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NEW EXACT SOLUTIONS FOR COUPLED SCHRÖDINGER-BOUSSINESQ EQUATIONS*

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Abstract Due to the importance of the coupled Schrödinger-Boussinesq equations (CSBEs) in applied physics, many mathematicians and physicists are interesting to CSBEs. One of the main tasks of studying CSBEs is to obtain the exact solutions for CSBEs. In this paper, we firstly use the coupled Riccati equations to change the polynomial expansion method. Secondly, CSBEs are changed into coupled ordinary differential equations by the traveling wave solution transformation. Then, we assume that the solutions for the coupled ordinary differential equations satisfy the coupled Riccati equations and substitute the solutions of the coupled Riccati equations into the coupled ordinary differential equations. By calculating the algebra system, we successfully construct more new exact traveling wave solutions for CSBEs with distinct physical structures. The exact solutions with arbitrary parameters are expressed by sech, sech^2, tanh, sinh, cosh, et al, functions, respectively. When the parameters are taken as special values, some examples are given to demonstrate the solutions and their physical meaning.

Keywords Nonlinear evolution equation, coupled Riccati equations, nonlinear partial differential equation, polynomial expansion method, solitary wave solution.

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1. Introduction

It is well known that most of the phenomena that arise in mathematical physics and engineering field can be described by partial differential equations (PDEs). For example,in the fields of applied mathematics, applied physics, fluid dynamics, quantum mechanics, electricity, plasma physics, ecology, human biology, propagation of shallow water waves, meteorology, zoology, botany, engineering, oceanography, medicine, and computer science, et al, most phenomena and models are well described by PDEs, see [1,2,5,8,23,29,30,39,42,44,58] and the references therein.

In 1926, Erwin Schrödinger, based on the three major principles, the de Broglie's hypothesis of matter wave, the law of conservation of energy and the classical plane wave equation, found a new equation which is called the time independent Schrödinger equation. The equation has sufficiently illuminated atomic phenomena and dynamical centerpiece of quantum wave mechanics. The Nonlinear Schrödinger

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equation is a prototypical dispersive nonlinear complex partial differential equation which has been derived and analyzed in many areas of physics and mathematics. For example, in [43], the authors studied the nonautonomous Schrödinger-Hirota equation with power-law nonlinearity via the unified method and found different types of optical wave solutions; in laser and plasma physics, the solution problem under interaction of a nonlinear complex Schrödinger field and a real Boussinesq field had been raised in [17] and the references therein; in [12], the authors studied the solitary waves for the generalized nonautonomous dual-power nonlinear Schrödinger equations. In addition, the approximate and exact solutions and the laws of conservation for the Schrödinger-Boussinesq dynamics system have been studied. Their applications in phenomenon of self-focusing and conditions under an electromagnetic beam can propagate without spreading in nonlinear media. In the general situations, an optical beam is broadened in a dielectric due to the diffraction. However, in materials whose dielectric constant increases with the field intensity, the critical angle for internal reflection at the beam's boundary can become greater than the angular divergence due to the diffraction and as a consequence the beam does not spread and can, in some situations, continue to focus into extremely high intensity spots.

Recently, the coupled nonlinear partial differential equations have often been proposed to describe the interaction of the long-waves with the short-wave packets in nonlinear dispersive media. It is well known that a high-frequency wave with a modulated-amplitude can lead to the excitation of an instability called the 'modulation instability' in dispersive media such as plasma. The nonlinear development of the instability is typically governed by the Schrödinger-like equation having a 'potential' which depends on the associated low-frequency perturbations. The latter is governed by a linear wave equation [45] or, in some cases, by the nonlinear Boussinesq equation [6], which is driven by the so-called ponderomotive force due to the high-frequency carrier wave [49]. For example, the nonlinear development of modulational instabilities associated with Langmuir field amplitude coupled to intense electromagnetic waves in dispersive media, such as plasmas, is known to be governed by CSBEs [21]

$$\begin{aligned} &iu_t - u_{xx} + uv = 0, \quad x \in \mathbb{R}, \quad t > 0, \\ &v_{tt} - v_{xx} + \alpha \left(v^2\right)_{xx} + \beta v_{xxxx} = \left(|u|^2\right)_{xx}, \quad x \in \mathbb{R}, \quad t > 0, \end{aligned} \tag{1.1}$$

where $i = \sqrt{-1}$, $u(x,t)$ represents the complex Schrödinger field, which provides a canonical description for the envelope dynamics of a quasi-monochromatic plane wave propagating in weakly nonlinear dispersive medium when dissipative processes are negligible. On short times and small propagation distances, the dynamics are linear, but cumulative nonlinear interactions result in a significant modulation of the wave amplitude on large spatial and temporal scales [50]; $v(x,t)$ represents the real Boussinesq field, which describes the propagation of long waves in shallow water under gravity propagation in both directions, and $v(x,t)$ also arises in other physical applications such as nonlinear lattice waves, iron sound waves in a plasma, and vibrations in a nonlinear string. It is used in many physical applications such as the percolation of water in porous subsurface of a horizontal layer of material [55]; α and β are real parameters, the subscripts t and x denote the partial differentiation with respect to time and space, respectively. The coupled equations (1.1) are considered as a model of the interactions between short and intermediate long waves, and are originated in describing the diatomic lattice system [57] and the dynamics of Langmuir solution formation, the interaction in a plasma [32, 59], etc. The system (1.1) is known to describe various physical processes in laser and plasma, such as formation, Langmuir field amplitude and intense electromagnetic waves and modulational instabilities.

Nowadays, the investigation of the exact solutions, especially the solitary wave solutions, of the complex nonlinear equations plays a considerable role due to the expectant effectuation in the real world, for example, in different aspects of mathematical and physical phenomena. Most complex phenomena arising in applied science, such as nuclear physics, chemical reactions, signal processing, optical fibers, fluid mechanics, plasma, nonlinear optics and ecology, etc, can be sometimes modeled and described by these equations. Hereby, a massive number of mathematicians and physicists have attempted to invent various approaches by which one can obtain the exact solutions of such equations [41]. The exact solutions include the solitary wave solutions, the shock wave solutions, the periodic wave solutions, etc. There exist several direct methods to find those solutions to nonlinear evolution equations, for example, the Hirota bilinear method [19], the Painlevé expansion method [56], the tanh-function method [9, 33], the homogeneous balance method [46, 52], the modified $\frac{G'}{G}$-polynomial expansion method by Riccati equation [27, 31], the Jacobian elliptic function expansion method [10, 11, 24, 28], the sub-ODE method [60], the truncated Painlevé expansion method [35], the F-expansion method [51], the Sine-Cosine function method [53], the Exp-function method [54], the generalized exponential rational function method for the extended Zakharov Kuzetsov equation with conformable derivative [14] and for the Fokas-Lenells equation in presence of perturbation terms [13], the Darboux transformation method [34], the trial equation method [18], the modified auxiliary equation method [37]. Seadawy et al. [48] proposed the sech-tanh method to solve the Olver equation and the fifth-order KdV equation and obtained traveling wave solutions. In addition, in [22], the authors combined the unified and the explicit exponential finite difference methods to obtain both analytical and numerical solutions for the Newell-Whitehead-SegelšCtype equations which are very important in mathematical biology; in [36], the authors investigated the complex Ginzburg-Landau equation by the generalized exponential function and the unified methods and obtained a variety of new complex waves solutions. The unified method is utilized to obtain various solitary wave solutions for these equations; in [3] and [4], the authors obtained the solitons to a generalized nonlinear Fokas-Lenells equation and to the generalized nonautonomous nonlinear Schrödinger equations in optical fibers via the Sine-Gordon expansion method, respectively. Using those methods, many exact solutions to nonlinear evolution equations are obtained. Those methods are very efficient, reliable, simple in solving many partial differential equations.

