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OPTIMAL CONTROL PROBLEMS FOR SPACE-FRACTIONAL WAVE EQUATIONS*

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Abstract In this paper, we study an optimal control problem for a space-fractional wave equation. First, we show the existence and uniqueness of weak solution by Galërkin approximate method. Then, we obtain an optimal control for the optimal control problem.

Keywords Optimal control, space-fractional wave equation, fractional Sobolev spaces.

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1. Introduction

There exist several phenomena that cannot be modeled by partial differential equations based on ordinary calculus, since they depend on the so called memory effect. In order to take account of this dependence, we may use fractional differential calculus. Fractional differential equation have gained considerable importance due to their application in various sciences, such as physics, mechanics, chemistry, engineering and so on. In recent years, there has been a significant development in fractional differential equations which may be ordinary or partial, see for example [4, 6–8, 10, 11, 13–15, 20] and the references therein.

The space-fractional wave equation is obtained from the classical wave equation, in which the Laplacian operator is replacing by the fractional Laplacian operator, see for examples [2, 5, 17]. Until now, the understanding of the dynamics of space-fractional wave equations is rather limited.

In this paper, we are concerned with an optimal control problem for the following space-fractional wave equation:

$$\begin{cases} u_{tt} + (-\Delta)^s u = f(x,t), & x \in \Omega, t > 0, \\ u(x,0) = \phi(x), \ u_t(x,0) = \psi(x), & x \in \Omega, \\ u(x,t) = 0, & x \in \mathbb{R}^N \setminus \Omega, t > 0, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $\phi \in X_0(\Omega)$, $\psi \in L^2(\Omega)$. For any $T > 0$, $Q_T := (0,T) \times \Omega$ and $f \in L^2(Q_T)$.

Motivated by [1, 3, 12, 16, 19], we intend to study optimal control problems for (1.1) due to the theory of Lions [9] in which the optimal control problems are surveyed on many types of linear partial differential equations. Let $X_0(\Omega)$ be a

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Hilbert space and let $\mathscr{U}$ be another Hilbert space of control variables, and B be a bounded linear operator from $\mathscr{U}$ into $L^2(Q_T)$, which is called a controller. We formulate our optimal control problems as follows:

$$u_{tt} + (-\Delta)^s u = f(x,t) + Bv, x \in \Omega, t > 0,$$

where $v \in \mathscr{U}$ is a control. As we know, there are almost no research works in this direction.

The structure of this paper is as follows. In Section 2, the definition of weak solutions besides notations and assumptions are stated.. In section 3, we first show the Galërkin approximate solution along with A priori estimates of $\{u_m\}_m$. Then, we prove the existence and uniqueness of weak solution. In section 4, we characterize the existence of an optimal control $u \in \mathscr{U}$ which minimizes the quadratic cost function.

2. Preliminaries

Let $s \in (0,1)$ and $2s < N$. The fractional Laplace operator for a function $\varphi \in C_0^\infty(\mathbb{R}^N)$ is defined pointwise by

$$(-\Delta)^s \varphi(x) = \frac{C(N,s)}{2} \int_{\mathbb{R}^N} \frac{\varphi(x) - \varphi(x+y) - \varphi(x-y)}{|y|^{N+2s}} dy,$$

for all $x \in \mathbb{R}^N$, where

$$\frac{1}{C(N,s)} = \int_{\mathbb{R}^N} \frac{1 - \cos \xi_1}{|\xi|^{N+2s}} d\xi.$$

The fractional Sobolev space $H^s(\mathbb{R}^N)$ is

$$H^s(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) : \frac{|u(x) - u(y)|}{|x-y|^{\frac{N}{2}+s}} \in L^2\left(\mathbb{R}^N \times \mathbb{R}^N\right) \right\}$$

endowed with the norm

$$\|u\|_{H^s(\mathbb{R}^N)} = \left(\|u\|^2_{L^2(\mathbb{R}^N)} + \int\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} dxdy \right)^{\frac{1}{2}}.$$

Let

$$X_0(\Omega) = \left\{ u \in H^s(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \backslash \Omega \right\}.$$

In the sequel we take

$$\|u\|_{X_0(\Omega)} = \left(\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} dxdy \right)^{1/2}$$

as norm on $X_0(\Omega)$. It is easily seen that $X_0(\Omega) = \left(X_0(\Omega), \|\cdot\|_{X_0(\Omega)}\right)$ is a Hilbert space with inner product

$$\langle u, v \rangle_{X_0(\Omega)} = \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y||^{N+2s}} dxdy.$$

For $u \in X_0(\Omega)$, we know that the norm and inner product can be extended to all $\mathbb{R}^N \times \mathbb{R}^N$.

Definition 2.1. A function $u(x,t)$ is a weak solution of problem (1.1) if for every $T>0$, u satisfies $u \in L^\infty(0,T;X_0(\Omega))$, $\frac{\partial u}{\partial t} \in L^\infty(0,T;L^2(\Omega))$, $\frac{\partial^2 u}{\partial t^2} \in L^2(0,T;X_0(\Omega))$ and for any $\varphi \in L^2(0,T;X_0(\Omega))$, a.e. $t \in [0,T]$,

$$\begin{aligned}
&\int_0^t \int_\Omega \frac{\partial^2 u}{\partial \tau^2}(x,\tau)\varphi(x,\tau)dxd\tau \\
&+ C(N,s)\int_0^t \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{(u(x,\tau)-u(y,\tau))(\varphi(x,\tau)-\varphi(y,\tau))}{|x-y|^{N+2s}}dxdyd\tau \\
=& \int_0^t \int_\Omega f(x,\tau)\varphi(x,\tau)dxd\tau
\end{aligned}$$

and for a.e. $x \in \Omega$,

$$u(x,0)=\phi(x),\ u_t(x,0)=\psi(x).$$

3. The existence and uniqueness of weak solution for (1.1)

In this section, we study the existence and uniqueness of weak solution for problem (1.1). First, via the results on eigenfunctions of fractional Laplace operators established in [18], we are able to apply the Galërkin method and construct finite-dimensional Galërkin approximations for (1.1). Then, we present a priori estimates, which allow us to pass to the limit and to obtain the desired weak solution u of (1.1).

