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HYERS-ULAM STABILITY FOR AN N^{TH} ORDER DIFFERENTIAL EQUATION USING FIXED POINT APPROACH

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Abstract In this paper, we prove the Hyers-Ulam stability and the Hyers-Ulam-Rassias stability of the n^{th} order differential equation of the form

$$x^{(n)}(t) = f(t, x(t))$$

and

$$x^{(n)}(t) = f\left(t, x(t), x'(t), x''(t), \cdots, x^{(n-1)}(t)\right)$$

with initial conditions

$$x(a) = x_0, x'(a) = x_1, x''(a) = x_2, \cdots, x^{(n-1)}(a) = x_{n-1}$$

for all $t \in I = [a, b] \subset \mathbb{R}$ and $x \in C^{(n)}(I)$ by using fixed point method in the sense of Cadariu and Radu.

Keywords Hyers-Ulam stability, Hyers-Ulam-Rassias stability, fixed point method, differential equation.

MSC(2010) 65Mxx, 49J20, 39B72, 47H10, 39B82.

1. Introduction

The Hyers-Ulam stability problem was introduced by Ulam [28] in 1940. Then in the next year, Hyers [13] was handled the issue of Ulam for Cauchy additive functional equation in Banach spaces. From that point forward, many mathematicians are interested in the Ulam problem, in particular, Aoki [4], Bourgin [5] and Rassias [26] are generalized the Hyers result. Starting there onwards, a number of authors have proved the Hyers-Ulam stability for various functional equation on different spaces (see [6, 22–24, 29]).

As of late, the Hyers-Ulam stability property was proposed by supplanting functional equations by differential equations. In 1998, Alsina *et al.* [3] are the primary authors who contemplated the Hyers-Ulam stability of $x'(t) = x(t)$. Then Takashi *et al.* [27] generalized the result reported in [3] for Banach space valued function.

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In 2003, Agarwal, Xu and Zhang [1] established the Hyers-Ulam stability for differential equations. The Hyers-Ulam stability theory of differential equations was developed in a series of papers [2, 10–12, 15, 17–20].

In 2010, Jung [16] investigated the Hyers-Ulam-Rassias stability of the differential equations $y'(x) = F(x, y(x))$ for a bounded and continuous function $F(x, y)$ in the sense of Cadariu and Radu [8, 9]. Recently, Murali and Ponmana Selvan [21] studied the Hyers-Ulam stability and Hyers-Ulam-Rassias stability of the differential equation of the form $u''(t) = h(t, u(t))$ using fixed point method. Motivated and connected by the above ideas in [16, 21] and in the sense of Cadariu and Radu [7, 9] by using fixed point method, we establish the Hyers-Ulam stability and Hyers-Ulam-Rassias stability of the n^{th} order differential equations of the form

$$x^{(n)}(t) = f(t, [x(t)]) \tag{1.1}$$

where $f(t, [x(t)])$ is a bounded and continuous function. Here $[x(t)] := (x(t), x'(t), \cdots, x^{(n-1)}(t)]$. We also investigate the Hyers-Ulam stability and Hyers-Ulam-Rassias stability of the n^{th} order differential equations of the form

$$x^{(n)}(t) = f\left(t, x(t), x'(t), x''(t), \cdots, x^{(n-1)}(t)\right) \tag{1.2}$$

with initial conditions

$$x(a) = x_0,\ x'(a) = x_1,\ x''(a) = x_2,\ \cdots,\ x^{(n-1)}(a) = x_{n-1} \tag{1.3}$$

for all $t \in I = [a, b]$, $x \in C^{(n)}(I)$, $(t, [x(t)]) \equiv \left(t, x(t), x'(t), x''(t), \cdots, x^{(n-1)}(t)\right)$ and $a, b \in \mathbb{R}$, and $f(t, [x(t)])$ is defined on a closed bounded set $X \subset \mathbb{R}^{n+1}$ that satisfies the condition

$$|f(t, [x(t)]) - f(t, [y(t)])| \leq w(t)\frac{|x(t) - y(t)|}{(b-a)^{n-1}} \tag{1.4}$$

where $w(t) : I \to (0, \infty)$ is an integrable function.

2. Preliminaries

The following definitions and theorem are very useful to our main results. For the reader's convenience and explicite later use, we will recall some fundamental results in fixed point theory.

Definition 2.1 ([7, 9])**.** Let X be a nonempty set. A function $\rho : X \times X \to [0, \infty]$ is called a generalized metric on X if and only if ρ satisfies the following conditions:

(M_1) $\rho(x, y) = 0$ if and only if $x = y$,
(M_2) $\rho(x, y) = \rho(y, x)$ for all $x, y \in X$,
(M_3) $\rho(x, y) \leq \rho(x, z) + \rho(z, x)$ for all $x, y, z \in X$.

We observe that the only one difference of the generalized metric from the usual metric is that the range of the former is allowed to include the infinity.

Example 2.1. Let X be a nonempty set. A function $\rho : X \times X \to [0, +\infty]$ is defined as follows: if $x \neq y$ then $\rho(x, y) = +\infty$ and if $x = y$ then $\rho(x, y) = 0$. Then ρ is a generalized metric on X, which is not a metric on X.

We now recall one of the fundamental results of fixed point theory. For the proof we can refer [9] (see also [25]). The following theorems will play an important role in proving our main results.

Theorem 2.1 ([25], Banach's contraction principle). *Let (X, ρ) be a complete metric space, and consider a mapping $T : X \to X$ which is strictly contractive, that is*

$$\rho(Tx, Ty) \leq L\ \rho(x, y)$$

for all $x, y \in X$ and for some Lipschitz constant $L < 1$. Then

(i). The mapping T has one and only one fixed point $x^* = T(x^*)$;

(ii). The fixed point x^* is globally attractive, that is,

$$\lim_{k\to\infty} T^k x = x^*$$

for any starting point $x \in X$;

(iii). One has the following estimation inequalities:

$$\begin{aligned}
&\rho(T^k x, x^*) \leq L^k\ \rho(x, x^*), && \forall\ x \in X;\\
&\rho(T^k x, x^*) \leq \frac{1}{1-L}\rho(T^k x,\ T^{k+1}x), && \forall\ k \geq 0, \quad x \in X;\\
&\rho(x, x^*) \leq \frac{1}{1-L}\ \rho(x,\ Tx) && \forall\ x \in X.
\end{aligned}$$

Theorem 2.2 ([7, 9]). *Suppose we are given a complete generalized metric space (X, ρ)-i.e., one for which ρ may assume infinite values and a strictly contractive operator $T : X \to X$ with the Lipschitz constant $L < 1$. If there exists a nonnegative integer k such that $\rho(T^{k+1}, T^k) < \infty$ for some $x \in X$, then the followings conditions are satisfied:*

(i) the sequence $\{T^n(x)\}$ converges to a fixed point x^* of T;
(ii) x^* is the unique fixed point of T in $X^* = \left\{y \in X/\rho(T^k x, y) < \infty\right\}$;
(iii) If $y \in X^*$, then $\rho(y, x^*) \leq \dfrac{1}{1-L}\rho(Ty,\ y)$.

