Journal of Applied Analysis and Computation
Volume 11, Number 2, April 2021, 632–655
Website:http://www.jaac-online.com
DOI:10.11948/20190154

ORLICZ MULTIPLE AFFINE QUERMASSINTEGRALS*

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Abstract In the paper, our main aim is to generalize the mixed affine quermassintegrals of j convex bodies to the Orlicz space. We find a new affine geometric quantity by calculating first-order variation and call it *Orlicz multiple affine quermassintegrals*. The mixed affine quermassintegrals and Aleksandrov-Fenchel inequality for the mixed affine quermassintegrals of j convex bodies are extended to an Orlicz setting. A new Orlicz-Aleksandrov-Fenchel inequality for the mixed affine quermassintegrals of j convex bodies is established. The new Orlicz-Aleksandrov-Fenchel inequality in special cases yield the classical Aleksandrov-Fenchel inequality for mixed volumes, the Aleksandrov-Fenchel inequality for the mixed affine quermassintegrals which is just built, and Zou's Orlicz Minkowski inequality for affine quermassintegrals, respectively. This new concept of L_p-multiple affine quermassintegrals and L_p-Aleksandrov-Fenchel inequality for the L_p-multiple affine quermassintegrals is also derived. Moreover, the Orlicz multiple mixed volumes and the Orlicz-Aleksandrov-Fenchel inequality for the mixed volumes are also included in our new conclusions. As an application, a new Orlicz-Brunn-Minkowski inequality for the mixed affine quermassintegrals of j convex bodies is proved.

Keywords Affine quermassintegral, L_p-addition, Orlicz addition, Orlicz affine quermassintegral, Orlicz multiple mixed volume, Orlicz-Aleksandrov-Fenchel inequality for the mixed volume.

MSC(2010) 46E30, 52A39, 52A40.

1. Introduction

One of the most important operations in geometry is vector addition. As an operation between sets K and L, defined by

$$K + L = \{x + y : x \in K, y \in L\},$$

it is usually called Minkowski addition and combine volume play an important role in the Brunn-Minkowski theory. During the last few decades, the theory has been extended to the L_p-Brunn-Minkowski theory. A set called as L_p-addition, introduced by Firey in [6,7]. Denoted by $+_p$, for $1 \leq p \leq \infty$, defined by

$$h(K +_p L, x)^p = h(K, x)^p + h(L, x)^p, \tag{1.1}$$

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*The author was supported by National Natural Science Foundation of China (10971205, 11371334).

for all $x \in \mathbb{R}^n$ and compact convex sets K and L in $\mathbb{R}^n$ containing the origin. Here the functions are the support functions. If K is a nonempty closed (not necessarily bounded) convex set in $\mathbb{R}^n$, then

$$h(K,x) = \max\{x \cdot y : y \in K\},$$

for $x \in \mathbb{R}^n$, defines the support function $h(K,x)$ of K. A nonempty closed convex set is uniquely determined by its support function. L_p-addition and inequalities are the fundamental and core content in the L_p-Brunn-Minkowski theory. For recent important results and more information from this theory, we refer to [11–14, 19, 22, 24, 27–31, 34, 35, 38, 39, 41, 42] and the references therein. In recent years, a new extension of the L_p-Brunn-Minkowski theory is to Orlicz-Brunn-Minkowski theory, initiated by Lutwak, Yang, and Zhang [32, 33]. Gardner, Hug and Weil [9] introduced first the Orlicz addition, Orlicz mixed volumes and established Orlicz Minkowski's, and Brunn-Minkowski's inequalities. Xi, Jin and Leng [43] have also established the same concepts and arguments by using the symmetry techniques of convex geometric. The other articles advance the theory and its dual theory can be found in literatures [10, 16–18, 20, 21, 36, 40, 44–52, 54, 55]. Gardner, Hug and Weil ([9]) introduced the Orlicz addition $K +_\varphi L$ of compact convex sets K and L in $\mathbb{R}^n$ containing the origin, implicitly, by

$$h(K +_\varphi L, u) = \inf\left\{\lambda > 0 : \varphi\left(\frac{h(K,u)}{\lambda}\right) + \varphi\left(\frac{h(L,u)}{\lambda}\right) \leq 1\right\}, \tag{1.2}$$

where, $u \in S^{n-1}$ for unit vectors, and the surface of the unit ball centered at the origin is S^{n-1}, and $\varphi : [0,\infty) \to (0,\infty)$ is a convex and increasing function such that $\varphi(1) = 1$ and $\varphi(0) = 0$, and let Φ denote the set of convex functions $\varphi : [0,\infty) \to [0,\infty)$ that is increasing and satisfy $\varphi(0) = 0$ and $\varphi(1) = 1$. When $p \geq 1$ and $\varphi(t) = t^p$, the Orlicz addition $K +_\varphi L$ becomes the L_p-addition $K +_p L$. Orlicz multiple mixed volumes of $(n+1)$ convex bodies with respect to the Orlicz addition was introduced by Zhao [50], denoted by $V_\varphi(K_1, \cdots, K_n, L_n)$, defined by

$$V_\varphi(K_1, \cdots, K_n, L_n) := \varphi'_-(1) \cdot \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0^+} V(K_1, \cdots, K_{n-1}, L_n +_\varphi \varepsilon \cdot K_n), \tag{1.3}$$

for $\varphi \in \Phi$ and $K_1, \cdots, K_n$ are convex bodies containing the origin, L_n is a convex body containing the origin in its interior, and $\varphi'_-(1)$ denotes the value of the left derivative of convex function φ at point 1. Here, $V(K_1, \cdots, K_n)$ is the usual mixed volume, defined by (see e.g. [8], p.353),

$$V(K_1, \ldots, K_n) = \frac{1}{n}\int_{S^{n-1}} h(K_n, u)\,dS(K_1, \ldots, K_{n-1}, u),$$

where $S(K_1, \ldots, K_{n-1}, \cdot)$ is a Borel measure on S^{n-1}, and called it mixed surface area measure of $K_1, \ldots, K_{n-1}$, and where $K_1, \ldots, K_{n-1}$ are convex bodies containing the origin.

Lutwak [25] proposed to define the affine quermassintegrals for a convex body K, $\Phi_0(K)$, $\Phi_1(K)$, $\ldots$, $\Phi_n(K)$, by taking $\Phi_0(K) := V(K), \Phi_n(K) := \omega_n$ and for $0 < j < n$,

$$\Phi_{n-j}(K) := \omega_n \left[\int_{G_{n,j}} \left(\frac{\mathrm{vol}_j(K|\xi)}{\omega_j}\right)^{-n} d\mu_j(\xi)\right]^{-1/n}, \tag{1.4}$$

where $G_{n,j}$ denotes the Grassman manifold of j-dimensional subspaces in $\mathbb{R}^n$, and μ_j denotes the gauge Haar measure on $G_{n,j}$, and $\mathrm{vol}_j(K|\xi)$ denotes the j-dimensional volume of the positive projection of K on j-dimensional subspace $\xi \subset \mathbb{R}^n$ and ω_j denotes the volume of j-dimensional unit ball. Lutwak showed the Brunn-Minkowski inequality for the affine quermassintegrals. If K and L are convex bodies and $0 < j < n$, then

$$\Phi_{n-j}(K+L)^{1/j} \geq \Phi_{n-j}(K)^{1/j} + \Phi_{n-j}(L)^{1/j}. \tag{1.5}$$

Lutwak [26] conjectured that

$$\omega_n^j \Phi_i(K)^{n-j} \leq \omega_n^i \Phi_j(K)^{n-i},$$

for $0 \leq i < j < n$ and K is a convex body.

In analogy to (1.4), one may also define mixed affine quermassintegrals of j convex bodies $K_1, \ldots, K_j$, denoted by $\Phi_{n-j}(K_1, \ldots, K_j)$, defined by (see [53])

$$\Phi_{n-j}(K_1, \ldots, K_j) := \omega_n \left[\int_{G_{n,j}} \left(\frac{\mathrm{vol}_j((K_1, \ldots, K_j)|\xi)}{\omega_j} \right)^{-n} d\mu_j(\xi) \right]^{-1/n}, \tag{1.6}$$

where $0 \leq j \leq n$, and $\mathrm{vol}_j((K_1, \ldots, K_j)|\xi)$ denotes $\mathrm{vol}_j(K_1|\xi, \ldots, K_j|\xi)$ is the j-dimensional mixed volume of $K_1|\xi, \ldots, K_j|\xi$, and by letting $\Phi_0(K_1, \ldots, K_j) := V(K_1, \ldots, K_n)$ and $\Phi_n(K_1, \ldots, K_j) := \omega_n$. The related inequalities on the mixed affine quermassintegrals are listed in Section 3.

In the paper, our main aim is to generalize the mixed affine quermassintegrals of j convex bodies to the Orlicz space. In this framework of the Orlicz-Brunn-Minkowski theory, we introduce a new affine geometric quantity by calculating Orlicz first-order variation of the mixed affine quermassintegrals of j convex bodies, and call it Orlicz multiple affine quermassintegrals. The fundamental notions and conclusions of the mixed affine quermassintegrals and Aleksandrov-Fenchel inequality for the mixed affine quermassintegrals of j convex bodies are extended to an Orlicz setting. A new Orlicz-Aleksandrov-Fenchel inequality for the mixed affine quermassintegrals of j convex bodies is established. The new Orlicz-Aleksandrov-Fenchel inequality in special cases yield the classical Aleksandrov-Fenchel inequality for mixed volumes, the Aleksandrov-Fenchel inequality for the mixed affine quermassintegrals which is just built, L_p-Aleksandrov-Fenchel inequality and Zou's Orlicz Minkowski inequality for affine quermassintegrals, respectively. This new concept of L_p-multiple affine quermassintegrals and L_p-Aleksandrov-Fenchel inequality for the L_p-multiple affine quermassintegrals is also derived. As a application, a new Orlicz-Brunn-Minkowski inequality for the mixed affine quermassintegrals of j convex bodies is proved, which implies Orlicz-Brunn-Minkowski inequalities for mixed volumes and quermassintegrals, and Zou's Orlicz Brunn-Minkowski ineqaulity for affine quermassintegrals, respectively.

