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SOME GEOMETRIC PROPERTIES OF MULTIVALENT ANALYTIC FUNCTIONS ASSOCIATED WITH THE LEMNISCATE OF BERNOULLI*

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Abstract By using the first-order differential subordination, a new class $\mathcal{M}_n(\alpha)$ of multivalent analytic functions associated with the lemniscate of Bernoulli is introduced. Several geometric properties of this class are given.

Keywords Analytic function, differential subordination, Lemniscate of Bernoulli.

MSC(2010) 30C45, 30C80.

1. Introduction

Let $\mathcal{A}_n(p)$ denote the class of functions of the form:

$$f(z) = z^p + \sum_{k=n}^{\infty} a_{p+k} z^{p+k} \quad (n, p \in \mathbb{N}) \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$. For $n = p = 1$, we write $\mathcal{A} := \mathcal{A}_1(1)$.

For functions $f(z)$ and $g(z)$ analytic in $\mathbb{U}$, we say that $f(z)$ is subordinate to $g(z)$ and write $f(z) \prec g(z)$ $(z \in \mathbb{U})$, if there exists an analytic function $w(z)$ in $\mathbb{U}$ such that

$$|w(z)| \leq |z| \quad \text{and} \quad f(z) = g(w(z)) \quad (z \in \mathbb{U}).$$

If the function $g(z)$ is univalent in $\mathbb{U}$, then

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Leftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

For functions $f_j(z) \in \mathcal{A}_n(p)$ $(j = 1, 2)$ given by

$$f_j(z) = z^p + \sum_{k=n}^{\infty} a_{k+p,j} z^{k+p},$$

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we define the Hadamard product (or convolution) of $f_1(z)$ and $f_2(z)$ by

$$(f_1 * f_2)(z) = z^p + \sum_{k=n}^{\infty} a_{k+p,1} a_{k+p,2} z^{k+p}.$$

A function $f(z) \in \mathcal{A}_n(p)$ is said to be convex of order β $(0 \leq \beta < p)$ if

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \beta \quad (z \in \mathbb{U}).$$

Let $\mathcal{SL}$ be the class of functions defined by

$$\mathcal{SL} := \left\{ f(z) \in \mathcal{A} : \left| \left(\frac{zf'(z)}{f(z)}\right)^2 - 1 \right| < 1 \right\}.$$

A function $f(z) \in \mathcal{SL}$ if $zf'(z)/f(z)$ lies in the region bounded by the right-half of the lemniscate of Bernoulli given by $|w^2 - 1| < 1$. In terms of differential subordination, the class $\mathcal{SL}$ consists of normalized analytic functions $f(z) \in \mathcal{A}$ satisfying $zf'(z)/f(z) \prec \sqrt{1+z}$. The class SL was first introduced by Sokól and Stankiewicz [10]. Recently, the $\mathcal{SL}$-radii for certain well-known classes of functions including the Janowski starlike functions were obtained in [1,9].

Motivated by the above and some recent works [2–7,11,12], we now introduce a new subclass of $\mathcal{A}_n(p)$.

Definition 1.1. A function $f(z) \in \mathcal{A}_n(p)$ $(p \geq 2)$ is said to be in the class $\mathcal{M}_n(\alpha)$ if it satisfies the second-order differential subordination:

$$\frac{1-\alpha}{p} z^{1-p} f'(z) + \frac{\alpha}{p(p-1)} z^{2-p} f''(z) \prec \sqrt{1+z} \quad (\alpha \geq 0;\ z \in \mathbb{U}). \tag{1.2}$$

In this note we obtain inclusion relation and coefficient estimate for functions $f(z)$ belonging to the class $\mathcal{M}_n(\alpha)$. Furthermore, we discuss the radius of convex for functions in $\mathcal{M}_n(0)$.

2. Geometric properties of functions in the class $\mathcal{M}_n(\alpha)$

In order to derive Theorem 1 below, we need the following lemma.

Lemma 2.1. *Let $g(z)$ be analytic in $\mathbb{U}$ and $h(z)$ be analytic and convex univalent in $\mathbb{U}$ with $h(0) = g(0)$. If*

$$g(z) + \frac{1}{\mu} zg'(z) \prec h(z),$$

where $\mathrm{Re}\mu \geq 0$ and $\mu \neq 0$, then $g(z) \prec h(z)$.