Now, we give the definitions of the Hyers-Ulam stability and the Hyers-Ulam-Rassias stability of the differential equations (1.1) and (1.2).

Definition 2.2. We say that the differential equation (1.1) has the Hyers-Ulam stability if there exists a constant $K > 0$ such that if for every $\epsilon > 0$, there exists an n times continuously differentiable function $x(t)$ satisfying the inequality

$$|x^{(n)}(t) - f(t, x(t))| \leq \epsilon,$$

then there exists some $y : (0, \infty) \to \mathbb{F}$ satisfying the differential equation (1.1) such that $|x(t) - y(t)| \leq K\epsilon$, for all $t > 0$. We call such K as the Hyers-Ulam stability constant for (1.1).

Definition 2.3. We say that the differential equation (1.2) has the Hyers-Ulam stability if there exists a constant $K > 0$ such that if for every $\epsilon > 0$, there exists an n times continuously differentiable function $x(t)$ satisfying the inequality

$$|x^{(n)}(t) - f\left(t, x(t), x'(t), x''(t), \cdots, x^{(n-1)}(t)\right)| \leq \epsilon,$$

then there exists some $y:(0,\infty)\to\mathbb{F}$ satisfying the differential equation (1.2) such that $|x(t)-y(t)|\le K\epsilon$ for all $t>0$. We call such K as the Hyers-Ulam stability constant for (1.2).

Definition 2.4. We say that the differential equation (1.1) has the Hyers-Ulam-Rassias stability if there exists a constant $K>0$ such that if for every $\epsilon>0$, there exist an n times continuously differentiable function $x(t)$ and $\phi:(0,\infty)\to(0,\infty)$ satisfying the inequality $|x^{(n)}(t)-f(t,x(t))|\le\phi(t)\epsilon$, then there exists some $y:(0,\infty)\to\mathbb{F}$ satisfying the differential equation (1.1) such that $|x(t)-y(t)|\le K\ \phi(t)\epsilon$ for all $t>0$. We call such K as Hyers-Ulam-Rassias stability constant for (1.1).

Definition 2.5. We say that the differential equation (1.2) has the Hyers-Ulam-Rassias stability if there exists a constant $K>0$ such that if for every $\epsilon>0$, there exist an n times continuously differentiable function $x(t)$ and $\phi:(0,\infty)\to(0,\infty)$ satisfying the inequality

$$|x^{(n)}(t)-f\left(t,x(t),x'(t),x''(t),\cdots,x^{(n-1)}(t)\right)|\le\phi(t)\epsilon,$$

then there exists some $y:(0,\infty)\to\mathbb{F}$ satisfying the differential equation (1.2) such that $|x(t)-y(t)|\le K\ \phi(t)\epsilon$ for all $t>0$. We call such K as Hyers-Ulam-Rassias stability constant for (1.2).

3. Hyers-Ulam stability

Cadariu and Radu [7] applied the fixed point method to the investigation of the Jensen's functional equation. Using such an idea, they could present a proof for the Hyers-Ulam stability of that equation.

In this section, by using the idea of Cadariu and Radu [7], we will prove the Hyers-Ulam stability of the differential equation (1.1) and (1.2) defined on a closed and bounded interval by using Theorem 2.2.

Theorem 3.1. *Let K and L be positive constants with $0<KL<1$. Assume that $f:I\times\mathbb{R}^n\to\mathbb{R}$ is a continuous function which satisfies the Lipschitz condition*

$$|f(t,x(t))-f(t,y(t))|\le L\,|x-y| \tag{3.1}$$

for all $t\in I$ and $x,y\in\mathbb{R}$. If $x\in C^n(I)$ satisfies the inequality

$$\left|x^{(n)}(t)-f(t,x(t))\right|\le\epsilon \tag{3.2}$$

for all $t\in I$, then there exists a unique solution $y(t)\in C^n(I)$ such that

$$|x(t)-y(t)|\le M\epsilon.$$

Proof. Let us assume that X is the set of all continuous functions $g:I\times\mathbb{R}\to\mathbb{R}$, i.e.,

$$X=\{g:I\times\mathbb{R}\to\mathbb{R}:g\text{ is continuous}\}.$$

Firstly, we define an operator $T:X\to X$ by

$$(Tg)(t)=\sum_{l=0}^{n-1}\frac{(t-a)^l x_l}{l!}+\frac{1}{(n-1)!}\int_a^t(t-s)^{n-1}f(s,g(s))\,ds \tag{3.3}$$

for all $t \in I$. Now, let us introduce the metric on X as follows:

$$\rho(g,h) = \inf\{c \in [0,\infty] : |g(t) - h(t)| \leq c\epsilon\} \tag{3.4}$$

for all $g, h \in X$ and $t \in I$. Now, we have to prove that ρ is a generalized metric on X. For that, we will here only prove the triangle inequality. Suppose that

$$\rho(g,h) > \rho(g,i) + \rho(i,h),$$

for some $g, h, i \in X$. Then we obtain that

$$\begin{aligned} |g(t) - h(t)| &= \rho(g,h) > \rho(g,i) + \rho(i,h) \\ &= |g(t) - i(t)| + |i(t) - h(t)|, \end{aligned}$$

a contradiction. Hence ρ is a generalized metric on X. Now, we claim that (X, ρ) is complete. Let $\{i_n\}$ be a Cauchy sequence in (X, ρ).