Following the basic spirit of Aleksandrov [2], Fenchel and Jessen [5] introduction of mixed quermassintegrals, and introduction of Lutwak's L_p-mixed quermassintegrals (see [23]), we are based on the study of the first order Orlicz variational of the affine quermassintegrals. We prove that Orlicz first order variation of mixed affine quermassintegral of j convex bodies can be expressed as: For $\varphi \in \Phi$ and $0 \leq j \leq n$,

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0^+} \Phi_{n-j}(K_1, \ldots, K_{j-1}, L_j +_\varphi \varepsilon \cdot K_j)$$

$$=\frac{1}{\varphi'_{-}(1)}\Phi_{n-j}(K_1,\ldots,K_{j-1},L_j)^{1+n}\Phi_{\varphi,n-j}(K_1,\ldots,K_j,L_j)^{-n}, \tag{1.7}$$

where $K_1,\cdots,K_j$ are convex bodies containing the origin, L_j is a convex body containing the origin in its interior, and $\varphi'_{-}(1)$ denotes the value of left derivative of convex function φ at point 1. Here, $K+_{\varphi}\varepsilon\cdot L$ denotes the Orlicz linear combination of convex bodies K and L. If K,L are convex bodies containing the origin, $\alpha,\beta\geq 0$ and $\varphi\in\Phi$, then Orlicz linear combination of K and L, denoted by $+_{\varphi}(K,L,\alpha,\beta)$, defined by ([9])

$$\alpha\cdot\varphi\left(\frac{h(K,x)}{h(+_{\varphi}(K,L,\alpha,\beta),x)}\right)+\beta\cdot\varphi\left(\frac{h(L,x)}{h(+_{\varphi}(K,L,\alpha,\beta),x)}\right)=1.$$

For $\alpha=1$ and $\beta=\varepsilon\geq 0$, the Orlicz linear combination $+_{\varphi}(K,L,1,\varepsilon)$ denoted by $K+_{\varphi}\varepsilon\cdot L$. In this first order variational equation (1.7), we find a new geometric quantity. Based on this, we extract the required geometric quantity, denoted by $\Phi_{\varphi,n-j}(K_1,\ldots,K_j,L_j)$ and call it Orlicz multiple affine quermassintegral of $(j+1)$ convex bodies $K_1,\ldots,K_j,L_j$, defined by

$$\begin{aligned}&\Phi_{\varphi,n-j}(K_1,\ldots,K_j,L_j)^{-n}\\:=&\frac{\varphi'_{-}(1)}{\Phi_{n-j}(K_1,\ldots,K_{j-1},L_j)^{1+n}}\cdot\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0^+}\Phi_{n-j}(K_1,\ldots,K_{j-1},L_j+_{\varphi}\varepsilon\cdot K_j),\end{aligned} \tag{1.8}$$

where $K_1,\ldots,K_j$ are convex bodies containing the origin, L_j is a convex body containing the origin in its interior, $0\leq j\leq n$ and $\varphi\in\Phi$. We prove also the new affine geometric quantity $\Phi_{\varphi,n-j}(K_1,\ldots,K_j,L_j)$ has an integral representation.

$$\begin{aligned}&\Phi_{\varphi,n-j}(K_1,\ldots,K_j,L_j)^{-n}\\=&\omega_n^{-n}\int_{G_{n,j}}\frac{V_{\varphi}^{(j)}((K_1,\ldots,K_j,L_j)|\xi)}{\mathrm{vol}_j((K_1,\ldots,K_{j-1},L_j)|\xi)}\left(\frac{\mathrm{vol}_j((K_1,\ldots,K_{j-1},L_j)|\xi)}{\omega_j}\right)^{-n}d\mu_j(\xi),\end{aligned} \tag{1.9}$$

where $\varphi\in\Phi$, $0\leq j\leq n$, and $K_1,\cdots,K_j$ are convex bodies containing the origin, L_j is a convex body containing the origin in its interior, and $V_{\varphi}^{(j)}((K_1,\ldots,K_j,L_j)|\xi)$ denotes the j-dimensional Orlicz multiple mixed volume of $K_1|\xi,\ldots,K_j|\xi,L_j|\xi$ (see [50], Definition 4.1). We show the affine invariance of Orlicz multiple affine quermassintegrals. For $0\leq j\leq n$, $\varphi\in\Phi$ and $g\in\mathrm{SL(n)}$,

$$\Phi_{\varphi,n-j}(gK_1,\ldots,gK_j,gL_j)=\Phi_{\varphi,n-j}(K_1,\ldots,K_j,L_j), \tag{1.10}$$

where $K_1,\cdots,K_j$ are convex bodies containing the origin, and L_j is a convex body containing the origin in its interior.

Because the Orlicz multiple affine quermassintegrals are extensions of the mixed affine quermassintegrals of j convex bodies and the Orlicz multiple mixed volumes, a very natural question is raised: is there a Aleksandrov-Fenchel type inequality for the Orlicz multiple affine quermassintegrals? in the Section 4, we give a positive answer to this question and establish an Orlicz-Aleksandrov-Fenchel inequality for the new affine geometric quantity.

Orlicz-Aleksandrov-Fenchel inequality for mixed affine quermassintegrals of j convex bodies *Let $\varphi\in\Phi$, $0\leq j\leq n$ and $0<r\leq j$. If $K_1,\cdots,K_j$*

are convex bodies containing the origin, L_j is a convex body containing the origin in its interior, then for $\varepsilon > 0$

$$\left(\frac{\Phi_{\varphi,n-j}(K_1,\ldots,K_j,L_j)}{\Phi_{n-j}(K_1,\ldots,K_{j-1},L_j)}\right)^{-n} \geq \varphi\left(\frac{\prod_{i=1}^{r}\Phi_{n-j}(K_i,\cdots,K_i,K_{r+1},\cdots,K_j)^{1/r}}{\Phi_{n-j}(K_1,\ldots,K_{j-1},L_j)}\right). \tag{1.11}$$

Obviously, the following classical Aleksandrov-Fenchel inequality (see [3]) for mixed volumes is a special case of (1.11).

$$V(K_1,\cdots,K_n) \geq \prod_{i=1}^{r} V(K_i\ldots,K_i,K_{r+1},\ldots,K_n)^{\frac{1}{r}}.$$

This new inequality in special case which yields also the following Orlicz-Aleksandrov-Fenchel inequality, which was recently established by Zhao [50]. If $K_1,\cdots,K_n$ are convex bodies containing the origin, L_n is a convex body containing the origin in its interior, $1 \leq r \leq n$ and $\varphi \in \Phi$, then for $\varepsilon > 0$

$$V_\varphi(K_1,\cdots,K_n,L_n) \geq V(K_1,\cdots,K_{n-1},L_n) \times \varphi\left(\frac{\prod_{i=1}^{r} V(K_i\ldots,K_i,K_{r+1},\ldots,K_n)^{\frac{1}{r}}}{V(K_1,\cdots,K_{n-1},L_n)}\right). \tag{1.12}$$

An important special case of (1.11) is the following result. If $0 \leq i < n$, $\varphi \in \Phi$, and K is a convex body containing the origin in its interior, L is a convex body containing the origin, then

$$W_{\varphi,i}(K,L) \geq W_i(K)\cdot\varphi\left(\left(\frac{W_i(L)}{W_i(K)}\right)^{1/(n-i)}\right). \tag{1.13}$$

If φ is strictly convex, equality holds if and only if K and L are homothetic. Here $W_i(K)$ denotes the usual quermassintegral and $W_{\varphi,i}(K,L)$ is the Orlicz mixed quermassintegral of K and L, defined by (see [46])

$$W_{\varphi,i}(K,L) = \frac{1}{n}\int_{S^{n-1}} \varphi\left(\frac{h(L,u)}{h(K,u)}\right) h(K,u)dS_i(K,u), \tag{1.14}$$

where $0 \leq i < n$.

Obviously, Zou's [55] the following result is a simpler special case of (1.11).

$$\left(\frac{\Phi_{\varphi,n-j}(K,L)}{\Phi_{n-j}(K)}\right)^{-n} \geq \varphi\left(\left(\frac{\Phi_{n-j}(L)}{\Phi_{n-j}(K)}\right)^{1/j}\right). \tag{1.15}$$

If φ is strictly convex, equality holds if and only if K and L are homothetic, where $\Phi_{\varphi,n-j}(K,L) = \Phi_{\varphi,n-j}(\underbrace{K,\ldots,K}_{j-1},L,K)$. Moreover, the Orlicz Minkowski inequality for i-th mixed affine quemassintegrals (see [46]) is also a special case of (1.11).

In the Section 5, we establish an Orlicz-Brunn-Minkoswki inequality for the mixed affine quermassintegrals.

Orlicz-Bruun-Minkowski inequality for mixed affine quermassintegrals of j convex bodies *Let $\varphi \in \Phi$ and $0 \leq j \leq n$. If $K_1,\cdots,K_j$ are convex bodies* *containing the origin, L_j is a convex body containing the origin in its interior, then for $\varepsilon > 0$*

$$1 \geq \varphi\left(\frac{\Phi_{n-j}(K_1,\ldots,K_{j-1},L_j)}{\Phi_{n-j}(K_1,\ldots,K_{j-1},L_j +_\varphi \varepsilon\cdot K_j)}\right) +\varepsilon\cdot\varphi\left(\frac{\Phi_{n-j}(K_1,\ldots,K_j)}{\Phi_{n-j}(K_1,\ldots,K_{j-1},L_j +_\varphi \varepsilon\cdot K_j)}\right). \tag{1.16}$$

If φ is strictly convex, equality holds if and only if K_j and L_j are homothetic.

This in special case which yields the following Orlicz-Brunn-Minkoswki inequality for mixed volumes, which was recently established by Zhao [50]. If $K_1,\cdots,K_j$ are convex bodies containing the origin, L_j is a convex body containing the origin in its interior, and $\varphi\in\Phi$, then for $\varepsilon>0$

$$1 \geq \varphi\left(\frac{V(K_1,\ldots,K_{n-1},L_n)}{V(K_1,\ldots,K_{n-1},L_n +_\varphi \varepsilon\cdot K_n)}\right)+\varepsilon\cdot\varphi\left(\frac{V(K_1,\ldots,K_n)}{V(K_1,\ldots,K_{n-1},L_n +_\varphi \varepsilon\cdot K_n)}\right). \tag{1.17}$$

If φ is strictly convex, equality holds if and only if K_n and L_n are homothetic.

An important special case of (1.16) is the following: If K, L are convex bodies containing the origin, $\varphi\in\Phi$ and $0\leq i<n$, then

$$1 \geq \varphi\left(\left(\frac{W_i(K)}{W_i(K +_\varphi L)}\right)^{1/(n-i)}\right) + \varphi\left(\left(\frac{W_i(L)}{W_i(K +_\varphi L)}\right)^{1/(n-i)}\right).$$

If φ is strictly convex, equality holds if and only if K and L are homothetic.

Obviously, Zou's [55] the following result is also a simpler special case of (1.16).

$$1 \geq \varphi\left(\left(\frac{\Phi_{n-j}(K)}{\Phi_{n-j}(K +_\varphi L)}\right)^{1/j}\right) + \varphi\left(\left(\frac{\Phi_{n-j}(L)}{\Phi_{n-j}(K +_\varphi L)}\right)^{1/j}\right). \tag{1.18}$$

If φ is strictly convex, equality holds if and only if K and L are homothetic. Moreover, the Orlicz-Brunn-Minkowski inequality for i-th mixed affine quemassintegrals (see [46]) is also a special case of (1.16).