Now, with the development of science and technology, nonlinear partial differential equations (NLPDEs) are used to describe numerous nonlinear physical phenomena in different branches of applied sciences. One of the most useful strategies for analyzing such nonlinear physical phenomena is to look for the exact solutions of NLPDEs [40]. For example, the coupled equation [21] is more and more important in the laser and plasma physics field, and the exact solutions for CSBEs have been considered by more and more scientists and engineers. So the study of CSBEs has been paid more and more attention, especially by mathematicians and physicists. One of the tasks of the study CSBEs is to find the exact solutions. In [21], the authors presented a multi-symplectic Hamiltonian formulation for CSBEs and the multi-symplectic scheme simulates the solitary waves for a long time was showed by numerical experiments; in [15], the authors studied the global existence of solutions and long time behavior of the finite dimensional behavior for weakly damped CSBEs; in [16], the authors proved the existence of the T-periodic solution for the weakly damped CSBEs; in [25] the authors consider the initial boundary value problems of dissipative CSBEs and proved the existence of global attractors and the finiteness of the Hausdorff and the fractal dimensions of the attractors; in [7], the authors using efficient mass-and energy-preserving schemes obtained the numerical results; in [62], the authors gave the N-order rogue waves with the determinants and studied the dynamical property of second-order and third-order rogue waves; in [20], the authors studied the homoclinic solutions and the analytic expressions of homoclinic orbits for CSBEs; in [47], the author proposed a new generalized Jacobian elliptic function expansion method and obtained several families of new generalized Jacobian double periodic elliptic function wave solutions; in [26], two orthogonal spline collocation schemes were formulated and got the numerical solutions for CSBEs, at the same time, the authors derived the conservation laws and investigated the convergence and stability of the nonlinear scheme. In [38], the authors investigated CSBEs with variable-coefficients using the unified method, and new non-autonomous complex wave solutions were obtained and classified into two categories, namely polynomial function solutions and rational function solutions. In this paper, we use the coupled Riccati equations and propose the modified polynomial expansion method, then by the method, we obtain more new exact solutions for the system (1.1).

The organization of this work is as follows. Section 1 gives an introduction. Section 2 gives brief description of the algorithm for CSBEs by using the coupled Riccati equations. Section 3 gives the exact solutions of (1.1). Section 4 gives some numerical results and their figures to illustrate the solutions, and gives the physical meaning for the numerical results. Finally, the paper ends with a conclusion and remark in Section 5.

2. Algorithm of the modified polynomial expansion method by the coupled Riccati equations

In this section, we describe the algorithm of polynomial expansion method by the coupled Riccati equations for finding the exact solutions of the coupled nonlinear evolution equations. Suppose that the coupled nonlinear equations, which have independent space variable x and time variable t, are given by

$$\begin{aligned} &P(u, v, u_x, v_x, u_t, v_t, u_x v, uv_x, u_t v, uv_t, u_{xx}, v_{xx}, u_{xt}, v_{xt}, u_{tt}, \cdots) = 0,\\ &Q(u, v, u_x, v_x, u_t, v_t, u_x v, uv_x, u_t v, uv_t, u_{xx}, v_{xx}, u_{xt}, v_{xt}, u_{tt}, \cdots) = 0, \end{aligned} \tag{2.1}$$

where $u = u(x,t), v = v(x,t)$ are unknown functions, P, Q are polynomials of $u(x,t), v(x,t)$ and their partial derivatives in which the highest order partial derivatives and the nonlinear terms are involved and the subscripts stand for the partial derivatives.

We will describe the leading steps of the algorithm of polynomial expansion method by the coupled Riccati equations as follows.

Step-1: Suppose that $u(x,t) = \phi(x-ct) = \phi(\xi), v(x,t) = \varphi(x-ct) = \varphi(\xi)$, where $\xi = x - ct$ and $c \in (\mathbb{R} - \{0\})$ is the wave speed. The equations (2.1) can be reduced to the coupled ordinary differential equations (ODEs) with variables $\phi(\xi), \varphi(\xi)$ and their derivatives

$$\begin{aligned} &P(\phi, \varphi, \phi', \varphi', \phi'', \varphi'', \cdots,) = 0, \\ &Q(\phi, \varphi, \phi', \varphi', \phi'', \varphi'', \cdots,) = 0, \end{aligned} \tag{2.2}$$

where the prime is the derivative with respect to ξ.

Step-2: Determination of the dominant terms. The positive integer N, M are usually attained by taking the homogeneous balance between the highest order nonlinear terms and the derivatives of the highest order appearing in the coupled equations (2.2).

Step-3: Suppose that the solutions of the coupled equation (2.2) can be expressed by the polynomials in $f(\xi)$ and $g(\xi)$ as follows,

$$\begin{aligned} \psi(\xi) &= \sum_{i_1=0, j_1=0}^{N} a_{i_1 j_1} f^{i_1}(\xi) g^{j_1}(\xi), \\ \varphi(\xi) &= \sum_{i_2=0, j_2=0}^{M} b_{i_2 j_2} f^{i_2}(\xi) g^{j_2}(\xi), \end{aligned} \tag{2.3}$$

where $a_{i_1 j_1}$ and $b_{i_2 j_2}$ are real constants to be determined, and the coefficients of the highest order terms of $\phi(\xi)$ and $\varphi(\xi)$ are not equal to zero. The functions $f(\xi)$ and $g(\xi)$ are the solutions to the coupled Riccati equations

$$\begin{aligned} f'(\xi) &= -kf(\xi)g(\xi), \\ g'(\xi) &= -k\left(1 - g^2(\xi) - rf(\xi)\right), \end{aligned} \tag{2.4}$$

where k and r are real constants. The coupled Riccati equations (2.4) have two coupled solutions (which are called basic soliton functions) [61]

$$\begin{aligned} f(\xi) &= \pm \frac{1}{\cosh[k(\xi+\xi_0)] + r}, \\ g(\xi) &= \frac{\sinh[k(\xi+\xi_0)]}{\cosh[k(\xi+\xi_0)] + r}, \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} f(\xi) &= \pm \frac{1}{\sinh[k(\xi+\xi_0)] + r}, \\ g(\xi) &= \frac{\cosh[k(\xi+\xi_0)]}{\sinh[k(\xi+\xi_0)] + r}, \end{aligned} \tag{2.6}$$

where ξ_0 is any constant (generally letting it be zero), k is the numbers of waves. Because the solutions (2.5) are regular soliton solutions, we suppose that $r \neq \pm 1$. From (2.5) and (2.6), we easily obtain, respectively, that

$$g^2 = 1 - 2rf + \left(r^2 - 1\right) f^2, \tag{2.7}$$

and

$$g^2 = 1 - 2rf + \left(r^2+1\right) f^2. \tag{2.8}$$

Step-4: The positive integer N and M are usually attained by taking the homogeneous balance between the highest order nonlinear terms and the derivatives of the highest order appearing in Equation (2.2).

Step-5: Substituting (2.5) into (2.2), collecting all terms with the same powers of he functions $f^{i_1}g^{j_1}$ and $f^{i_2}g^{j_2}$, and finding the algebraic systems of equations for the coefficients $a_{i_1j_1}$ and $b_{i_2j_2}$ with the parameters k, r. Solving this system we get the values of unknown coefficients.

Step-6: Substituting the solutions (2.5) of the Coupled Riccati equations (2.4) into (2.2), we obtain the exact wave solutions to the coupled partial differential equations (1.1).

3. The exact wave solutions for CSBEs

We will employ the modified polynomial expansion method for obtaining the exact solutions for the system (1.1). At first, the traveling wave transformations are introduced

$$\begin{aligned} u(x,t) =& u(x-ct)\exp\left(-\frac{ic}{2}(c-\mu t)\right) = \psi(\xi)\exp\left(-\frac{ic}{2}(c-\mu t)\right), \\ v(x,t) =& v(x-ct) = \varphi(\xi), \end{aligned} \tag{3.1}$$

where $\xi = x - ct$ and $c \in (\mathbb{R} - \{0\})$ is the wave velocity.

Substituting the transformations (3.1) into (1.1) and integrating twice for the second equation of (1.1). we obtain the following coupled ordinary differential equation system

$$\begin{aligned} &\psi'' + \frac{c}{2}\left(\mu + \frac{c}{2}\right)\psi - \psi\varphi = 0, \\ &\beta\varphi'' + \alpha\varphi^2 + (c^2-1)\varphi = \psi^2, \end{aligned} \tag{3.2}$$

where the prime is the derivative with respect to ξ and taking the integral constants are zeroes.

Substituting the solutions (2.5) of the coupled Riccati equation into (3.2), balancing the highest orders and we obtain that $N = 2, M = 2$. So

$$\begin{aligned} \psi &= a_{00} + a_{10}f + a_{01}g + a_{20}f^2 + a_{11}fg, \\ \varphi &= b_{00} + b_{10}f + b_{01}g + b_{20}f^2 + b_{11}fg, \end{aligned} \tag{3.3}$$

because of f and g satisfying the relation (2.7), the highest order of g in (3.3) is 1. Substituting (3.3) into (3.2), and according to (2.4), (2.5), and (2.7), we can obtain the parameters $a_{00}, a_{10}, a_{01}, a_{20}, a_{11}, b_{00}, b_{10}, b_{01}, b_{20}, b_{11}, k, r, c$ and μ, the specific algorithms can be found in the appendix. There are 19 groups of solutions for all the parameters.