Then for any $\epsilon > 0$, there exists an integer $N_\epsilon > 0$ such that

$$\rho(i_m, i_n) \leq \epsilon$$

for all $m, n \geq N_\epsilon$. By using (3.4), we have that for every $\epsilon > 0$ there exists an integer $N_\epsilon > 0$ such that

$$|i_m(t) - i_n(t)| \leq \epsilon \tag{3.5}$$

for all $m, n \geq N_\epsilon$, $t \in I$. If t is fixed, then (3.5) implies that $\{i_n\}$ is a Cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ is complete, $\{i_n\}$ converges for all $t \in I$.

Thus we define a function $i : I \times \mathbb{R} \to \mathbb{R}$ such that

$$i(t) = \lim_{n\to\infty} i_n(t).$$

Using (3.4), we get for every $\epsilon > 0$ there exists an integer $N_\epsilon \in N$ such that

$$|i(t) - i_n(t)| \leq \epsilon. \tag{3.6}$$

That is, $\{i_n\}$ converges uniformly to i. Hence i is a continuous function and $i \in X$. Then by using (3.6), for any $\epsilon > 0$, there exists an integer $N_\epsilon \in N$ such that $\rho(i, i_n) \leq \epsilon$ for all $n \geq N_\epsilon$. That is, the Cauchy sequence $\{i_n\}$ converges to i on X. Hence (X, ρ) is complete. Since f and g are continuous, by Fundamental Theorem of Calculus, Tg is n times continuously differentiable on I.

Hence we can conclude that $Tg \in X$. Now, we assert that T is a strictly contractive mapping on X. For any $g, h \in X$, let $c_{gh} \in [0,\infty]$ be an arbitrary constant with $\rho(g,h) \leq c_{gh}\epsilon$. Then by using (3.4), we have

$$|g(t) - h(t)| \leq c_{gh}\epsilon \tag{3.7}$$

for all $t \in I$. Then using (3.1), (3.3), (3.4) and (3.7), we have

$$\begin{aligned} \rho(Tg, Th) &= |Tg(t) - Th(t)| \\ &= \left| \frac{1}{(n-1)!} \int_a^t (t-s)^{n-1} f(s, g(s))\, ds - \frac{1}{(n-1)!} \int_a^t (t-s)^{n-1} f(s, h(s))\, ds \right| \end{aligned}$$

$$\leq \frac{1}{(n-1)!}\int_a^t |t-s|\,|f(s,g(s))-f(s,h(s))|\ ds$$

$$\leq \frac{L}{(n-1)!}\int_a^t |t-s|\,|g(s)-h(s)|\ ds$$

$$\leq \frac{L}{(n-1)!}c_{gh}\epsilon\int_a^t |t-s|\ ds,$$

$$\rho(Tg,Th) \leq KLc_{gh}\epsilon = KL\rho(g,h)$$

for all $g, h \in X$ and $0 < KL < 1$. It follow from (3.3) that for any arbitrary $h_0 \in X$, there exists a constant $0 < c < \infty$ with

$$|Th_0(t) - h_0(t)| = \left|\sum_{l=0}^{n-1}\frac{(t-a)^l x_l}{l!} + \frac{1}{(n-1)!}\int_a^t (t-s)^{n-1} f(s,h_0(s))\ ds - h_0(t)\right| < c\epsilon$$

for all $t \in I$. Since $f(s, h_0(s))$ and $h_0(t)$ are bounded on I, $\rho(Th_0, h_0) < \infty$. Hence by using Theorem 2.2, there exists a continuous function $y_0(t) \in C^n(I)$ such that $Th_0 \to y_0$ in (X, ρ) and $Ty_0 = y_0$. That is, y_0 is a solution of

$$x(t) = \sum_{l=0}^{n-1}\frac{(t-a)^l x_l}{l!} + \frac{1}{(n-1)!}\int_a^t (t-s)^{n-1} f(s,x(s))\ ds.$$

Now, we have to prove that $\{h \in X : \rho(h_0, h) < \infty\}$. For all $h \in X$, since h_0 and h are bounded on I, there exists a constant $c_h \in [0, \infty]$ such that

$$\rho(h_0, h) = |h_0(t) - h(t)| \leq c_h\epsilon$$

for all $t \in I$. Thus we have $\rho(h_0, h) < \infty$ for all $h \in X$. Hence by Theorem 2.2, y_0 is the unique continuous function with the property (3.4). Now, from the inequality (3.2), we get

$$-\epsilon \leq x^{(n)}(t) - f(t, x(t)) \leq \epsilon \tag{3.8}$$

for all $t \in I$. Now, we integrate (3.8) n times from a to t, we obtain

$$\left|x(t) - \sum_{l=0}^{n-1}\frac{(t-a)^l x_l}{l!} + \frac{1}{(n-1)!}\int_a^t (t-s)^{n-1} f(s,x(s))\ ds\right| \leq \frac{(b-a)^n}{n!}\epsilon$$

for all $t \in I$. Thus

$$\rho(Tx, x) \leq M\epsilon. \tag{3.9}$$

Finally, by using Theorem 2.2 and (3.9), we obtain that

$$\rho(x, y_0) \leq \frac{1}{1-KL}\rho(Tx, x) \leq \frac{M\epsilon}{1-KL} = H\epsilon.$$

Hence by virtue of Definition 2.2, the differential equation (1.1) has the Hyers-Ulam stability. □