2. Preliminaries

The setting for this paper is n-dimensional Euclidean space $\mathbb{R}^n$. Let $\mathcal{K}^n$ denote the set of convex bodies (compact convex subsets with nonempty interiors) in $\mathbb{R}^n$, let $\mathcal{K}^n_o$ be the class of members of $\mathcal{K}^n$ containing the origin, and let $\mathcal{K}^n_{oo}$ be those sets in $\mathcal{K}^n$ containing the origin in their interiors. We reserve the letter $u\in S^{n-1}$ for unit vectors, and the letter B for the unit ball centered at the origin. The surface of B is S^{n-1}. For a compact set K, we write $V(K)$ for the (n-dimensional) Lebesgue measure of K and call this the volume of K. If K is a nonempty closed (not necessarily bounded) convex set, then

$$h(K,x)=\sup\{x\cdot y : y\in K\},$$

for $x\in\mathbb{R}^n$, defines the support function of K, where $x\cdot y$ denotes the usual inner product x and y in $\mathbb{R}^n$. A nonempty closed convex set is uniquely determined by its support function. Support function is homogeneous of degree 1, that is,

$$h(K,rx)=rh(K,x),$$

for all $x \in \mathbb{R}^n$ and $r \geq 0$ (see e.g. [3]). Let d denote the Hausdorff metric on $\mathcal{K}^n$, i.e., for $K, L \in \mathcal{K}^n$,

$$d(K, L) = |h(K, u) - h(L, u)|_\infty,$$

where $|\cdot|_\infty$ denotes the sup-norm on the space of continuous functions $C(S^{n-1})$. Let $K \subset \mathbb{R}^n$ be a nonempty closed convex set. If ξ is a subspace of $\mathbb{R}^n$, then it is easy to show that

$$h(K|\xi, x) = h(K, x|\xi),$$

for $x \in \mathbb{R}^n$. The formula (see [8], p.18)

$$h(AK, x) = h(K, A^t x), \tag{2.1}$$

for $x \in \mathbb{R}^n$ and a linear transformation $A : \mathbb{R}^n \to \mathbb{R}^n$, gives the change in a support function under A, where A^t denotes the transpose of A. Equation (2.1) is proved in [8], p.18] for compact sets and $A \in GL(n)$, but the proof is the same if K is unbounded or A is singular.

2.1 Mixed volumes

If $K_i \in \mathcal{K}^n$ $(i = 1, 2, \ldots, r)$ and λ_i $(i = 1, 2, \ldots, r)$ are nonnegative real numbers, then of fundamental importance is the fact that the volume of $\sum_{i=1}^r \lambda_i K_i$ is a homogeneous polynomial in λ_i given by (see e.g. [37])

$$V(\lambda_1 K_1 + \cdots + \lambda_n K_n) = \sum_{i_1, \ldots, i_n} \lambda_{i_1} \ldots \lambda_{i_n} V_{i_1 \ldots i_n}, \tag{2.2}$$

where the sum is taken over all n-tuples $(i_1, \ldots, i_n)$ of positive integers not exceeding r. The coefficient $V_{i_1 \ldots i_n}$ depends only on the bodies $K_{i_1}, \ldots, K_{i_n}$ and is uniquely determined by (2.2), it is called the mixed volume of $K_i, \ldots, K_{i_n}$, and is written as $V(K_{i_1}, \ldots, K_{i_n})$. Let $K_1 = \ldots = K_{n-i} = K$ and $K_{n-i+1} = \ldots = K_n = L$, then the mixed volume $V(K_1, \ldots, K_n)$ is written as $V_i(K, L)$. If $K_1 = \cdots = K_{n-i} = K$, $K_{n-i+1} = \cdots = K_n = B$ The mixed volumes $V_i(K, B)$ is written as $W_i(K)$ and call as quermassintegrals (or ith mixed quermassintegrals) of K. We write $W_i(K, L)$ for the mixed volume $V(K, \cdots, K, \underbrace{B, \cdots, B}_{i}, L)$ and call as mixed quermassintegrals of K and L. Aleksandrov [1] and Fenchel and Jessen [5] (also see Busemann [4] and Schneider [37]) have shown that for $K \in \mathcal{K}^n$, and $i = 0, 1, \ldots, n-1$, there exists a regular Borel measure $S_i(K, \cdot)$ on S^{n-1}, such that the mixed quermassintegrals $W_i(K, L)$ has the following representation:

$$W_i(K, L) = \frac{1}{n-i} \lim_{\varepsilon \to 0^+} \frac{W_i(K + \varepsilon \cdot L) - W_i(K)}{\varepsilon} = \frac{1}{n} \int_{S^{n-1}} h(L, u) dS_i(K, u). \tag{2.3}$$

Associated with $K_1, \ldots, K_{n-1} \in \mathcal{K}^n$ is a Borel measure $S(K_1, \ldots, K_{n-1}, \cdot)$ on S^{n-1}, called the mixed surface area measure of $K_1, \ldots, K_{n-1}$, which has the property that for each $K \in \mathcal{K}^n$ (see e.g. [8], p.353),

$$V(K_1, \ldots, K_{n-1}, K) = \frac{1}{n} \int_{S^{n-1}} h(K, u) dS(K_1, \ldots, K_{n-1}, u). \tag{2.4}$$

In fact, the measure $S(K_1, \ldots, K_{n-1}, \cdot)$ can be defined by the propter that (2.4) holds for all $K \in \mathcal{K}^n$. Let $K_1 = \ldots = K_{n-i-1} = K$ and $K_{n-i} = \ldots = K_{n-1} = L$, then the mixed surface area measure $S(K_1,\ldots,K_{n-1},\cdot)$ is written as $S_i(K,L,\cdot)$. When $L=B$, $S_i(K,B,\cdot)$ is written as $S_i(K,\cdot)$ and called as ith mixed surface area measure. A fundamental inequality for mixed quermassintegrals stats that: If $K,L\in\mathcal{K}^n$ and $0\leq i<n-1$, then

$$W_i(K,L)^{n-i}\geq W_i(K)^{n-i-1}W_i(L), \tag{2.5}$$

with equality if and only if K and L are homothetic and $L=\{o\}$. Good general references for this material are [4, 9].

2.2 p-mixed quermassintegrals

Mixed quermassintegrals are the first variation of the ordinary quermassintegrals, with respect to Minkowski addition. The p-mixed quermassintegrals $W_{p,0}(K,L)$, $W_{p,1}(K,L),\ldots,W_{p,n-1}(K,L)$, as the first variation of the ordinary quermassintegrals, with respect to Firey addition: For $K,L\in\mathcal{K}^n_o$, and real $p\geq 1$, defined by (see e.g. [23])

$$W_{p,i}(K,L)=\frac{p}{n-i}\lim_{\varepsilon\to 0^+}\frac{W_i(K+_p\varepsilon\cdot L)-W_i(K)}{\varepsilon}. \tag{2.6}$$

The mixed p-quermassintegrals $W_{p,i}(K,L)$, for all $K,L\in\mathcal{K}^n_o$, has the following integral representation:

$$W_{p,i}(K,L)=\frac{1}{n}\int_{S^{n-1}}h(L,u)^p dS_{p,i}(K,u), \tag{2.7}$$

where $S_{p,i}(K,\cdot)$ denotes the Boel measure on S^{n-1}. The measure $S_{p,i}(K,\cdot)$ is absolutely continuous with respect to $S_i(K,\cdot)$, and has Radon-Nikodym derivative

$$\frac{dS_{p,i}(K,\cdot)}{dS_i(K,\cdot)}=h(K,\cdot)^{1-p}, \tag{2.8}$$

where $S_i(K,\cdot)$ is a regular Boel measure on S^{n-1}. The measure $S_{n-1}(K,\cdot)$ is independent of the body K, and is just ordinary Lebesgue measure, S, on S^{n-1}. $S_i(B,\cdot)$ denotes the i-th surface area measure of the unit ball in $\mathbb{R}^n$. In fact, $S_i(B,\cdot)=S$ for all i. The surface area measure $S_0(K,\cdot)$ just is $S(K,\cdot)$. When $i=0$, $S_{p,i}(K,\cdot)$ is written as $S_p(K,\cdot)$ (see [29, 30]). A fundamental inequality for mixed p-quermassintegrals stats that: For $K,L\in\mathcal{K}^n_o, p>1$ and $0\leq i<n-1$,

$$W_{p,i}(K,L)^{n-i}\geq W_i(K)^{n-i-p}W_i(L)^p, \tag{2.9}$$

with equality if and only if K and L are homothetic. L_p-Brunn-Minkowski inequality for the quermassintegrals established by Lutwak [23]. If $K,L\in\mathcal{K}^n_o$ and $p\geq 1$ and $0\leq i\leq n$, then

$$W_i(K+_p L)^{p/(n-i)}\geq W_i(K)^{p/(n-i)}+W_i(L)^{p/(n-i)}, \tag{2.10}$$

with equality if and only if K and L are homothetic or $L=\{o\}$. Obviously, putting $i=0$ in (2.7), the mixed p-quermassintegrals $W_{p,i}(K,L)$ become the well-known L_p-mixed volume $V_p(K,L)$, defined by (see e.g. [29])

$$V_p(K,L)=\frac{1}{n}\int_{S^{n-1}}h(L,u)^p dS_p(K,u). \tag{2.11}$$

2.3 Orlicz addition

Throughout the paper, let Φ_m, $m \in \mathbb{N}$, denote the set of convex functions $\varphi : [0,\infty)^m \to [0,\infty)$ that are strictly increasing in each variable and satisfy $\varphi(0,\ldots,0) = 0$ and $\varphi(0,\ldots,1,\ldots,0) = 1$. When $m = 1$, we shall write Φ instead of Φ_1.

Let $m \geq 2, \varphi \in \Phi_m$, $K_j \in \mathcal{K}_o^n$ and $j = 1,\ldots,m$, the Orlicz addition of $K_1,\ldots,K_m$, denoted by $+_\varphi(K_1,\ldots,K_m)$, is defined by ([9])

$$h(+_\varphi(K_1,\ldots,K_m),x) = \inf\left\{\lambda > 0 : \varphi\left(\frac{h(K_1,x)}{\lambda},\ldots,\frac{h(K_m,x)}{\lambda}\right) \leq 1\right\}, \quad (2.12)$$

for $x \in \mathbb{R}^n$.

