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EXISTENCE AND ASYMPTOTIC BEHAVIOR OF TRAVELING WAVES IN A HOST-VECTOR EPIDEMIC MODEL*

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Abstract In this paper, we are concerned with a diffusive host-vector epidemic model with a nonlocal spatiotemporal interaction. When the delay kernel takes some special form, by employing linear chain techniques and geometric singular perturbation theory, we establish the existence of travelling front solutions connecting the two spatially uniform steady states for sufficiently small delays. Furthermore, by employing standard asymptotic theory, we also obtain the asymptotic behavior of traveling wave fronts of this model.

Keywords Host-vector epidemic model, traveling waves, linear chain technique, geometric singular perturbation theory.

MSC(2010) 34E15, 34D15, 35Q53, 35C07.

1. Introduction

Recently, Wu and Weng [16] introduced the following diffusive host-vector epidemic model

$$\frac{\partial u(t,x)}{\partial t} = \frac{\partial^2 u}{\partial x^2} - au + \frac{(N-u)\int_{-\infty}^{t}\int_{\Omega} F(t,s,x,y)u(s,y)dyds}{1+\beta\int_{-\infty}^{t}\int_{\Omega} F(t,s,x,y)u(s,y)dyds}, t \geq 0, x \in \Omega \quad (1.1)$$

where $u(t,x)$ represents the population density of infective host at time t and location x, a is the recovery rate, β is a positive constant related to the saturation incidence rate, and N is a constant representing the total host population (i.e. the birth rate equals to the death rate of the host population). When the domain Ω is a bounded set of $\mathbb{R}$, the authors obtained the global asymptotic stability of steady states of eq.(1.1), by using the ideas introduced by Pozio [13]. When $\Omega = \mathbb{R}$, the authors also established the existence of traveling wavefronts for (1.1) by employing

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a new approach based on a combination of perturbation methods, the Fredholm theory, and the Banach fixed point theorem [12].

In (1.1), the convolution kernel $F(t,s,x,y)$ is a positive continuous function in its variables $t \in \mathbb{R}, s \in \mathbb{R}^+, x, y \in \Omega$, which satisfies the usual normalization assumption as follows

$$\int_0^{+\infty} \int_\Omega F(t,s,x,y)dyds = 1. \tag{1.2}$$

so that the kernel does not affect the two spatially uniform steady states $u = 0$ and $u = \frac{N-a}{1+a\beta}$.

If the kernel F is taken to be

$$F(t,s,x,y) = \delta(t-s)\delta(x-y),$$

where $\delta(x)$ is Dirac delta function, then (1.1) becomes the reaction-diffusion equation without delay

$$\frac{\partial u(t,x)}{\partial t} = \frac{\partial^2 u}{\partial x^2} - au + \frac{(N-u)u}{1+\beta u}. \tag{1.3}$$

If the kernel F is taken to be

$$F(t,s,x,y) = \delta(t-s-\tau)\delta(x-y),$$

then (1.1) becomes the reaction-diffusion equation with discrete delay

$$\frac{\partial u(t,x)}{\partial t} = \frac{\partial^2 u(t,x)}{\partial x^2} - au(t,x) + \frac{(N-u(t,x))u(t-\tau,x)}{1+\beta u(t-\tau,x)}. \tag{1.4}$$

If the kernel F is taken to be

$$F(t,s,x,y) = \frac{1}{\sqrt{4\pi(t-s)}} e^{-\frac{(x-y)^2}{4(t-s)}} G(t-s),$$

where

$$G(t) = \frac{1}{\tau} e^{-\frac{t}{\tau}} \quad or \quad G(t) = \frac{t}{\tau^2} e^{-\frac{t}{\tau}}, \tag{1.5}$$

then (1.1) becomes a reaction diffusion equation with both distributed delay and spatial averaging. The parameter τ is representative of the delay and the two kernel functions G in (1.5) are used frequently in the literature on delay differential equations [5, 6]. The first kernel function $G(t) = \frac{1}{\tau} e^{-\frac{t}{\tau}}$ is sometimes called the weak generic kernel because it implies that the importance of events in the past decreases exponentially. The second kernel function $G(t) = \frac{t}{\tau^2} e^{-\frac{t}{\tau}}$ is called the strong generic kernel because it implies that the population density τ time units ago is more important than any other since this kernel achieves its unique maximum when $t = \tau$.