Now, we use the equations (3.1), (3.3) and the solution sets from **1**) to **19**) (in the Appendix) of system (3.2) obtaining the solutions of CSBEs (1.1).

For solutions 1), we have

$$\psi = a_{20} f^2 = a_{20}\text{sech}^2[k(x-ct)+\xi_0],$$
$$\varphi = b_{20} f^2 = -6k^2\text{sech}^2[k(x-ct)+\xi_0],$$

thus, the solution for (1.1) is

$$\begin{cases} u_1(x,t) = a_{20}\text{sech}^2\left[k(x-ct)+\xi_0\right]\exp\left[-\dfrac{ic}{2}\left(x+\dfrac{c^2+16k^2}{2c}t\right)\right], \\ v_1(x,t) = 6k^2\text{sech}^2\left[k(x-ct)+\xi_0\right], \end{cases} \tag{3.4}$$

where a_{20}, c, k and ξ_0 are arbitrary constants.

Similarly, for 2), we have the solution for (1.1)

$$\begin{cases} u_2(x,t) = a_{20}\text{sech}^2\left[k(x-ct)+\xi_0\right] \\ \qquad\qquad \times\exp\left[-\dfrac{ic}{2}\left(x+\dfrac{12k^2(c^2-4k^2)(c^2-1)+a_{20}^2(c^2+16k^2)}{2c(12k^2(1-c^2)+a_{20}^2)}t\right)\right], \\ v_2(x,t) = \dfrac{36k^4(1-c^2)}{12k^2(1-c^2)+a_{20}^2} - 6k^2\text{sech}^2\left[k(x-ct)+\xi_0\right], \end{cases} \tag{3.5}$$

where a_{20}, c, k and ξ_0 are arbitrary constants.

For 3), we have the solution for (1.1)

$$\begin{cases} u_3(x,t) = 2\sqrt{3(1-c^2)}k\,\text{sech}^2\left[k(x-ct)+\xi_0\right] \\ \qquad\qquad \times\exp\left[-\dfrac{ic}{2}\left(x+\dfrac{c^2+16k^2}{2c}t\right)\right], \\ v_3(x,t) = \dfrac{30k^2}{7}\text{sech}^2\left[k(x-ct)+\xi_0\right], \end{cases} \tag{3.6}$$

where $|c| \le 1, k$ and ξ_0 are arbitrary constants.

For 4), we have the solution for (1.1)

$$\begin{cases} u_4(x,t) = \left[\pm\dfrac{3k\sqrt{14(c^2-1)}}{7\left(\cosh\left[k(x-ct)+\xi_0\right]+\sqrt{\frac{2}{7}}\right)} + \right. \\ \qquad\qquad \left. \dfrac{6k\sqrt{c^2-1}}{7}\left(\dfrac{1}{\cosh\left[k(x-ct)+\xi_0\right]+\sqrt{\frac{2}{7}}}\right)^2\right] \\ \qquad\qquad \times\exp\left[-\dfrac{ic}{2}\left(x+\dfrac{7c^2-272k^2}{14c}t\right)\right], \\ v_4(x,t) = \dfrac{75k^2}{7} - \dfrac{30k^2}{7}\left(\dfrac{1}{\cosh\left[k(x-ct)+\xi_0\right]+\sqrt{\frac{2}{7}}}\right)^2, \end{cases} \tag{3.7}$$

where $|c| \ge 1, k$ and ξ_0 are arbitrary constants.

For 5), we have the solution for (1.1)

$$\begin{cases} u_5(x,t) = \left[\pm \dfrac{3k\sqrt{14(1-c^2)}}{7\left(\cosh\left[k(x-ct)+\xi_0\right]+\sqrt{\frac{2}{7}}\right)} + \dfrac{6k\sqrt{1-c^2}}{7} \right. \\ \qquad \left. \times \left(\dfrac{1}{\cosh\left[k(x-ct)+\xi_0\right]+\sqrt{\frac{2}{7}}}\right)^2 \right] \exp\left[-\dfrac{ic}{2}\left(x+\dfrac{c^2+4k^2}{2c}t\right)\right], \\ v_5(x,t) = -\dfrac{30k^2}{7}\left(\dfrac{1}{\cosh\left[k(x-ct)+\xi_0\right]+\sqrt{\frac{2}{7}}}\right)^2, \end{cases} \tag{3.8}$$

where $|c| \le 1, k$ and ξ_0 are arbitrary constants.

For 6), we have the solution for (1.1)

$$\begin{cases} u_6(x,t) = \pm\,\text{sech}\left[k(x-ct)+\xi_0\right]\exp\left[-\dfrac{ic}{2}\left(x+\dfrac{c^2+4k^2}{2c}t\right)\right], \\ v_6(x,t) = -\,2k^2\text{sech}^2\left[k(x-ct)+\xi_0\right], \end{cases} \tag{3.9}$$

where a_{10}, c, k and ξ_0 are arbitrary constants.

For 7), we have the solution for (1.1)

$$\begin{cases} u_7(x,t) = \pm\,\text{sech}\left[k(x-ct)+\xi_0\right] \\ \qquad \times \exp\left[-\dfrac{ic}{2}\left(x+\dfrac{2k^2c^2(1-c^2)+a_{10}^2(c^2+4k^2)}{2c(2k^2(1-c^2)+a_{10}^2)}t\right)\right], \\ v_7(x,t) = -\dfrac{2k^4(c^2-1)}{2k^2(c^2-1)k^2+a_{10}^2} - 2k^2\text{sech}^2\left[k(x-ct)+\xi_0\right], \end{cases} \tag{3.10}$$

where a_{10}, c, k and ξ_0 are arbitrary constants.

For 8), we have the solution for (1.1)

$$\begin{cases} u_8(x,t) = \left[\pm \dfrac{\sqrt{6}a_{20}}{46\left(\cosh\left[k(x-ct)+\xi_0\right]-\frac{17\sqrt{6}}{3}\right)} \right. \\ \qquad \left. +a_{20}\left(\dfrac{1}{\cosh\left[k(x-ct)+\xi_0\right]-\frac{17\sqrt{6}}{3}}\right)^2 \right] \\ \qquad \times \exp\left[-\dfrac{ic}{2}\left(x+\dfrac{(10580000k^4+264500k^2-a_{20}^2)\sqrt{5}}{2300k\sqrt{264500k^2-a_{20}^2}}t\right)\right], \\ v_8(x,t) = \pm\dfrac{40\sqrt{6}k^2}{\cosh\left[k(x-ct)+\xi_0\right]-\frac{17\sqrt{6}}{3}} \\ \qquad + 1150k^2\left(\dfrac{1}{\cosh\left[k(x-ct)+\xi_0\right]-\frac{17\sqrt{6}}{3}}\right)^2, \end{cases} \tag{3.11}$$

where a_{20}, k, ξ_0 are arbitrary constants and $c = \frac{1}{230k}\sqrt{\frac{264500k^2-a_{20}^2}{5}}$, $264500k^2 > a_{20}^2$.

For 9), we have the solution for (1.1)

$$\begin{cases} u_9(x,t) = \pm \dfrac{a_{11}\sinh\left[k(x-ct)+\xi_0\right]}{\cosh^2\left[k(x-ct)+\xi_0\right]}\exp\left[-\dfrac{ic}{2}\left(x+\dfrac{c^2+4k^2}{2c}t\right)\right], \\ v_9(x,t) = -6k^2\mathrm{sech}^2\left[k(x-ct)+\xi_0\right], \end{cases} \tag{3.12}$$

where a_{11}, c, k and ξ_0 are arbitrary constants.

For 10), we have the solution for (1.1)

$$\begin{cases} u_{10}(x,t) = \pm \dfrac{a_{11}\sinh\left[k(x-ct)+\xi_0\right]}{\cosh^2\left[k(x-ct)+\xi_0\right]} \\ \qquad\qquad \times\exp\left[-\dfrac{ic}{2}\left(x+\dfrac{12k^2(c^2-1)(8k^2-c^2)+a_{11}^2(c^2+4k^2)}{2c(12k^2(1-c^2)+a_{11}^2)}t\right)\right], \\ v_{10}(x,t) = \dfrac{36k^4(1-c^2)}{12k^2(1-c^2)+a_{11}^2} - 6k^2\mathrm{sech}^2\left[k(x-ct)+\xi_0\right], \end{cases} \tag{3.13}$$

where a_{11}, c, k and ξ_0 are arbitrary constants.