Equivalently, the Orlicz radial addition $+_\varphi(K_1,\ldots,K_m)$ can be defined implicitly (and uniquely) by

$$\varphi\left(\frac{h(K_1,x)}{h(+_\varphi(K_1,\ldots,K_m),x)},\ldots,\frac{h(K_m,x)}{h(+_\varphi(K_1,\ldots,K_m),x)}\right) = 1, \quad (2.13)$$

if $h(K_1,x) + \cdots + h(K_m,x) > 0$ and by $h(+_\varphi(K_1,\ldots,K_m),x) = 0$, if $h(K_1,x) = \cdots = h(K_m,x) = 0$, for all $x \in \mathbb{R}^n$.

An important special case is obtained when

$$\varphi(x_1,\ldots,x_m) = \sum_{j=1}^m \varphi(x_j), \quad (2.14)$$

for some fixed $\varphi \in \Phi$ such that $\varphi(1) = 1$, and in this case write $+_\varphi(K_1,\ldots,K_m) = K_1 +_\varphi \cdots +_\varphi K_m$. This means that $K_1 +_\varphi \cdots +_\varphi K_m$ is defined either by

$$h(K_1 +_\varphi \cdots +_\varphi K_m,x) = \inf\left\{\lambda > 0 : \sum_{j=1}^m \varphi\left(\frac{h(K_j,x)}{\lambda}\right) \leq 1\right\}, \quad (2.15)$$

for all $x \in \mathbb{R}^n$, or by the corresponding special case of (2.13). From (2.15), it follows easy that

$$\sum_{j=1}^m \varphi\left(\frac{h(K_j,x)}{\lambda}\right) = 1,$$

if and only if

$$\lambda = h(K_1 +_\varphi \cdots +_\varphi K_m,x).$$

The Orlicz addition $+_\varphi$ is continuous, monotonic, $GL(n)$ covariant and projection covariant. The Orlicz linear combination was defined by (see [9])

$$h(+_\varphi(K_1,\ldots,K_m,\alpha_1,\ldots,\alpha_m),x) = \inf\left\{\lambda > 0 : \sum_{j=1}^m \alpha_j\varphi\left(\frac{h(K_j,x)}{\lambda}\right) \leq 1\right\},$$

for all $x \in \mathbb{R}^n$, where, $K_1,\ldots,K_m \in \mathcal{K}_o^n$, $\alpha_j \geq 0$ and $\varphi \in \Phi$.

3. Orlicz multiple affine querlmassintegrals

In order to define the Orlicz multiple affine querlmassintegrals, we need recall the Orlicz multiple mixed volumes and define mixed affine quermassintegrals of j convex bodies.

Definition 3.1 (see [50]). (Orlicz multiple mixed volumes) For $\varphi \in \Phi$, $K_1, \ldots, K_n \in \mathcal{K}_o^n$ and $L_n \in \mathcal{K}_{oo}^n$, the Orlicz multiple mixed volume of $(n+1)$ convex bodies $K_1, \ldots, K_n, L_n$, denoted by $V_\varphi(K_1, \cdots, K_n, L_n)$, defined by

$$V_\varphi(K_1, \cdots, K_n, L_n) := \frac{1}{n}\int_{S^{n-1}} \varphi\left(\frac{h(K_n, u)}{h(L_n, u)}\right) h(L_n, u) dS(K_1, \ldots, K_{n-1}; u). \tag{3.1}$$

Lemma 3.1. *If $K_1, \ldots, K_n \in \mathcal{K}_o^n$, $L_n \in \mathcal{K}_{oo}^n$ and $\varphi \in \Phi$, then for $\varepsilon > 0$*

$$\begin{aligned} V(K_1, \cdots, K_{n-1}, L_n +_\varphi \varepsilon \cdot K_n) &= V_\varphi(K_1, \cdots, K_n, L_n +_\varphi \varepsilon \cdot K_n) \\ &\quad + \varepsilon \cdot V_\varphi(K_1, \cdots, K_{n-1}, L_n, L_n +_\varphi \varepsilon \cdot K_n). \end{aligned} \tag{3.2}$$

Lemma 3.2. *If $K_1, \cdots, K_n \in \mathcal{K}_o^n$, $L_n \in \mathcal{K}_{oo}^n$ and $\varphi \in \Phi$, then*

$$\frac{V_\varphi(K_1, \cdots, K_n, L_n)}{V(K_1, \cdots, K_{n-1}, L_n)} \geq \varphi\left(\frac{V(K_1, \cdots, K_n)}{V(K_1, \cdots, K_{n-1}, L_n)}\right). \tag{3.3}$$

If φ is strictly convex, equality holds if and only if K_n and L_n are homothetic.

Lemma 3.3. *If $\varphi \in \Phi$, $K_1, \ldots, K_n \in \mathcal{K}_o^n$ and $L_n \in \mathcal{K}_{oo}^n$, then for $\varepsilon > 0$*

$$\begin{aligned} &\varphi'_-(1) \cdot \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0^+} V(K_1, \cdots, K_{n-1}, L_n +_\varphi \varepsilon \cdot K_n) \\ &= \frac{1}{n}\int_{S^{n-1}} \varphi\left(\frac{h(K_n, u)}{h(L_n, u)}\right) h(L_n, u) dS(K_1, \ldots, K_{n-1}; u). \end{aligned} \tag{3.4}$$

Lemmas 3.1-3.3 have been published in reference [50].

Lemma 3.4 ([9]). *If $K, L \in \mathcal{K}_o^n$ and $\varphi \in \Phi$, then for $\varepsilon > 0$*

$$K +_\varphi \varepsilon \cdot L \to K, \tag{3.5}$$

in the Hausdorff metric as $\varepsilon \to 0^+$.

Lemma 3.5 ([9]). *If $K, L \in \mathcal{K}_o^n$, $\varepsilon > 0$ and $\varphi \in \Phi$, then*

$$(K +_\varphi \varepsilon \cdot L)|\xi = K|\xi +_\varphi \varepsilon \cdot L|\xi. \tag{3.6}$$

Here, for the following statement, we list first the definition of the mixed affine quermassintegrals of j convex bodies.

Definition 3.2 ([53]). (The mixed affine quermassintegrals of j convex bodies) The mixed affine quermassintegral of j convex bodies $K_1, \ldots, K_j$, denoted by $\Phi_{n-j}(K_1, \ldots, K_j)$, defined by

$$\Phi_{n-j}(K_1, \ldots, K_j) := \omega_n \left[\int_{G_{n,j}} \left(\frac{\mathrm{vol}_j((K_1, \ldots, K_j)|\xi)}{\omega_j}\right)^{-n} d\mu_j(\xi)\right]^{-1/n}, \tag{3.7}$$

where $0 \leq j \leq n$.

When $K_1 = \cdots = K_j = K$, $\Phi_{n-j}(K_1, \ldots, K_j)$ becomes Lutwak's affine quermassintegral $\Phi_{n-j}(K)$. When $K_1 = \cdots = K_{j-1} = K$ and $K_j = L$, $\Phi_{n-j}(K_1, \ldots, K_j)$ becomes a new affine geometric quantity, denoted by $\Phi_{n-j}(K, L)$ and call it mixed affine quermassintegral of K and L.

In order to define the Orlicz multiple affine quermassintegrals, we need calculate the first-order variation of the mixed affine quermassintegrals of j convex bodies.

Lemma 3.6. *Let $\varphi \in \Phi$ and $0 \le j \le n$. If $K_1, \ldots, K_j \in \mathcal{K}_o^n$ and $L_j \in \mathcal{K}_{oo}^n$, then for $\varepsilon > 0$*

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0^+} \Phi_{n-j}(K_1, \ldots, K_{j-1}, L_j +_\varphi \varepsilon \cdot K_j) = \frac{1}{\varphi'_-(1)} \Phi_{n-j}(K_1, \ldots, K_{j-1}, L_j)^{1+n} \times \Phi_{\varphi, n-j}(K_1, \ldots, K_j, L_j)^{-n}. \quad (3.8)$$

Proof. From (3.1), Lemma 3.3 and Lemma 3.4, we have

$$\begin{aligned}
&\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0^+} \int_{G_{n,j}} \mathrm{vol}_j((K_1, \ldots, K_{j-1}, L_j +_\varphi \varepsilon \cdot K_j)|\xi)^{-n} d\mu_j(\xi) \\
&= \lim_{\varepsilon \to 0^+} \int_{G_{n,j}} \frac{\mathrm{vol}_j((K_1, \ldots, K_{j-1}, L_j +_\varphi \varepsilon \cdot K_j)|\xi)^{-n} - \mathrm{vol}_j((K_1, \ldots, K_{j-1}, L_j)|\xi)^{-n}}{\varepsilon} d\mu_j(\xi) \\
&= -n \int_{G_{n,j}} \Bigg(\mathrm{vol}_j((K_1, \ldots, K_{j-1}, L_j)|\xi)^{-n-1} \\
&\quad \times \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0^+} \mathrm{vol}_j((K_1, \ldots, K_{j-1}, L_j +_\varphi \varepsilon \cdot K_j)|\xi) \Bigg) d\mu_j(\xi) \\
&= \frac{-n}{\varphi'_-(1)} \int_{G_{n,j}} \mathrm{vol}_j((K_1, \ldots, K_{j-1}, L_j)|\xi)^{-n-1} \cdot V_\varphi^{(j)}((K_1, \ldots, K_j, L_j)|\xi) d\mu_j(\xi).
\end{aligned} \tag{3.9}$$

On the other hand, from (1.9), (3.6), (3.7) and (3.9), we obtain

$$\begin{aligned}
&\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0^+} \Phi_{n-j}(K_1, \ldots, K_{j-1}, L_j +_\varphi \varepsilon \cdot K_j) \\
&= \frac{\omega_n}{\omega_j} \cdot \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0^+} \left[\int_{G_{n,j}} \mathrm{vol}_j((K_1, \ldots, K_{j-1}, L_j +_\varphi \varepsilon \cdot K_j)|\xi)^{-n} d\mu_j(\xi) \right]^{-1/n} \\
&= \frac{\omega_n}{-n\omega_j} \left(\int_{G_{n,j}} \mathrm{vol}_j((K_1, \ldots, K_{j-1}, L_j)|\xi)^{-n} d\mu_j(\xi) \right)^{-(n+1)/n} \\
&\quad \times \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0^+} \int_{G_{n,j}} \mathrm{vol}_j((K_1, \ldots, K_{j-1}, L_j +_\varphi \varepsilon \cdot K_j)|\xi)^{-n} d\mu_j(\xi) \\
&= \frac{\omega_n}{\varphi'_-(1)\omega_j} \left(\int_{G_{n,j}} \mathrm{vol}_j((K_1, \ldots, K_{j-1}, L_j)|\xi)^{-n} d\mu_j(\xi) \right)^{-(n+1)/n} \\
&\quad \times \int_{G_{n,j}} \mathrm{vol}_j((K_1, \ldots, K_{j-1}, L_j)|\xi)^{-n-1} V_\varphi^{(j)}((K_1, \ldots, K_j, L_j)|\xi) d\mu_j(\xi) \\
&= \frac{1}{\varphi'_-(1)} \Phi_{n-j}(K_1, \ldots, K_{j-1}, L_j)^{1+n} \Phi_{\varphi, n-j}(K_1, \ldots, K_j, L_j)^{-n}.
\end{aligned}$$