In the present paper, we shall study the existence of travelling front solutions of (1.1) with the weak generic kernel function G in (1.5). The method we employ is geometric singular perturbation theory and Fredholm alternative [4, 9]. It should be remarked that geometric singular perturbation method has been successfully used to traveling wave problem for various reaction-diffusion equations with spatially and temporally nonlocal terms in the form of the convolution of a kernel (see, for example [1, 3, 7, 8, 10, 17–20]). Moreover, we shall also study the asymptotic behavior of the traveling wave fronts of (1.1) by using the standard asymptotic theory.

This article is organized as follows. In Sect.2, the existence of traveling wave fronts for Eq.(1.1) with the weak generic kernel is established. In Sect.3, the corresponding asymptotic behavior of the traveling wave front is obtained. Finally, some concluding remarks are presented in Section 4.

2. Existence of traveling wavefront solutions in the host-vector epidemic model

In this section, we shall show that Eq.(1.1) has a travelling wavefront solution connecting two non-negative uniform states $u = 0$ and $u = u^* = \frac{N-a}{1+a\beta}$ for the weak generic delay kernel

$$G(t) = \frac{1}{\tau} e^{-\frac{t}{\tau}}. \tag{2.1}$$

Firstly, if we define

$$v(x,t) = \int_{-\infty}^{t} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{4\pi(t-s)}} e^{-\frac{(x-y)^2}{4(t-s)}} \frac{1}{\tau} e^{-\frac{t-s}{\tau}} u(s,y) dy ds, \tag{2.2}$$

it is straightforward to see that v satisfies

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + \frac{1}{\tau}(u - v), \tag{2.3}$$

and then Eq.(1.1) can be reformulated as the following system

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} - au + \dfrac{(N-u)v}{1+\beta v}, \\ \dfrac{\partial v}{\partial t} = \dfrac{\partial^2 v}{\partial x^2} + \dfrac{1}{\tau}(u - v). \end{cases} \tag{2.4}$$

Note that our purpose is to establish the existence of travelling wavefront solutions of system (2.4) connecting the two uniform steady-states $(u, v) = (0, 0)$ and (u^*, u^*), for sufficiently small delay. To this end, we first need to establish the existence of traveling wavefront solutions when the delay is zero. However, it can be easily seen that $v \to u$ when $\tau \to 0$ by examining (2.2). Thus, in this limit, system (2.4) is reduced to the model (1.3) without delay. Making travelling wave transform in the model (1.3) by setting $u(x,t) = U(z)$ with $z = x - ct, c > 0$, yields the following second-order ODE for $U(z)$

$$\frac{d^2U(z)}{dz^2} + c\frac{dU(z)}{dz} - aU(z) + \frac{(N - U(z))U(z)}{1 + \beta U(z)} = 0. \tag{2.5}$$

By using phase-plane techniques, Wu and Weng [16] have established the following result for Eq.(2.5):

Lemma 2.1. *If $N > a$ and $c \geq 2\sqrt{N-a}$, then there exists a function $U(z)$ that satisfies (2.5), together with $U(-\infty) = u^*$ and $U(+\infty) = 0$, and which is strictly monotonically decreasing for all $z \in \mathbb{R}$.*