Step 1. The existence of Galërkin approximate solutions of (1.1). As $C_0^\infty(Q_T)$ is dense in $L^2(Q_T)$, for $f \in L^2(Q_T)$, there exist $\{f_m\}_m \subset C_0^\infty(Q_T)$ such that $f_m \to f$ in $L^2(Q_T)$, as $m\to\infty$. Similarly, we take $\{\phi_m\}_m,\ \{\psi_m\}_m \subset C_0^\infty(\Omega)$ such that $\phi_m \to \phi$ in $X_0(\Omega)$ and $\psi_m \to \psi$ in $L^2(\Omega)$, as $m\to\infty$.

Let $\{e_k\}_k$ be the eigenfunctions corresponding to the sequence $\{\lambda_k\}_k$ of eigenvalues of the fractional Laplace operator $(-\Delta)^s$. $\{e_k\}_k$ is an orthonormal basis of $L^2(\Omega)$ and an orthogonal basis of $X_0(\Omega)$.

Define $I_m : \mathbb{R}^m \to \mathbb{R}^m$ as follows:

$$(I_m(\eta))_i = C(N,s)\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{\sum_{j=1}^m \eta_j \cdot (e_j(x)-e_j(y))(e_i(x)-e_i(y))}{|x-y|^{N+2s}}dxdy,$$

where $i=1,2,\cdots,m$. In the following, denote

$$\begin{aligned}
&(F_m(t))_i = \int_\Omega f_m(x,t)e_i(x)dx, \\
&(u_{0m})_i = \int_\Omega \phi_m(x)e_i(x)dx, \quad (u_{1m})_i = \int_\Omega \psi_m(x)e_i(x)dx.
\end{aligned}$$

Next, we consider the following Cauchy problem

$$\begin{cases} \eta''(t) + I_m(\eta(t)) = F_m(t), \\ \eta(0)=u_{0m},\ \eta'(0)=u_{1m}. \end{cases} \tag{3.1}$$

Denote $X(t)=\eta'(t)$, $Y(t)=(\eta(t),\eta'(t))$ and $J_m(t,Y(t))=(X(t),F_m(t)-I_m(\eta(t)))$. Then, the Cauchy problem (3.1) is equivalent to the following problem

$$\begin{cases} Y'(t)=J_m(t,Y(t)),\\ Y(0)=(u_{0m},u_{1m}). \end{cases} \tag{3.2}$$

Multiplying (3.2) by $Y(t)$, we have

$$Y'(t)Y(t)=\eta(t)\eta'(t)+\left(F_m(t)-I_m(\eta(t))\right)\eta'(t). \tag{3.3}$$

Since

$$\begin{aligned} I_m(\eta(t))\eta'(t)=&C(N,s)\\ &\times\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{\left(\sum\limits_{j=1}^{m}\eta_j(t)(e_j(x)-e_j(y))\right)\left(\sum\limits_{j=1}^{m}\eta_j'(t)(e_j(x)-e_j(y))\right)}{|x-y|^{N+2s}}dxdy,\\ F_m(t)\eta'(t)=&\int_\Omega f_m(x,t)\sum_{i=1}^{m}\eta_i'(t)e_i(x)dx, \end{aligned}$$

from (3.3) we get

$$\begin{aligned} &\frac{1}{2}\frac{d|Y|^2}{dt}+C(N,s)\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{\left(\sum\limits_{j=1}^{m}\eta_j(t)(e_j(x)-e_j(y))\right)\left(\sum\limits_{j=1}^{m}\eta_j'(t)(e_j(x)-e_j(y))\right)}{|x-y|^{N+2s}}dxdy\\ =&\eta(t)\eta'(t)+\int_\Omega f_m(x,t)\sum_{i=1}^{m}\eta_i'(t)e_i(x)dx\\ \le&|\eta(t)|^2+|\eta'(t)|^2+\int_\Omega(\sum_{j=1}^{m}\eta_j'(t)e_j(x))^2dx+\int_\Omega f_m^2(x,t)dx\\ =&|\eta(t)|^2+2|\eta'(t)|^2+\int_\Omega f_m^2(x,t)dx\\ \le&2|Y|^2+\int_\Omega f_m^2(x,t)dx. \end{aligned} \tag{3.4}$$

Note that

$$\begin{aligned} &\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{\left(\sum\limits_{j=1}^{m}\eta_j(t)(e_j(x)-e_j(y))\right)\left(\sum\limits_{j=1}^{m}\eta_j'(t)(e_j(x)-e_j(y))\right)}{|x-y|^{N+2s}}dxdy\\ =&\frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{d}{dt}\frac{\left(\sum\limits_{j=1}^{m}\eta_j(t)(e_j(x)-e_j(y))\right)^2}{|x-y|^{N+2s}}dxdy \end{aligned}$$

and

$$\sum_{j=1}^{m}\eta_j(0)e_j(x)=\sum_{j=1}^{m}(u_{0m})_je_j(x)=\phi_m(x),$$

$$\sum_{j=1}^{m}(\eta'(0))_j e_j(x) = \sum_{j=1}^{m}(u_{1m})_j e_j(x) = \psi_m(x),$$

integrating (3.4) with respect to t, we have

$$\frac{1}{2}|Y(t)|^2 - \frac{1}{2}|Y(0)|^2 + \frac{C(N,s)}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{\left(\sum\limits_{j=1}^{m}\eta_j(t)(e_j(x)-e_j(y))\right)^2}{|x-y|^{N+2s}}dxdy$$
$$\leq 2\int_0^t |Y(\tau)|^2 d\tau + \int_0^t\int_\Omega f_m^2(x,\tau)dxd\tau + \frac{C(N,s)}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{(\phi_m(x)-\phi_m(y))^2}{|x-y|^{N+2s}}dxdy.$$

As $f_m \to f$ in $L^2(Q_T)$, $\phi_m \to \phi$ in $X_0(\Omega)$ and $\psi_m \to \psi$ in $L^2(\Omega)$, by Gronwall Inequality, for any $t \in [0,T]$,

$$|Y(t)| \leq C(T).$$

From Peano Theorem, for any $m \in \mathbb{N}$, there exists $Y_m \in C^1([0,T])$ satisfying

$$\begin{cases} Y_m'(t) = J_m(t, Y_m(t)), \\ Y_m(0) = (u_{0m}, u_{1m}), \end{cases}$$

which implies that there exists $\eta_m \in C^2([0,T])$ satisfying

$$\begin{cases} \eta_m''(t) + I_m(\eta_m(t)) = F_m(t), \\ \eta_m(0) = u_{0m}, \quad \eta_m'(0) = u_{1m}. \end{cases} \tag{3.5}$$

Denote

$$u_m(x,t) = \sum_{j=1}^{m}(\eta_m(t))_j\, e_j(x), \quad m = 1, 2, \cdots,$$

then $\{u_m\}_m$ are the Galërkin approximate solutions of (1.1).

