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EXISTENCE AND MULTIPLICITY OF POSITIVE PERIODIC SOLUTIONS FOR A CLASS OF SECOND ORDER DAMPED FUNCTIONAL DIFFERENTIAL EQUATIONS WITH MULTIPLE DELAYS

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Abstract By using the Krasnoselskii fixed point theorem, sufficient conditions are obtained for the existence and multiplicity of positive periodic solutions for a class of second order damped functional differential equations with multiple delays. Our results are a further expansion of the previous research results.

Keywords Periodic solutions, Green's function, Krasnoselskii fixed point theorem.

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1. Introduction

For the following equation

$$x'' = f(t, x(t)), \qquad t \in \mathbb{R},$$

where $f \in C(\mathbb{R}/\mathbb{T}\mathbb{Z}, (0, +\infty))$, there are many results [3,18] on the periodic solution of this equation.

However, the systems controlled by feedback loops in engineering, predator-prey models in ecosystems [8,12], and value laws in economics in real life all have the influence of delay factors, so the research on functional differential equations has already stepped into a climax period [1,15,17]. At the same time, many research methods have been considered, such as the upper and lower solutions method and monotone iterative technique [10,16], fixed point theorems [11,13,21] and so on [5,9,14,19,20].

Jiang et al. [10] studied the following periodic problem

$$-x'' = f(t, x(t), x(t - \tau(t))), \qquad t \in \mathbb{R},$$

where $f \in C(\mathbb{R}^3, \mathbb{R})$, $\tau \in C(\mathbb{R}, [0, +\infty))$, and they are T-periodic functions. They established the existence results of T-periodic solutions by using monotone iterative technique.

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However, for many problems in real life, we only need to consider the properties of its positive periodic solution. In [21], Wu obtained the existence and multiplicity of the solutions to the following equation

$$x'' + a(t)x = \lambda f(t, x(t-\tau_0(t)), x(t-\tau_1(t)), ..., x(t-\tau_n(t))), \qquad t \in \mathbb{R},$$

where $a \in C(\mathbb{R}/\mathbb{T}\mathbb{Z}, (0,+\infty))$, $f \in C\big((\mathbb{R}/\mathbb{T}\mathbb{Z}) \times [0,+\infty)^{n+1}, [0,+\infty)\big)$, $\tau_i(t) \in C(\mathbb{R}/\mathbb{T}\mathbb{Z}, \mathbb{R})$, and $a(t)$ satisfies the condition that $0 < a(t) < \frac{\pi^2}{T}$ for every $t \in \mathbb{R}$.

Li et al. studied the following equation in [13]

$$x'' + a(t)x = f(t, x(t), x(t-\tau_1(t)), ..., x(t-\tau_n(t))), \qquad t \in \mathbb{R},$$

where $a \in C(\mathbb{R}/\mathbb{T}\mathbb{Z}, (0,+\infty))$, $f \in C\big((\mathbb{R}/\mathbb{T}\mathbb{Z}) \times [0,+\infty)^{n+1}, [0,+\infty)\big)$, $\tau_i(t) \in C(\mathbb{R}/\mathbb{T}\mathbb{Z}, [0,+\infty))$, they obtained the existence of positive periodic solution by using the first eigenvalue corresponding to the relevant linear operator and fixed-point index theory in cones.

In [11], Kang et al. considered the following equation with damped term

$$x'' + h(t)x' + a(t)x = \lambda g(t) f(t, x(t-\tau(t))), \qquad t \in \mathbb{R},$$

where $h \in C(\mathbb{R}/\mathbb{T}\mathbb{Z}, [0,+\infty))$, $a \in C(\mathbb{R}/\mathbb{T}\mathbb{Z}, [0,+\infty))$, $f \in C((\mathbb{R}/\mathbb{T}\mathbb{Z}) \times \mathbb{R}, [0,+\infty))$, $\tau_i(t) \in C(\mathbb{R}/\mathbb{T}\mathbb{Z}, \mathbb{R})$, $g \in C(\mathbb{R}/\mathbb{T}\mathbb{Z}, [0,+\infty))$. They obtained the existence and multiplicity of positive periodic solutions when the coefficients $h(t)$, $a(t)$ and $g(t)$ satisfy $\int_0^T h(\xi)d\xi > 0$, $\int_0^T a(\xi)d\xi > 0$ and $\int_0^T g(\xi)d\xi > 0$, respectively, moveover, f is nondecreasing in the second variable.