Theorem 2.1. *Let $0 \leq \alpha_1 < \alpha_2$. Then $\mathcal{M}_n(\alpha_2) \subset \mathcal{M}_n(\alpha_1)$.*

Proof. Suppose that

$$g(z) = \frac{f'(z)}{pz^{p-1}} \tag{2.1}$$

for $f(z) \in \mathcal{M}_n(\alpha_2)$. Then $g(z)$ is analytic in $\mathbb{U}$ and $g(0) = 1$. By using (1.2) and (2.1), we have

$$\frac{1-\alpha_2}{p} z^{1-p} f'(z) + \frac{\alpha_2}{p(p-1)} z^{2-p} f''(z) = g(z) + \frac{\alpha_2}{p-1} z g'(z) \prec \sqrt{1+z}. \tag{2.2}$$

An application of Lemma yields

$$g(z) \prec \sqrt{1+z}. \tag{2.3}$$

Noting that $0 \leq \frac{\alpha_1}{\alpha_2} < 1$ and that the function $\sqrt{1+z}$ is convex univalent in $\mathbb{U}$, it follows from (2.1), (2.2) and (2.3) that

$$\begin{aligned} &\frac{1-\alpha_1}{p} z^{1-p} f'(z) + \frac{\alpha_1}{p(p-1)} z^{2-p} f''(z) \\ &= \frac{\alpha_1}{\alpha_2} \left(\frac{1-\alpha_2}{p} z^{1-p} f'(z) + \frac{\alpha_2}{p(p-1)} z^{2-p} f''(z) \right) + \left(1 - \frac{\alpha_1}{\alpha_2} \right) g(z) \\ &\prec \sqrt{1+z}. \end{aligned}$$

This shows that $f(z) \in \mathcal{M}_n(\alpha_1)$. The proof of Theorem 1 is completed. □

Theorem 2.2. *Let* $f(z) \in \mathcal{M}_n(\alpha)$, $g(z) \in \mathcal{A}_n(p)$ *and*

$$\operatorname{Re}\left(z^{-p} g(z)\right) > \frac{1}{2} \quad (z \in \mathbb{U}). \tag{2.4}$$

Then $(f * g)(z) \in \mathcal{M}_n(\alpha)$.

Proof. For $f(z) \in \mathcal{M}_n(\alpha)$ and $g(z) \in \mathcal{A}_n(p)$, we have

$$\begin{aligned} &\frac{1-\alpha}{p} z^{1-p} (f*g)'(z) + \frac{\alpha}{p(p-1)} z^{2-p} (f*g)''(z) \\ &= \frac{1-\alpha}{p} \left(z^{1-p} f'(z)\right) * \left(z^{-p} g(z)\right) + \frac{\alpha}{p(p-1)} \left(z^{2-p} f''(z)\right) * \left(z^{-p} g(z)\right) \\ &= h(z) * \left(z^{-p} g(z)\right), \end{aligned} \tag{2.5}$$

where

$$h(z) = \frac{1-\alpha}{p} z^{1-p} f'(z) + \frac{\alpha}{p(p-1)} z^{2-p} f''(z) \prec \sqrt{1+z} \quad (z \in \mathbb{U}). \tag{2.6}$$

From (2.4), we can see that the function $z^{-p} g(z)$ has Herglotz representation:

$$z^{-p} g(z) = \int_{|x|=1} \frac{d\mu(x)}{1 - xz} \quad (z \in \mathbb{U}), \tag{2.7}$$

where $\mu(x)$ is a probability measure on the unit circle $|x| = 1$ and $\int_{|x|=1} d\mu(x) = 1$.

In view of the function $\sqrt{1+z}$ is convex univalent in $\mathbb{U}$, it follows from (2.5), (2.6) and (2.7) that

$$\begin{aligned} &\frac{1-\alpha}{p} z^{1-p} (f*g)'(z) + \frac{\alpha}{p(p-1)} z^{2-p} (f*g)''(z) \\ &= \int_{|x|=1} h(xz) d\mu(x) \prec \sqrt{1+z} \quad (z \in \mathbb{U}). \end{aligned}$$

This shows that $(f * g)(z) \in \mathcal{M}_n(\alpha)$. The proof of Theorem 2 is completed. □

Theorem 2.3. *Let*

$$f(z) = z^p + \sum_{k=n}^{\infty} a_{p+k} z^{p+k} \in \mathcal{M}_n(\alpha). \tag{2.8}$$

Then

$$|a_{p+k}| \leq \frac{p(p-1)}{2(p+k)(\alpha k + p - 1)} \quad (k \geq n). \tag{2.9}$$

The result is sharp for each $k \geq n$.