Step 2. A priori estimates of $\{u_m\}_m$.

Multiplying (3.5) by $\eta_m'(t)$, we have

$$\eta_m''(t)\eta_m'(t) + I_m(\eta_m(t))\eta_m'(t) = F_m(t)\eta_m'(t),$$

which implies

$$\begin{aligned} &\int_\Omega \frac{\partial^2 u_m}{\partial t^2}\frac{\partial u_m}{\partial t}dx + \frac{C(N,s)}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{d}{dt}\frac{(u_m(x,t)-u_m(y,t))^2}{|x-y|^{N+2s}}dxdy \\ &= \int_\Omega f_m(x,t)\frac{\partial u_m}{\partial t}dx. \end{aligned} \tag{3.6}$$

Note that

$$\int_\Omega \frac{\partial^2 u_m}{\partial t^2}\frac{\partial u_m}{\partial t}dx = \frac{1}{2}\int_\Omega \frac{\partial}{\partial t}(\frac{\partial u_m}{\partial t})^2 dx,$$
$$\frac{\partial u_m}{\partial t}(x,0) = \sum_{j=1}^{m}(\eta_m'(0))_j e_j(x) = \sum_{j=1}^{m}(u_{1m})_j e_j(x) = \psi_m(x),$$

then, integrating (3.6) with respect to t, we have

$$\begin{aligned}
&\frac{1}{2}\int_\Omega (\frac{\partial u_m}{\partial t}(x,t))^2dx + \frac{C(N,s)}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{(u_m(x,\tau)-u_m(y,\tau))^2}{|x-y|^{N+2s}}dxdy\\
=&\frac{1}{2}\int_\Omega \psi_m^2(x)dx + \frac{C(N,s)}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{(\phi_m(x)-\phi_m(y))^2}{|x-y|^{N+2s}}dxdy + \int_0^t\int_\Omega f_m(x,\tau)\frac{\partial u_m}{\partial \tau}dxd\tau\\
\leq&\frac{1}{2}\int_\Omega \psi_m^2(x)dx + \frac{C(N,s)}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{(\phi_m(x)-\phi_m(y))^2}{|x-y|^{N+2s}}dxdy\\
&+\frac{1}{2}\int_0^t\int_\Omega f_m^2(x,\tau)dxd\tau + \frac{1}{2}\int_0^t\int_\Omega(\frac{\partial u_m}{\partial \tau})^2dxd\tau.
\end{aligned} \tag{3.7}$$

Thus

$$\int_\Omega(\frac{\partial u_m}{\partial t}(x,t))^2dx \leq \int_0^t\int_\Omega(\frac{\partial u_m}{\partial \tau}(x,\tau))^2dxd\tau + C(T).$$

By Gronwall Inequality, for any $t\in[0,T]$,

$$\int_\Omega(\frac{\partial u_m}{\partial t}(x,t))^2dx \leq C(T).$$

It follows from (3.7) that for any $t\in[0,T]$,

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|u_m(x,t)-u_m(y,t)|^2}{|x-y|^{N+2s}}dxdy \leq C(T).$$

Then $\{u_m\}_m$ is bounded in $L^\infty(0,T;X_0(\Omega))$ and $\{\frac{\partial u_m}{\partial t}\}_m$ is bounded in $L^\infty\left(0,T;L^2(\Omega)\right)$.

For any $i\in\mathbb{N}$, from (3.5) we have

$$\begin{aligned}
&\int_\Omega\frac{\partial^2 u_m}{\partial t^2}e_i(x)dx + C(N,s)\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{u_m(x,t)-u_m(y,t)}{|x-y|^{N+2s}}(e_i(x)-e_i(y))dxdy\\
=&\int_\Omega f_m(x,t)e_i(x)dx.
\end{aligned}$$

Denote

$$V_n = \text{span}\{e_1,e_2,\cdots,e_n\}.$$

For any $\varphi\in\bigcup\limits_{n=1}^{\infty}V_n$,

$$\begin{aligned}
&\int_\Omega\frac{\partial^2 u_m}{\partial t^2}\varphi(x)dx + C(N,s)\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{u_m(x,t)-u_m(y,t)}{|x-y|^{N+2s}}(\varphi(x)-\varphi(y))dxdy\\
=&\int_\Omega f_m(x,t)\varphi(x)dx.
\end{aligned}$$

Then, for any $\varphi\in L^2(0,T;X_0(\Omega))$,

$$\int_0^t\int_\Omega\frac{\partial^2 u_m}{\partial t^2}\varphi dxd\tau + C(N,s)\int_0^t\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{u_m(x,\tau)-u_m(y,\tau)}{|x-y|^{N+2s}}(\varphi(x,\tau)-\varphi(y,\tau))dxdyd\tau$$

$$=\int_0^t \int_\Omega f_m(x,\tau)\varphi(x,\tau)dxd\tau, \tag{3.8}$$

which implies

$$\int_0^t \int_\Omega \frac{\partial^2 u_m}{\partial \tau^2}\varphi dxd\tau \le C(N,S)\int_0^t \|u_m\|_{X_0(\Omega)}\|\varphi\|_{X_0(\Omega)}d\tau + \|f_m\|_{L^2(Q_T)}\|\varphi\|_{L^2(Q_T)}$$
$$\le C(N,s)\cdot T^{\frac{1}{2}} \sup_{t\in[0,T]} \|u_m\|_{X_0(\Omega)}\|\varphi\|_{L^2(0,T;X_0(\Omega))} + \|f_m\|_{L^2(Q_T)}\|\varphi\|_{L^2(Q_T)}$$
$$\le C(N,s,T)\|\varphi\|_{L^2(0,T;X_0(\Omega))}. \tag{3.9}$$

Thus, $\{\frac{\partial^2 u_m}{\partial t^2}\}_m$ is bounded in $\big(L^2\big(0,T;(X_0(\Omega))^*\big)\big)^* = L^2(0,T;X_0(\Omega))$.