Motivated by the above papers, in this paper, we study the existence, multiplicity of positive periodic solutions for the following equation

$$x'' + h\,(t)\,x' + a(t)x = \lambda g(t) f(t, x(t-\tau_0(t)), x(t-\tau_1(t)), ..., x(t-\tau_n(t))), \quad (1.1)$$

where $h \in C(\mathbb{R}/\mathbb{T}\mathbb{Z}, \mathbb{R})$, $a \in C(\mathbb{R}/\mathbb{T}\mathbb{Z}, \mathbb{R})$, $f \in C\big((\mathbb{R}/\mathbb{T}\mathbb{Z}) \times [0,+\infty)^{n+1}, [0,+\infty)\big)$ and $f(t, x_0, x_1, ..., x_n) > 0$ for $(x_i \geq 0, 0 \leq i \leq n, (x_0, x_1, ..., x_n) \neq (0, 0, ..., 0))$, $\tau_i(t) \in C(\mathbb{R}/\mathbb{T}\mathbb{Z}, \mathbb{R})$, $g \in C(\mathbb{R}/\mathbb{T}\mathbb{Z}, [0,+\infty))$ and $\int_0^T g(\xi)d\xi > 0$, $\lambda > 0$ is a parameter.

Three highlights should be pointed out. Firstly, compared with the equation studied in [13, 21], we add the damping term $h(t)x'$. Secondly, different from [11], the equation we studied has multiple delays. Thirdly, we relax the restrictions for the coefficients $h(t)$ and $a(t)$ in [11].

2. Preliminaries

If the unique solution of linear equation

$$x'' + h\,(t)\,x' + a(t)x = 0, \quad (2.1)$$

associated to periodic boundary conditions

$$x(0) = x(T), \quad x'(0) = x'(T) \quad (2.2)$$

is trivial, then it is nonresonant. By Fredholm's alternative theorem, we know that when (2.1)-(2.2) is nonresonant,

$$x'' + h(t)x' + a(t)x = l(t) \tag{2.3}$$

has a unique solution and it can be expressed as

$$x(t) = \int_0^T G(t,\xi)l(\xi)d\xi,$$

where $G(t,\xi)$ is the Green's function of (2.1)-(2.2).

Next we assume that:

$(A0)$ The Green's function $G(t,\xi)$ of system (2.1)-(2.2), is positive for all $(t,\xi) \in [0,T] \times [0,T]$.

In general, condition $(A0)$ is difficult to establish. However, through the anti-maximum principle established by Hakl and Torres (see [7]), Chu, Fan and Torres obtained that $(A0)$ is true in [2]. Describe the above criterion by defining the following function

$$\sigma(h)(t) = \exp(\int_0^t h(\xi)d\xi),$$

and

$$\sigma_1(h)(t) = \sigma(h)(T)\int_0^t \sigma(h)(\xi)d\xi + \int_t^T \sigma(h)(\xi)d\xi.$$

Lemma 2.1 (Corollary 2.6, [7]). *If $a(t) \not\equiv 0$ and the following two inequalities*

$$\int_0^T a(\xi)\sigma(h)(\xi)\sigma_1(-h)(\xi)d\xi \geq 0, \tag{H1}$$

and

$$\sup_{0\leq t\leq T}\left\{\int_t^{t+T}\sigma(-h)(\xi)d\xi\int_t^{t+T}[a(\xi)]_+\sigma(h)(\xi)d\xi\right\} \leq 4 \tag{H2}$$

are satisfied, where $[a(\xi)]_+ = \max\{a(\xi), 0\}$. Then $(A0)$ holds.

When $(A0)$ holds, we always denote

$$A = \min_{0\leq\xi,t\leq T} G(t,\xi), \quad B = \max_{0\leq\xi,t\leq T} G(t,\xi), \quad \sigma = A/B. \tag{2.4}$$

Obviously $B > A > 0$ and $0 < \sigma < 1$.