Proof. It is known that, if

$$\varphi(z) = \sum_{j=1}^{\infty} b_j z^j \prec \psi(z) \quad (z \in \mathbb{U}),$$

where $\varphi(z)$ is analytic in $\mathbb{U}$ and $\psi(z) = z + \cdots$ is analytic and convex univalent in $\mathbb{U}$, then $|b_j| \leq 1$ $(j \in \mathbb{N})$.

By (2.8) we have

$$\begin{aligned} &2\left(\frac{1-\alpha}{p} z^{1-p} f'(z) + \frac{\alpha}{p(p-1)} z^{2-p} f''(z) - 1\right) \\ &= \frac{2}{p(p-1)} \sum_{k=n}^{\infty} (p+k)(\alpha k + p - 1) a_{p+k} z^k \\ &\prec 2(\sqrt{1+z} - 1) \quad (z \in \mathbb{U}). \end{aligned} \tag{2.10}$$

In view of the function $\psi(z) = 2(\sqrt{1+z} - 1) = z + \cdots$ is analytic and convex univalent in $\mathbb{U}$, it follows from (2.10) that

$$\frac{2(p+k)(\alpha k + p - 1)}{p(p-1)} |a_{p+k}| \leq 1 \quad (k \geq n),$$

which gives (2.9).

Next we consider the function $f_k(z)$ given by

$$f_k(z) = z^p + \sum_{m=1}^{\infty} \frac{p(p-1)\begin{pmatrix} \frac{1}{2} \\ m \end{pmatrix}}{(km+p)(\alpha km + p - 1)} z^{km+p} \quad (k \geq n;\ z \in \mathbb{U}),$$

where

$$\begin{pmatrix} \gamma \\ m \end{pmatrix} = \frac{\gamma(\gamma-1)\cdots(\gamma-m+1)}{m!}.$$

Since

$$\frac{1-\alpha}{p} z^{1-p} f_k'(z) + \frac{\alpha}{p(p-1)} z^{2-p} f_k''(z) = \sqrt{1+z^k} \prec \sqrt{1+z} \quad (z \in \mathbb{U})$$

and

$$f_k(z) = z^p + \frac{p(p-1)}{2(k+p)(\alpha k + p - 1)} z^{p+k} + \cdots$$

for each $k \geq n$, the proof of Theorem 3 is completed. □

Theorem 2.4. *Let $0 \le \rho < p$ and $\frac{\rho}{p} < \delta \le 1$. If $f(z) \in \mathcal{M}_n(0)$, then*

$$\operatorname{Re}\left\{(1-\delta)\left(\frac{f'(z)}{pz^{p-1}}\right)^2 + \delta\left(1+\frac{zf''(z)}{f'(z)}\right)\right\} > \rho \quad (|z| < r_0), \tag{2.11}$$

where r_0 is the smallest root in $(0,1)$ of the equation

$$(1-\delta)r^{2n} - \left(2-\rho+\delta\left(\frac{n}{2}+p-2\right)\right)r^n + 1 - \rho + \delta(p-1) = 0.$$

The result is sharp.

Proof. For $f(z) \in \mathcal{M}_n(0)$ we can write

$$\left(\frac{f'(z)}{pz^{p-1}}\right)^2 = 1 + z^n\varphi(z), \tag{2.12}$$

where $\varphi(z)$ is analytic and $|\varphi(z)| \le 1$ in $\mathbb{U}$. Differentiating both sides of (2.12) logarithmically, we have

$$1 + \frac{zf''(z)}{f'(z)} = p + \frac{nz^n\varphi(z) + z^{n+1}\varphi'(z)}{2(1+z^n\varphi(z))} \quad (z \in \mathbb{U}). \tag{2.13}$$