Step 3. The existence of weak solution for (1.1).

Let

$$u_m \to u \text{ weakly}-* \text{ in } L^\infty(0,T;X_0(\Omega)) \tag{3.10}$$

and

$$\frac{\partial u_m}{\partial t} \to w \text{ weakly}-* \text{ in } L^\infty\left(0,T;L^2(\Omega)\right).$$

We have $w=\frac{\partial u}{\partial t}$. In fact, for any $\varphi\in C_0^\infty(Q_T)$,

$$\int_{Q_T} \frac{\partial u_m}{\partial t}\varphi dxdt = -\int_{Q_T} u_m \frac{\partial \varphi}{\partial t}dxdt$$

and

$$\int_{Q_T} \frac{\partial u_m}{\partial t}\varphi dxdt \to \int_{Q_T} w\varphi dxdt,$$
$$\int_{Q_T} u_m \frac{\partial \varphi}{\partial t}dxdt \to \int_{Q_T} u\frac{\partial \varphi}{\partial t}dxdt.$$

Thus,

$$\int_{Q_T} w\varphi dxdt = -\int_{Q_T} u\frac{\partial \varphi}{\partial t}dxdt,$$

which implies $w=\frac{\partial u}{\partial t}$. Then,

$$\frac{\partial u_m}{\partial t} \to \frac{\partial u}{\partial t} \text{ weakly}-* \text{ in } L^\infty(0,T;L^2(\Omega)). \tag{3.11}$$

Similarly, we could get that

$$\frac{\partial^2 u_m}{\partial t^2} \to \frac{\partial^2 u}{\partial t^2} \text{ weakly in } L^2(0,T;X_0(\Omega)).$$

Let $m\to\infty$, from (3.8) we have

$$\int_0^t\int_\Omega \frac{\partial^2 u}{\partial \tau^2}\varphi dxd\tau + C(N,s)\int_0^t\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{u(x,\tau)-u(y,\tau)}{|x-y|^{N+2s}}(\varphi(x,\tau)-\varphi(y,\tau))dxdyd\tau$$
$$=\int_0^t \int_\Omega f(x,\tau)\varphi(x,\tau)dxd\tau, \tag{3.12}$$

for any $\varphi\in L^2(0,T;X_0(\Omega))$.

In the following, we will verify that $u(x,0)=\phi(x)$ and $\frac{\partial u}{\partial t}(x,0)=\psi(x)$ a.e. in Ω.

In fact, for any $\varphi\in C_0^\infty(\Omega)$,

$$\int_\Omega (u_m(x,0)-u(x,0))\varphi(x)dx=\int_\Omega (u_m(x,t)-u(x,t))\varphi(x)dx - \int_0^t\int_\Omega (\frac{\partial u_m}{\partial \tau}-\frac{\partial u}{\partial \tau})\varphi(x)dxd\tau,$$

which implies

$$\begin{aligned} &T\int_\Omega (u_m(x,0)-u(x,0))\varphi(x)dx\\ =&\int_0^T\int_\Omega (u_m(x,t)-u(x,t))\varphi(x)dxdt-\int_0^T\int_0^t\int_\Omega (\frac{\partial u_m}{\partial \tau}-\frac{\partial u}{\partial \tau})\varphi(x)dxd\tau dt. \end{aligned} \tag{3.13}$$

From (3.10) and (3.11),

$$\int_0^T\int_\Omega (u_m(x,t)-u(x,t))\varphi(x)dxdt\to 0$$

and for any $t\in[0,T]$,

$$\int_0^t\int_\Omega (\frac{\partial u_m}{\partial \tau}-\frac{\partial u}{\partial \tau})\varphi(x)dxd\tau\to 0.$$

In addition,

$$\int_0^t\int_\Omega (\frac{\partial u_m}{\partial \tau}-\frac{\partial u}{\partial \tau})\varphi(x)dxd\tau\le\int_0^t\left\|\frac{\partial u_m}{\partial \tau}-\frac{\partial u}{\partial \tau}\right\|_{L^2(\Omega)}\|\varphi\|_{L^2(\Omega)}d\tau\le C(T).$$

By using Lebesgue Dominated Convergence Theorem, we obtain

$$\int_0^T\int_0^t\int_\Omega (\frac{\partial u_m}{\partial \tau}-\frac{\partial u}{\partial \tau})\varphi(x)dxd\tau dt\to 0.$$

From (3.13), for any $\varphi\in L^2(\Omega)$,

$$\int_\Omega (u_m(x,0)-u(x,0))\varphi(x)dx\to 0.$$

As $u_m(x,0)=\phi_m(x)\to\phi$ in $X_0(\Omega)$, we get

$$u(x,0)=\phi(x) \text{ a.e. in } \Omega.$$

Similarly, we could verifty that

$$\frac{\partial u}{\partial t}(x,0)=\psi(x) \text{ a.e. in } \Omega.$$

Then u is a weak solution of (1.1).

Step 4. The uniqueness of weak solution for (1.1).

In this part, we denote

$$U(x,t)=\int_0^t u(x,\tau)d\tau.$$

Then, $\frac{\partial U}{\partial t}=u$, $\frac{\partial^2 U}{\partial t^2}=\frac{\partial u}{\partial t}$, $U(x,0)=0$, $\frac{\partial U}{\partial t}(x,0)=\phi(x)$, $\frac{\partial^2 U}{\partial t^2}(x,0)=\psi(x)$.