Then, let $X = C(\mathbb{R}/T\mathbb{Z}, \mathbb{R})$, $\| x \| = \max\{|x(t)| \,\big|\, x(t) \in X,\ t \in [0,T]\}$, and $P = \{x(t) \in X : x(t) \geq \sigma \| x \|,\ t \in [0,T]\}$. Moreover, for $r > 0$, let $\Omega_r = \{x \in X, \| x \| < r\}$ and

$$m(r) = \min\{f(t, x_0, x_1, ..., x_n) :\ 0 \leq t \leq T,\ \sigma r \leq x_i \leq r,\ 0 \leq i \leq n\};$$
$$M(r) = \max\{f(t, x_0, x_1, ..., x_n) :\ 0 \leq t \leq T,\ 0 \leq x_i \leq r,\ 0 \leq i \leq n\}.$$

Define operator:

$$Q_\lambda x(t) = \lambda\int_0^T G(t,\xi)g(\xi)f(\xi, x(\xi-\tau_0(\xi)), x(\xi-\tau_1(\xi)), ..., x(\xi-\tau_n(\xi)))d\xi.$$

Therefore, the fixed point of the operator equation $x = Q_\lambda x$ is the T-periodic solution of (1.1).

Lemma 2.2. *$Q_\lambda : P \to P$ is completely continuous and $Q_\lambda(P) \subset P$.*

Proof. Since

$$Q_\lambda x(t) \geq \lambda A \int_0^T g(\xi) f(\xi, x(\xi - \tau_0(\xi)), x(\xi - \tau_1(\xi)), ..., x(\xi - \tau_n(\xi))) d\xi,$$

and

$$\| Q_\lambda x(t) \| \leq \lambda B \int_0^T g(\xi) f(\xi, x(\xi - \tau_0(\xi)), x(\xi - \tau_1(\xi)), ..., x(\xi - \tau_n(\xi))) d\xi,$$

therefore

$$Q_\lambda x(t) \geq \lambda A \frac{\| Q_\lambda x(t) \|}{\lambda B} = \sigma \| Q_\lambda x(t) \| .$$

Then, according to the Arscoli-Arzele theorem, Q_λ is completely continuous. The proof is completed. □

Lemma 2.3. *If $x \in P \cap \partial\Omega_r$ for $r > 0$, then $\lambda A m(r) \int_0^T g(\xi) d\xi \leq \| Q_\lambda x(t) \| \leq \lambda B M(r) \int_0^T g(\xi) d\xi$.*

Proof. Since $x \in P \cap \partial\Omega_r$, it is clear that $\sigma r \leq x(t) \leq r$, that is

$$Q_\lambda x(t) \geq \lambda A \int_0^T g(\xi) m(r) d\xi$$
$$= \lambda A m(r) \int_0^T g(\xi) d\xi,$$

hence $\| Q_\lambda x(t) \| \geq \lambda A m(r) \int_0^T g(\xi) d\xi$. And

$$Q_\lambda x(t) \leq \lambda B \int_0^T g(\xi) M(r) d\xi$$
$$= \lambda B M(r) \int_0^T g(\xi) d\xi,$$

thus $\| Q_\lambda x(t) \| \leq \lambda B M(r) \int_0^T g(\xi) d\xi$. The proof is finished. □

Lemma 2.4 ([4,6]). *Let X be a Banach space and P be a close convex cone in X. Ω_1, Ω_2 are bounded open subsets of X, $\theta \in \Omega_1$, $\overline{\Omega_1} \subset \Omega_2$. $Q : P \cap (\overline{\Omega_2} \setminus \Omega_1) \to P$ is a completely continuous operator. Assume that Q satisfies one of the following conditions:*

(i) $\| Qx \| \geq \| x \|$ for $x \in P \cap \partial\Omega_1$, $\| Qx \| \leq \| x \|$ for $x \in P \cap \partial\Omega_2$;

(ii) $\| Qx \| \leq \| x \|$ for $x \in P \cap \partial\Omega_1$, $\| Qx \| \geq \| x \|$ for $x \in P \cap \partial\Omega_2$.

Then Q has at least one fixed point in $P \cap (\overline{\Omega_2} \setminus \Omega_1)$.

3. Main results

Let $x = (x_0, x_1, ..., x_n) \in [0, +\infty)^{n+1}$, $\overline{x} \triangleq \max\{x_0, x_1, ..., x_n\}$.

