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COMPLETE INVARIANT FUZZY METRICS ON SEMIGROUPS AND GROUPS*

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Abstract In this paper, we study the Raĭkov completion of invariant fuzzy metric groups and complete fuzzy metric semigroups (in the sense of Kramosil and Michael). We establish that: (1) if $(G, M, *)$ is a fuzzy metric group such that $(M, *)$ is invariant, then the Raĭkov completion ϱG of (G, τ_M) is a fuzzy metric group $(\varrho G, \widetilde{M}, *)$ such that $(\widetilde{M}, *)$ is invariant on ϱG and $\widetilde{M}_{|G\times G\times[0,\infty)} = M$; (2) if $(G, M, *)$ is a fuzzy metric semigroup such that $(M, *)$ is invariant, then a fuzzy metric completion $(\widetilde{G}, \widetilde{M}, *)$ of $(G, M, *)$ is a fuzzy metric semigroup and $(\widetilde{M}, *)$ is invariant.

Keywords Fuzzy metric, topological group, topological semigroup, Raĭkov completion.

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1. Introduction

In 1975, Kramosil and Michalek introduced a notion of metric fuzziness [10], which become an important tool in fuzzy normed spaces (among others, the interested reader can consult [2,4,9,11]). In fuzzy Topological Algebra, fuzzy metric topological groups are considered (see [8, 12]). In [5], Gregori and Romaguera remove the symmetric condition in the definition of a fuzzy metric (in the sense of Kramosil and Michalek). This allows us to consider nonsymmetric structures which fit in the realm of fuzzy nonsymmetric topology. For example, fuzzy quasi-metric spaces and fuzzy quasi-normed spaces (see [2, 6, 7]). Recently, Sánchez and Sanchis [14] found sufficient conditions in order that a topological algebraic structure (in particular a nonsymmetric structure) become a stronger topological structure (in particular, a symmetric structure). For example:

Theorem 1.1 ([14, Thoerem 3.2]). *If $(G, M, *)$ is a fuzzy pseudometric right topological group such that $(M, *)$ is left-invariant, then $(G, M, *)$ is a fuzzy topological group.*

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Recently, Sánchez and Sanchis studied complete invariant fuzzy metrics (in the sense of Kramosil and Michalek) on groups. They proved that:

Theorem 1.2 ([13, Theorem 2.2]). *if $(G, M, *)$ is a fuzzy metric group such that $(M, *)$ is invariant, then a fuzzy metric completion $(\tilde{G}, \tilde{M}*)$ of $(G, M, *)$ is a fuzzy metric group and $(\tilde{M}, *)$ is invariant.*

In this paper, we consider the follow two questions: (1) Let $(G, M, *)$ be a fuzzy metric group such that $(M, *)$ is invariant. Can the invariant fuzzy metric $(M, *)$ extend on ϱG? ϱG is the Raĭkov completion of (G, τ_M), where the topology τ_M is induced by the fuzzy metric $(M, *)$ on G. (2) Does Theorem 1.2 hold for fuzzy metric semigroups?

We shall answer the two questions above. Firstly, some notations and definitions are stated.

Recall that a binary operation $* : [0,1]\times[0,1] \to [0,1]$ is a *continuous t-norm* [15] if $*$ satisfies the following conditions: (i) $*$ is associative and commutative; (ii) $*$ is continuous; (iii) $a * 1 = a$ for all $a \in [0,1]$ and (iv) $a * b \le c * d$ whenever $a \le c$ and $b \le d$, with $a, b, c, d \in [0,1]$.

Three paradigmatic examples of continuous t-norms are $\wedge$, $\cdot$ and $*_L$ (the Lukasiewicz t-norm), which are defined by $a \wedge b = \min\{a, b\}$, $a \cdot b = ab$ and $a *_L b = \max\{a + b - 1, 0\}$, respectively. It is well known that $* \le \wedge$ for every continuous t-norm $*$.