From (3.12), for any $\varphi\in\bigcup_{n=1}^{\infty}V_n$,

$$\begin{aligned}&\int_\Omega \frac{\partial^2 U}{\partial t^2}(x,t)\varphi(x)dx-\int_\Omega \psi(x)\varphi(x)dx\\&+C(N,s)\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{U(x,t)-U(y,t)}{|x-y|^{N+2s}}(\varphi(x)-\varphi(y))dxdy\\=&\int_\Omega\Big(\int_0^t f(x,\tau)d\tau\Big)\varphi(x)dx.\end{aligned}\tag{3.14}$$

Next, we denote $F(x,t)=\int_0^t f(x,\tau)d\tau$. Then, for any $\varphi\in L^2(0,T;X_0(\Omega))$,

$$\begin{aligned}&\int_0^t\int_\Omega \frac{\partial^2 U}{\partial \tau^2}(x,\tau)\varphi(x,\tau)dxd\tau-\int_0^t\int_\Omega \psi(x)\varphi(x,\tau)dxd\tau\\&+C(N,s)\int_0^t\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{U(x,\tau)-U(y,\tau)}{|x-y|^{N+2s}}(\varphi(x,\tau)-\varphi(y,\tau))dxdyd\tau\\=&\int_0^t\int_\Omega F(x,\tau)\varphi(x,\tau)dxd\tau.\end{aligned}\tag{3.15}$$

Assume that u_i $(i=1,2)$ are weak solutions of

$$\begin{cases}\frac{\partial^2 u_i}{\partial t^2}+(-\Delta)^s u_i=f_i(x,t), & x\in\Omega, t>0,\\ u_i(x,0)=\phi_i(x),\ \frac{\partial u_i}{\partial t}(x,0)=\psi_i(x), & x\in\Omega,\\ u_i(x,t)=0, & x\in\mathbb{R}^N\setminus\Omega, t>0.\end{cases}$$

We denote

$$F_i(x,t)=\int_0^t f_i(x,\tau)d\tau,\quad U_i(x,t)=\int_0^t u_i(x,\tau)d\tau,$$

then, for U_i $(i=1,2)$ we have (3.15).

Denote $W=U_1-U_2$. For any $\varphi\in L^2(0,T;X_0(\Omega))$,

$$\begin{aligned}&\int_0^t\int_\Omega \frac{\partial^2 W}{\partial \tau^2}(x,\tau)\varphi(x,\tau)dxd\tau-\int_0^t\int_\Omega (\psi_1(x)-\psi_2(x))\varphi(x,\tau)dxd\tau\\&+C(N,s)\int_0^t\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{W(x,\tau)-W(y,\tau)}{|x-y|^{N+2s}}(\varphi(x,\tau)-\varphi(y,\tau))dxdyd\tau\\=&\int_0^t\int_\Omega (F_1(x,\tau)-F_2(x,\tau))\varphi(x,\tau)dxd\tau.\end{aligned}$$

Take $\varphi = \frac{\partial U_1}{\partial t} - \frac{\partial U_2}{\partial t}$,

$$\begin{aligned}
&\frac{1}{2}\int_0^t\int_\Omega \frac{\partial}{\partial\tau}(\frac{\partial W}{\partial\tau})^2 dxd\tau - \int_0^t\int_\Omega (\psi_1(x)-\psi_2(x))\frac{\partial W}{\partial\tau}dxd\tau\\
&+\frac{C(N,s)}{2}\int_0^t\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{\partial}{\partial\tau}\frac{(W(x,\tau)-W(y,\tau))^2}{|x-y|^{N+2s}}dxdyd\tau\\
=&\int_0^t\int_\Omega (F_1(x,\tau)-F_2(x,\tau))\frac{\partial W}{\partial\tau}dxd\tau,
\end{aligned}$$

which implies

$$\begin{aligned}
&\frac{1}{2}\int_\Omega(\frac{\partial W}{\partial t}(x,t))^2dx + \frac{C(N,s)}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{(W(x,t)-W(y,t))^2}{|x-y|^{N+2s}}dxdy\\
=&\frac{1}{2}\int_\Omega(\phi_1(x)-\phi_2(x))^2dx + \int_0^t\int_\Omega(\psi_1(x)-\psi_2(x))\frac{\partial W}{\partial\tau}dxd\tau\\
&+\int_0^t\int_\Omega (F_1(x,\tau)-F_2(x,\tau))\frac{\partial W}{\partial\tau}dxd\tau\\
\le&\int_0^t\int_\Omega(\frac{\partial W}{\partial\tau})^2dxd\tau + \frac{1}{2}\int_\Omega(\phi_1(x)-\phi_2(x))^2dx\\
&+\frac{1}{2}\int_0^t\int_\Omega(\psi_1(x)-\psi_2(x))^2dxd\tau + \frac{1}{2}\int_0^t\int_\Omega (F_1(x,\tau)-F_2(x,\tau))^2dxd\tau.
\end{aligned}$$

From Gronwall Inequality, for any $t\in[0,T]$,

$$\begin{aligned}
&\int_\Omega(u_1-u_2)^2dx = \int_\Omega(\frac{\partial U_1}{\partial t}-\frac{\partial U_2}{\partial t})^2dx = \int_\Omega(\frac{\partial W}{\partial t})^2dx\\
\le& C(T)\Big(\int_\Omega(\phi_1(x)-\phi_2(x))^2dx + \int_0^T\int_\Omega(F_1(x,\tau)-F_2(x,\tau))^2dxd\tau\\
&+\int_\Omega(\psi_1(x)-\psi_2(x))^2dx\Big).
\end{aligned} \tag{3.16}$$

From the above inequality, we could get the uniqueness of weak solution for (1.1).