Put $|z| = r < 1$ and $\left(\frac{f'(z)}{pz^{p-1}}\right)^2 = u + iv$ $(u, v \in \mathbb{R})$. Then (2.12) implies that

$$z^n\varphi(z) = u - 1 + iv \tag{2.14}$$

and

$$1 - r^n \le u \le 1 + r^n. \tag{2.15}$$

With the help of the Carathéodory inequality:

$$|\varphi'(z)| \le \frac{1-|\varphi(z)|^2}{1-r^2},$$

it follows from (2.14) and (2.15) that

$$\begin{aligned}
&\operatorname{Re}\left\{(1-\delta)\left(\frac{f'(z)}{pz^{p-1}}\right)^2 + \delta\left(1+\frac{zf''(z)}{f'(z)}\right)\right\}\\
&\ge (1-\delta)u + p\delta + \frac{n\delta}{2}\operatorname{Re}\left\{\frac{z^n\varphi(z)}{1+z^n\varphi(z)}\right\} - \frac{\delta}{2}\left|\frac{z^{n+1}\varphi'(z)}{1+z^n\varphi(z)}\right|\\
&\ge (1-\delta)u + p\delta + \frac{n\delta}{2}\left(1 - \frac{u}{u^2+v^2}\right) + \frac{\delta}{2}\frac{(u-1)^2+v^2-r^{2n}}{r^{n-1}(1-r^2)(u^2+v^2)^{\frac{1}{2}}}\\
&=: H_n(u,v)
\end{aligned}$$

and

$$\frac{\partial H_n(u,v)}{\partial v} = \frac{\delta v}{2}P_n(u,v), \tag{2.16}$$

where

$$P_n(u,v) := \frac{2nu}{(u^2+v^2)^2} + \frac{2}{r^{n-1}(1-r^2)(u^2+v^2)^{\frac{1}{2}}}$$

$$+\frac{r^{2n}-((u-1)^2+v^2)}{r^{n-1}(1-r^2)(u^2+v^2)^{\frac{3}{2}}}>0 \tag{2.17}$$

because of (2.14), (2.15) and $|\varphi(z)| \leq 1$ ($z \in \mathbb{U}$). In view of (2.16) and (2.17) we see that

$$\begin{aligned} H_n(u,v) &\geq H_n(u,0) \\ &= (1-\delta)u + p\delta + \frac{n\delta}{2}\left(1-\frac{1}{u}\right) + \frac{\delta((u-1)^2-r^{2n})}{2r^{n-1}(1-r^2)u}. \end{aligned} \tag{2.18}$$

Next we calculate the minimum value of $H_n(u,0)$ on the closed interval $[1-r^n, 1+r^n]$. From (2.18) we deduce that

$$\begin{aligned} \frac{d}{du}H_n(u,0) &= 1-\delta+\frac{\delta}{2u^2}\left(n+\frac{r^{2n}+u^2-1}{r^{n-1}(1-r^2)}\right) \\ &\geq 1-\delta+\frac{\delta}{2(1+r^n)^2}\left(n-\frac{2r(1-r^n)}{1-r^2}\right) \\ &= 1-\delta+\frac{\delta}{2(1+r^n)^2}I_n(r), \end{aligned} \tag{2.19}$$

where

$$I_n(r) := n - \frac{2r(1-r^n)}{1-r^2}.$$

Note that $I_1(r) = \frac{1-r}{1+r} > 0$. Suppose that $I_n(r) > 0$. Then

$$\begin{aligned} I_{n+1}(r) &= n+1-\frac{2r(1-r^{n+1})}{1-r^2} \\ &= I_n(r) + \frac{1-r^{n+1}+r(1-r^n)}{1+r} > 0. \end{aligned}$$