□

Definition 3.3 (Orlicz multiple affine querlmassintegrals)**.** Let $\varphi \in \Phi$ and $0 \le j \le n$. If $K_1, \ldots, K_j \in \mathcal{K}_o^n$ and $L_j \in \mathcal{K}_{oo}^n$, the Orlicz multiple affine querlmassintegral of $K_1, \ldots, K_j, L_j$, denoted by $\Phi_{\varphi,n-j}(K_1, \ldots, K_j, L_j)$, defined by

$$\Phi_{\varphi,n-j}(K_1,\ldots,K_j,L_j) := \omega_n \left[\int_{G_{n,j}} \frac{V_\varphi^{(j)}((K_1,\ldots,K_j,L_j)|\xi)}{\mathrm{vol}_j((K_1,\ldots,K_{j-1},L_j)|\xi)} \times \left(\frac{\mathrm{vol}_j((K_1,\ldots,K_{j-1},L_j)|\xi)}{\omega_j} \right)^{-n} d\mu_j(\xi) \right]^{-1/n}. \quad (3.10)$$

When $K_1 = \cdots = K_{j-1} = K$, $K_j = L$ and $L_j = K$, writing $\Phi_{\varphi,n-j}(K_1, \ldots, K_j, L_j)$ as $\Phi_{\varphi,n-j}(K, L)$ and call it Orlicz mixed affine querlmass-integral of K and L. When $K_1 = \cdots = K_{j-i-1} = K$, $K_{j-i} = L, K_{j-i+1} = \cdots = K_j = B$, and $L_j = K$, where $0 \le i < j \le n$, writing $\Phi_{\varphi,n-j}(K_1, \ldots, K_j, L_j)$ as $\Phi_{\varphi,n-j,i}(K, L)$ and call it i-th Orlicz mixed affine querlmassintegral of K and L.

Specifically, for $j = n$, we agreed:

$$\Phi_{\varphi,0}(K_1,\ldots,K_n,L_n) = \left(\frac{V_\varphi(K_1,\ldots,K_n,L_n)}{V(K_1,\ldots,K_{n-1},L_n)} \right)^{-1/n} V(K_1,\ldots,K_{n-1},L_n).$$

Lemma 3.7. *Let $\varphi \in \Phi$ and $0 \le j \le n$. If $K_1, \ldots, K_j \in \mathcal{K}_o^n$, then*

$$\Phi_{\varphi,n-j}(K_1,\ldots,K_{j-1},K_j,K_j) = \Phi_{n-j}(K_1,\ldots,K_j). \quad (3.11)$$

Proof. From the Definitions 3.1, 3.2 and 3.3, (3.11) easy follows. □

Remark 3.1. When $\varphi(t) = t^p, 1 \le p < \infty$, we write $\Phi_{\varphi,n-j}(K_1, \ldots, K_j, L_j)$ as $\Phi_{p,n-j}(K_1, \ldots, K_j, L_j)$, and call it L_p-multiple affine quermassintegral of K and L, and

$$\Phi_{p,n-j}(K_1,\ldots,K_j,L_j) = \omega_n \left[\int_{G_{n,j}} \frac{V_p^{(j)}((K_1,\ldots,K_j,L_j)|\xi)}{\mathrm{vol}_j((K_1,\ldots,K_{j-1},L_j)|\xi)} \times \left(\frac{\mathrm{vol}_j((K_1,\ldots,K_{j-1},L_j)|\xi)}{\omega_j} \right)^{-n} d\mu_j(\xi) \right]^{-1/n},$$

where $V_p^{(j)}((K_1, \ldots, K_j, L_j)|\xi)$ denotes the j-dimensional L_p-multiple mixed volumes (see [50]).

When $K_1 = \cdots = K_{j-1} = K$, $K_j = L$ and $L_j = K$, writing $\Phi_{p,n-j}(K_1, \ldots, K_j, L_j)$ as $\Phi_{p,n-j}(K, L)$ and call it L_p mixed affine querlmassintegral of K and L. When $K_1 = \cdots = K_{j-i-1} = K$, $K_{j-i} = \cdots = K_{j-1} = B$, $K_j = L$ and $L_j = K$, where $0 \le i < j \le n$, writing $\Phi_{p,n-j}(K_1, \ldots, K_j, L_j)$ as $\Phi_{p,n-j,i}(K, L)$ and call it i-th L_p mixed affine querlmassintegral of K and L.

Lemma 3.8 ([9])**.** *If $K, L \in \mathcal{K}_o^n$, $\varphi \in \Phi$ and any $g \in \mathrm{SL(n)}$, then for $\varepsilon > 0$*

$$g(K +_\varphi \varepsilon \cdot L) = (gK) +_\varphi \varepsilon \cdot (gL). \quad (3.12)$$

In the following, we prove that the Orlicz multiple affine querlmassintegral is invariant under simultaneous unimodular centro-affine transformation.

Lemma 3.9. *Let $\varphi \in \Phi$ and $0 \le j \le n$. If $K_1, \ldots, K_j \in \mathcal{K}_o^n$ and $L_j \in \mathcal{K}_{oo}^n$, then for any $g \in \mathrm{SL(n)}$*

$$\Phi_{\varphi,n-j}(gK_1, \ldots, gK_j, gL_j) = \Phi_{\varphi,n-j}(K_1, \ldots, K_j, L_j).$$

Proof. From Lemma 3.6 and Lemma 3.8, we have for $g \in \mathrm{SL(n)}$,

$$\begin{aligned}
&\Phi_{\varphi,n-j}(gK_1, \ldots, gK_j, gL_j)\\
=&\left(\frac{\varphi'_-(1)}{\Phi_{n-j}(gK_1,\ldots,gK_j)^{1+n}} \cdot \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0^+} \Phi_{n-j}(gK_1,\ldots,gK_j, gK_j +_\varphi \varepsilon \cdot gL_j)\right)^{-1/n}\\
=&\left(\frac{\varphi'_-(1)}{\Phi_{n-j}(gK_1,\ldots,gK_j)^{1+n}} \cdot \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0^+} \Phi_{n-j}(gK_1,\ldots,gK_j, g(K_j +_\varphi \varepsilon \cdot L_j))\right)^{-1/n}\\
=&\left(\frac{\varphi'_-(1)}{\Phi_{n-j}(K_1,\ldots,K_j)^{1+n}} \cdot \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0^+} \Phi_{n-j}(K_1,\ldots,K_j, K_j +_\varphi \varepsilon \cdot L_j)\right)^{-1/n}\\
=&\Phi_{\varphi,n-j}(K_1,\ldots,K_j,L_j).
\end{aligned}$$

□

Lemma 3.10 (Jensen's inequality)**.** *Let μ be a probability measure on a space X and $g : X \to I \subset \mathbb{R}$ is a μ-integrable function, where I is a possibly infinite interval. If $\phi : I \to \mathbb{R}$ is a convex function, then*

$$\int_X \phi(g(x))d\mu(x) \ge \phi\left(\int_X g(x)d\mu(x)\right).$$

If ϕ is strictly convex, equality holds if and only if $g(x)$ is constant for μ-almost all $x \in X$ (see [15], p.165).

Lemma 3.11 ([53])**.** *If $K_1, \cdots, K_j \in \mathcal{K}_o^n$, $0 \le j \le n$ and $0 < r \le j$, then*

$$\Phi_{n-j}(K_1, \cdots, K_j) \ge \prod_{i=1}^{r} \Phi_{n-j}(K_i, \cdots, K_i, K_{r+1}, \cdots, K_j)^{1/r}. \tag{3.13}$$

Obviously, a specisl case of (3.13) is the following: If $K_1, \cdots, K_j \in \mathcal{K}_o^n$ and $0 \le j \le n$, then

$$\Phi_{n-j}(K_1, \cdots, K_j)^j \ge \Phi_{n-j}(K_1) \cdots \Phi_{n-j}(K_j), \tag{3.14}$$

with equality if and only if $K_1, \cdots, K_j$ are homothetic. Another specisl case of (3.13) is the following: If $K, L \in \mathcal{K}_o^n$ and $0 \le j \le n$, then

$$\Phi_{n-j}(K, L)^j \ge \Phi_{n-j}(K)^{j-1}\Phi_{n-j}(L), \tag{3.15}$$

with equality if and only if K and L are homothetic.

4. Orlicz-Aleksandrov-Fenchel inequality for mixed affine quermassintegrals

Theorem 4.1. *Let $\varphi \in \Phi$ and $0 \le j \le n$. If $K_1, \ldots, K_j \in \mathcal{K}_o^n$ and $L_j \in \mathcal{K}_{oo}^n$, then*

$$\left(\frac{\Phi_{\varphi,n-j}(K_1,\ldots,K_j,L_j)}{\Phi_{n-j}(K_1,\ldots,K_{j-1},L_j)}\right)^{-n} \ge \varphi\left(\frac{\Phi_{n-j}(K_1,\ldots,K_j)}{\Phi_{n-j}(K_1,\ldots,K_{j-1},L_j)}\right). \tag{4.1}$$

If φ is strictly convex, equality holds if and only if K_j and L_j are homothetic.

Proof. When $j = n$, (4.1) becomes the Orlicz-Aleksandrov-Fenchel inequality (3.3) for mixed volumes, hence we assume $0 \le j < n$. Let

$$d\nu(\xi) = \frac{\mathrm{vol}_j((K_1,\ldots,K_{j-1},L_j)|\xi)^{-n}}{\int_{G_{n,j}} \mathrm{vol}_j((K_1,\ldots,K_{j-1},L_j)|\xi)^{-n} d\mu_j(\xi)} d\mu_j(\xi).$$

Obviously, this defines a Borel probability measure ν on $G_{n,j}$.