4. The existence of an optimal control

We consider the following nonlinear control problem:

$$\begin{cases}
\frac{\partial^2 u}{\partial t^2} + (-\Delta)^s u = f(x,t) + Bv, & x\in\Omega, t>0,\\
u(v;x,0)=\phi(x),\ \frac{\partial u}{\partial t}(v;x,0)=\psi(x), & x\in\Omega,\\
u(v;x,t)=0, & x\in\mathbb{R}^N\setminus\Omega, t>0,
\end{cases} \tag{4.1}$$

where $\phi\in X_0(\Omega)$, $\psi\in L^2(\Omega)$, $f\in L^2(Q_T)$, $v\in\mathscr{U}$ is a control. From Section 3, we can define uniquely the solution map for (4.1):

$$\begin{aligned}
&\mathscr{U}\to L^\infty(0,T;X_0(\Omega)),\\
&v\mapsto u(v;x,t)\triangleq u(v).
\end{aligned}$$

We call the solution $u(v;x,t)$ the state of the control system (4.1). The observation of the state is given by

$$z(v) = Cu(v),$$

where $C \in \mathscr{L}(L^2(Q_T), M)$ is called the observer and M is a Hilbert space of observation variables.

For $v \in \mathscr{U}$, the quadratic cost function associated with (4.1) is given by

$$J(v) = \|Cu(v) - z_d\|_M^2 + (Rv, v)_{\mathscr{U}},$$

where $z_d \in M$ is a desired value of $u(v)$ and $R \in \mathscr{L}(\mathscr{U}, \mathscr{U})$ is symmetric and positive, i.e.,

$$(Rv, v)_{\mathscr{U}} = (v, Rv)_{\mathscr{U}} \geq d\|v\|_{\mathscr{U}}^2,$$

for some $d > 0$. Let U_{ad} be an admissible set (a closed convex subset of $\mathscr{U}$). An element $v_0 \in U_{ad}$ is called an optimal control for the cost function $J(v)$ if

$$J(v_0) = \min_{v \in U_{ad}} J(v).$$

Theorem 4.1. *There exists at least one optimal control v_0 for the control problem (4.1).*

Proof. For any $v \in U_{ad}$, $J(v) \geq 0$. There exists $\{v_n\}_n \subset U_{ad}$ such that

$$J(v_n) \to \inf_{v \in U_{ad}} J(v) \triangleq J \geq 0,$$

which implies that $\{J(v_n)\}_n$ is bounded. As

$$J(v_n) \geq (Rv_n, v_n)_{\mathscr{U}} \geq d\|v_n\|_{\mathscr{U}}^2,$$

$\{v_n\}_n$ is bounded in $\mathscr{U}$. We assume that

$$v_n \to v_0 \quad \text{weakly in } \mathscr{U}.$$

In the following, let $u_n := u(v_n; x, t) \in L^\infty(0, T; X_0(\Omega))$ be the solution for

$$\begin{cases} \frac{\partial^2 u_n}{\partial t^2} + (-\Delta)^s u_n = f(x,t) + Bv_n, & x \in \Omega, t > 0, \\ u_n(x,0) = \phi(x), \ \frac{\partial u_n}{\partial t}(x,0) = \psi(x), & x \in \Omega, \\ u_n(x,t) = 0, & x \in \mathbb{R}^N \setminus \Omega, t > 0. \end{cases}$$

Next, we denote

$$U_n(x,t) = \int_0^t u_n(x,\tau)d\tau.$$

Then, for any $\varphi \in L^2(0,T; X_0(\Omega))$,

$$\begin{aligned} &\int_0^t \int_\Omega \frac{\partial^2 U_n}{\partial \tau^2}(x,\tau)\varphi(x,\tau)dxd\tau - \int_0^t \int_\Omega \psi(x)\varphi(x,\tau)dxd\tau \\ &+ C(N,s)\int_0^t \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{U_n(x,\tau) - U_n(y,\tau)}{|x-y|^{N+2s}}(\varphi(x,\tau) - \varphi(y,\tau))dxdyd\tau \qquad (4.2) \\ =& \int_0^t \int_\Omega \left(\int_0^\tau (f(x,s) + Bv_n)ds\right)\varphi(x,\tau)dxd\tau. \end{aligned}$$

As the operator B is bounded,

$$\|Bv_n\|_{L^2(Q_T)} \le \|B\|\|v_n\|_{\mathscr{U}} \le C.$$

From (3.16),

$$\begin{aligned}&\int_\Omega (\frac{\partial U_n}{\partial t}(x,t))^2 dx + \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{(U_n(x,t)-U_n(y,t))^2}{|x-y|^{N+2s}}dxdy\\ \le& C\left(\int_\Omega \phi^2(x)dx + \int_\Omega \psi^2(x)dx + \int_0^t\int_\Omega\left(\int_0^\tau (f(x,s)+Bv_n)ds\right)^2 dxd\tau\right).\end{aligned}$$

As

$$\left(\int_0^t Bv_n ds\right)^2 \le t\int_0^t (Bv_n)^2 ds,$$

we have

$$\int_{Q_T}\left(\int_0^t Bv_n ds\right)^2 dxdt \le \int_{Q_T} t\int_0^t (Bv_n)^2 dsdxdt \le C.$$

Then, $\{\int_0^t Bv_n ds\}$ is bounded in $L^2(Q_T)$. Thus $\{U_n\}_n$ is bounded in $L^\infty(0,T;X_0(\Omega))$ and $\{\frac{\partial U_n}{\partial t}\}_n$ is bounded in $L^\infty(0,T;L^2(\Omega))$.

From (3.14), for any $\varphi \in X_0(\Omega)$,

$$\begin{aligned}&\int_\Omega \frac{\partial^2 U_n}{\partial t^2}(x,t)\varphi(x)dx - \int_\Omega \psi(x)\varphi(x)dx\\ &+ C(N,s)\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{U_n(x,t)-U_n(y,t)}{|x-y|^{N+2s}}(\varphi(x)-\varphi(y))dxdy\\ =&\int_\Omega\big(\int_0^t (f(x,\tau)+Bv_n)d\tau\big)\varphi(x)dx. \end{aligned}\tag{4.3}$$

Similarly to the discussion of (3.9), we could verify that $\{\frac{\partial^2 U_n}{\partial t^2}\}_n$ is bounded in $L^\infty(0,T;(X_0(\Omega))^*)$.