Theorem 3.2. *Let K and L be positive constants with $0 < KL < 1$. Assume that $f : I \times \mathbb{R}^n \to \mathbb{R}$ is a continuous function which satisfies the Lipschitz condition*

$$|f(t,[x(t)]) - f(t,[y(t)])| \leq L \sum_{\ell=0}^{n-1} \left| x^{(\ell)}(t) - y^{(\ell)}(t) \right| \tag{3.10}$$

for all $t \in I$. If $x \in C^n(I)$ satisfies the inequality

$$\left| x^{(n)}(t) - f\left(t, x(t), x'(t), x''(t), ..., x^{(n-1)}(t)\right) \right| \leq \epsilon \tag{3.11}$$

for all $t \in I$, then there exists a unique solution $y(t) \in C^{(n)}(I)$ such that

$$|x(t) - y(t)| \leq H\epsilon.$$

Proof. Let us assume that X is the set of all continuous functions $g : I \times \mathbb{R}^n \to \mathbb{R}$, i.e., $X = \{g : I \times \mathbb{R}^n \to \mathbb{R} : g \text{ is continuous}\}$. Firstly, we introduce an operator $T : X \to X$ by

$$(Tg)(t) = \sum_{l=0}^{n-1} \frac{(t-a)^l x_l}{l!} + \frac{1}{(n-1)!} \int_a^t (t-s)^{n-1} f(s,[g(s)])\, ds \tag{3.12}$$

for all $t \in I$. Now, let us introduce the metric on X as follows:

$$\rho(g,h) = \inf\{M \in [0,\infty] : |g(t) - h(t)| \leq M\epsilon\} \tag{3.13}$$

for all $g, h \in X$ and $t \in I$. Now, we have to prove that ρ is a generalized metric on X. For that, we will here only prove the triangle inequality. Suppose that $\rho(g,h) > \rho(g,i) + \rho(i,h)$ for some $g, h, i \in X$. Then we obtain that

$$|g(t) - h(t)| = \rho(g,h) > \rho(g,i) + \rho(i,h) = |g(t) - i(t)| + |i(t) - h(t)|,$$

a contradiction. Hence ρ is a generalized metric on X. Now, we claim that (X, ρ) is complete. Let $\{i_k(t)\}$ be a Cauchy sequence in (X, ρ). Then for any $\epsilon > 0$, there exists an integer $N_\epsilon > 0$ such that $\rho(i_k, i_j) \leq \epsilon$ for all $j, k \geq N_\epsilon$. By using (3.13), we have for every $\epsilon > 0$ there exists an integer $N_\epsilon > 0$ such that

$$|i_j(t) - i_k(t)| \leq \epsilon \tag{3.14}$$

for all $j, k \geq N_\epsilon$, $t \in I$. If t is fixed, then (3.14) implies that $\{i_k\}$ is a Cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ is complete, $\{i_k\}$ converges for all $t \in I$. Thus we define a function $i(t) : I \times \mathbb{R}^n \to$ such that $i(t) = \lim_{k\to\infty} i_k(t)$.

If $k \to \infty$ and using (3.13), we get for every $\epsilon > 0$ there exists an integer $N_\epsilon \in N$ such that

$$|i(t) - i_k(t)| \leq \epsilon. \tag{3.15}$$

That is, $\{i_k\}$ converges uniformly to i. Hence $i(t)$ is a continuous function and $i(t) \in X$. Then by using (3.15), for any $\epsilon > 0$ there exists an integer $N_\epsilon \in N$ such that $\rho(i(t), i_k(t)) \leq \epsilon$ for all $k \geq N_\epsilon$. That is, the Cauchy sequence $\{i_k\}$ converges to i on X. Hence (X, ρ) is complete. Since f and g are continuous, by Fundamental

Theorem of Calculus, Tg is n times continuously differentiable on I. Hence we can conclude that $Tg \in X$.

Now, we assert that T is a strictly contractive mapping on X. For any $g, h \in X$, let $M_{gh} \in [0, \infty]$ be an arbitrary constant with $\rho(g,h) \le M_{gh}\epsilon$. Then by using (3.13), we have

$$|g(t) - h(t)| \le M_{gh}\epsilon \tag{3.16}$$

for all $t \in I$. Then using (3.10), (3.12), (3.13) and (3.16), we have

$$\begin{aligned}
&\rho(Tg, Th)\\
&= \left| \frac{1}{(n-1)!} \int_a^t (t-s)^{n-1} f(s,[g(s)])\, ds - \frac{1}{(n-1)!} \int_a^t (t-s)^{n-1} f(s,[h(s)])\, ds \right|\\
&\le \frac{1}{(n-1)!} \int_a^t \left|(t-s)^{n-1}\right| \left|f(s,[g(s)]) - f(s,[h(s)])\right|\, ds\\
&\le \frac{L}{(n-1)!} \int_a^t \left|(b-a)^{n-1}\right| w(s) \frac{|g(s)-h(s)|}{(b-a)^{n-1}}\, ds\\
&\le \frac{L}{(n-1)!}\epsilon \int_a^t w(s) M_{gh}\, ds.
\end{aligned}$$

Hence $\rho(Tg, Th) \le KLM_{gh}\epsilon = KL\rho(g,h)$, where $K = \dfrac{1}{(n-1)!} \int_a^t w(s)\, ds$ for all $g, h \in X$ and $0 < KL < 1$. It follow from (3.12) that for any arbitrary $h_0 \in X$, there exists a constant $0 < M < \infty$ with

$$\begin{aligned}
|Th_0(t) - h_0(t)| &= \left| \sum_{l=0}^{n-1} \frac{(t-a)^l x_l}{l!} + \frac{1}{(n-1)!} \int_a^t (t-s)^{n-1} f(s,[h_0(s)])\, ds - h_0(t) \right|\\
&< M\epsilon
\end{aligned}$$

for all $t \in I$. Since $f(s,[h_0(s)])$ and $h_0(t)$ are bounded on I, $\rho(Th_0, h_0) < \infty$. Hence by using Theorem 2.2, there exists a continuous function $h_0(t) \in C^n(I)$ such that $Th_0 \to h_0$ in (X, ρ) and $Ty_0 = y_0$. That is, y_0 is a solution of

$$x(t) = \sum_{l=0}^{n-1} \frac{(t-a)^l x_l}{l!} + \frac{1}{(n-1)!} \int_a^t (t-s)^{n-1} f(s,[x(s)])\, ds.$$