Next, make the following assumptions about f:

$$f^0 = \limsup_{\overline{x}\to 0^+} \max_{t\in[0,T]} \frac{f(t,x)}{\overline{x}}, \quad f_\infty = \liminf_{\overline{x}\to +\infty} \min_{t\in[0,T]} \frac{f(t,x)}{\overline{x}},$$
$$f_0 = \liminf_{\overline{x}\to 0^+} \min_{t\in[0,T]} \frac{f(t,x)}{\overline{x}}, \quad f^\infty = \limsup_{\overline{x}\to +\infty} \max_{t\in[0,T]} \frac{f(t,x)}{\overline{x}}.$$

Assume that:

$j_0 =$ the number of zeros in set $\{f^0, f^\infty\}$; $j_\infty =$ the number of infinities in set $\{f^0, f^\infty\}$;

$j'_0 =$ the number of zeros in set $\{f_0, f_\infty\}$; $j'_\infty =$ the number of infinities in set $\{f_0, f_\infty\}$.

Theorem 3.1. *Suppose that* $(A0)$ *holds.*
(1) If $j_0 = 1$ *or* 2*, when* $\lambda > \frac{1}{Am(1)\int_0^T g(\xi)d\xi} > 0$*, equation* (1.1) *has at least* j_0 *positive* T*-periodic solution(s).*
(2) If $j'_\infty = 1$ *or* 2*, when* $0 < \lambda < \frac{1}{BM(1)\int_0^T g(\xi)d\xi}$*, equation* (1.1) *has at least* j'_∞ *positive* T*-periodic solution(s).*
(3) If $j'_0 = 0$ *or* $j_\infty = 0$*, there is no positive* T*-periodic solution to equation* (1.1) *for sufficiently large or sufficiently small* $\lambda > 0$*, respectively.*

Proof. For $\phi \in P \cap \partial\Omega_r$, define

$$\Phi(t) = (\phi(t-\tau_0(t)), \phi(t-\tau_1(t)), ..., \phi(t-\tau_n(t)))$$

and $\overline{\Phi(t)} = \max_{0\le i\le n}\{\phi(t-\tau_i(t))\}$.

(1) Let $r_1 = 1$, by Lemma 2.3, we can obtain that there exists $\lambda_0 = \frac{1}{Am(1)\int_0^T g(\xi)d\xi} > 0$, such that

$$\| Q_\lambda\phi \| \ge \lambda Am(1)\int_0^T g(\xi)d\xi > \| \phi \|, \quad \phi \in P\cap\partial\Omega_1, \quad \lambda > \lambda_0.$$

If $f^0 = 0$, then we have $f(t,x) \le \varepsilon\overline{x}$ for $0 < \overline{x} \le r_2$ and $t \in [0,T]$, where $\varepsilon > 0$ satisfies $\lambda\varepsilon B\int_0^T g(\xi)d\xi < 1$, and $0 < r_2 < r_1 = 1$, obviously, $\Omega_{r_2} \subset \Omega_1$.

Then $0 < \sigma r_2 = \sigma \| \phi \| \le \overline{\Phi(t)} \le \| \phi \| = r_2$, for all $\phi \in P\cap\partial\Omega_{r_2}$, $t \in [0,T]$, thus

$$f(t,\Phi(t)) \le \varepsilon\overline{\Phi(t)}.$$

From the definition of Q_λ, for $\phi \in P\cap\partial\Omega_{r_2}$, we can obtain

$$\begin{aligned}\| Q_\lambda\phi \| &\le \lambda\varepsilon B\int_0^T g(\xi)\overline{\Phi(\xi)}d\xi \\ &\le \lambda\varepsilon B \| \phi \| \int_0^T g(\xi)d\xi < \| \phi \| .\end{aligned} \tag{3.1}$$

Thus, by Lemma 2.4(ii), the operator Q_λ has at least one fixed point in $P\cap(\overline{\Omega_1}\backslash\Omega_{r_2})$.

If $f^\infty = 0$, then there exists $H > 0$, such that $f(t,x) \le \varepsilon\overline{x}$ for $\overline{x} \ge H$ and $t \in [0,T]$, where $\varepsilon > 0$ still satisfies $\lambda\varepsilon B\int_0^T g(\xi)d\xi < 1$. Moreover, select $r_3 = \max\{2, \frac{H}{\sigma}\}$, obviously, $\Omega_1 \subset \Omega_{r_3}$.