From (3.3), (3.7), (3.10), and by using the Jensen inequality and Hölder inequality, we obtain

$$\begin{aligned}
&\left(\frac{\Phi_{\varphi,n-j}(K_1,\ldots,K_j,L_j)}{\Phi_{n-j}(K_1,\ldots,K_{j-1},L_j)}\right)^{-n}\\
=&\int_{G_{n,j}} \frac{V_\varphi^{(j)}((K_1,\ldots,K_j,L_j)|\xi)}{\mathrm{vol}_j((K_1,\ldots,K_{j-1},L_j)|\xi)} d\nu(\xi)\\
\ge&\int_{G_{n,j}} \varphi\left(\frac{\mathrm{vol}_j((K_1,\ldots,K_j)|\xi)}{\mathrm{vol}_j((K_1,\ldots,K_{j-1},L_j)|\xi)}\right) d\nu(\xi)\\
\ge&\varphi\left(\int_{G_{n,j}} \frac{\mathrm{vol}_j((K_1,\ldots,K_j)|\xi)\cdot \mathrm{vol}_j((K_1,\ldots,K_{j-1},L_j)|\xi)^{-n-1}}{\int_{G_{n,j}} \mathrm{vol}_j((K_1,\ldots,K_{j-1},L_j)|\xi)^{-n} d\mu_j(\xi)} d\mu_j(\xi)\right)\\
\ge&\varphi\left(\left(\frac{\int_{G_{n,j}} \mathrm{vol}_j((K_1,\ldots,K_j)|\xi)^{-n} d\mu_j(\xi)}{\int_{G_{n,j}} \mathrm{vol}_j((K_1,\ldots,K_{j-1},L_j)|\xi)^{-n} d\mu_j(\xi)}\right)^{-1/n}\right)\\
=&\varphi\left(\frac{\Phi_{n-j}(K_1,\ldots,K_j)}{\Phi_{n-j}(K_1,\ldots,K_{j-1},L_j)}\right).
\end{aligned}$$

This implies inequality (4.1) holds.

On the other hand, suppose the equality holds in (4.1), then these three inequalities in the above must satisfy the equal. The first inequality is following:

$$\frac{V_\varphi^{(j)}((K_1,\ldots,K_{j-1},K_j,L_j)|\xi)}{\mathrm{vol}_j((K_1,\ldots,K_{j-1},L_j)|\xi)} \ge \varphi\left(\frac{\mathrm{vol}_j((K_1,\ldots,K_j)|\xi)}{\mathrm{vol}_j((K_1,\ldots,K_{j-1},L_j)|\xi)}\right).$$

When φ is strictly convex, form the equality of inequality (3.3), it yields that $K_j|\xi$ and $L_j|\xi$ must be homothetic. The second inequality is following:

$$\begin{aligned}
&\int_{G_{n,j}} \varphi\left(\frac{\mathrm{vol}_j((K_1,\ldots,K_j)|\xi)}{\mathrm{vol}_j((K_1,\ldots,K_{j-1},L_j)|\xi)}\right) d\nu\\
\ge&\varphi\left(\int_{G_{n,j}} \left(\frac{\mathrm{vol}_j((K_1,\ldots,K_j)|\xi)}{\mathrm{vol}_j((K_1,\ldots,K_{j-1},L_j)|\xi)}\right) d\nu\right).
\end{aligned}$$

When φ is strictly convex, form the equality condition of Jensen's inequality, then $\frac{\mathrm{vol}_j((K_1,\ldots,K_j)|\xi)}{\mathrm{vol}_j((K_1,\ldots,K_{j-1},L_j)|\xi)}$ must be a constant, this yields that $K_j|\xi$ and $L_j|\xi$ must be homothetic. The third inequality is obtained by applying the Hölder inequality. Form the equality condition of Hölder inequality, this yields that equality holds $\mathrm{vol}_j((K_1,\ldots,K_j)|\xi)$ and $\mathrm{vol}_j((K_1,\ldots,K_{j-1},L_j)|\xi)$ must be proportional, namely $K_j|\xi$ and $L_j|\xi$ be homothetic. To sum up, if φ is strictly convex, it follows that equality in (4.1) holds if and only if K_j and L_j are homothetic. □

Theorem 4.2 (Orlicz-Aleksandrov-Fenchel inequality for mixed affine quermassintegrals of j convex bodies)**.** *Let $\varphi \in \Phi$, $0 \le j \le n$ and $0 < r \le j$. If $K_1, \ldots, K_j \in \mathcal{K}_o^n$ and $L_j \in \mathcal{K}_{oo}^n$, then*

$$\left(\frac{\Phi_{\varphi,n-j}(K_1,\ldots,K_j,L_j)}{\Phi_{n-j}(K_1,\ldots,K_{j-1},L_j)}\right)^{-n} \ge \varphi\left(\frac{\prod_{i=1}^r \Phi_{n-j}(K_i,\cdots,K_i,K_{r+1},\cdots,K_j)^{1/r}}{\Phi_{n-j}(K_1,\ldots,K_{j-1},L_j)}\right). \tag{4.2}$$

Proof. This yields immediately from Lemma 3.11 and Theorem 4.1. □

Unfortunately, the equality conditions of the Orlicz-Aleksandrov-Fenchel inequality are, in general, unknown.

When $j = n$ and $K_j = L_j$, (4.2) becomes the classical Aleksandrov-Fenchel inequality for mixed volumes. When $K_j = L_j$, (4.2) becomes the Aleksandrov-Fenchel inequality (3.13) for the mixed affine quermassintegrals. A special case of (4.2) is the following Orlicz-Aleksandrov-Fenchel inequality for mixed volumes established by Zhao [50].

Corollary 4.1 (Orlicz-Aleksandrov-Fenchel inequality for mixed volumes)**.** *Let $0 < r \le n$ and $\varphi \in \Phi$. If $K_1, \cdots, K_n \in \mathcal{K}_o^n$, $L_n \in \mathcal{K}_{oo}^n$, then*

$$\begin{aligned} V_\varphi(K_1,\cdots,K_n,L_n) \ge & V(K_1,\cdots,K_{n-1},L_n) \\ & \times \varphi\left(\frac{\prod_{i=1}^r V(K_i\ldots,K_i,K_{r+1},\ldots,K_n)^{\frac{1}{r}}}{V(K_1,\cdots,K_{n-1},L_n)}\right). \end{aligned}$$

Proof. This follows immediately from Definitions 3.1 and 3.3, and (4.2) with $j = n$. □

Corollary 4.2 (L_p-Aleksandrov-Fenchel inequality for the mixed affine quermassintegrals)**.** *Let $p \ge 1$ and $0 \le j \le n$. If $K_1, \ldots, K_j \in \mathcal{K}_o^n$ and $L_j \in \mathcal{K}_{oo}^n$, then*

$$\Phi_{p,n-j}(K_1,\ldots,,K_j,L_j)^{-n} \ge \frac{\prod_{i=1}^r \Phi_{n-j}(K_i\ldots,K_i,K_{r+1},\ldots,K_j)^{p/r}}{\Phi_{n-j}(K_1,\ldots,K_{j-1},L_j)^{n+p}}. \tag{4.3}$$

Proof. This follows immediately from (4.2) with $\varphi(t) = t^p$ and $1 \le p < \infty$. □

Remark 4.1. Obviously, a special case of (4.3) is the following L_p-Aleksandrov-Fenchel inequality for mixed volumes established by Zhao [50].

L_p-Aleksandrov-Fenchel inequality for mixed volumes If $K_1, \ldots, K_j \in \mathcal{K}_o^n$ and $L_j \in \mathcal{K}_{oo}^n$, $1 \le r \le n$ and $p \ge 1$, then

$$V_p(K_1,\cdots,K_n,L_n) \ge \frac{\prod\limits_{i=1}^r V(K_i\ldots,K_i,K_{r+1},\ldots,K_n)^{p/r}}{V(K_1,\cdots,K_{n-1},L_n)^{p-1}}.$$

Corollary 4.3. *Let $\varphi \in \Phi$, $0 \le j \le n$ and $0 < r \le j$. If $K_1, \ldots, K_j \in \mathcal{K}_o^n$ and $L_j \in \mathcal{K}_{oo}^n$, then*

$$\left(\frac{\Phi_{\varphi,n-j}(K_1,\ldots,K_j,L_j)}{\Phi_{n-j}(K_1,\ldots,K_{j-1},L_j)}\right)^{-n} \ge \varphi\left(\frac{\Big(\Phi_{n-j}(K_1)\cdots\Phi_{n-j}(K_j)\Big)^{1/j}}{\Phi_{n-j}(K_1,\ldots,K_{j-1},L_j)}\right). \tag{4.4}$$

If φ is strictly convex, equality holds if and only if $K_1, \cdots, K_j, L_j$ are homothetic.

Proof. This yields immediately from (3.14) and Theorem 4.1.

Next, we discuss the equality condition of (4.4). When $r = j$, the Aleksandrov-Fenchel inequality (3.13) becomes

$$\Phi_{n-j}(K_1, \ldots, K_j)^j \geq \Phi_{n-j}(K_1) \cdots \Phi_{n-j}(K_j). \tag{4.5}$$

Although, the precise equality for the Aleksandrov-Fenchel inequality (3.13) are unknown in general, but it is well known that the equality in (4.5) holds if and only if $K_1, \ldots, K_n$ are all homothetic of each other (see Lemma 3.11). Hence, if φ is strictly convex, combining the equality conditions of Theorem 4.1, it follows that the equality in (4.4) holds if and only $K_1, \ldots, K_j, L_j$ are all homothetic of each other. □

Another special case of (4.2) is the following Orlicz Minkowski inequality for volumes established by Gardner, Hug and Weil [9] and Xi, Jin and Leng [43], respectively.

Corollary 4.4. (Orlicz-Minkowski inequality) *If $K, L \in \mathcal{K}^n$ and $\varphi \in \Phi$, then*

$$V_\varphi(K, L) \geq V(K)\varphi\left(\left(\frac{V(L)}{V(K)}\right)^{1/n}\right).$$

If φ is strictly convex, equality holds if and only if K and L are homothetic.

The following uniqueness is a direct consequence of the Orlicz-Aleksandrov-Fenchel inequality for the mixed affine quermassintegrals of j convex bodies.

Theorem 4.3. *If $K_1, \cdots, K_j, L_j \in \mathcal{M} \subset \mathcal{K}_o^n$, $0 \leq j \leq n$, and $\varphi \in \Phi$ be strictly convex, and if either*

$$\Phi_{\varphi,n-j}(K_1, \ldots, K_{j-1}, L_j, Q) = \Phi_{\varphi,n-j}(K_1, \ldots, K_{j-1}, K_j, Q), \quad for\ all \quad Q \in \mathcal{M}, \tag{4.6}$$

or

$$\frac{\Phi_{\varphi,n-j}(K_1, \ldots, K_{j-1}, Q, L_j)}{\Phi_{n-j}(K_1, \ldots, K_{j-1}, L_j)} = \frac{\Phi_{\varphi,n-j}(K_1, \ldots, K_{j-1}, Q, K_j)}{\Phi_{n-j}(K_1, \ldots, K_j)}, \quad for\ all \quad Q \in \mathcal{M}, \tag{4.7}$$

then $K_j = L_j$.