For 11), we have the solution for (1.1)

$$\begin{cases} u_{11}(x,t) = \pm \dfrac{2\sqrt{3(c^2-1)}k\sinh\left[k(x-ct)+\xi_0\right]}{\cosh^2\left[k(x-ct)+\xi_0\right]}\exp\left[-\dfrac{ic}{2}\left(x+\dfrac{c^2+4k^2}{2c}t\right)\right], \\ v_{11}(x,t) = -6k^2\mathrm{sech}^2\left[k(x-ct)+\xi_0\right], \end{cases} \tag{3.14}$$

where $|c| \geq 1, k$ and ξ_0 are arbitrary constants.

For 12), we have the solution for (1.1)

$$\begin{cases} u_{12}(x,t) = \pm \dfrac{\sqrt{6(1-r^2)(1-c^2)}k\sinh\left[k(x-ct)+\xi_0\right]}{\cosh^2\left[k(x-ct)+\xi_0\right]} \\ \qquad\qquad \times\exp\left[-\dfrac{ic}{2}\left(x+\dfrac{c^2+4k^2}{2c}t\right)\right], \\ v_{12}(x,t) = \mp 6k^2r\mathrm{sech}\left[k(x-ct)+\xi_0\right] \\ \qquad\qquad + 6k^2(r^2-1)\mathrm{sech}^2\left[k(x-ct)+\xi_0\right], \end{cases} \tag{3.15}$$

where c, k, r are arbitrary constants and $(1-r^2)(1-c^2) \geq 0$.

For 13), we have the solution for (1.1)

$$\begin{cases} u_{13}(x,t) = \pm \dfrac{\sqrt{6(r^2-1)(1-c^2)}k\sinh\left[k(x-ct)+\xi_0\right]}{\cosh^2\left[k(x-ct)+\xi_0\right]} \\ \qquad\qquad \times\exp\left[-\dfrac{ic}{2}\left(x+\dfrac{c^2+4k^2}{2c}t\right)\right], \\ v_{13}(x,t) = \mp 6k^2r\mathrm{sech}\left[k(x-ct)+\xi_0\right] \\ \qquad\qquad + 6k^2(r^2-1)\mathrm{sech}^2\left[k(x-ct)+\xi_0\right], \end{cases} \tag{3.16}$$

where c, k, r are arbitrary constants and $(r^2-1)(1-c^2) \geq 0$.

For 14), we have the solution for (1.1)

$$\begin{cases} u_{14}(x,t) = \pm \dfrac{\sqrt{6(r^2-1)(1-c^2)}k\sinh\left[k(x-ct)+\xi_0\right]}{\cosh^2\left[k(x-ct)+\xi_0\right]} \\ \qquad\qquad \times \exp\left[-\dfrac{ic}{2}\left(x+\dfrac{c^2-4k^2}{2c}t\right)\right], \\ v_{14}(x,t) = 2k^2 \mp 6k^2 r\operatorname{sech}\left[k(x-ct)+\xi_0\right] \\ \qquad\qquad + 6k^2(r^2-1)\operatorname{sech}^2\left[k(x-ct)+\xi_0\right], \end{cases} \tag{3.17}$$

where c, k, r are arbitrary constants and $(r^2-1)(1-c^2) \geq 0$.

For 15), we have the solution for (1.1)

$$\begin{cases} u_{15}(x,t) = \pm \dfrac{3\sqrt{2(c^2-1)}k\sinh\left[k(x-ct)+\xi_0\right]}{2\left(\cosh\left[k(x-ct)+\xi_0\right]+\frac{1}{2}\right)^2} \\ \qquad\qquad \times \exp\left[-\dfrac{ic}{2}\left(x+\dfrac{c^2+4k^2}{2c}t\right)\right], \\ v_{15}(x,t) = \mp \dfrac{3k^2}{\cosh\left[k(x-ct)+\xi_0\right]+\frac{1}{2}} \\ \qquad\qquad - \dfrac{9k^2}{2\left(\cosh\left[k(x-ct)+\xi_0\right]+\frac{1}{2}\right)^2}, \end{cases} \tag{3.18}$$

where c, k, ξ_0 are arbitrary constants and $|c| \geq 1$.

For 16), we have the solution for (1.1)

$$\begin{cases} u_{16}(x,t) = \pm \dfrac{3\sqrt{2(1-c^2)}k\sinh\left[k(x-ct)+\xi_0\right]}{2\left(\cosh\left[k(x-ct)+\xi_0\right]+\frac{1}{2}\right)^2} \\ \qquad\qquad \times \exp\left[-\dfrac{ic}{2}\left(x+\dfrac{c^2-4k^2}{2c}t\right)\right], \\ v_{16}(x,t) = 2k^2 \mp \dfrac{3k^2}{\cosh\left[k(x-ct)+\xi_0\right]+\frac{1}{2}} \\ \qquad\qquad - \dfrac{9k^2}{2\left(\cosh\left[k(x-ct)+\xi_0\right]+\frac{1}{2}\right)^2}, \end{cases} \tag{3.19}$$

where c, k, ξ_0 are arbitrary constants and $|c| \leq 1$.

For 17), we have the solution for (1.1)

$$\begin{cases} u_{17}(x,t) = \pm \dfrac{3\sqrt{2(c^2-1)}k\sinh\left[k(x-ct)+\xi_0\right]}{2\left(\cosh\left[k(x-ct)+\xi_0\right]-\frac{1}{2}\right)^2} \\ \qquad\qquad \times \exp\left[-\dfrac{ic}{2}\left(x+\dfrac{c^2+4k^2}{2c}t\right)\right], \\ v_{17}(x,t) = \pm \dfrac{3k^2}{\cosh\left[k(x-ct)+\xi_0\right]-\frac{1}{2}} \\ \qquad\qquad - \dfrac{9k^2}{2\left(\cosh\left[k(x-ct)+\xi_0\right]-\frac{1}{2}\right)^2}, \end{cases} \tag{3.20}$$

where c, k, ξ_0 are arbitrary constants and $|c| \geq 1$.

For 18), we have the solution for (1.1)

$$\begin{cases} u_{18}(x,t) = \pm \dfrac{3\sqrt{2(1-c^2)}k\sinh\left[k(x-ct)+\xi_0\right]}{2\left(\cosh\left[k(x-ct)+\xi_0\right]-\frac{1}{2}\right)^2} \\ \qquad\qquad \times \exp\left[-\dfrac{ic}{2}\left(x+\dfrac{c^2-4k^2}{2c}t\right)\right], \\ v_{18}(x,t) = 2k^2 \pm \dfrac{3k^2}{\cosh\left[k(x-ct)+\xi_0\right]-\frac{1}{2}} \\ \qquad\qquad - \dfrac{9k^2}{2\left(\cosh\left[k(x-ct)+\xi_0\right]-\frac{1}{2}\right)^2}, \end{cases} \tag{3.21}$$

where c, k, ξ_0 are arbitrary constants and $|c| \leq 1$.

For 19), we have the solution for (1.1)

$$\begin{cases} u_{19}(x,t) = \left(\pm \dfrac{\sqrt{6}a_{20}}{46\cosh\left[k(x-ct)+\xi_0\right]-\frac{17\sqrt{6}}{3}} + \dfrac{a_{20}}{\left(\cosh\left[k(x-ct)+\xi_0\right]-\frac{17\sqrt{6}}{3}\right)^2}\right) \\ \qquad\qquad \times \exp\left[-\dfrac{ic}{2}\left(x+\dfrac{(1851500k^2-8464000k^4+7a_{20}^2)\sqrt{5}}{16100k\sqrt{a_{20}^2+264500k^2}}t\right)\right], \\ v_{19}(x,t) = \dfrac{15k^2}{7} \pm \dfrac{40\sqrt{6}k^2}{\cosh\left[k(x-ct)+\xi_0\right]-\frac{17\sqrt{6}}{3}} \\ \qquad\qquad - \dfrac{1150k^2}{\left(\cosh\left[k(x-ct)+\xi_0\right]-\frac{17\sqrt{6}}{3}\right)^2}, \end{cases} \tag{3.22}$$

where a_{20}, k, ξ_0 are arbitrary constants and $c = \frac{\sqrt{a_{20}^2+264500k^2}}{230\sqrt{5}k}$.