When the delay τ is non-zero, we will show that Eq. (1.1) has traveling wave fronts for sufficiently small $\tau > 0$ by applying the geometric singular perturbation theory. By setting $u(x,t) = U(z), v(x,t) = V(z), z = x - ct, c > 0$, and substituting it to Eq.(2.4), then we can obtain the corresponding traveling wave system

$$\begin{cases} U'' + cU' - aU + \dfrac{(N-U)V}{1+\beta V} = 0, \\ V'' + cV' + \dfrac{1}{\tau}(U - V) = 0, \end{cases} \tag{2.6}$$

where the prime denotes derivative with respect to z. In order to seek solutions satisfying $(U,V)(-\infty) = (u^*, u^*), (U,V)(+\infty) = (0,0)$, it is convenient to introduce the new variables

$$\hat{U} = U' + \frac{1}{2}cU, \quad \hat{V} = V' + \frac{1}{2}cV,$$

in terms of which (2.6) can be reformulated as

$$\begin{cases} U' = \hat{U} - \dfrac{1}{2}cU, \\ \hat{U}' = \dfrac{1}{4}c^2U - \dfrac{1}{2}c\hat{U} + aU - \dfrac{(N-U)V}{1+\beta V}, \\ V' = \hat{V} - \dfrac{1}{2}cV, \\ \hat{V}' = \dfrac{1}{4}c^2V - \dfrac{1}{2}c\hat{V} - \dfrac{1}{\tau}(U - V). \end{cases} \tag{2.7}$$

This system has two equilibria, both of which are independent of τ, namely

$$(U, \hat{U}, V, \hat{V}) = (0, 0, 0, 0)$$

and

$$(U, \hat{U}, V, \hat{V}) = (u^*, \frac{1}{2}cu^*, u^*, \frac{1}{2}cu^*).$$

Our aim now is to establish a heteroclinic connection between these two equilibrium points of system (2.7) and it corresponds to the travelling front solutions of system (2.4).

We introduce the small parameter $\epsilon = \sqrt{\tau}$ and define

$$u = U, \hat{u} = \hat{U}, v = V, \hat{v} = \epsilon\hat{V}.$$

With this notation, system (2.7) becomes

$$\begin{cases} u_z = \hat{u} - \dfrac{1}{2}cu, \\ \hat{u}_z = \dfrac{1}{4}c^2u - \dfrac{1}{2}c\hat{u} + au - \dfrac{(N-u)v}{1+\beta v}, \\ \epsilon v_z = \hat{v} - \dfrac{1}{2}\epsilon cv, \\ \epsilon\hat{v}_z = \dfrac{1}{4}\epsilon^2c^2v - \dfrac{1}{2}\epsilon c\hat{v} + v - u. \end{cases} \tag{2.8}$$

Obviously, when $\epsilon = 0$ system (2.8) reduces to the second-order planar dynamical system which is equivalent to Eq.(2.5). Note that when τ is very small, system (2.8) is a singularly perturbed system. Let $z = \epsilon\eta$, then system (2.8) becomes

$$\begin{cases} u_\eta = \epsilon(\hat{u} - \frac{1}{2}cu), \\ \hat{u}_\eta = \epsilon\left(\frac{1}{4}c^2u - \frac{1}{2}c\hat{u} + au - \frac{(N-u)v}{1+\beta v}\right), \\ v_\eta = \hat{v} - \frac{1}{2}\epsilon cv, \\ \hat{v}_\eta = \frac{1}{4}\epsilon^2c^2v - \frac{1}{2}\epsilon c\hat{v} + v - u. \end{cases} \tag{2.9}$$

These two systems are equivalent for $\tau > 0$, the different time-scales give rise to two different limiting systems. Letting $\tau \to 0$ in (2.8), we obtain

$$\begin{cases} u_z = \hat{u} - \frac{1}{2}cu, \\ \hat{u}_z = \frac{1}{4}c^2u - \frac{1}{2}c\hat{u} + au - \frac{(N-u)v}{1+\beta v}, \\ 0 = \hat{v}, \\ 0 = v - u. \end{cases} \tag{2.10}$$

Thus the flow of system (2.10) is confined to the set

$$\mathcal{M}_0 = \left\{(u, \hat{u}, v, \hat{v}) \in \mathbb{R}^4 \,|\, \hat{v} = 0, \quad v = u\right\}, \tag{2.11}$$

which is a two-dimensional invariant manifold for (2.8) with $\epsilon = 0$. And it is very interesting that the dynamics of system (2.10) are determined by the first two equations only. On the other hand, setting $\tau \to 0$ in (2.9) yields the system

$$\begin{cases} u_\eta = 0, \\ \hat{u}_\eta = 0, \\ v_\eta = \hat{v}, \\ \hat{v}_\eta = v - u. \end{cases} \tag{2.12}$$

