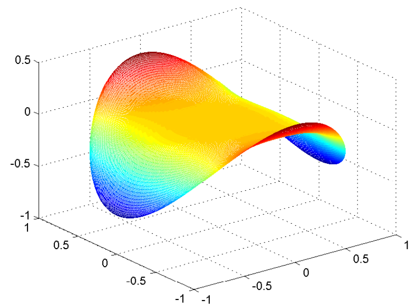

Figure 8. Mesh image of $real(u_2^0(r)Y_2^0(\pi/2,\phi))$.

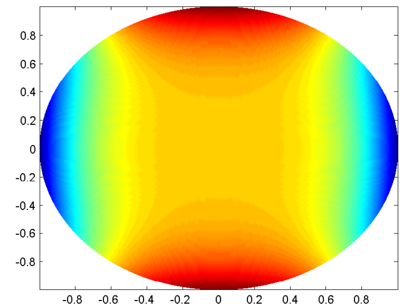

Figure 9. Contour image of $real(u_2^0(r)Y_2^0(\pi/2,\phi))$.

6. Conclusions

We present in this paper a high precision numerical method based on Legendre-Galerkin approximation for the Steklov eigenvalue problem in spherical domain. Firstly, each one-dimensional eigenvalue problem is independent of each other and can be solved in parallel, which greatly reduces the computational time and memory capacity. Secondly, we observe from the numerical results that each one-dimensional eigenvalue problem has only one meaningful physical eigenvalue. Finally, by combining with a spectral element method or a boundary perturbation algorithm, the method developed in this paper can be extended to more complex eigenvalue problems which will be the subject of our future work.

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