Now, we have to prove that $\{h \in X : \rho(h_0, h) < \infty\}$. For all $h \in X$, since h_0 and h are bounded on I, there exists a constant $M \in [0, \infty]$ such that

$$\rho(h_0, h) = |h_0(t) - h(t)| \le M\epsilon$$

for all $t \in I$. Hence we have $\rho(h_0, h) < \infty$ for all $h \in X$. Thus by Theorem 2.2, y_0 is the unique continuous function with the property (3.13). Now, from the inequality (3.11), we get

$$-\epsilon \le x^{(n)}(t) - f(t,[x(t)]) \le \epsilon \tag{3.17}$$

for all $t \in I$. Now, we integrate (3.17) n times from a to t, we obtain

$$\left| x(t) - \sum_{l=0}^{n-1} \frac{(t-a)^l x_l}{l!} + \frac{1}{(n-1)!} \int_a^t (t-s)^{n-1} f(s, [h_0(s)])\, ds \right| \leq \frac{(b-a)^n}{n!} \epsilon$$

for all $t \in I$. Thus we have $|x(t) - (Tx)(t)| \leq \left| \frac{(b-a)^n}{n!} \epsilon \right| \leq S\epsilon$ and so

$$\rho(Tx, x) \leq S\epsilon. \tag{3.18}$$

Hence, by using Theorem 2.2 and (3.18), we have

$$\rho(x, y_0) \leq \frac{1}{1-KL} \rho(Tx, x) \leq \frac{S\epsilon}{1-KL} = H\epsilon.$$

Then by the virtue of Definition 2.3, the differential equation (1.2) has the Hyers-Ulam stability, as desired. □

4. Hyers-Ulam-Rassias stability

In this section, we prove the Hyers-Ulam-Rassias stability of the differential equations (1.1) and (1.2) defined on a closed and bounded interval by using the fixed point theorem.

Theorem 4.1. *Let K and L be positive constants with $0 < KL < 1$. Assume that $f : I \times \mathbb{R}^n \to \mathbb{R}$ is a continuous function which satisfies the Lipschitz condition*

$$|f(t, x(t)) - f(t, y(t))| \leq L\, |x - y| \tag{4.1}$$

for all $t \in I$ and $x, y \in \mathbb{R}$. If there exists $\phi : (0, \infty) \to (0, \infty)$ such that $x \in C^n(I)$ satisfies the inequality

$$\left| x^{(n)}(t) - f(t, x(t)) \right| \leq \phi(t)\epsilon \tag{4.2}$$

for all $t \in I$, then there exists a unique solution $y(t) \in C^n(I)$ such that

$$|x(t) - y(t)| \leq M\phi(t)\epsilon.$$

Proof. Let us assume that X is the set of all continuous functions $g : I \times \mathbb{R} \to \mathbb{R}$, i.e., $X = \{g : I \times \mathbb{R} \to \mathbb{R} : g \text{ is continuous}\}$. Now, we define an operator $T : X \to X$ by

$$(Tg)(t) = \sum_{l=0}^{n-1} \frac{(t-a)^l x_l}{l!} + \frac{1}{(n-1)!} \int_a^t (t-s)^{n-1} f(s, g(s))\, ds \tag{4.3}$$

for all $t \in I$. Now, let us introduce the metric on X as follows:

$$\rho(g, h) = \inf \{c \in [0, \infty] : |g(t) - h(t)| \leq c\phi(t)\epsilon\} \tag{4.4}$$

for all $g, h \in X$ and $t \in I$, where $\phi : (0, \infty) \to (0, \infty)$ with

$$\left| \int_a^t \phi(t) dt \right| \leq K\phi(t). \tag{4.5}$$

Now, we have to prove that ρ is a generalized metric on X. For that, we will here only prove the triangle inequality. Suppose that $\rho(g,h) > \rho(g,i) + \rho(i,h)$ for some $g, h, i \in X$. Then we obtain that

$$|g(t) - h(t)| = \rho(g,h) > \rho(g,i) + \rho(i,h) = |g(t) - i(t)| + |i(t) - h(t)|,$$

a contradiction. Hence ρ is a generalized metric on X. Now, we claim that (X, ρ) is complete. Let $\{i_n\}$ be a Cauchy sequence in (X, ρ). Then for any $\epsilon > 0$, there exists an integer $N_\epsilon > 0$ such that $\rho(i_m, i_n) \leq \epsilon$ for all $m, n \geq N_\epsilon$. By using (4.4), we have for every $\epsilon > 0$ there exists an integer $N_\epsilon > 0$ such that

$$|i_m(t) - i_n(t)| \leq \phi(t)\epsilon \tag{4.6}$$

for all $m, n \geq N_\epsilon$, $t \in I$. If t is fixed, then (4.6) implies that $\{i_n\}$ is a Cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ is complete, $\{i_n\}$ converges for all $t \in I$. Thus we define a function $i : I \times \mathbb{R} \to \mathbb{R}$ such that $i(t) = \lim_{n\to\infty} i_n(t)$. If $m \to \infty$ and using (4.4), we get for every $\epsilon > 0$ there exists an integer $N_\epsilon \in N$ such that

$$|i(t) - i_n(t)| \leq \phi(t)\epsilon. \tag{4.7}$$

That is, $\{i_n(t)\}$ converges uniformly to $i(t)$. Hence $i(t)$ is a continuous function and $i \in X$. Then by using (4.7), for any $\epsilon > 0$ there exists an integer $N_\epsilon \in N$ such that $\rho(i, i_n) \leq \phi(t)\epsilon$ for all $n \geq N_\epsilon$. That is, the Cauchy sequence $\{i_n\}$ converges to i on X. Hence (X, ρ) is complete. Since f and g are continuous, by Fundamental Theorem of Calculus, Tg is n times continuously differentiable on I. Hence we can conclude that $Tg \in X$. Now, we assert that T is a strictly contractive mapping on X. For any $g, h \in X$, let $c_{gh} \in [0, \infty]$ be an arbitrary constant with $\rho(g,h) \leq c_{gh}\phi(t)\epsilon$. Then by using (4.4), we have

$$|g(t) - h(t)| \leq c_{gh}\phi(t)\epsilon \tag{4.8}$$

for all $t \in I$. Then using (4.1), (4.3), (4.4) and (4.8), we have

$$\begin{aligned}
\rho(Tg, Th) &= |Tg(t) - Th(t)| \\
&= \left| \frac{1}{(n-1)!} \int_a^t (t-s)^{n-1} f(s, g(s))\ ds - \frac{1}{(n-1)!} \int_a^t (t-s)^{n-1} f(s, h(s))\ ds \right| \\
&\leq \frac{1}{(n-1)!} \int_a^t |t-s|\, |f(s, g(s)) - f(s, h(s))|\ ds \\
&\leq \frac{L}{(n-1)!} \int_a^t |t-s|\, |g(s) - h(s)|\ ds \\
&\leq \frac{L}{(n-1)!} c_{gh}\phi(t)\epsilon \int_a^t |t-s|\ ds,
\end{aligned}$$