4. Numerical experiments and the physical explanation

In this section, we investigate some of the numerical results for CSBEs and interpret some of the CSBEs model wave solutions in the perspective of their physical meaning.

4.1. Illustrative examples and their figures

At first, we provide simple numerical examples to confirm our main results and demonstrate the system (1.1) as follows.

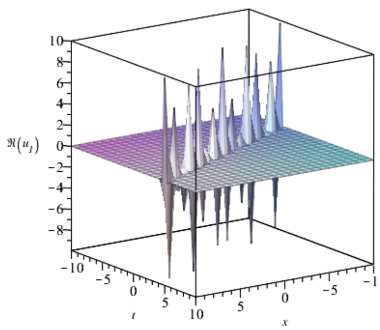
Example 4.1. In this example, for the first solution (3.4), we assume the following parameters:

$$a_{20} = 10,\ k = 3,\ c = 2,\ \xi_0 = 0,$$

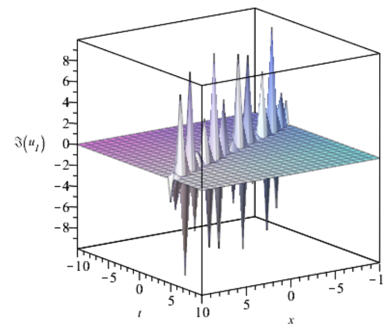
thus, the solution (3.4) becomes

$$\begin{cases} u_1(x,t) = 10\text{sech}^2\left(3(x-2t)\right)\exp\left[-i\left(x+37t\right)\right], \\ v_1(x,t) = 6\text{sech}^2\left(2(x-2t)\right), \end{cases}$$

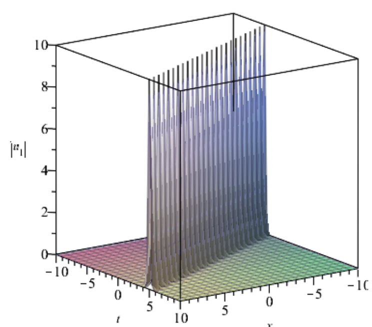
so the figures of $u_1(x,t)$ and $v_1(x,t)$ for the system (1.1) are like to the Figure 1.



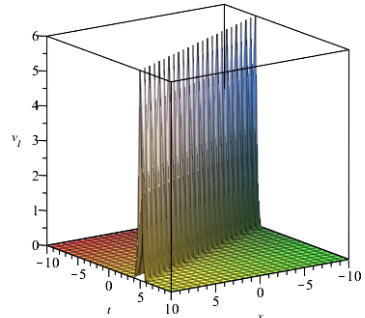
(a) The real part of the solution u_1



(b) The imaginary part of the solution u_1



(c) The norm of the solution u_1



(d) The solution v_1

Figure 1. The real part of the solution $u_1(x,t)$ as shown in (a), the imaginary part of the solution $u_1(x,t)$ as shown in (b), the norm of the solution $u_1(x,t)$ as shown in (c), and the solution $v_1(x,t)$ as shown in (d).

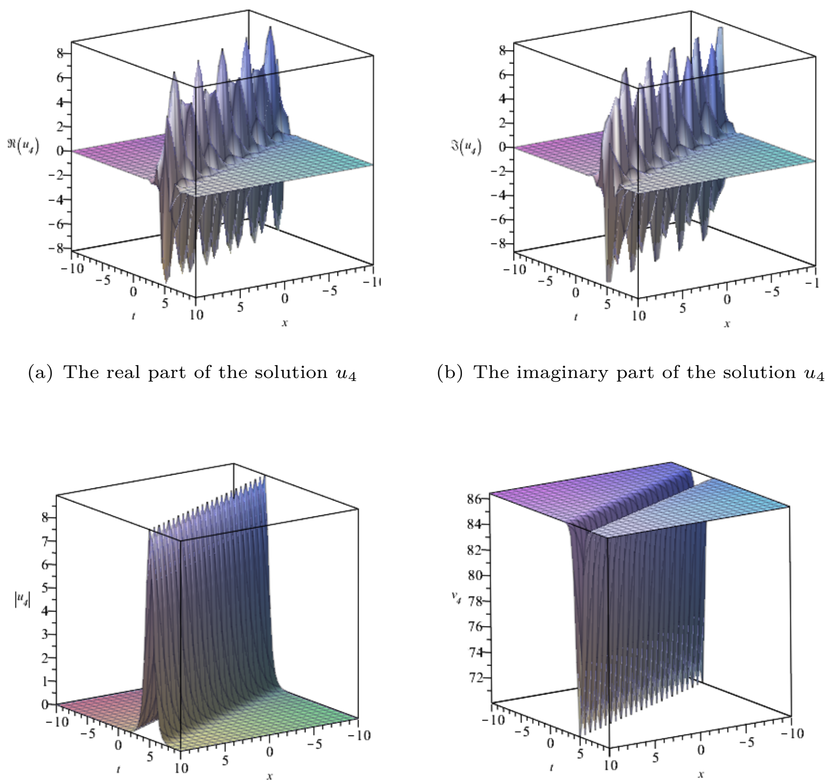
Example 4.2. In this example, for the fourth solution (3.7), we assume the following parameters:

$$k = 3,\ c = 2,\ \xi_0 = 0,$$

and we take the positive sign for the solution $u_4(x,t)$, then the solution (3.7) becomes

$$\begin{cases} u_4(x,t) = \left[\dfrac{9\sqrt{42}}{7\cosh(x-2t)+\sqrt{14}} + \dfrac{63\sqrt{42}}{\left(7\cosh(x-2t)+\sqrt{14}\right)^2}\right] \\ \qquad \times \exp\left[-i\left(x - \dfrac{605}{7}t\right)\right], \\ v_4(x,t) = \dfrac{605}{7} - \dfrac{1890}{\left(7\cosh(x-2t)+\sqrt{14}\right)^2}, \end{cases}$$

so the figures of $u_4(x,t)$ and $v_4(x,t)$ for the system (1.1) are like to the Figure 2.



(a) The real part of the solution u_4

(b) The imaginary part of the solution u_4

(c) The norm of the solution u_4

(d) The solution v_4

Figure 2. The real part of the solution $u_4(x,t)$ as shown in (a), the imaginary part of the solution $u_4(x,t)$ as shown in (b), the norm of the solution $u_4(x,t)$ as shown in (c), and the solution $v_4(x,t)$ as shown in (d).

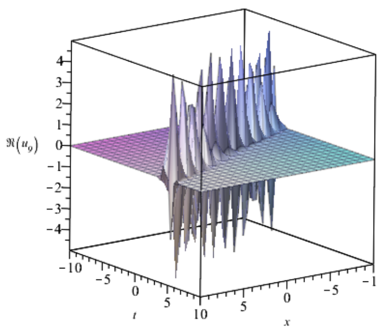
Example 4.3. In this example, for the ninth solution (3.12), we assume the following parameters:

$$a_{21} = 10,\ k = 2,\ c = 2,\ \xi_0 = 0,$$

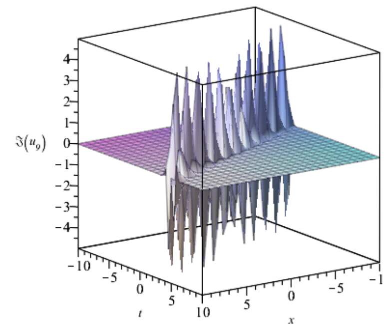
thus, the solution (3.12) becomes

$$\begin{cases} u_9(x,t) = 10\text{sinh}\left(2(x-2t)\right)\text{sech}^2\left(2(x-2t)\right)\exp\left[-i\left(x+5t\right)\right], \\ v_9(x,t) = -24\text{sech}^2\left(2(x-2t)\right), \end{cases}$$

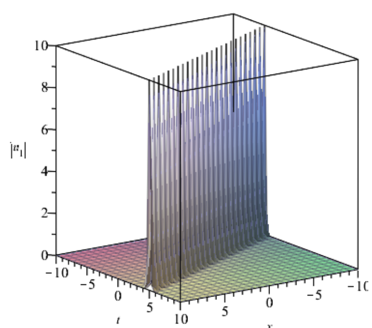
so the figures of $u_9(x,t)$ and $v_9(x,t)$ for the system (1.1) are like to the Figure 3.