Note that any points in $\mathcal{M}_0$ are the equilibria of system (2.12). Generally, system (2.8) is referred to as the slow system, since the time-scale z is slow, and system (2.9) is referred to as the fast system, since the time-scale η is fast. While systems (2.10) and (2.12) are referred to as the limiting slow (or reduced problem) and limiting fast subsystem (or layer problem), respectively. $\mathcal{M}_0$ is called the critical manifold.

Recall that $\mathcal{M}_0$ is a normally hyperbolic manifold if the linearization of the fast system (2.9), restricted to $\mathcal{M}_0$, has exactly $dim\mathcal{M}_0$ eigenvalues with zero real part. In fact, the linearization of (2.9) restricted to $\mathcal{M}_0$ is given by the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{pmatrix} \tag{2.13}$$

the eigenvalues of which are $0, 0, 1, -1$. Thus we verified that $\mathcal{M}_0$ is normally hyperbolic.

According to the geometric singular perturbation theory of Fenichel [4], for $\epsilon > 0$ sufficiently small, there exists a two-dimensional manifold $\mathcal{M}_\epsilon$, which is close to $\mathcal{M}_0$ and which is locally invariant for the flow. $\mathcal{M}_\epsilon$ is called the slow manifold. Moreover, there exists local stable and unstable manifolds of $\mathcal{M}_\epsilon$, $W^s_{loc}(\mathcal{M}_\epsilon)$ and $W^u_{loc}(\mathcal{M}_\epsilon)$, that lie within $\mathcal{O}(\epsilon)$ of and are diffeomorphic to $W^s(\mathcal{M}_0)$ and $W^u(\mathcal{M}_0)$. As a consequence, the dynamics in the vicinity of the slow manifold are completely determined by the one on the slow manifold. Therefore, we only need to study the flow of the slow system (2.8) restricted to $\mathcal{M}_\epsilon$ and show that the two-dimensional reduced system has a heteroclinic orbit.

By Fenichel's theory, we know that a two-dimensional invariant manifold $\mathcal{M}_\epsilon$ can be written in the form

$$\mathcal{M}_\epsilon := \{(u, \hat{u}, v, \hat{v}) \in \mathbb{R}^4 | \hat{v} = g(u, \hat{u}; \epsilon), v = u + h(u, \hat{u}; \epsilon)\},$$

where the functions $g(u, \hat{u}; \epsilon)$ and $h(u, \hat{u}; \epsilon)$ satisfy

$$g(u, \hat{u}; 0) = h(u, \hat{u}; 0) = 0.$$

Note the fact that $\mathcal{M}_\epsilon$ is an invariant manifold for the flow of slow system (2.8). Substituting it to (2.8) yields

$$\begin{cases} \epsilon\left[(1+\dfrac{\partial h}{\partial u})(\hat{u}-\dfrac{1}{2}cu)+\dfrac{\partial h}{\partial \hat{u}}(\dfrac{1}{4}c^2u-\dfrac{1}{2}c\hat{u}+au-\dfrac{(N-u)(u+h)}{1+\beta(u+h)})\right] \\ =g-\dfrac{1}{2}\epsilon c(u+h), \\ \epsilon\left[\dfrac{\partial g}{\partial u}(\hat{u}-\dfrac{1}{2}cu)+\dfrac{\partial g}{\partial \hat{u}}(\dfrac{1}{4}c^2u-\dfrac{1}{2}c\hat{u}+au-\dfrac{(N-u)(u+h)}{1+\beta(u+h)})\right] \\ =\dfrac{1}{4}\epsilon^2c^2(u+h)-\dfrac{1}{2}\epsilon cg+h. \end{cases} \tag{2.14}$$