In the following, we assume that

$$\int_0^t Bv_n ds \to w \text{ weakly in } L^2(Q_T).$$

For any $\varphi \in C_0^\infty(Q_T)$,

$$\int_{Q_T}\left(\int_0^t Bv_n ds\right)\frac{\partial\varphi}{\partial t}dxdt = -\int_{Q_T} Bv_n\cdot\varphi dxdt$$

and

$$\int_{Q_T}\left(\int_0^t Bv_n ds\right)\frac{\partial\varphi}{\partial t}dxdt \to \int_{Q_T} w\frac{\partial\varphi}{\partial t}dxdt.$$

Note that $Bv_n \to Bv_0$ weakly in $L^2(Q_T)$, we have

$$\int_{Q_T} Bv_n\cdot\varphi dxdt \to \int_{Q_T} Bv_0\cdot\varphi dxdt.$$

Then,

$$-\int_{Q_T} w\frac{\partial \varphi}{\partial t}dxdt = \int_{Q_T} Bv_0 \cdot \varphi dxdt,$$

which implies

$$\frac{\partial w}{\partial t} = Bv_0.$$

Thus $w = \int_0^t Bv_0 ds$. We obtain

$$\int_0^t Bv_n ds \to \int_0^t Bv_0 ds \text{ weakly in } L^2(Q_T).$$

we assume that

$$U_n \to U \text{ weakly}-* \text{ in } L^\infty\left(0,T; X_0(\Omega)\right).$$

$$\frac{\partial U_n}{\partial t} \to \frac{\partial U}{\partial t} \text{ weakly}-* \text{ in } L^\infty\left(0,T; L^2(\Omega)\right).$$

Then, similar to the discussion in Section 3, we could verify that

$$\frac{\partial U}{\partial t}(x,0) = u(x,0) = \phi(x) \text{ a.e. in } \Omega$$

and

$$\frac{\partial u}{\partial t}(x,0) = \frac{\partial^2 U}{\partial t^2}(x,0) = \psi(x) \text{ a.e. in } \Omega.$$

Let $n \to \infty$, from (4.3) we get that for any $\varphi \in X_0(\Omega)$,

$$\begin{aligned}
&\int_\Omega \frac{\partial^2 U}{\partial t^2}(x,t)\varphi(x)dx - \int_\Omega \psi(x)\varphi(x)dx \\
&+ C(N,s)\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{U(x,t)-U(y,t)}{|x-y|^{N+2s}}(\varphi(x)-\varphi(y))dxdy \\
= &\int_\Omega \left(\int_0^t (f(x,s)+Bv_0)ds\right)\varphi(x)dx.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\int_\Omega \frac{\partial u}{\partial t}(x,t)\varphi(x)dx - \int_\Omega \psi(x)\varphi(x)dx \\
&+ C(N,s)\int_0^t\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{u(x,\tau)-u(y,\tau)}{|x-y|^{N+2s}}(\varphi(x)-\varphi(y))dxdyd\tau \\
= &\int_\Omega \left(\int_0^t (f(x,s)+Bv_0)ds\right)\varphi(x)dx.
\end{aligned}$$

Differentiating the above equality with respect to t, we get

$$\begin{aligned}
&\int_\Omega \frac{\partial^2 u}{\partial t^2}(x,t)\varphi(x)dx + C(N,s)\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{u(x,t)-u(y,t)}{|x-y|^{N+2s}}(\varphi(x)-\varphi(y))dxdy \\
= &\int_\Omega (f(x,s)+Bv_0)\varphi(x)dx.
\end{aligned}$$

Then, for any $\varphi \in L^2(0,T;X_0(\Omega))$, we have

$$\int_0^t \int_\Omega \frac{\partial^2 u}{\partial \tau^2}(x,\tau)\varphi(x,\tau)dxd\tau$$
$$+ C(N,s)\int_0^t \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{u(x,\tau)-u(y,\tau)}{|x-y|^{N+2s}}(\varphi(x,\tau)-\varphi(y,\tau))dxdyd\tau$$
$$= \int_0^t \int_\Omega (f(x,\tau)+Bv_0)\varphi(x,\tau)dxd\tau.$$

For any $\varphi \in C_0^\infty(\Omega)$,

$$\int_0^t \int_\Omega (\frac{\partial^2 U_n}{\partial \tau^2} - \frac{\partial^2 U_m}{\partial \tau^2})\varphi dxd\tau = \int_\Omega (\frac{\partial U_n}{\partial \tau} - \frac{\partial U_m}{\partial \tau})\varphi dx.$$

Thus $\{\frac{\partial U_n}{\partial t}\}_n$ is a Cauchy sequence in the weak topology of $L^2(\Omega)$, for any $t \in [0,T]$.

For any $\varphi \in C_0^\infty(Q_T)$,

$$\int_0^T \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{(\frac{\partial}{\partial \tau}U_n(x,\tau) - \frac{\partial}{\partial \tau}U_n(y,\tau))}{|x-y|^{N+2s}}(\varphi(x,\tau)-\varphi(y,\tau))dxdyd\tau$$
$$= -\int_0^T \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{(U_n(x,\tau)-U_n(y,\tau))}{|x-y|^{N+2s}}(\frac{\partial}{\partial \tau}\varphi(x,\tau) - \frac{\partial}{\partial \tau}\varphi(y,\tau))dxdyd\tau,$$

which implies that $\{\frac{\partial U_n}{\partial t}\}_n$ is a Cauchy sequence in the weak topology of $L^2(0,T;X_0(\Omega))$. Then, $\{\frac{\partial U_n}{\partial t}\}_n$ is bounded in $L^2(0,T;X_0(\Omega))$. We could verify that $\frac{\partial U_n}{\partial t} \to \frac{\partial U}{\partial t}$ in $L^2(Q_T)$ (see Appendix), i.e. $u_n \to u$ in $L^2(Q_T)$.