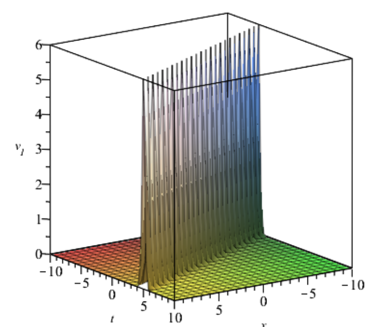
(a) The real part of the solution u_9



(b) The imaginary part of the solution u_9



(c) The norm of the solution u_9



(d) The solution v_9

Figure 3. The real part of the solution $u_9(x,t)$ as shown in (a), the imaginary part of the solution $u_9(x,t)$ as shown in (b), the norm of the solution $u_9(x,t)$ as shown in (c), and the solution $v_9(x,t)$ as shown in (d).

Unfortunately, it does not seem mathematically tractable to determine the figures of the other sixteen types solutions to the equations (1.1), however, there are only tedious algebraic calculation process, thus, we omit the examples and the figures about them.

4.2. Physical meaning

In this part, we interpret some of the complex wave solutions $u(x,t)$ and real wave solution $v(x,t)$ to the SCHBs model in the perspective of their physical meaning. CSBEs have often been proposed to describe the interaction of long-waves with short-wave packets in nonlinear dispersive media. It is well known that a high-frequency wave with modulated-amplitude can lead to the excitation of an instability called the 'modulation instability' in dispersive media such as plasma. The nonlinear development of the instability is typically governed by the Schrödinger equation having a 'potential' which depends on the associated low-frequency perturbations. The latter is governed by the nonlinear Boussinesq equation [6], which is driven by the so-called ponderomotive force due to the high-frequency carrier wave [49]. Figure 1 depicts the 3-dimension solution given by $u_1(x,t)$ and $v_1(x,t)$ with the parameters $a_{20}=10, k=3, c=2$, and $\xi_0=0$. In Figure 1, (a) represents the real part of the complex wave solution $u_1(x,t)$, (b) represents the imaginary parts of the complex wave solution $u_1(x,t)$, (c) represents the norm of the complex wave solution $u_1(x,t)$, and (d) represents the bright soliton wave solution $v_1(x,t)$. Figure 2 depicts the 3-dimension solution given by $u_4(x,t)$ and $v_4(x,t)$ with the parameters $k=3, c=2$, and $\xi_0=0$. In Figure 2, (a) represents the real part of the complex wave solution $u_4(x,t)$, (b) represents the imaginary parts of the complex wave solution $u_4(x,t)$, (c) represents the norm of the complex wave solution $u_4(x,t)$, and (d) represents the dark soliton wave solution $v_4(x,t)$. Figure 3 depicts the 3-dimension solution given by $u_9(x,t)$ and $v_9(x,t)$ with the parameters $k=3, c=2$, and $\xi_0=0$. In Figure 3, (a) represents the real part of the complex wave solution $u_9(x,t)$, (b) represents the imaginary parts of the complex wave solution $u_94(x,t)$, (c) represents the norm of the complex wave solution $u_9(x,t)$, and (d) represents the bright solition wave solution $v_9(x,t)$. Likely examples 1, examples 2, and examples 3, we can obtain the physical meaning of other solutions and we omit them.

5. Conclusions and remarks

In this work, the coupled Schrödinger-Boussinesq equations were investigated. The others¡¯ works were focused on the peakons solutions, peaked solitary wave solutions. Here, a solution set of the coupled Riccati equations is introduced to formally derive abundant solutions for CSBEs with distinct physical structures. All the solutions included one or more sech, sech^2, tanh, sinh, cosh, et al, functions, therefore, the solution are solitons, solitary patterns solutions, periodic solutions, compactons, and peakons solutions. The obtained results complement the useful works of others for this important physical model.

We, here, proposed the efficient modified polynomial expansion method by the coupled Riccati equations and obtained more new exact wave solutions for CSBEs (1.1). On comparing with the polynomial expansion method and other methods in handling a huge number of nonlinear dispersive and dissipative equations, the proposed scheme is more effective, powerful and reliable to be used in identical nonlinear dispersive models. Moreover, the modified polynomial expansion method can be used to solve any coupled high-order degree partial differential equations. Using the modified polynomial expansion method we get a set of nonlinear algebraic equations that can be solved by the Maple software. Also, the Maple software was applied over for both the graphical impersonation and the emulation. Finally, we can say that the method is a very strong scheme to find more new exact solutions of CSBEs.

In addition, CSBEs plays an important role in applied mathematics and applied physics, however, we only use the coupled Riccati equations obtaining more new exact solutions in this paper, and according to some special parameter values, we give the images of special solutions and their physical meanings. In the future research, we will use other methods to study the structure and properties of solutions for CSBEs and the application of the solutions in practice.

Remark: In this paper, using the solution (2.5) of the coupled Riccati equations (2.4) and the relation (2.7) between f and g, we obtain all above solutions for CSBEs (1.1). If using the solutions (2.6) and the relation (2.8), we can obtain singular soliton solutions for CSBEs (1.1). Unfortunately, all the calculation is tedious algebraic process and is similar to that of Section 3.2. Thus, we only give one couple solution for CSBEs (1.1) and omit others solutions and their computation.

$$\begin{cases} u(x,t) = \left[\pm \dfrac{2k\sqrt{14(1+c^2)}}{5\left(\sinh\left[k(x-ct)+\xi_0\right]+\sqrt{\frac{3}{5}}\right)} + \dfrac{4k\sqrt{1+c^2}}{5} \right. \\ \qquad\qquad \left. \times \left(\dfrac{1}{\sinh\left[k(x-ct)+\xi_0\right]+\sqrt{\frac{3}{5}}} \right)^2 \right] \exp\left[-\dfrac{ic}{2}\left(x + \dfrac{2c^2+3k^2}{3c} t \right) \right], \\ v(x,t) = \dfrac{27k^2}{5} \left(\dfrac{1}{\sinh\left[k(x-ct)+\xi_0\right]+\sqrt{\frac{3}{5}}} \right)^2, \end{cases}$$

where c, k and ξ_0 are arbitrary constants.

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Appendix

A. Analysis to the algorithm of the modified polynomial expansion method

In this section, we will construct the algorithm of polynomial expansion method by the coupled Riccati equations to obtain the exact solutions for all the parameters. From the coupled Riccati equation (2.4) and (3.3), we obtain

$$\begin{aligned} \psi' &= a_{10}f' + a_{01}g' + 2a_{20}ff' + a_{11}(f'g + fg') \\ &= k(a_{11}rf^2 - 2a_{20}f^2g + a_{01}rf + a_{01}g^2 - a_{10}fg - a_{11}f - a_{01}), \\ \varphi' &= b_{10}f' + b_{01}g' + 2b_{20}ff' + b_{11}(f'g + fg') \\ &= k(b_{11}rf^2 - 2b_{20}f^2g + b_{01}rf + b_{01}g^2 - b_{10}fg - b_{11}f - b_{01}), \end{aligned} \tag{A.1}$$

and

$$
\begin{aligned}
\psi'' =& -k\left(2a_{11}rff' - 4a_{20}ff'g - 2a_{20}f^2g' + a_{01}rf'\right.\\
&\left.+2a_{01}gg' - a_{10}f'g - a_{10}fg' - a_{11}f'\right)\\
=& -k^2\left(a_{20}f^3 - 2a_{20}f^2g^2 + 2a_{11}rf^2g + (a_{10}r - 2a_{20})f^2\right.\\
&\left.-(a_{01}r - a_{11})fg - a_{10}f - 2a_{01}g^3 + 2a_{01}g\right),\\
\varphi'' =& -k\left(2b_{11}rff' - 4b_{20}ff'g - 2b_{20}f^2g' + b_{01}rf'\right.\\
&\left.+2b_{01}gg' - b_{10}f'g - b_{10}fg' - b_{11}f'\right)\\
=& -k^2\left(b_{20}f^3 - 2b_{20}f^2g^2 + 2b_{11}rf^2g + (b_{10}r - 2b_{20})f^2\right.\\
&\left.-(b_{01}r - b_{11})fg - b_{10}f - 2b_{01}g^3 + 2b_{01}g\right).
\end{aligned}
\tag{A.2}
$$