$$\rho(Tg, Th) \leq KLc_{gh}\phi(t)\epsilon = KL\rho(g, h)$$

for all $g, h \in X$ and $0 < KL < 1$. It follow from (4.3) that for any arbitrary $h_0 \in X$, there exists a constant $0 < c < \infty$ with

$$\begin{aligned} |Th_0(t) - h_0(t)| &= \left| \sum_{l=0}^{n-1} \frac{(t-a)^l x_l}{l!} + \frac{1}{(n-1)!} \int_a^t (t-s)^{n-1} f(s, h_0(s))\ ds - h_0(t) \right| \\ &< c\phi(t)\epsilon \end{aligned}$$

for all $t \in I$. Since $f(s, h_0(s))$ and $h_0(t)$ are bounded on I, $\rho(Th_0, h_0) < \infty$. Hence by using Theorem 2.2, there exists a continuous function $y_0(t) \in C^n(I)$ such that $Th_0 \to y_0$ in (X, ρ) and $Ty_0 = y_0$. That is, y_0 is a solution of

$$x(t) = \sum_{l=0}^{n-1} \frac{(t-a)^l x_l}{l!} + \frac{1}{(n-1)!} \int_a^t (t-s)^{n-1} f(s, x(s))\ ds.$$

Now, we have to prove that $\{h \in X : \rho(h_0, h) < \infty\}$. For all $h \in X$, since h_0 and h are bounded on I, there exists a constant $c_h \in [0, \infty]$ such that

$$\rho(h_0, h) = |h_0(t) - h(t)| \leq c_h\phi(t)\epsilon$$

for all $t \in I$. Hence we have $\rho(h_0, h) < \infty$ for all $h \in X$. Thus by Theorem 2.2, y_0 is the unique continuous function with the property (3.4). Now, from the inequality (3.2), we get

$$-\phi(t)\epsilon \leq x^{(n)}(t) - f(t, x(t)) \leq \phi(t)\epsilon \tag{4.9}$$

for all $t \in I$. Now, we integrate (4.9) n times from a to t, we obtain

$$\left| x(t) - \sum_{l=0}^{n-1} \frac{(t-a)^l x_l}{l!} + \frac{1}{(n-1)!} \int_a^t (t-s)^{n-1} f(s, x(s))\ ds \right| \leq \frac{(b-a)^n}{n!}\epsilon \left| \int_a^t \phi(t)dt \right|$$

for all $t \in I$. Thus using the above inequality with (4.3) and (4.5), we have

$$|Tx(t) - x(t)| \leq \frac{(b-a)^n}{n!}\epsilon \left| \int_a^t \phi(t)dt \right| \leq MK\phi(t)\epsilon$$

for all $t \in I$ and so

$$\rho(Tx, x) \leq MK\phi(t)\epsilon. \tag{4.10}$$

Finally, by Theorem 2.2 and (4.10), we obtain

$$\rho(x, y_0) \leq \frac{1}{1-KL}\rho(Tx, x) \leq \frac{MK\phi(t)\epsilon}{1-KL} = H\phi(t)\epsilon.$$

Hence by the virtue of Definition 2.4 the differential equation (1.1) has the Hyers-Ulam-Rassias stability. □

Theorem 4.2. *Let K, J and L be positive constants with $0 < KJL < 1$. Assume that $f : I \times \mathbb{R}^n \to \mathbb{R}$ is a continuous function which satisfies the Lipschitz condition*

$$|f(t,[x(t)]) - f(t,[y(t)])| \leq L \sum_{\ell=0}^{n-1} \left| x^{(\ell)}(t) - y^{(\ell)}(t) \right| \tag{4.11}$$

for all $t \in I$. If there exists $\phi : (0, \infty) \to (0, \infty)$, and $x \in C^n(I)$ satisfying

$$\left| x^{(n)}(t) - f\left(t, x(t), x'(t), x''(t), ..., x^{(n-1)}(t)\right) \right| \leq \phi(t)\epsilon \tag{4.12}$$

for all $t \in I$, then there exists a unique solution $y(t) \in C^{(n)}(I)$ such that

$$|x(t) - y(t)| \leq H\phi(t)\epsilon.$$

Proof. Let us assume that X is the set of all continuous functions $g : I \times \mathbb{R}^n \to \mathbb{R}$, i.e., $X = \{g : I \times \mathbb{R}^n \to \mathbb{R} : g \text{ is continuous}\}$. Firstly, we introduce an operator $T : X \to X$ by

$$(Tg)(t) = \sum_{l=0}^{n-1} \frac{(t-a)^l x_l}{l!} + \frac{1}{(n-1)!} \int_a^t (t-s)^{n-1} f(s, [g(s)])\, ds \tag{4.13}$$

for all $t \in I$. Now, let us introduce the metric on X as follows:

$$\rho(g,h) = \inf\{M \in [0, \infty] : |g(t) - h(t)| \leq M\phi(t)\epsilon\} \tag{4.14}$$

for all $g, h \in X$ and $t \in I$, where $\phi : (0, \infty) \to (0, \infty)$ with

$$\left| \int_a^t \phi(t)dt \right| \leq K\phi(t). \tag{4.15}$$

Now, we have to prove that ρ is a generalized metric on X. For that, we will here only prove the triangle inequality. Suppose that $\rho(g,h) > \rho(g,i) + \rho(i,h)$, for some $g, h, i \in X$. Then we obtain that

$$|g(t) - h(t)| = \rho(g,h) > \rho(g,i) + \rho(i,h) = |g(t) - i(t)| + |g(t) - h(t)|,$$

a contradiction. Hence ρ is a generalized metric on X. Now, we claim that (X, ρ) is complete. Let $\{i_k(t)\}$ be a Cauchy sequence in (X, ρ). Then for any $\epsilon > 0$, there exists an integer $N_\epsilon > 0$ such that $\rho(i_k, i_j) \leq \phi(t)\epsilon$ for all $j, k \geq N_\epsilon$. By using (4.14), we have for every $\epsilon > 0$ there exists an integer $N_\epsilon > 0$ such that