Definition 1.1 ([10]). A *fuzzy metric* (in the sense of Kramosil and Michalek) on a set X is a pair $(M, *)$ such that M is a fuzzy set in $X \times X \times [0, \infty)$ and $*$ is a continuous t-norm satisfying for all $x, y, z \in X$:

(i) $M(x, y, 0) = 0$;

(ii) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$;

(iii) $M(x, y, t + s) \ge M(x, z, t) * M(z, y, s)$ for all $t, s > 0$;

(iv) $M(x, y, \_\,) : [0, \infty) \to [0, 1]$ is a left continuous function;

(v) $M(x, y, t) = M(y, x, t)$.

Definition 1.2 ([14, Definition 2.4]). A fuzzy metric $(M, *)$ on a semigroup G is *left-invariant* (respectively, *right-invariant*) if $M(x, y, t) = M(ax, ay, t)$ (respectively, $M(x, y, t) = M(xa, ya, t)$) whenever $a, x, y \in G$ and $t > 0$. We say that $(M, *)$ is invariant if it is both left-invariant and right-invariant.

By a *fuzzy metric space* we mean a triple $(X, M, *)$ such that X is a set and $(M, *)$ is a fuzzy metric on X. Every fuzzy metric $(M, *)$ on a set X induces a topology τ_M on X, which has as a base the family of open sets of the form $\{B_M(x, \varepsilon, t) : x \in X, \varepsilon \in (0, 1), t > 0\}$, where $B_M(x, \varepsilon, t) = \{y \in X : M(x, y, t) > 1 - \varepsilon\}$ for all $x \in X$, $\varepsilon \in (0, 1)$, $t > 0$.

By a *fuzzy metric group* (resp., *fuzzy metric semigroup*) we mean a 4-tuple $(G, \cdot, M, *)$ such that $(G, M, *)$ is a fuzzy metric space and (G, τ_M) is a topological group (resp., topological semigroup).

A sequence $(x_n)_{n\in\mathbb{N}}$ in a fuzzy metric space $(X, M, *)$ is said to be a *Cauchy sequence* provided that for each $\varepsilon \in (0, 1)$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for every $n, m \ge n_0$. A fuzzy metric space $(X, M, *)$ where every Cauchy sequence converges are called *completeness*.

2. Main results

A *filter* on a set X is a family η of non-empty subsets of X satisfying the next two conditions: (i) If U and V are in η, then $U \cap V$ is also in η; (ii) If $U \in \eta$ and $U \subseteq W \subseteq X$, then $W \in \eta$.

Let G be a topological group with the identity e. A filter η of a topological group G is said to be a *Cauchy filter* if for every open neighbourhood V of e in G, there exist $a, b \in G$ and $A, B \in \eta$ such that $A \subseteq aV$ and $B \subseteq Vb$. A topological group G such that every Cauchy filter on G converges is called *Raĭkov complete*. Next we shall investigate the Raĭkov complete of the group topologies induced by invariant fuzzy metrics on groups.

Proposition 2.1. *Let $(G, M, *)$ be a fuzzy metric group. If $(G, M, *)$ is complete, then (G, τ_M) is Raĭkov complete.*

Proof. Suppose that $(G, M, *)$ is complete. Take an arbitrary Cauchy filter η in G. Then for each $n \in \mathbb{N}$ there are $F'_n \in \eta$ and $x_n \in G$ such that $F'_n \subseteq B_M(x_n, \frac{1}{n}, \frac{1}{n})$. Put $F_n = \bigcap_{i=1}^{n} F'_i$. Clearly, $F_n \subseteq B_M(x_n, \frac{1}{n}, \frac{1}{n})$ holds for each $n \in \mathbb{N}$. Take $y_n \in F_n$ for each $n \in \mathbb{N}$. Then the sequence $\{y_n\}$ is a Cauchy sequence in $(G, M, *)$. In fact, for each $n \in \mathbb{N}$, since t-norm $*$ is continuous, there is $n_0 \in \mathbb{N}$ such that $(1 - \frac{1}{n_0}) * (1 - \frac{1}{n_0}) > (1 - \frac{1}{n})$ and $\frac{1}{n_0} \leq \frac{1}{2n}$. Clearly, $y_i, y_j \in F_{n_0} \subseteq B_M(x_{n_0}, \frac{1}{n_0}, \frac{1}{n_0})$ whenever $i, j \in \mathbb{N}$ and $i, j > n_0$. Thus