Substituting (2.4), (3.3), and (A.2) into (3.2), we obtain the polynomials about f^ig^j. Collecting all the terms with the same power of f^ig^j and equate this expressions to zero, we get the algebraic equation system for a_{00}, a_{10}, a_{01}, a_{20}, a_{11}, b_{00}, b_{10}, b_{01}, b_{20}, b_{11}, k, r, c, α, β, and μ as follows:

$$
\begin{aligned}
&6a_{20}k^2r^2 - a_{11}b_{11}r^2 - 6a_{20}k^2 + a_{11}b_{11} - a_{20}b_{20} = 0,\\
&6a_{11}k^2r^2 - 6a_{11}k^2 - a_{11}b_{20} - a_{20}b_{11} = 0,\\
&2a_{10}k^2r^2 - a_{01}b_{11}r^2 - a_{11}b_{01}r^2 - 10a_{20}k^2r - 2a_{10}k^2 + 2a_{11}b_{11}r\\
&+ a_{01}b_{11} - a_{10}b_{20} + a_{11}b_{01} - a_{20}b_{10} = 0,\\
&2a_{01}k^2r^2 - 6a_{11}k^2r - 2a_{01}k^2 - a_{01}b_{20} - a_{10}b_{11} - a_{11}b_{10} - a_{20}b_{01} = 0,\\
&4a_{20}k^2 - 3a_{10}k^2r + \frac{1}{2}ca_{20}\mu - a_{00}b_{20} - a_{20}b_{00} + \frac{1}{4}a_{20}c^2 - a_{10}b_{10}\\
&+ a_{01}b_{01} + 2a_{11}b_{01}r + 2a_{01}b_{11}r - a_{01}b_{01}r^2 - a_{11}b_{11} = 0,\\
&a_{11}k^2 + \frac{1}{4}a_{11}c^2 - a_{00}b_{11} - b_{10}a_{01} - a_{10}b_{01} - a_{11}b_{00} + \frac{1}{2}ca_{11}\mu - k^2a_{01}r = 0,\\
&a_{10}k^2 + \frac{1}{2}ca_{10}\mu - a_{00}b_{10} + \frac{1}{4}a_{10}c^2 - a_{10}b_{00} + 2a_{01}b_{01}r - a_{01}b_{11} - a_{11}b_{01} = 0,\\
&\frac{1}{2}ca_{01}\mu + \frac{1}{4}a_{01}c^2 - a_{00}b_{01} - a_{01}b_{00} = 0,\\
&\frac{1}{4}a_{00}c^2 - a_{00}b_{00} + \frac{1}{2}ca_{00}\mu - a_{01}b_{01} = 0,\\
&(b_{11}^2r^2 - b_{11}^2 + b_{20}^2)\alpha + (4b_{20}\beta k^2 - a_{11}^2)r^2 - 4b_{20}\beta k^2 - a_{20}^2 + a_{11}^2 = 0,\\
&6b_{11}\beta k^2r^2 - 6b_{11}\beta k^2 + 2\alpha b_{11}b_{20} - 2a_{11}a_{20} = 0,\\
&(2b_{01}b_{11}r^2 - 2b_11^2r - 2b_{01}b_{11} + 2b_{10}b_{20})\alpha + (2b_{10}\beta k^2 - 2a_{01}a_{11})r^2\\
&+ (-6b_{20}\beta k^2 + 2a_{11}^2)r - 2b_{10}\beta k^2 - 2a_{10}a_{20} + 2a_{01}a_{11} = 0,\\
&(2b_{01}b_{20} + 2b_{10}b_{11})\alpha + 2b_{01}\beta k^2r^2 - 6b_{11}\beta k^2r - 2b_{01}\beta k^2\\
&- 2a_{01}a_{20} - 2a_{10}a_{11} = 0,\\
&(-4b_{01}b_{11}r + 2b_{00}b_{20} + b_{10}^2 + b_{11}^2)\alpha - a_{01}^2r^2 + (-3b_{10}\beta k^2 + 4a_{01}a_{11})r\\
&+ 2b_{20}\beta k^2 + a_{01}^2 - a_{11}^2 + (c^2 - 1)b_{20} - 2a_{00}a_{20} - a_{10}^2 = 0,\\
&(2b_{00}b_{11} + 2b_{01}b_{10})\alpha - b_{01}\beta k^2r + (\beta k^2 + c^2 - 1)b_{11} - 2a_{00}a_{11} - 2a_{01}a_{10} = 0,\\
&(2b_{00}b_{10} + 2b_{01}b_{11})\alpha + b_{10}\beta k^2 + b_{10}c^2 + 2a_{01}^2r - 2a_{00}a_{10} - 2a_{01}a_{11} - b_{10} = 0,\\
&2\alpha b_{00}b_{01} + b_{01}c^2 - 2a_{00}a_{01} - b_{01} = 0,
\end{aligned}
$$

$$\alpha b_{00}^2 + b_{00}c^2 - a_{00}^2 - a_{01}^2 - b_{00} = 0.$$

Solving the above algebraic equation system by Maple, we obtained nineteen types of solutions as follows:

$$1)\ \ a_{00}=0, a_{01}=0, a_{10}=0, a_{11}=0, a_{20}=a_{20}, \alpha=\frac{12k^2(c^2-1)+a_{20}^2}{36k^4},$$
$$b_{00}=0, b_{01}=0, b_{10}=0, b_{11}=0, b_{20}=-6k^2, \beta=-\frac{c^2-1}{2k^2}, c=c,$$
$$k=k, \mu=-\frac{c^2+16k^2}{2c}, r=0,$$

where a_{20}, c and k are arbitrary constants.

$$2)\ a_{00}=0, a_{01}=0, a_{10}=0, a_{11}=0, a_{20}=a_{20}, \alpha=\frac{12k^2(1-c^2)+a_{20}^2}{36k^4},$$
$$b_{00}=\frac{36k^4(1-c^2)}{12k^2(1-c^2)+a_{20}^2}, b_{01}=0, b_{10}=0, b_{11}=0, b_{20}=-6k^2, c=c,$$
$$\beta=\frac{c^2-1}{2k^2}, k=k, \mu=-\frac{12k^2(c^2-4k^2)(c^2-1)+a_{20}^2(c^2+16k^2)}{2c(12k^2(1-c^2)+a_{20}^2)}, r=0,$$

where a_{20}, c and k are arbitrary constants.

$$3)\ a_{00}=0, a_{01}=0, a_{10}=0, a_{11}=0, a_{20}=2\sqrt{3(1-c^2)}k, \alpha=0, b_{00}=0,$$
$$b_{01}=0, b_{10}=0, b_{11}=0, b_{20}=-6k^2, \beta=\frac{1-c^2}{2k^2}, c=c, k=k,$$
$$\mu=-\frac{c^2+16k^2}{2c}, r=0,$$

where c and k are arbitrary constants.

$$4)\ a_{00}=0, a_{01}=0, a_{10}=\frac{3k\sqrt{14(c^2-1)}}{7}, a_{11}=0, a_{20}=\frac{6k\sqrt{c^2-1}}{7},$$
$$\alpha=-\frac{7(c^2-1)}{75k^2}, b_{00}=\frac{75k^2}{7}, b_{01}=0, b_{10}=0, b_{11}=0, b_{20}=-\frac{30k^2}{7},$$
$$\beta=\frac{c^2-1}{5k^2}, c=c, k=k, \mu=-\frac{7c^2-272k^2}{14c}, r=\sqrt{\frac{2}{7}},$$

where c and k are arbitrary constants.

$$5)\ a_{00}=0, a_{01}=0, a_{10}=\frac{3k\sqrt{14(1-c^2)}}{7}, a_{11}=0, a_{20}=\frac{6k\sqrt{1-c^2}}{7},$$
$$\alpha=\frac{7(c^2-1)}{75k^2}, b_{00}=0, b_{01}=0, b_{10}=0, b_{11}=0, b_{20}=-\frac{30k^2}{7},$$
$$\beta=\frac{1-c^2}{5k^2}, c=c, k=k, \mu=-\frac{c^2+4k^2}{2c}, r=\sqrt{\frac{2}{7}},$$

where c and k are arbitrary constants.

$$6)\ a_{00}=0, a_{01}=0, a_{10}=a_{10}, a_{11}=0, a_{20}=0, \alpha=\frac{2k^2(c^2-1)+a_{10}^2}{2k^4},$$
$$b_{00}=0, b_{01}=0, b_{10}=0, b_{11}=0, b_{20}=-2k^2, \beta=-\frac{2k^2(c^2-1)+a_{10}^2}{4k^4},$$
$$c=c, k=k, \mu=-\frac{c^2+4k^2}{2c}, r=0,$$

where a_{10}, c and k are arbitrary constants.