$$|i_j(t) - i_k(t)| \leq \phi(t)\epsilon \tag{4.16}$$

for all $j, k \geq N_\epsilon$, $t \in I$. If t is fixed, then (4.16) implies that $\{i_k\}$ is a Cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ is complete, $\{i_k\}$ converges for all $t \in I$. Thus we define a function $i(t) : I \times \mathbb{R}^n \to$ such that $i(t) = \lim_{k\to\infty} i_k(t)$. Using (4.14), we get for every $\epsilon > 0$ there exists an integer $N_\epsilon \in N$ such that

$$|i(t) - i_k(t)| \leq \phi(t)\epsilon. \tag{4.17}$$

That is, $\{i_k\}$ converges uniformly to $i(t)$. Hence $i(t)$ is a continuous function and $i(t) \in X$. Then by using (4.17), for any $\epsilon > 0$ there exists an integer $N_\epsilon \in N$ such that $\rho(i(t), i_k(t)) \leq M\phi(t)\epsilon$ for all $k \geq N_\epsilon$. That is, the Cauchy sequence $\{i_k\}$ converges to $i(t)$ on X. Hence (X, ρ) is complete. Since f and g are continuous, by Fundamental Theorem of Calculus, Tg is n times continuously differentiable on I.

Hence we can conclude that $Tg \in X$. Now, we assert that T is a strictly contractive mapping on X. For any $g, h \in X$, let $M_{gh} \in [0, \infty]$ be an arbitrary constant with $\rho(g, h) \leq M_{gh}\phi(t)\epsilon$. Then by using (4.14), we have

$$|g(t) - h(t)| \leq M_{gh}\phi(t)\epsilon \tag{4.18}$$

for all $t \in I$. Then using (4.11), (4.13), (4.14), (4.15) and (4.18), we have

$$\begin{aligned}
&\rho(Tg, Th) \\
&= \left| \frac{1}{(n-1)!} \int_a^t (t-s)^{n-1} f(s, [g(s)])\, ds - \frac{1}{(n-1)!} \int_a^t (t-s)^{n-1} f(s, [h(s)])\, ds \right| \\
&\leq \frac{1}{(n-1)!} \int_a^t \left|(t-s)^{n-1}\right| |f(s, [g(s)]) - f(s, [h(s)])|\, ds \\
&\leq \frac{L}{(n-1)!} \int_a^t \left|(b-a)^{n-1}\right| w(s) \frac{|g(s) - h(s)|}{(b-a)^{n-1}}\, ds \\
&\leq \frac{L}{(n-1)!}\epsilon \int_a^t w(s) M_{gh}\phi(s)\, ds,
\end{aligned}$$

and so $\rho(Tg, Th) \leq KJLM_{gh}\phi(t)\epsilon = KJL\rho(g, h)$, where $J = \frac{1}{(n-1)!} \int_a^t w(s)\, ds$ for all $g, h \in X$ and $0 < KJL < 1$. It follow from (4.13) that for any arbitrary $h_0 \in X$, there exists a constant $0 < M < \infty$ with

$$\begin{aligned}
|Th_0(t) - h_0(t)| &= \left| \sum_{l=0}^{n-1} \frac{(t-a)^l x_l}{l!} + \frac{1}{(n-1)!} \int_a^t (t-s)^{n-1} f(s, [h_0(s)])\, ds - h_0(t) \right| \\
&< M\phi(t)\epsilon
\end{aligned}$$

for all $t \in I$. Since $f(s, [h_0(s)])$ and $h_0(t)$ are bounded on I, $\rho(Th_0, h_0) < \infty$. Hence by using Theorem 2.2, there exists a continuous function $h_0(t) \in C^n(I)$ such that $Th_0 \to h_0$ in (X, ρ) and $Ty_0 = y_0$. That is, y_0 is a solution of

$$x(t) = \sum_{l=0}^{n-1} \frac{(t-a)^l x_l}{l!} + \frac{1}{(n-1)!} \int_a^t (t-s)^{n-1} f(s, [x(s)])\, ds.$$

Now, we have to prove that $\{h \in X : \rho(h_0, h) < \infty\}$. For all $h \in X$, since h_0 and h are bounded on I, there exists a constant $C_n \in [0, \infty]$ such that

$$\rho(h_0, h) = |h_0(t) - h(t)| \leq M\phi(t)\epsilon$$

for all $t \in I$. Hence we have $\rho(h_0, h) < \infty$ for all $h \in X$. Thus by Theorem 2.2, y_0 is the unique continuous function with the property (4.14). Now, from the inequality (4.12), we get

$$-\phi(t)\epsilon \leq x^{(n)}(t) - f(t, [x(t)]) \leq \phi(t)\epsilon \tag{4.19}$$

for all $t \in I$. Now, we integrate (4.19) n times from a to t, we obtain

$$\left| x(t) - \sum_{l=0}^{n-1} \frac{(t-a)^l x_l}{l!} + \frac{1}{(n-1)!} \int_a^t (t-s)^{n-1} f(s, [h_0(s)])\ ds \right|$$
$$\leq \frac{(b-a)^n}{n!} \left| \int_a^t \phi(s)\ ds \right| \epsilon$$

for all $t \in I$. Thus we have $|x(t) - (Tx)(t)| \leq \left| \frac{(b-a)^n}{n!} K\phi(t)\epsilon \right| \leq SK\phi(t)\epsilon$,

$$\rho(Tx, x) \leq SK\phi(t)\epsilon. \tag{4.20}$$

Finally, by Theorem 2.2 and (4.20), we obtain

$$\rho(x, y_0) \leq \frac{1}{1 - KJL} \rho(Tx, x) \leq \frac{SK\phi(t)\epsilon}{1 - KJL} = H\phi(t)\epsilon.$$

Then by the virtue of Definition 2.5 the differential equation (1.2) has the Hyers-Ulam-Rassias stability. □

5. Some examples

In this section, we provide some examples to illustrate the main results.