By employing the smallness of ϵ, we can express the functions g and h as the following form of regular perturbation series with respect to ϵ

$$\begin{aligned} g(u, \hat{u}; \epsilon) &= \epsilon g_1(u, \hat{u}) + \epsilon^2 g_2(u, \hat{u}) + \cdots , \\ h(u, \hat{u}; \epsilon) &= \epsilon h_1(u, \hat{u}) + \epsilon^2 h_2(u, \hat{u}) + \cdots . \end{aligned}$$

Substituting and comparing coefficients of ϵ and ϵ^2 on both sides of (2.14) leads to

$$\begin{cases} g_1(u, \hat{u}) = \hat{u}, \quad g_2(u, \hat{u}) = 0, \\ h_1(u, \hat{u}) = 0, \quad h_2(u, \hat{u}) = au - \dfrac{(N-u)u}{1+\beta u}. \end{cases} \tag{2.15}$$

Thus, we have

$$\begin{cases} g(u, \hat{u}; \epsilon) = \epsilon\hat{u} + \mathcal{O}(\epsilon^3), \\ h(u, \hat{u}; \epsilon) = \epsilon^2\left[au - \dfrac{(N-u)u}{1+\beta u}\right] + \mathcal{O}(\epsilon^3). \end{cases} \tag{2.16}$$

Then the slow system restricted to $\mathcal{M}_\epsilon$ is given by

$$\begin{cases} u_z = \hat{u} - \dfrac{1}{2}cu, \\ \hat{u}_z = \dfrac{1}{4}c^2u - \dfrac{1}{2}c\hat{u} + au - \dfrac{(N-u)(u+h)}{1+\beta(u+h)}, \end{cases} \tag{2.17}$$

where h is given by (2.16). It is easy to verify that when $\epsilon = 0$, system (2.17) reduces to the corresponding ODE (2.5) for travelling fronts of the non-delay problem. Moreover, for any $\epsilon > 0$, system (2.17) has equilibrium points $(u, \hat{u}) = (0, 0)$ and $(u^*, \frac{1}{2}cu^*)$. Now, we wish to establish the existence of a heteroclinic connection between these two critical points. From Lemma 2.1, we know that such a connection exists when $\epsilon = 0$ and we shall seek a solution of (2.17) that is only a perturbation of this heteroclinic connection. Let $(u_0, \hat{u}_0)$ be the solution of (2.17) when $\epsilon = 0$. To solve the system for $\epsilon > 0$ sufficiently small, we set

$$u = u_0 + \epsilon^2\phi + \cdots, \quad \hat{u} = \hat{u}_0 + \epsilon^2\psi + \cdots.$$

Substituting it to (2.17) yields that the differential system determining ϕ and ψ (to lowest order) is

$$\frac{d}{dz}\begin{pmatrix}\phi\\ \psi\end{pmatrix} + \begin{pmatrix}\frac{1}{2}c & -1\\ \frac{N-2u_0-\beta u_0^2}{(1+\beta u_0)^2} - a - \frac{1}{4}c^2 & \frac{1}{2}c\end{pmatrix}\begin{pmatrix}\phi\\ \psi\end{pmatrix} = \begin{pmatrix}0\\ -\frac{(N-u_0)H(u_0)}{(1+\beta u_0)^2}\end{pmatrix} \tag{2.18}$$

where $H(u_0) = au_0 - \frac{(N-u_0)u_0}{1+\beta u_0}$ and we shall prove that this system has a solution satisfying $\phi(\pm\infty) = 0$ and $\psi(\pm\infty) = 0$.