$$7)\ a_{00}=0, a_{01}=0, a_{10}=a_{10}, a_{11}=0, a_{20}=0, \alpha=\frac{2k^2(1-c^2)+a_{10}^2}{2k^4},$$
$$b_{00}=-\frac{2k^4(c^2-1)}{2k^2(c^2-1)k^2+a_{10}^2}, b_{01}=0, b_{10}=0, b_{11}=0, b_{20}=-2k^2,$$
$$\beta=-\frac{2k^2(1-c^2)+a_{10}^2}{4k^4}, c=c, \mu=-\frac{2k^2c^2(1-c^2)+a_{10}^2(c^2+4k^2)}{2c(2k^2(1-c^2)+a_{10}^2)},$$
$$k=k, r=0,$$

where a_{10}, c and k are arbitrary constants.

$$8)\ a_{00}=0, a_{01}=0, a_{10}=\frac{\sqrt{6}a_{20}}{46}, a_{11}=0, a_{20}=a_{20}, \alpha=-\frac{7a_{20}^2}{3967500k^4},$$
$$b_{00}=0, b_{01}=0, b_{10}=40\sqrt{6}k^2, b_{11}=0, b_{20}=1150k^2, \beta=\frac{a_{20}^2}{264500k^4},$$
$$c=\frac{1}{230k}\sqrt{\frac{264500k^2-a_{20}^2}{5}}, \mu=-\frac{(10580000k^4+264500k^2-a_{20}^2)\sqrt{5}}{2300k\sqrt{264500k^2-a_{20}^2}},$$
$$k=k, r=-\frac{17\sqrt{6}}{3},$$

where a_{20} and k are arbitrary constants.

$$9)\ a_{00}=0, a_{01}=0, a_{10}=0, a_{11}=a_{11}, a_{20}=0, \alpha=\frac{12k^2(c^2-1)+a_{11}^2}{36k^4},$$
$$b_{00}=0, b_{01}=0, b_{10}=0, b_{11}=0, b_{20}=-6k^2, \beta=-\frac{6k^2(c^2-1)+a_{11}^2}{12k^4},$$
$$c=c, k=k, \mu=-\frac{c^2+4k^2}{2c}, r=0,$$

where a_{11}, c and k are arbitrary constants.

$$10)\ a_{00}=0, a_{01}=0, a_{10}=0, a_{11}=a_{11}, a_{20}=0, \alpha=\frac{12k^2(1-c^2)+a_{11}^2}{36k^4},$$
$$b_{00}=\frac{36k^4(1-c^2)}{12k^2(1-c^2)+a_{11}^2}, b_{01}=0, b_{10}=0, b_{11}=0, b_{20}=-6k^2,$$
$$\beta=-\frac{6k^2(1-c^2)+a_{11}^2}{12k^4}, \mu=-\frac{12k^2(c^2-1)(8k^2-c^2)+a_{11}^2(c^2+4k^2)}{2c(12k^2(1-c^2)+a_{11}^2)},$$
$$c=c, k=k, r=0,$$

where a_{11}, c and k are arbitrary constants.

$$11)\ a_{00}=0, a_{01}=0, a_{10}=0, a_{11}=2\sqrt{3(c^2-1)}k, a_{20}=0, \alpha=0,$$
$$b_{00}=0, b_{01}=0, b_{10}=0, b_{11}=0, b_{20}=-6k^2, \beta=\frac{c^2-1}{2k^2},$$
$$c=c, k=k, \mu=-\frac{c^2+4k^2}{2c}, r=0,$$

where c and k are arbitrary constants.

$$12)\ a_{00}=0, a_{01}=0, a_{10}=0, a_{11}=\sqrt{6(1-r^2)(1-c^2)}k, a_{20}=0,$$
$$\alpha=\frac{c^2-1}{2k^2}, b_{00}=0, b_{01}=0, b_{10}=-6k^2r, b_{11}=0, b_{20}=6k^2(r^2-1),$$
$$\beta=\frac{1-c^2}{k^2}, c=c, k=k, \mu=-\frac{c^2+4k^2}{2c}, r=r,$$

where c, k and r are arbitrary constants.

$$13)\ a_{00}=0, a_{01}=0, 10=0, a_{11}=\sqrt{6(r^2-1)(1-c^2)}k, a_{20}=0,$$
$$\alpha=\frac{1-c^2}{2k^2}, b_{00}=0, b_{01}=0, b_{10}=-6k^2r, b_{11}=0, b_{20}=6k^2(r^2-1),$$
$$\beta=\frac{1-c^2}{k^2}, c=c, k=k, \mu=-\frac{c^2+4k^2}{2c}, r=r,$$

where c, k and r are arbitrary constants.

$$14)\ a_{00}=0, a_{01}=0, a_{10}=0, a_{11}=\sqrt{6(r^2-1)(c^2-1)}k, a_{20}=0,$$
$$\alpha=\frac{1-c^2}{2k^2}, b_{00}=2k^2, b_{01}=0, b_{10}=-6k^2r, b_{11}=0,$$
$$b_{20}=6k^2(r^2-1), \beta=\frac{c^2-1}{k^2}, c=c, k=k, \mu=-\frac{c^2-4k^2}{2c}, r=r,$$

where c, k and r are arbitrary constants.

$$15)\ a_{00}=0, a_{01}=0, a_{10}=0, a_{11}=\frac{3\sqrt{2(c^2-1)}k}{2}, a_{20}=0,$$
$$\alpha=\frac{c^2-1}{2k^2}, b_{00}=0, b_{01}=0, b_{10}=-3k^2, b_{11}=0, b_{20}=-\frac{9k^2}{2},$$
$$\beta=-\frac{c^2-1}{k^2}, c=c, k=k, \mu=-\frac{c^2+4k^2}{2c}, r=\frac{1}{2},$$

where c and k are arbitrary constants.

$$16)\ a_{00}=0, a_{01}=0, a_{10}=0, a_{11}=\frac{3\sqrt{2(1-c^2)}k}{2}, a_{20}=0,$$
$$\alpha=\frac{1-c^2}{2k^2}, b_{00}=2k^2, b_{01}=0, b_{10}=-3k^2, b_{11}=0, b_{20}=-\frac{9k^2}{2},$$
$$\beta=\frac{c^2-1}{k^2}, c=c, k=k, \mu=-\frac{c^2-4k^2}{2c}, r=\frac{1}{2},$$

where c and k are arbitrary constants.

$$17)\ a_{00}=0, a_{01}=0, a_{10}=0, a_{11}=\frac{3\sqrt{2(c^2-1)}k}{2}, a_{20}=0,$$
$$\alpha=\frac{c^2-1}{2k^2}, b_{00}=0, b_{01}=0, b_{10}=3k^2, b_{11}=0, b_{20}=-\frac{9k^2}{2},$$
$$\beta=\frac{1-c^2}{k^2}, c=c, k=k, \mu=-\frac{c^2+4k^2}{2c}, r=-\frac{1}{2},$$

where c and k are arbitrary constants.

$$18)\ a_{00}=0, a_{01}=0, a_{10}=0, a_{11}=\frac{3\sqrt{2(1-c^2)}k}{2}, a_{20}=0,$$
$$\alpha=\frac{1-c^2}{2k^2}, b_{00}=2k^2, b_{01}=0, b_{10}=3k^2, b_{11}=0, b_{20}=-\frac{9k^2}{2},$$
$$\beta=\frac{c^2-1}{k^2}, c=c, k=k, \mu=-\frac{c^2-4k^2}{2c}, r=-\frac{1}{2},$$

where c and k are arbitrary constants.

$$19)\ a_{00}=0, a_{01}=0, a_{10}=\frac{\sqrt{6}a_{20}}{46}, a_{11}=0, a_{20}=a_{20}, \alpha=-\frac{7a_{20}^2}{3967500k^4},$$
$$b_{00}=\frac{15k^2}{7}, b_{01}=0, b_{10}=40\sqrt{6}k^2, b_{11}=0, b_{20}=1150k^2, \beta=\frac{a_{20}^2}{264500k^4},$$
$$c=\frac{\sqrt{a_{20}^2+264500k^2}}{230\sqrt{5}k}, k=k, r=-\frac{17\sqrt{6}}{3},$$
$$\mu=-\frac{(-8464000k^4+7a_{20}^2+1851500k^2)\sqrt{5}}{16100k\sqrt{a_{20}^2+264500k^2}},$$

where a_{20} and k are arbitrary constants.

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