Example 5.1. Let η and L be positive constants with $\eta L < 1$. Let

$$I = \{t \in \mathbb{R} |\ \nu - \eta \leq \mathrm{t} \leq \nu + \eta\}$$

for some real number ν. Suppose that x is an n times continuously differentiable function satisfying the inequality

$$\left| x^{(n)}(t) - L\ x(t) - \xi(t) \right| \leq \epsilon$$

for all $t \in I$ and for some $\epsilon \geq 0$, where $\xi(t)$ is a polynomial. Then by Theorem 3.1, there exists a unique n times continuously differentiable function $y_0 \in C^n(I)$ such that

$$y_0(t) = \sum_{l=0}^{n-1} \frac{(t-a)^l x_l}{l!} + \frac{1}{(n-1)!} \int_a^t (t-s)^{n-1}\ \ [L\ x(s) - \xi(s)]\ ds$$

and

$$|x(t) - y_0(t)| \leq \frac{M\ \epsilon}{1 - KL} = H\ \epsilon.$$

for all $t \in I$.

Example 5.2. Let $I = [0, 5K - \epsilon]$ be a closed interval for positive numbers ϵ and K with $\epsilon < 5K$. For a given polynomial $\ell(t)$, we assume that $x \in C^n(I)$ satisfies the inequality

$$\left| x^{(n)}(t) - L\ x(t) - \ell(t) \right| \leq \epsilon$$

for all $t \in I$. If we set $f(t, x(t)) = L\ x(t) + \ell(t)$, according to Theorem 3.1, there exists a unique n times continuously differentiable function $y_0(t)$ such that

$$y_0(t) = \sum_{l=0}^{n-1} \frac{t^l x_l}{l!} + \frac{1}{(n-1)!} \int_a^t (t-s)^{n-1}\ \{L\ x(s) - \ell(s)\}\ ds$$

and

$$|x(t) - y_0(t)| \leq \frac{M\ \epsilon}{1 - KL} = H\ \epsilon.$$

for all $t \in I$.

Example 5.3. Assume that K and L be positive constants with $KL < 1$. Let $I = [0, 3K - \epsilon]$ be a closed interval for a positive number $\epsilon < 3K$. For a given polynomial $\ell(t)$, we assume that $x \in C^n(I)$ satisfies the inequality

$$\left| x^{(n)}(t) - L\ x(t) - \ell(t) \right| \leq t^2\ \epsilon$$

for all $t \in I$. If we set $f(t, x(t)) = L\ x(t) + \ell(t)$ and $\phi(t) = t^2$, then the above inequality has the identical form with (4.2). Moreover, we obtain

$$\left| \int_0^t \phi(s)\ ds \right| = \frac{t^3}{3} \leq K\ \phi(t)$$

for all $t \in I$, since $K\ \phi(t) - \dfrac{t^3}{3} \geq 0$ for all $t \in I$. According to Theorem 4.1, there exists a unique n times continuously differentiable function $y_0(t)$ such that

$$y_0(t) = \sum_{l=0}^{n-1} \frac{t^l x_l}{l!} + \frac{1}{(n-1)!} \int_a^t (t-s)^{n-1}\ \{L\ x(s) - \ell(s)\}\ ds$$

and

$$|x(t) - y_0(t)| \leq \frac{M\ K\ \phi(t)\epsilon}{1 - KL} = H\ \phi(t)\epsilon.$$

for all $t \in I$.

Example 5.4. Assume that K and L be positive constants with $KL < 1$. Let $I = [0, 2K - \epsilon]$ be a closed interval for a positive number $\epsilon < 2K$. For a given polynomial $\ell(t)$, we assume that $x \in C^n(I)$ satisfies the inequality

$$\left| x^{(n)}(t) - L\ x(t) - \ell(t) \right| \leq \alpha\ t^n\ \epsilon$$

for all $t \in I$. If we set $f(t, x(t)) = L\ x(t) + \ell(t)$ and $\phi(t) = \alpha\ t^n$, where n is a positive integer and α is a constant, then the above inequality has the identical form with

(4.2). Moreover, we obtain

$$\left|\int_0^t \phi(s)\ ds\right| = \frac{\alpha\ t^{n+1}}{n+1} \leq K\ \phi(t)$$

for all $t \in I$, since $K\ \phi(t) - \dfrac{\alpha\ t^{n+1}}{n+1} \geq 0$ for all $t \in I$. According to Theorem 4.1, there exists a unique n times continuously differentiable function $y_0(t)$ such that

$$y_0(t) = \sum_{l=0}^{n-1} \frac{t^l x_l}{l!} + \frac{1}{(n-1)!} \int_a^t (t-s)^{n-1}\ \{L\ x(s) - \ell(s)\}\ ds$$

and

$$|x(t) - y_0(t)| \leq \frac{M\ K\ \phi(t)\epsilon}{1-KL} = H\ \phi(t)\epsilon.$$

for all $t \in I$.

Example 5.5. Let $\vartheta > 1$ be a constant and η be a constant with $0 < \eta < \ln\vartheta$.Let $I = [0, \infty)$ and $\xi(t)$ be a polynomial. Suppose that $x \in C^n(I)$ satisfies the inequation

$$\left|x^{(n)}(t) - \eta\,(3\ x'''(t) - 2\ x''(t) - 6\ x(t)) - \xi(t)\right| \leq \beta\ \vartheta^t\ \epsilon$$

for all $t \in I$. If we set $f\,(t, [y(t)]) = \eta\,(3\ x'''(t) - 2\ x''(t) - 6\ x(t)) - \xi(t)$ and $\phi(t) = \beta\ \vartheta^t$, β is a constant, then the above inequality has the identical form with (4.12). Moreover, we obtain

$$\left|\int_0^t \phi(s)\ ds\right| = \frac{\beta\ \vartheta^t}{\ln\vartheta} \leq K\ \phi(t)$$

for all $t \in I$, since $K\ \phi(t) - \dfrac{\beta\ \vartheta^t}{\ln\vartheta} \geq 0$ for all $t \in I$. According to Theorem 4.2, there exists a unique n times continuously differentiable function $y_0(t)$ such that

$$y_0(t) = \sum_{l=0}^{n-1} \frac{t^l x_l}{l!} + \frac{1}{(n-1)!} \int_a^t (t-s)^{n-1}\ \{\eta\,(3\ x'''(t) - 2\ x''(t) - 6\ x(t)) - \xi(s)\}\ ds$$

and

$$|x(t) - y_0(t)| \leq \frac{S\ K\ \phi(t)\epsilon}{\ln\vartheta - KJL} = H\ \phi(t)\epsilon.$$

for all $t \in I$.

Declarations

Availablity of data and materials

Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors equally conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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