Then $\overline{\Phi(t)} \geq \sigma \parallel \phi \parallel = \sigma r_3 \geq H$, for all $\phi \in P \cap \partial\Omega_{r_3}$, $t \in [0,T]$, thus

$$f(t, \Phi(t)) \leq \varepsilon\overline{\Phi(t)}.$$

Then for $\phi \in P \cap \partial\Omega_{r_3}$, we can obtain

$$\parallel Q_\lambda\phi \parallel \leq \lambda\varepsilon B \parallel \phi \parallel \int_0^T g(\xi)d\xi < \parallel \phi \parallel .$$

Thus, by Lemma 2.4(i), the operator Q_λ has at least one fixed point in $P \cap (\overline{\Omega_{r_3}} \setminus \Omega_1)$.

Above all, if $f^0 = 0$ and $f^\infty = 0$, the operator Q_λ has at least two fixed points in $P \cap (\overline{\Omega_{r_3}} \setminus \Omega_{r_2})$, that is, (1.1) has at least two positive T-periodic solutions for $\lambda > \lambda_0$.

(2) Let $r_1 = 1$, by Lemma 2.3, we can obtain that there exists $\lambda_0 = \frac{1}{BM(1)\int_0^T g(\xi)d\xi}$ > 0, such that

$$\parallel Q_\lambda\phi \parallel \leq \lambda BM(1) \int_0^T g(\xi)d\xi < \parallel \phi \parallel, \quad \phi \in P \cap \partial\Omega_1, \quad 0 < \lambda < \lambda_0.$$

If $f_0 = \infty$, then we have $f(t,x) \geq \eta\overline{x}$ for $0 < \overline{x} \leq r_2$ and $t \in [0,T]$, where $\eta > 0$ satisfies $\lambda\eta\sigma A \int_0^T g(\xi)d\xi > 1$, and $0 < r_2 < r_1 = 1$, obviously, $\Omega_{r_2} \subset \Omega_1$.

Then $0 < \sigma r_2 = \sigma \parallel \phi \parallel \leq \overline{\Phi(t)} \leq \parallel \phi \parallel = r_2$, for all $\phi \in P \cap \partial\Omega_{r_2}$, $t \in [0,T]$, thus

$$f(t, \Phi(t)) \geq \eta\overline{\Phi(t)}.$$

From the definition of Q_λ, for $\phi \in P \cap \partial\Omega_{r_2}$, we can obtain

$$\begin{aligned} \parallel Q_\lambda\phi \parallel &\geq \lambda\eta A \int_0^T g(\xi)\overline{\Phi(\xi)}d\xi \\ &\geq \lambda\eta\sigma A \parallel \phi \parallel \int_0^T g(\xi)d\xi > \parallel \phi \parallel . \end{aligned} \tag{3.2}$$

Thus, by Lemma 2.4(i), the operator Q_λ has at least one fixed point in $P \cap (\overline{\Omega_1} \setminus \Omega_{r_2})$.

If $f_\infty = \infty$, then there exists $H' > 0$, such that $f(t,x) \geq \eta\overline{x}$ for $\overline{x} \geq H'$ and $t \in [0,T]$, where $\eta > 0$ still satisfies $\lambda\eta\sigma A \int_0^T g(\xi)d\xi > 1$. Moreover, select $r_3 = \max\{2, \frac{H'}{\sigma}\}$, obviously, $\Omega_1 \subset \Omega_{r_3}$.

Then $\overline{\Phi(t)} \geq \sigma \parallel \phi \parallel = \sigma r_3 \geq H'$, for all $\phi \in P \cap \partial\Omega_{r_3}$, $t \in [0,T]$, thus

$$f(t, \Phi(t)) \geq \eta\overline{\Phi(t)}.$$

Then for $\phi \in P \cap \partial\Omega_{r_3}$, we can obtain

$$\parallel Q_\lambda\phi \parallel \geq \lambda\eta\sigma A \parallel \phi \parallel \int_0^T g(\xi)d\xi > \parallel \phi \parallel .$$

Thus, by Lemma 2.4(ii), the operator Q_λ has at least one fixed point in $P \cap (\overline{\Omega_{r_3}} \setminus \Omega_1)$.