$$
\begin{aligned}
M(y_i, y_j, \frac{1}{n}) &\geq M(y_i, x_{n_0}, \frac{1}{2n}) * M(x_{n_0}, y_j, \frac{1}{2n}) \\
&\geq M(y_i, x_{n_0}, \frac{1}{n_0}) * M(x_{n_0}, y_j, \frac{1}{n_0}) > (1 - \frac{1}{n_0}) * (1 - \frac{1}{n_0}) > (1 - \frac{1}{n})
\end{aligned}
$$

whenever $i, j > n_0$. This show that $\{y_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence. Since $(G, M, *)$ is complete, $\{y_n\}_{n \in \mathbb{N}}$ converges to some $y \in G$. We shall show that the Cauchy filter η converges to y, which implies that (G, τ_M) is Raĭkov complete.

Take any open neighbourhood V of y. Without loss of generality, we assume that $V = B_M(y, \frac{1}{n}, \frac{1}{n})$. Since t-norm is continuous, there is $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} \leq \frac{1}{2n}$ and $(1 - \frac{1}{n_0}) * (1 - \frac{1}{n_0}) > (1 - \frac{1}{n})$. Note that $\{y_i\}_{i \in \mathbb{N}}$ converges to y, then there is $n' \in \mathbb{N}$ such that $\frac{1}{2n} \geq \frac{2}{n'}$, $(1 - \frac{1}{n'}) * (1 - \frac{1}{n'}) > (1 - \frac{1}{n_0})$ and $M(y, y_{n'}, \frac{1}{n_0}) > (1 - \frac{1}{n_0})$. Then for each $x \in F_{n'}$, noting that $y_{n'} \in F_{n'} \subseteq B_M(x_{n'}, \frac{1}{n'}, \frac{1}{n'})$, we have

$$
\begin{aligned}
M(y, x, \frac{1}{n}) &\geq M(y, y_{n'}, \frac{1}{2n}) * M(y_{n'}, x, \frac{1}{2n}) \geq M(y, y_{n'}, \frac{1}{n_0}) * M(y_{n'}, x, \frac{2}{n'}) \\
&\geq M(y, y_{n'}, \frac{1}{n_0}) * M(y_{n'}, x_{n'}, \frac{1}{n'}) * M(x_{n'}, x, \frac{1}{n'}) \\
&\geq (1 - \frac{1}{n_0}) * (1 - \frac{1}{n'}) * (1 - \frac{1}{n'}) \geq (1 - \frac{1}{n_0}) * (1 - \frac{1}{n_0}) \geq (1 - \frac{1}{n}).
\end{aligned}
$$

This implies that $x \in B_M(y, \frac{1}{n}, \frac{1}{n})$, i.e., $F_{n'} \subseteq B_M(y, \frac{1}{n}, \frac{1}{n})$. Clearly, $F_{n'} \in \eta$, thus we have proved that η converges to y. $\square$

Theorem 2.1. *If $(G, M, *)$ is a fuzzy metric group such that $(M, *)$ is invariant, then the Raĭkov completion ϱG of (G, τ_M) is a fuzzy metric group $(\varrho G, \widetilde{M}, *)$ such that $(\widetilde{M}, *)$ is invariant on ϱG and $\widetilde{M}_{|G \times G \times [0,\infty)} = M$.*