Hence, by virtue of the mathematical induction, we have $I_n(r) > 0$ for all $n \in \mathbb{N}$ and $0 \leq r < 1$. This implies that

$$\frac{d}{du}H_n(u,0) > 0 \quad (1-r^n \leq u \leq 1+r^n). \tag{2.20}$$

Further, it follows from (2.18) and (2.20) that

$$\begin{aligned} &\operatorname{Re}\left\{(1-\delta)\left(\frac{f'(z)}{pz^{p-1}}\right)^2 + \delta\left(1+\frac{zf''(z)}{f'(z)}\right)\right\} - \rho \\ &\geq H_n(1-r^n,0) - \rho \\ &= (1-\delta)(1-r^n) + p\delta - \frac{n\delta r^n}{2(1-r^n)} - \rho \\ &= \frac{(1-\delta)r^{2n} - \left(2-\rho+\delta\left(\frac{n}{2}+p-2\right)\right)r^n + 1 - \rho + \delta(p-1)}{1-r^n} \\ &=: \frac{J_n(r)}{1-r^n}. \end{aligned} \tag{2.21}$$

From the hypotheses of the theorem we can see that $J_n(0) = 1-\rho+\delta(p-1) > 0$ and $J_n(1) = -\frac{n\delta}{2} < 0$. If we let r_0 denote the smallest root in $(0,1)$ of the equation $J_n(r) = 0$, then (2.21) yields the desired result (2.11).

To see that the bound r_0 is the best possible, we consider the function

$$f(z) = p\int_0^z t^{p-1}(1-t^n)^{\frac{1}{2}}dt \in \mathcal{M}_n(0). \tag{2.22}$$

It is clear that for $z = r \in (r_0, 1)$,

$$(1-\delta)\left(\frac{f'(r)}{pr^{p-1}}\right)^2 + \delta\left(1 + \frac{rf''(r)}{f'(r)}\right) - \rho = \frac{J_n(r)}{1-r^n} < 0,$$

which shows that r_0 can not be increased. The proof of Theorem 4 is completed.

Setting $\delta = 1$, Theorem 4 reduces to the following result. □

Corollary 2.1. *Let $f(z) \in \mathcal{M}_n(0)$ and $0 \le \rho < p$. Then $f(z)$ is convex of order ρ in*

$$|z| < \left(\frac{p-\rho}{p+\frac{n}{2}-\rho}\right)^{\frac{1}{n}}.$$

The result is sharp.

References

[1] R. M. Ali, N. K. Jain and V. Ravichandran, *Radii of starlikeness associated with the lemniscate of Bernoulli and the left-half plane*, Appl. Math. Comput., 2012, 218, 6557–6565.

[2] M. K. Aouf, A. O. Mostafa and H. M. Zayed, *Mapping properties for convolution involving hypergeometric series*, Ukrainian. Math. J., 2019, 70, 1688–1699.

[3] N. E. Cho, V. Kumar, S. S. Kumar and V. Ravichandran, *Radius problems for starlike functions associated with the sine function*, Bull. Iran. Math. Soc., 2019, 45, 213–232.

[4] N. E. Cho, H. J. Lee, J. H. Park and R. Srivastava, *Some applications of the first-order differential subordination*, Filomat, 2016, 30, 1456–1474.

[5] J. Dziok, *Classes of meromorphic functions associated with conic regions*, Acta Math. Sci. Ser. B Engl. Ed., 2012, 32, 765–774.

[6] J. Liu and R. Srivastava, *A linear operator associated with the Mittag-Leffler function and related conformal mappings*, J. Appl. Anal. Comput., 2018, 8, 1886–1892.

[7] S. Mahmood and J. Sokól, *New subclass of analytic functions in conical domain associated with Ruscheweyh q-differential operator*, Results Math., 2017, 71, 1345–1357.

[8] S. S. Miller and P. T. Mocanu, *Differential subordination and univalent functions*, Michigan Math. J., 1981, 28, 157–171.

[9] V. Ravichandran and K. Sharma, *Sufficient conditions for starlikeness*, J. Korean Math. Soc., 2015, 52, 727–749.

[10] J. Sokól and J. Stankiewicz, *Radius of convexity of some subclasses of strongly starlike functions*, Zeszyty Nauk. Politech. Rzeszowskiej Mat., 1996, 19, 101–105.

[11] H. M. Srivastava, M. K. Aouf, A. O. Mostafa and H. M. Zayed, *Certain subordination-preserving family of integral operators associated with p-valent functions*, Appl. Math. Inform. Sci., 2017, 11, 951–960.

[12] H. M. Srivastava, B. Khan, N. Khan and Q. Z. Ahmad, *Coefficient inequalities for q-starlike functions associated with the Janowski functions*, Hokkaido Math. J., 2019, 48, 407–425.