Proof. Suppose (4.6) hold. Taking L_j for Q, then from Definition 3.2 and Theorem 4.1, we obtain

$$\begin{aligned}\Phi_{n-j}(K_1, \ldots, K_{j-1}, L_j) &= \Phi_{\varphi,n-j}(K_1, \ldots, K_{j-1}, K_j, L_j)\\ &\leq \Phi_{n-j}(K_1, \ldots, K_{j-1}, L_j)\\ &\quad \times\varphi\left(\frac{\Phi_{n-j}(K_1, \ldots, K_j)}{\Phi_{n-j}(K_1, \ldots, K_{j-1}, L_j)}\right)^{-1/n},\end{aligned}$$

with equality if and only if K_j and L_j are homothetic. Hence

$$1 \geq \varphi\left(\frac{\Phi_{n-j}(K_1, \ldots, K_j)}{\Phi_{n-j}(K_1, \ldots, K_{j-1}, L_j)}\right),$$

with equality if and only if K_j and L_j are homothetic. Since φ is increasing function on $(0, \infty)$, this follows that

$$\Phi_{n-j}(K_1, \ldots, K_j) \leq \Phi_{n-j}(K_1, \ldots, K_{j-1}, L_j),$$

with equality if and only if K_j and L_j are homothetic. On the other hand, if taking K_j for Q, we similar get $\Phi_{n-j}(K_1,\ldots,K_j) \geq \Phi_{n-j}(K_1,\ldots,K_{j-1},L_j)$, with equality if and only if K_j and L_j are homothetic. Hence $\Phi_{n-j}(K_1,\ldots,K_j) = \Phi_{n-j}(K_1,\ldots,K_{j-1},L_j)$, and K_j and L_j are homothetic, it follows that K_j and L_j must be equal.

Suppose (4.7) hold. Taking K_j for Q, then from Definition 3.2 and Theorem 4.1, we obtain

$$1 = \frac{\Phi_{\varphi,n-j}(K_1,\ldots,K_j,L_j)}{\Phi_{n-j}(K_1,\ldots,K_{j-1},L_j)} \leq \varphi\left(\frac{\Phi_{n-j}(K_1,\ldots,K_j)}{\Phi_{n-j}(K_1,\ldots,K_{j-1},L_j)}\right)^{-1/n},$$

with equality if and only if K_j and L_j are homothetic. Since φ is increasing function on $(0,\infty)$, this follows that

$$\Phi_{n-j}(K_1,\ldots,K_j) \leq \Phi_{n-j}(K_1,\ldots,K_{j-1},L_j),$$

with equality if and only if K_j and L_j are homothetic. On the other hand, if taking L_j for Q, we similar get

$$\Phi_{n-j}(K_1,\ldots,K_j) \geq \Phi_{n-j}(K_1,\ldots,K_{j-1},L_j),$$

with equality if and only if K_j and L_j are homothetic. Hence $\Phi_{n-j}(K_1,\ldots,K_j) = \Phi_{n-j}(K_1,\ldots,K_{j-1},L_j)$, and K_j and L_j are homothetic, it follows that K_j and L_j must be equal. □

Corollary 4.5. *If $K_1,\cdots,K_n,L_n \in \mathcal{M} \subset \mathcal{K}_o^n$ and $\varphi \in \Phi$ be strictly convex, and if either*

$$V_\varphi(K_1,\ldots,K_{n-1},L_n,Q) = V_\varphi(K_1,\ldots,K_{n-1},K_n,Q), \quad for\ all\ \ Q \in \mathcal{M},$$

or

$$\frac{V_\varphi(K_1,\ldots,K_{n-1},Q,L_n)}{V(K_1,\ldots,K_{n-1},L_n)} = \frac{V_\varphi(K_1,\ldots,K_{n-1},Q,K_n)}{V(K_1,\ldots,K_n)}, \quad for\ all\ \ Q \in \mathcal{M},$$

then $K_n = L_n$.

Proof. This follows immediately from Theorem 4.3 with $j = n$. □

Corollary 4.6. *If $\varphi \in \Phi$ and is strictly convex and $\mathcal{M} \subset \mathcal{K}_o^n$ such that $K, L \in \mathcal{M}$, and $0 \leq i < n$. If*

$$W_{\varphi,i}(M,K) = W_{\varphi,i}(M,L), \quad for\ all\ M \in \mathcal{M}$$

or

$$\frac{W_{\varphi,i}(K,M)}{W_i(K)} = \frac{W_{\varphi,i}(L,M)}{W_i(L)}, \quad for\ all\ M \in \mathcal{M}$$

then $K = L$.

Corollary 4.7. *If $\varphi \in \Phi$ and is strictly convex, $0 \leq j \leq n$ and $\mathcal{M} \subset \mathcal{K}_o^n$ such that $K, L \in \mathcal{M}$. If*

$$\Phi_{\varphi,n-j}(M,K) = \Phi_{\varphi,n-j}(M,L), \quad for\ all\ M \in \mathcal{M}$$

or

$$\frac{\Phi_{\varphi,n-j}(K,M)}{\Phi_{n-j}(K)} = \frac{\Phi_{\varphi,n-j}(L,M)}{\Phi_{n-j}(L)}, \quad for\ all\ M \in \mathcal{M}$$

then $K = L$.

Corollary 4.8. *If $K, L \in \mathcal{K}_o^n$, $\varphi \in \Phi$ and $0 \le j \le n$, then*

$$\left(\frac{\Phi_{\varphi,n-j}(K,L)}{\Phi_{n-j}(K)}\right)^{-n} \ge \varphi\left(\left(\frac{\Phi_{n-j}(L)}{\Phi_{n-j}(K)}\right)^{1/j}\right). \tag{4.8}$$

If φ is strictly convex, equality holds if and only if K and L are homothetic (see [55]).

Proof. Putting $K_1 = \cdots = K_{j-1} = L_j = K$ and $K_j = L$ in (4.1) and in view of (3.15), we have

$$\begin{aligned}\left(\frac{\Phi_{\varphi,n-j}(K,L)}{\Phi_{n-j}(K)}\right)^{-n} &\ge \varphi\left(\frac{\Phi_{n-j}(K,L)}{\Phi_{n-j}(K)}\right)\\ &\ge \varphi\left(\left(\frac{\Phi_{n-j}(L)}{\Phi_{n-j}(K)}\right)^{1/j}\right).\end{aligned}$$

If φ is strictly convex, equality holds if and only if K and L are homothetic. □

5. Orlicz-Brunn-Minkoswki inequality for mixed affine quermassintegrals

Lemma 5.1. *Let $\varphi \in \Phi$ and $0 \le j \le n$. If $K_1, \ldots, K_j \in \mathcal{K}_o^n$ and $L_j \in \mathcal{K}_{oo}^n$, then for $\varepsilon > 0$*

$$\begin{aligned}1 =& \left(\frac{\Phi_{\varphi,n-j}(K_1,\ldots,K_j,L_j +_\varphi \varepsilon\cdot K_j)}{\Phi_{n-j}(K_1,\ldots,K_{j-1},L_j +_\varphi \varepsilon\cdot K_j)}\right)^{-n}\\ &+ \varepsilon\cdot\left(\frac{\Phi_{\varphi,n-j}(K_1,\ldots,K_{j-1},L_j,L_j +_\varphi \varepsilon\cdot K_j)}{\Phi_{n-j}(K_1,\ldots,K_{j-1},L_j +_\varphi \varepsilon\cdot K_j)}\right)^{-n}.\end{aligned} \tag{5.1}$$

Proof. From Lemma 3.1 and Lemma 3.5, we have for $\varepsilon > 0$

$$\begin{aligned}&V_\varphi^{(j)}((K_1,\ldots,K_j,L_j +_\varphi \varepsilon\cdot K_j)|\xi) + \varepsilon\cdot V_\varphi^{(j)}((K_1,\ldots,K_{j-1},L_j,L_j +_\varphi \varepsilon\cdot K_j)|\xi)\\ =&V_\varphi^{(j)}(K_1|\xi,\ldots,K_j|\xi,L_j|\xi +_\varphi \varepsilon\cdot K_j|\xi)\\ &+ \varepsilon\cdot V_\varphi^{(j)}(K_1|\xi,\ldots,K_{j-1}|\xi,L_j|\xi,L_j|\xi +_\varphi \varepsilon\cdot K_j|\xi)\\ =&\mathrm{vol}_j(K_1|\xi,\ldots,K_{j-1}|\xi,L_j|\xi +_\varphi \varepsilon\cdot K_j|\xi)\\ =&\mathrm{vol}_j((K_1,\ldots,K_{j-1},L_j +_\varphi \varepsilon\cdot K_j)|\xi).\end{aligned} \tag{5.2}$$

Let $Q = L_j +_\varphi \varepsilon\cdot K_j$, from (3.6), (3.10) and (5.2), we have

$$\begin{aligned}&\Phi_{\varphi,n-j}(K_1,\ldots,K_j,Q)^{-n} + \varepsilon\cdot\Phi_{\varphi,n-j}(K_1,\ldots,K_{j-1},L_j,Q)^{-n}\\ =&\omega_n^{-n}\int_{G_{n,j}} \frac{V_\varphi^{(j)}((K_1,\ldots,K_j,Q)|\xi) + \varepsilon\cdot V_\varphi^{(j)}((K_1,\ldots,K_{j-1},L_j,Q)|\xi)}{\mathrm{vol}_j((K_1,\ldots,K_{j-1},Q)|\xi)}\\ &\times\left(\frac{\mathrm{vol}_j((K_1,\ldots,K_{j-1},Q)|\xi)}{\omega_j}\right)^{-n} d\mu_j(\xi)\\ =&\omega_n^{-n}\int_{G_{n,j}}\left(\frac{\mathrm{vol}_j((K_1,\ldots,K_{j-1},Q)|\xi)}{\omega_j}\right)^{-n} d\mu_j(\xi)\end{aligned}$$

$$=\Phi_{n-j}(K_1,\ldots,K_{j-1},Q)^{-n}.$$

The proof is complete. □

Lemma 5.2 ([55]). *Let $K, L \in \mathcal{K}_o^n$, $\varepsilon > 0$ and $\varphi \in \Phi$.*

(1) If K and L are homothetic, then K and $K +_\varphi \varepsilon \cdot L$ are homothetic.

(2) If K and $K +_\varphi \varepsilon \cdot L$ are homothetic, then K and L are homothetic.

Theorem 5.1. *(Orlicz-Brunn-Minkowski inequality for mixed affine quermassintegrals) Let $\varphi \in \Phi$ and $0 \le j \le n$. If $K_1, \ldots, K_j \in \mathcal{K}_o^n$ and $L_j \in \mathcal{K}_{oo}^n$, then for $\varepsilon > 0$*

$$\begin{aligned}1 \ge &\varphi\left(\frac{\Phi_{n-j}(K_1,\ldots,K_{j-1},L_j)}{\Phi_{n-j}(K_1,\ldots,K_{j-1},L_j +_\varphi \varepsilon\cdot K_j)}\right)\\ &+\varepsilon\cdot\varphi\left(\frac{\Phi_{n-j}(K_1,\ldots,K_j)}{\Phi_{n-j}(K_1,\ldots,K_{j-1},L_j +_\varphi \varepsilon\cdot K_j)}\right).\end{aligned} \tag{5.3}$$

If φ is strictly convex, equality holds if and only if K_j and L_j are homothetic.