Proof. Let $(\widetilde{G}, \hat{M}, *)$ be a fuzzy metric completion of $(G, M, *)$. Then according to Theorem 1.2, $(\widetilde{G}, \hat{M}, *)$ is a fuzzy metric group $(\widetilde{G}, \hat{M}, *)$ satisfying: $(\hat{M}, *)$ is invariant on $\widetilde{G}$ and $\hat{M}_{|G\times G\times[0,\infty)} = M$. Then according to Proposition 2.1 it follows that $(\widetilde{G}, \tau_{\hat{M}})$ is Raĭkov complete and G is a dense subgroup in $\widetilde{G}$. Then according to [3, Theorem 3.6.14] there is a topological isomorphism $\varphi : \varrho G \to \widetilde{G}$ such that $\varphi(g) = g$ for each $g \in G$. Thus we can define $\widetilde{M} : \varrho G \times \varrho G \times [0, \infty) \to [0, 1]$ as following: $\widetilde{M}(x, y, t) = \hat{M}(\varphi(x), \varphi(y), t)$ for each $(x, y, t) \in \varrho G \times \varrho G \times [0, \infty)$. One easily show that fuzzy metric $(\widetilde{M}, *)$ is required. This completes the proof. □

Proposition 2.2. *Let $(G, M, *)$ be a fuzzy metric group with $(M, *)$ being invariant. If (G, τ_M) is Raĭkov complete, then $(G, M, *)$ is complete.*

Proof. Suppose that (G, τ_M) is Raĭkov complete. Take arbitrary Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ of $(G, M, *)$. Put $\eta = \{A \subseteq G : F_n \subseteq A$ for some $F_n\}$, where $F_n = \{x_i : i \geq n\}$ for each $n \in \mathbb{N}$. Now we shall prove that η is a Cauchy filter of G. Take any $B_M(e, \frac{1}{n}, \frac{1}{n})$, where e is the identity of G. Since $\{x_i : i \in \mathbb{N}\}$ is a Cauchy sequence, there is $n_0 \in \mathbb{N}$ such that $M(x_k, x_m, \frac{1}{n}) > 1 - \frac{1}{n}$ whenever $k, m \geq n_0$. This implies that $x_k \in B(x_{n_0}, \frac{1}{n}, \frac{1}{n})$ whenever $k \geq n_0$. Hence, $F_{n_0} \subseteq B(x_{n_0}, \frac{1}{n}, \frac{1}{n})$. Noting that M is invariant, so $B_M(x_{n_0}, \frac{1}{n}, \frac{1}{n}) = x_{n_0}B_M(e, \frac{1}{n}, \frac{1}{n}) = B_M(e, \frac{1}{n}, \frac{1}{n})x_{n_0}$, so $F_{n_0} \subseteq x_{n_0}B_M(e, \frac{1}{n}, \frac{1}{n})$ and $F_{n_0} \subseteq B_M(e, \frac{1}{n}, \frac{1}{n})x_{n_0}$. This implies that η is a Cauchy filter. Since (G, τ_M) is Raĭkov complete, the Cauchy filter η converges to a point g in G. Then one can easily show that the Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to g. This implies that $(G, M, *)$ is complete. □

By Theorem 2.1 and Proposition 2.2 we have the following:

Corollary 2.1. *Let $(G, M, *)$ be a fuzzy metric group such that $(M, *)$ is invariant on G. Then $(G, M, *)$ is complete if and only if (G, τ_M) is Raĭkov complete.*

Since every Abelian fuzzy metric group $(G, M, *)$ satisfies that $(M, *)$ is invariant on G, by Corollary 2.1 we have the following:

Corollary 2.2. *Every Abelian fuzzy metric group $(G, M, *)$ is complete if and only if (G, τ_M) is Raĭkov complete.*

Next we shall show that Theorem 1.2 holds for semigroups.