Working in the space L^2 of square integrable functions, with inner product

$$\int_{-\infty}^{+\infty}(x(z), y(z))dz,$$

where $(\cdot, \cdot)$ denotes the Euclidean inner product on $\mathbb{R}^2$. Fredholm theory [15] states that (2.18) have a solution if and only if

$$\int_{-\infty}^{+\infty}\left(x(z), \begin{pmatrix}0\\ -\frac{(N-u_0)H(u_0)}{(1+\beta u_0)^2}\end{pmatrix}\right)dz = 0$$

for all functions $x(z)$ in the kernel of the adjoint of the operator L defined by the left-hand side of (2.18). It is easy to verify that the adjoint operator L^* is given by

$$L^* = -\frac{d}{dz} + \begin{pmatrix}\frac{1}{2}c & \frac{N-2u_0-\beta u_0^2}{(1+\beta u_0)^2} - a - \frac{1}{4}c^2\\ -1 & \frac{1}{2}c\end{pmatrix},$$

and thus to compute $kerL^*$ we have to find all $x(z)$ satisfying

$$\frac{dx}{dz} = \begin{pmatrix}\frac{1}{2}c & \frac{N-2u_0-\beta u_0^2}{(1+\beta u_0)^2} - a - \frac{1}{4}c^2\\ -1 & \frac{1}{2}c\end{pmatrix}x, \tag{2.19}$$

the general solution of which will be difficult to find because the matrix is non-constant. However, we are only looking for solutions satisfying $x(\pm\infty) = 0$. In fact, we can derive that the only such solution is the zero solution. Recall that $u_0(z)$ is the solution of the unperturbed problem (2.5) and although we have no explicit expression for it, we do know that it tends to zero as $z \to +\infty$. Letting $z \to +\infty$ in (2.19), the matrix becomes a constant matrix, the eigenvalues λ of which satisfy

$$\lambda^2 - c\lambda + N - a = 0.$$

It is easy to see that when $N > a$ and $c \geq 2\sqrt{N-a}$, the eigenvalues are both real and positive. Then we can derive that any solution of (2.19) other than the zero solution must be growing exponentially for large z. So the only solution satisfying $x(\pm\infty) = 0$ is the zero solution. This implies that the Fredholm orthogonality condition trivially holds and so the solutions of (2.18) satisfying $\phi(\pm\infty) = 0$ and $\psi(\pm\infty) = 0$ exist. In other words, this means that there exists a heteroclinic connection between the two equilibria $(0,0,0,0)$ and $(u^*, \frac{1}{2}u^*, u^*, \frac{1}{2}u^*)$ of system (2.7).

To summarize, we have the first main theorem.

Theorem 2.1. *If $N > a$ and $c \geq 2\sqrt{N-a}$, Eq. (1.1) with the weak generic delay kernel (2.1) has a traveling wave front $u(x,t) = U(x-ct)$ satisfying $U(-\infty) = u^*$ and $U(+\infty) = 0$, provided the time delay τ is sufficiently small.*

3. Asymptotic behavior

In this section, by using the standard asymptotic theory, we shall obtain the asymptotic behavior of traveling wave front which has been shown in Sect. 2. Here we use the method which is similar to Smith and Zhao [14], Lv and Wang [11]. Let $\Phi(z) = (U(z), V(z))^T$ be the traveling wave front of Eq.(2.6). Differentiating (2.6) with respect to z, and denoting $\Phi'(z) = (U_1(z), V_1(z))^T$, then we have

$$\begin{cases} U_1'' + cU_1' - aU_1 + \dfrac{NV_1 - U_1V - UV_1}{1+\beta V} - \dfrac{\beta V(N-U)}{(1+\beta V)^2}V_1 = 0, \\ V_1'' + cV_1' + \dfrac{1}{\tau}(U_1 - V_1) = 0. \end{cases} \tag{3.1}$$

In view of $U(+\infty) = V(+\infty) = 0$, the limiting system for system (3.1) as $z \to +\infty$ is

$$\begin{cases} U_{1+}'' + cU_{1+}' - aU_{1+} + NV_{1+} = 0, \\ V_{1+}'' + cV_{1+}' + \dfrac{1}{\tau}(U_{1+} - V_{1+}) = 0, \end{cases} \tag{3.2}$$

where $(U_{1+}, V_{1+})^T$ is the traveling wave front of system (3.2) as $z \to +\infty$. By setting $U_{1+}' = U_{2+}, V_{1+}' = V_{2+}$, then the system (3.2) can be written as a first order system of ordinary differential equation

$$Z' = AZ, \quad Z = (U_{1+}, U_{2+}, V_{1+}, V_{2+})^T, \tag{3.3}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ a & -c & -N & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{1}{\tau} & 0 & \frac{1}{\tau} & -c \end{bmatrix}. \tag{3.4}$$