As $C \in \mathscr{L}(L^2(Q_T), M)$, we have $Cu_n \to Cu$. Then,

$$\lim_{n\to\infty} \|Cu_n - z_d\|_M \geq \|Cu - z_d\|_M.$$

As $\lim_{n\to\infty}(Rv_n, v_n)_{\mathscr{U}} \geq (Rv_0, v_0)_{\mathscr{U}}$,

$$J = \lim_{n\to\infty} J(v_n) \geq J(v_0).$$

Then, by the definition of J,

$$J = J(v_0) = \inf_{v\in U_{ad}} J(v).$$

□

Appendix

Theorem 4.2. *Let $\{u_n\}_n$ be a bounded sequence in $L^2(0,T;X_0(\Omega))$ and $L^\infty(0,T;L^1(\Omega))$, $u_n(x,t) \to u(x,t)$ weakly in $L^1(\Omega)$ for a.e. $t \in [0,T]$, then $u_n \to u$ in $L^2(Q_T)$.*

Proof. In the following, we denote

$$u_\delta(x,t) = \int_{\mathbb{R}^{\mathbb{N}}} \rho_\delta(x-y)u(y,t)dy,$$

$$u_{n\delta}(x,t)=\int_{\mathbb{R}^{\mathbb{N}}}\rho_\delta(x-y)u_n(y,t)dy,$$

where $\delta>0$ and $\rho_\delta(x)=\rho(\frac{x}{\delta})$ is the mollifier.

As $u_n(x,t)\to u(x,t)$ weakly in $L^1(\Omega)$ for a.e. $t\in[0,T]$ and $\rho_\delta\in L^\infty(\Omega)$, we have $u_{n\delta}(x,t)\to u_\delta(x,t)$ a.e. in Q_T, as $n\to\infty$. By Hölder inequality,

$$\begin{aligned}&u_{n\delta}(x,t)-u_n(x,t)\\&=\int_{\mathbb{R}^{\mathbb{N}}}\frac{u_n(y,t)-u_n(x,t)}{|x-y|^{\frac{N}{2}+s}}|x-y|^{\frac{N}{2}+s}\rho_\delta(x-y)dy\\&\le\left(\int_{\mathbb{R}^{\mathbb{N}}}\frac{\left(u_n(y,t)-u_n(x,t)\right)^2}{|x-y|^{N+2s}}dy\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^{\mathbb{N}}}|x-y|^{N+2s}\rho_\delta^2(x-y)dy\right)^{\frac{1}{2}}.\end{aligned}$$

Note that

$$\int_{\mathbb{R}^N}|x-y|^{N+2s}\rho_\delta^2(x-y)dy=\int_{|z|\le 1}\delta^{2s}|z|^{N+2s}\rho(z)dz\le\delta^{2s},$$

which implies

$$\begin{aligned}&\int_{\mathbb{R}^{\mathbb{N}}}\left(u_{n\delta}(x,t)-u_n(x,t)\right)^2dx\\\le&\delta^{2s}\iint_{\mathbb{R}^{\mathbb{N}}\times\mathbb{R}^{\mathbb{N}}}\frac{\left(u_n(x,t)-u_n(y,t)\right)^2}{|x-y|^{N+2s}}dxdy\\\le&\delta^{2s}\|u_n\|_{X_0(\Omega)}^2.\end{aligned}$$

Thus,

$$\|u_{n\delta}-u_n\|_{L^2(\Omega)}\le\delta^s\|u_n\|_{X_0(\Omega)}.$$

For any $\varepsilon>0$, we have

$$\begin{aligned}&\|u_{n\delta}-u_{m\delta}\|_{L^2(\Omega)}\\&\le\varepsilon\|u_{n\delta}-u_{m\delta}\|_{L^{2_s^*}(\Omega)}+C\|u_{n\delta}-u_{m\delta}\|_{L^1(\Omega)}\\&\le\varepsilon\left(\|u_{n\delta}\|_{L^{2_s^*}(\Omega)}+\|u_{m\delta}\|_{L^{2_s^*}(\Omega)}\right)+C\|u_{n\delta}-u_{m\delta}\|_{L^1(\Omega)}\\&\le C\varepsilon\left(\|u_n\|_{L^{2_s^*}(\Omega)}+\|u_m\|_{L^{2_s^*}(\Omega)}\right)+C\|u_{n\delta}-u_{m\delta}\|_{L^1(\Omega)}\\&\le C\varepsilon\left(\|u_n\|_{X_0(\Omega)}+\|u_m\|_{X_0(\Omega)}\right)+C\|u_{n\delta}-u_{m\delta}\|_{L^1(\Omega)},\end{aligned}$$

which implies

$$\begin{aligned}&\int_0^t\|u_{n\delta}-u_{m\delta}\|_{L^2(\Omega)}^2d\tau\\&\le C\varepsilon^2\int_0^t\left(\|u_n\|_{X_0(\Omega)}^2+\|u_m\|_{X_0(\Omega)}^2\right)dt+C\int_0^t\|u_{n\delta}-u_{m\delta}\|_{L^1(\Omega)}^2d\tau.\end{aligned}$$

As

$$u_{n\delta}(x,t)\to u_\delta(x,t)\text{ a.e. in }Q_T,\ \text{as }n\to\infty,$$

for fixed $\delta>0$,

$$\int_\Omega|u_{n\delta}(x,t)|dx=\int_\Omega\Big|\int_{\mathbb{R}^{\mathbb{N}}}u_n(y,t)\rho_\delta(x-y)\Big|dx\le C.$$

Then, from Lebesgue Dominated Convergence Theorem,

$$\int_0^t \|u_{n\delta} - u_{m\delta}\|^2_{L^1(\Omega)} d\tau \to 0,$$

as $m, n \to \infty$. Note that

$$\begin{aligned}
&\int_0^t \|u_n - u_m\|^2_{L^2(\Omega)} d\tau \\
\leq &\int_0^t \|u_n - u_{n\delta}\|^2_{L^2(\Omega)} d\tau + \int_0^t \|u_{n\delta} - u_{m\delta}\|^2_{L^2(\Omega)} d\tau + \int_0^t \|u_{m\delta} - u_m\|^2_{L^2(\Omega)} d\tau \\
\leq &\, C\delta^{2\delta} + C\varepsilon^2 + \int_0^t \|u_{n\delta} - u_{m\delta}\|^2_{L^1(\Omega)} d\tau.
\end{aligned}$$

As ε and δ are arbitrary, we could verify that $\{u_n\}_n$ is a Cauchy sequence in $L^2(Q_T)$. □

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