Theorem 2.2. *If $(G, M, *)$ is a fuzzy metric semigroup such that $(M, *)$ is invariant, then a fuzzy metric completion $(\widetilde{G}, \widetilde{M}, *)$ of $(G, M, *)$ is a fuzzy metric semigroup and $(\widetilde{M}, *)$ is invariant.*

Proof. Firstly, we shall prove the following Claim 1.

Claim 1. Let $(a_n)_n$ and $(b_n)_n$ be Cauchy sequences in G. Then $(a_nb_n)_n$ is also a Cauchy sequence in G.

Fix $\epsilon \in (0, 1)$ and $t > 0$. Since $*$ is a continuous t-norm, there is $s > 0$ such that $(1 - s) * (1 - s) > 1 - \epsilon$. Observing $(a_n)_n$ and $(b_n)_n$ are Cauchy sequences in G, so there exists $n_0 \in \mathbb{N}$ such that $M(a_n, a_m, \frac{t}{2}) > 1 - s$ and $M(b_n, b_m, \frac{t}{2}) > 1 - s$ whenever $n, m > n_0$. Since M is invariant in G, we have

$$\begin{aligned} M(a_nb_n, a_mb_m, t) &\geq M(a_nb_n, a_mb_n, \frac{t}{2}) * M(a_mb_n, a_mb_m, \frac{t}{2}) \\ &= M(a_n, a_m, \frac{t}{2}) * M(b_n, b_m, \frac{t}{2}) > (1 - s) * (1 - s) > 1 - \epsilon \end{aligned}$$

whenever $n, m > n_0$.

This implies that $(a_n b_n)_n$ is a Cauchy sequence in G.

Now we define a binary operation $\cdot$ on $\widetilde{G}$ as follows: $(a_n)_n \cdot (b_n)_n = (a_n b_n)_n$ for each pair Cauchy sequences $(a_n)_n$ and $(b_n)_n$ in G. Let us show that $\cdot$ is well defined. According to Claim 1 it is enough to show the following Claim 2.

Claim 2. Let $(a_n)_n$, $(b_n)_n$ and $(a'_n)_n$, $(b'_n)_n$ be Cauchy sequences in G such that $\widetilde{M}((a_n)_n, (a'_n)_n, t) = \lim_{n\to\infty} M(a_n, a'_n, t) = 1$ and $\widetilde{M}((b_n)_n, (b'_n)_n, t) = \lim_{n\to\infty} M(b_n, b'_n, t) = 1$ for all $t > 0$. Then $\widetilde{M}((a_n b_n)_n, (a'_n b'_n)_n, t) = \lim_{n\to\infty} M(a_n b_n, a'_n b'_n, t) = 1$ for all $t > 0$.

Fix $t > 0$, taking $\epsilon \in (0,1)$. Then there exists $\epsilon' \in (0,1)$ such that $(1-\epsilon') * (1-\epsilon') > 1-\epsilon$. For ϵ', since $\lim_{n\to\infty} M(a_n, a'_n, \frac{t}{2}) = 1$ and $\lim_{n\to\infty} M(b_n, b'_n, \frac{t}{2}) = 1$, there is $n_0 \in \mathbb{N}$ such that $M(b_n, b'_n, \frac{t}{2}) > 1-\epsilon'$ and $M(b_n, b'_n, \frac{t}{2}) > 1-\epsilon'$ whenever $n > n_0$. Observing $(M, *)$ is invariant, so we have:

$$\begin{aligned} M(a_n b_n, a'_n b'_n, t) &\geq M(a_n b_n, a'_n b_n, \frac{t}{2}) * M(a'_n b_n, a'_n b'_n, \frac{t}{2}) \\ &= M(a_n, a'_n, \frac{t}{2}) * M(b_n, b'_n, \frac{t}{2}) > (1-\epsilon') * (1-\epsilon') > 1-\epsilon \end{aligned}$$

whenever $n > n_0$. This implies that $\lim_{n\to\infty} M(a_n b_n, a'_n b'_n, t) = 1$ for all $t > 0$. Thus, the binary operation $\cdot$ is well defined.