Solving the system (3.3), we get

$$Z = (U_{1+}, U_{2+}, V_{1+}, V_{2+})^T = \sum_{i=1}^{4} c_i h_i e^{\lambda_i z}, \tag{3.5}$$

where

$$\begin{cases} \lambda_1 = \dfrac{-c+\sqrt{c^2+4\Theta_0}}{2}, \quad \lambda_2 = \dfrac{-c-\sqrt{c^2+4\Theta_0}}{2}, \\ \lambda_3 = \dfrac{-c+\sqrt{c^2+4\Theta_1}}{2}, \quad \lambda_4 = \dfrac{-c-\sqrt{c^2+4\Theta_1}}{2}, \\ \Theta_0 = \dfrac{a+\frac{1}{\tau}+\sqrt{(a+\frac{1}{\tau})^2+\frac{4(N-a)}{\tau}}}{2}, \\ \Theta_1 = \dfrac{a+\frac{1}{\tau}-\sqrt{(a+\frac{1}{\tau})^2+\frac{4(N-a)}{\tau}}}{2}. \end{cases} \tag{3.6}$$

$h_i (i = 1,2,3,4)$ are eigenvectors of the matrix A with λ_i as the corresponding eigenvalues, and c_i are arbitrary constants. It is easy to see that $\lambda_1 > 0, \lambda_2 < 0, \lambda_3 < 0, \lambda_4 < 0$. Note that $(U_{1+}, U_{2+}, V_{1+}, V_{2+})^T \to (0,0,0,0)^T$ as $z \to +\infty$, then we can derive from (3.5) that $c_1 = 0$ and

$$(U_{1+}, U_{2+}, V_{1+}, V_{2+})^T = \sum_{i=2}^{4} c_i h_i e^{\lambda_i z}.$$

Then we deduce the following asymptotic behavior as $z \to +\infty$

$$\begin{cases} U_1(z) = \alpha_1(m_1+o(1))e^{\lambda_2 z} + \alpha_2(m_2+o(1))e^{\lambda_3 z} + \alpha_3(m_3+o(1))e^{\lambda_4 z}, \\ V_1(z) = \alpha_1(n_1+o(1))e^{\lambda_2 z} + \alpha_2(n_2+o(1))e^{\lambda_3 z} + \alpha_3(n_3+o(1))e^{\lambda_4 z}, \end{cases} \tag{3.7}$$

where $m_i, n_i (i = 1,2,3)$ are constants, and $\alpha_i (i = 1,2,3)$ cannot be zero simultaneously. By making the similar discussion as [2], we can easily claim that $m_i \neq 0, n_i \neq 0$ in (3.7).

Similarly, $U(-\infty) = V(-\infty) = u^*$, the limiting system for system (3.1) as $z \to -\infty$ is

$$\begin{cases} U_{1-}'' + cU_{1-}' - \dfrac{N}{1+\beta u^*}U_{1-} + \dfrac{a}{1+\beta u^*}V_{1-} = 0, \\ V_{1-}'' + cV_{1-}' + \dfrac{1}{\tau}(U_{1-} - V_{1-}) = 0, \end{cases} \tag{3.8}$$

where $(U_{1-}, V_{1-})^T$ is the traveling wave front of system (3.8) as $z \to -\infty$. By setting $U_{1-}' = U_{2-}, V_{1-}' = V_{2-}$, then the system (3.8) can be written as a first order system of ordinary differential equation

$$Z' = BZ, \quad Z = (U_{1-}, U_{2-}, V_{1-}, V_{2-})^T, \tag{3.9}$$

where

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{N}{1+\beta u^*} & -c & -\frac{a}{1+\beta u^*} & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{1}{\tau} & 0 & \frac{1}{\tau} & -c \end{bmatrix}. \tag{3.10}$$