Proof. From Lemma 5.1 and Theorem 4.1, we obtain for $\varepsilon > 0$

$$\begin{aligned}1 =&\left(\frac{\Phi_{\varphi,n-j}(K_1,\ldots,K_{j-1},L_j,L_j +_\varphi \varepsilon\cdot K_j)}{\Phi_{n-j}(K_1,\ldots,K_{j-1},L_j +_\varphi \varepsilon\cdot K_j)}\right)^{-n}\\ &+\varepsilon\cdot\left(\frac{\Phi_{\varphi,n-j}(K_1,\ldots,K_j,L_j +_\varphi \varepsilon\cdot K_j)}{\Phi_{n-j}(K_1,\ldots,K_{j-1},L_j +_\varphi \varepsilon\cdot K_j)}\right)^{-n}\\ \ge&\ \varphi\left(\frac{\Phi_{n-j}(K_1,\ldots,K_{j-1},L_j)}{\Phi_{n-j}(K_1,\ldots,K_{j-1},L_j +_\varphi \varepsilon\cdot K_j)}\right)\\ &+\varepsilon\cdot\varphi\left(\frac{\Phi_{n-j}(K_1,\ldots,K_j)}{\Phi_{n-j}(K_1,\ldots,K_{j-1},L_j +_\varphi \varepsilon\cdot K_j)}\right).\end{aligned}$$

If φ is strictly convex, from equality condition of the Orlicz-Minkowski inequality (4.1), the equality holds if and only if L_j and $L_j +_\varphi \varepsilon \cdot K_j$ are homothetic, and K_j and $L_j +_\varphi \varepsilon \cdot K_j$ are homothetic and combine with Lemma 5.2, this yields that if φ is strictly convex, equality holds in (5.3) if and only if K_j and L_j are homothetic. □

Corollary 5.1. *If $K, L \in \mathcal{K}_o^n$, $\varphi \in \Phi$ and $0 \le j \le n$, then for $\varepsilon > 0$*

$$1 \ge \varphi\left(\left(\frac{\Phi_{n-j}(K)}{\Phi_{n-j}(K +_\varphi \varepsilon\cdot L)}\right)^{1/j}\right) + \varepsilon\cdot\varphi\left(\left(\frac{\Phi_{n-j}(L)}{\Phi_{n-j}(K +_\varphi \varepsilon\cdot L)}\right)^{1/j}\right). \tag{5.4}$$

If φ is strictly convex, equality holds if and only if K and L are homothetic. (see [55]).

Proof. Putting $K_1 = \cdots = K_{j-1} = L_j +_\varphi \varepsilon \cdot K_j$ in (6.3), we have for $\varepsilon > 0$

$$1 \ge \varphi\left(\frac{\Phi_{n-j}(L_j +_\varphi \varepsilon\cdot K_j, K_j)}{\Phi_{n-j}(L_j +_\varphi \varepsilon\cdot K_j)}\right) + \varepsilon\cdot\varphi\left(\frac{\Phi_{n-j}(L_j +_\varphi \varepsilon\cdot K_j, L_j)}{\Phi_{n-j}(L_j +_\varphi \varepsilon\cdot K_j)}\right). \tag{5.5}$$

If φ is strictly convex, equality holds if and only if K_j and L_j are homothetic.

From (3.15) and (5.5), (5.4) follows easily. □

Corollary 5.2 (L_p- Brunn-Minkowski inequality for mixed affine quermassintegrals)**.** *Let $p \geq 1$ and $0 \leq j \leq n$. If $K_1, \ldots, K_j \in \mathcal{K}_o^n$ and $L_j \in \mathcal{K}_{oo}^n$, then for $\varepsilon > 0$*

$$\Phi_{n-j}(K_1, \ldots, K_{j-1}, L_j +_\varphi \varepsilon \cdot K_j)^p \geq \Phi_{n-j}(K_1, \ldots, K_j)^p + \varepsilon \cdot \Phi_{n-j}(K_1, \ldots, K_{j-1}, L_j)^p, \tag{5.6}$$

with equality if and only if K_j and L_j are homothetic.

Proof. This follows immediately from (5.3) with $\varphi(t) = t^p$ and $1 \leq p < \infty$. □

Theorem 5.2 ([9, 43])**.** (Orlicz Brunn-Minkowski inequality for mixed volumes) *If $K, L \in \mathcal{K}^n$ and $\varphi \in \Phi$, then*

$$1 \geq \varphi\left(\left(\frac{V(K)}{V(K +_\varphi L)}\right)^{1/n}\right) + \left(\left(\frac{V(L)}{V(K +_\varphi L)}\right)^{1/n}\right). \tag{5.7}$$

If φ is strictly convex, equality holds if and only if K and L are homothetic.

Corollary 5.3. *Let $\varphi \in \Phi$ and $0 \leq j \leq n$. If $K_1, \ldots, K_j \in \mathcal{K}_o^n$ and $L_j \in \mathcal{K}_{oo}^n$, then*

$$\left(\frac{\Phi_{\varphi,n-j}(K_1, \ldots, K_j, L_j)}{\Phi_{n-j}(K_1, \ldots, K_{j-1}, L_j)}\right)^{-n} \geq \varphi\left(\frac{\Phi_{n-j}(K_1, \ldots, K_j)}{\Phi_{n-j}(K_1, \ldots, K_{j-1}, L_j)}\right). \tag{5.8}$$

If φ is strictly convex, equality holds if and only if K_j and L_j are homothetic.

Proof. Let

$$K_\varepsilon = L_j +_\varphi \varepsilon \cdot K_j,$$

where $0 \leq j \leq n$. From Lemma 3.6 and Theorem 5.1, we obtain

$$\begin{aligned}
&\frac{1}{\varphi'_-(1)} \Phi_{n-j}(K_1, \ldots, K_{j-1}, L_j)^{1+n} \Phi_{\varphi,n-j}(K_1, \ldots, K_j, L_j)^{-n} \\
=&\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0^+} \Phi_{n-j}(K_1, \cdots, K_{j-1}, K_\varepsilon) \\
=&\lim_{\varepsilon \to 0^+} \frac{\Phi_{n-j}(K_1, \cdots, K_{j-1}, K_\varepsilon) - \Phi_{n-j}(K_1, \cdots, K_{j-1}, L_j)}{\varepsilon} \\
=&\lim_{\varepsilon \to 0^+} \frac{1 - \dfrac{\Phi_{n-j}(K_1, \cdots, K_{j-1}, L_j)}{\Phi_{n-j}(K_1, \cdots, K_{j-1}, K_\varepsilon)}}{1 - \varphi\left(\dfrac{\Phi_{n-j}(K_1, \cdots, K_{j-1}, L_j)}{V(K_1, \cdots, K_{j-1}, K_\varepsilon)}\right)} \\
&\times \frac{1 - \varphi\left(\dfrac{\Phi_{n-j}(K_1, \cdots, K_{j-1}, L_j)}{\Phi_{n-j}(K_1, \cdots, K_{j-1}, K_\varepsilon)}\right)}{\varepsilon} \cdot \Phi_{n-j}(K_1, \cdots, K_{j-1}, K_\varepsilon) \\
=&\lim_{t \to 0^+} \frac{1-t}{\varphi(1) - \varphi(t)} \cdot \lim_{\varepsilon \to 0^+} \frac{1 - \varphi\left(\dfrac{\Phi_{n-j}(K_1, \cdots, K_{j-1}, L_j)}{\Phi_{n-j}(K_1, \cdots, K_{j-1}, K_\varepsilon)}\right)}{\varepsilon} \\
&\times \lim_{\varepsilon \to 0^+} \Phi_{n-j}(K_1, \cdots, K_{j-1}, K_\varepsilon) \\
\geq&\frac{1}{\varphi'_-(1)} \cdot \varphi\left(\frac{\Phi_{n-j}(K_1, \cdots, K_j)}{\Phi_{n-j}(K_1, \cdots, K_{j-1}, L_j)}\right) \cdot \Phi_{n-j}(K_1, \cdots, K_{j-1}, L_j).
\end{aligned} \tag{5.4}$$

From (5.9), (5.8) easy follows. This proof is complete. □

Through the proof of Theorem 5.1 and Corollary 5.3, it is not difficult to seebinequality (5.3) is equivalent to inequality (4.1).

Corollary 5.4. *If $K_1, \cdots, K_j, L_j \in \mathcal{K}_o^n$, $0 \le i, j \le n$, $0 < r \le n$ and $\varphi \in \Phi$, then for $\varepsilon > 0$*

$$1 \ge \varphi\left(\frac{\prod_{i=1}^{r} \Phi_{n-j}(K_i, \ldots, K_i, K_{r+1}, \ldots, K_j)^{1/r}}{\Phi_{n-j}(K_1, \cdots, K_{j-1}, K_j +_{\varphi,\varepsilon} L_j)}\right) + \varepsilon \cdot \varphi\left(\frac{\prod_{k=1}^{r} \Phi_{n-j}(K_k, \ldots, K_k, K_{r+1}, \ldots, K_{j-1}, L_j)^{1/r}}{\Phi_{n-j}(K_1, \cdots, K_{j-1}, K_j +_{\varphi,\varepsilon} L_j)}\right). \tag{5.10}$$

Proof. This follows immediately from Theorem 5.1 combining the Aleksandrov-Fenchel inequality (3.13). □

Similarly, we see also that inequality (5.10) is equivalent to inequality (4.2).

Corollary 5.5. *If $K_1, \cdots, K_j, L_j \in \mathcal{K}_o^n$, $0 \le j \le n$ and $\varphi \in \Phi$, then for $\varepsilon > 0$*

$$1 \ge \varphi\left(\left(\frac{\Phi_{n-j}(K_1) \cdots \Phi_{n-j}(K_j)}{\Phi_{n-j}(K_1, \cdots, K_{j-1}, K_j +_{\varphi,\varepsilon} L_j)^j}\right)^{1/j}\right) + \varepsilon \cdot \left(\left(\frac{\Phi_{n-j}(K_1) \cdots \Phi_{n-j}(K_{j-1})\Phi_{n-j}(L_j)}{\Phi_{n-j}(K_1, \cdots, K_{j-1}, K_j +_{\varphi,\varepsilon} L_j)^j}\right)^{1/j}\right). \tag{5.11}$$

If φ is strictly convex, equality holds if and only if $K_1, \ldots, K_j, L_j$ are all homothetic of each other.

Proof. This follows immediately from (5.10) with $r = j$. □

Acknowledgements

The author expresses his thanks to the referee for his valuable comments and suggestions which improved the presentation of this manuscript.

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