Since G is a semigroup, one can easily show that $((a_n)_n \cdot (b_n)_n) \cdot (c_n)_n = (a_n)_n \cdot ((b_n)_n \cdot (c_n)_n)$. Thus $(\widetilde{G}, \cdot)$ is a semigroup. Now, we shall show that $(\widetilde{M}, *)$ is invariant on $(\widetilde{G}, \cdot)$.

Since $(M, *)$ is invariant, $\widetilde{M}((c_n)_n \cdot (a_n)_n, (c_n)_n \cdot (b_n)_n, t) = \widetilde{M}((c_n a_n)_n, (c_n b_n)_n, t) = \lim_{n\to\infty} M(c_n a_n, c_n b_n, t) = \lim_{n\to\infty} M(a_n, b_n, t) = \widetilde{M}((a_n)_n, (b_n)_n, t)$. This implies that $(\widetilde{M}, *)$ is left invariant on $(\widetilde{G}, \cdot)$. Similarly, one can show that $(\widetilde{M}, *)$ is right invariant on $(\widetilde{G}, \cdot)$. Thus $(\widetilde{M}, *)$ is invariant.

Finally, we shall show that $(\widetilde{G}, \widetilde{M}, *)$ is a fuzzy metric semigroup. Let $(a_n)_n$ and $(b_n)_n$ be Cauchy sequences in G. Take any open neighborhood U of $(a_n)_n \cdot (b_n)_n$. Then there exist $\epsilon \in (0,1)$ and $t > 0$ such that $B_{\widetilde{M}}((a_n)_n \cdot (b_n)_n, \epsilon, t) \subseteq U$. Since t-norm is continuous, there exists $\epsilon' \in (0,1)$ such that $(1-\epsilon') * (1-\epsilon') > 1-\epsilon$. Now we claim that

$$B_{\widetilde{M}}((a_n)_n, \epsilon', \frac{t}{2}) \cdot B_{\widetilde{M}}((b_n)_n, \epsilon', \frac{t}{2}) \subseteq B_{\widetilde{M}}((a_n)_n \cdot (b_n)_n, \epsilon, t).$$

This implies that the binary operation $\cdot$ is joint continuous on $(\widetilde{G}, \widetilde{M}, *)$. Thus, $(\widetilde{G}, \widetilde{M}, *)$ is a fuzzy metric semigroup.

In fact, take any $(c_n)_n \in B_{\widetilde{M}}((a_n)_n, \epsilon', \frac{t}{2})$ and $(c'_n)_n \in B_{\widetilde{M}}((b_n)_n, \epsilon', \frac{t}{2})$. Then

$$\begin{aligned} &\widetilde{M}((a_n)_n \cdot (b_n)_n, (c_n)_n \cdot (c'_n)_n, t) = \widetilde{M}((a_n b_n)_n, (c_n c'_n)_n, t) \\ &= \lim_{n\to\infty} M(a_n b_n, c_n c'_n, t) \geq \lim_{n\to\infty} (M(a_n b_n, c_n b_n, \frac{t}{2}) * M(c_n b_n, c_n c'_n, \frac{t}{2})) \\ &= \lim_{n\to\infty} (M(a_n, c_n, \frac{t}{2}) * M(b_n, c'_n, \frac{t}{2})) = \lim_{n\to\infty} M(a_n, c_n, \frac{t}{2}) * \lim_{n\to\infty} M(b_n, c'_n, \frac{t}{2}) \\ &= \widetilde{M}((a_n)_n, (c_n)_n, \frac{t}{2}) * \widetilde{M}((b_n)_n, (c'_n)_n, \frac{t}{2}) > (1-\epsilon') * (1-\epsilon') > 1-\epsilon. \end{aligned}$$

This completes the proof. $\square$

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