Solving the system (3.9), we get

$$Z = (U_{1-}, U_{2-}, V_{1-}, V_{2-})^T = \sum_{i=1}^{4} d_i f_i e^{\Lambda_i z}, \tag{3.11}$$

where

$$
\begin{cases}
\Lambda_1 = \dfrac{-c+\sqrt{c^2+4\Delta_0}}{2}, \quad \Lambda_2 = \dfrac{-c+\sqrt{c^2+4\Delta_1}}{2}, \\
\Lambda_3 = \dfrac{-c-\sqrt{c^2+4\Delta_0}}{2}, \quad \Lambda_4 = \dfrac{-c-\sqrt{c^2+4\Delta_1}}{2}, \\
\Delta_0 = \dfrac{\frac{N}{1+\beta u^*}+\frac{1}{\tau}+\sqrt{(\frac{N}{1+\beta u^*}+\frac{1}{\tau})^2-\frac{4(N-a)}{(1+\beta u^*)\tau}}}{2}, \\
\Delta_1 = \dfrac{\frac{N}{1+\beta u^*}+\frac{1}{\tau}-\sqrt{(\frac{N}{1+\beta u^*}+\frac{1}{\tau})^2-\frac{4(N-a)}{(1+\beta u^*)\tau}}}{2}.
\end{cases}
\tag{3.12}
$$

$f_i (i = 1, 2, 3, 4)$ are eigenvectors of the matrix B with Λ_i as the corresponding eigenvalues, and d_i are arbitrary constants. It is easy to see that $\Lambda_1 > 0, \Lambda_2 > 0, \Lambda_3 < 0, \Lambda_4 < 0$. Note that $(U_{1-}, U_{2-}, V_{1-}, V_{2-})^T \to (u^*, 0, u^*, 0)^T$ as $z \to -\infty$, then we can derive from (3.11) that $d_3 = d_4 = 0$ and

$$(U_{1-}, U_{2-}, V_{1-}, V_{2-})^T = \sum_{i=1}^{2} d_i f_i e^{\Lambda_i z}.$$

Then we deduce the following asymptotic behavior as $z \to -\infty$

$$
\begin{cases}
U_1(z) = \gamma_1(p_1 + o(1))e^{\Lambda_1 z} + \gamma_2(p_2 + o(1))e^{\Lambda_2 z}, \\
V_1(z) = \gamma_1(q_1 + o(1))e^{\Lambda_1 z} + \gamma_2(q_2 + o(1))e^{\Lambda_2 z},
\end{cases}
\tag{3.13}
$$

where $p_i, q_i (i = 1, 2)$ are constants and $\gamma_i (i = 1, 2)$ cannot be zero. From the above discussion, we obtain the second main theorem.

Theorem 3.1. *Under the assumptions of Theorem 2.1, there exist positive constants A and B such that Eq. (1.1) with the weak generic delay kernel (2.1) has a traveling wave front $U(z), z = x - ct$, which satisfies the following properties*

$$U(z) = (A + o(1))e^{\lambda z}, \quad z \to +\infty, \tag{3.14}$$

and

$$U(z) = u^* - (B + o(1))e^{\Lambda z}, \quad z \to -\infty, \tag{3.15}$$

where λ may be one of the λ_2, λ_3 and λ_4 presented in (3.6), and Λ may be one of the Λ_1 and Λ_2 presented in (3.12).

4. Conclusions

In this paper, we have investigated a host-vector epidemic model with a particular kernel known as the weak generic delay kernel. For this particular kernel, by using linear chain techniques the traveling wave equation can be recasted as a singular perturbed system of ODEs, in which the time delay in the kernel appears as a small coefficient. Then the travelling fronts of the original host-vector epidemic model, which correspond to heterclinic orbits, are shown to exist by employing geometric singular perturbation theory, together with the Fredholm alternative. Furthermore, by applying the standard asymptotic theory, we also obtained the asymptotic behavior of the corresponding traveling wave fronts. Certainly our methods can still be applied to the strong generic delay case. It should be emphasized the fact here that only much more complicated calculations are needed, and we believe that the similar conclusion still holds.

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