Above all, if $f_0 = \infty$ and $f_\infty = \infty$, the operator Q_λ has at least two fixed points $P \cap (\overline{\Omega_{r_3}} \setminus \Omega_{r_2})$, that is, (1.1) has at least two positive T-periodic solutions for $0 < \lambda < \lambda_0$.

(3) If $j'_0 = 0$, then $f_0 > 0$ and $f_\infty > 0$, that is, there exist positive constants ω_1, ω_2, r_1, r_2, where $r_1 < r_2$, such that

$$f(t,x) \geq \omega_1 \overline{x}, \quad \overline{x} \in [0, r_1], \quad t \in [0,T];$$
$$f(t,x) \geq \omega_2 \overline{x}, \quad \overline{x} \in [r_2, +\infty), \quad t \in [0,T].$$

Select $c_1 = \min\Big\{\omega_1, \omega_2, \min\{\frac{f(t,x)}{\overline{x}} : t \in [0,T], \overline{x} \in [r_1, r_2]\}\Big\}$. Thus $c_1 > 0$, and

$$f(t,x) \geq c_1 \overline{x}, \quad \forall x \in [0, +\infty)^{n+1}, \quad t \in [0,T].$$

Assume $\varphi(t)$ is the fixed point of the operator Q_λ, then $Q_\lambda \varphi(t) = \varphi(t)$, $t \in [0,T]$. Moreover, define $\varphi' = (\varphi(t-\tau_0(t)), \varphi(t-\tau_1(t)), ..., \varphi(t-\tau_n(t)))$, thus $f(t,\varphi') \geq c_1 \overline{\varphi'}$.

On the other hand, there exists $\lambda_0 = \frac{1}{c_1 \sigma A \int_0^T g(\xi) d\xi}$, such that

$$\| \varphi \| = \| Q_\lambda \varphi \| \geq \lambda c_1 \sigma A \| \varphi \| \int_0^T g(\xi) d\xi > \| \varphi \| ,$$

for $\lambda > \lambda_0$. This is contradictory.

If $j_\infty = 0$, then $f^0 < \infty$ and $f^\infty < \infty$, that is, there exist positive constants ζ_1, ζ_2, r_1, r_2, where $r_1 < r_2$, such that

$$f(t,x) \leq \zeta_1 \overline{x}, \quad \overline{x} \in [0, r_1], \quad t \in [0,T];$$
$$f(t,x) \leq \zeta_2 \overline{x}, \quad \overline{x} \in [r_2, +\infty), \quad t \in [0,T].$$

Select $c_2 = \max\{\zeta_1, \zeta_2, \max\{\frac{f(t,x)}{\overline{x}} : t \in [0,T], \overline{x} \in [r_1, r_2]\}\}$. Thus $c_2 > 0$, and

$$f(t,x) \leq c_2 \overline{x}, \quad \forall x \in [0, +\infty)^{n+1}, \quad t \in [0,T].$$

Assume $\psi(t)$ is the fixed point of the operator Q_λ, then $Q_\lambda \psi(t) = \psi(t)$, $t \in [0,T]$. Moreover, define $\psi' = (\psi(t-\tau_0(t)), \psi(t-\tau_1(t)), ..., \psi(t-\tau_n(t)))$, thus $f(t,\psi') \leq c_2 \overline{\psi'}$.

On the other hand, there exists $\lambda_0 = \frac{1}{c_2 B \int_0^T g(\xi) d\xi}$, such that

$$\| \psi \| = \| Q_\lambda \psi \| \leq \lambda c_2 B \| \psi \| \int_0^T g(\xi) d\xi < \| \psi \| ,$$

for $0 < \lambda < \lambda_0$. This is also contradictory.

This proves the theorem. □

Corollary 3.1. *Suppose that* (*A*0) *holds.*
(1) If there exists a $c_1 > 0$ such that $f(t,x) \geq c_1 \overline{x}$ for $t \in [0,T]$, $x \in [0,+\infty)^{n+1}$, when $\lambda > \frac{1}{c_1 \sigma A \int_0^T g(\xi) d\xi}$, equation (1.1) *has no positive T-periodic solution.*
(2) If there exists a $c_2 > 0$ such that $f(t,x) \leq c_2 \overline{x}$ for $t \in [0,T]$, $x \in [0,+\infty)^{n+1}$, when $0 < \lambda < \frac{1}{c_2 B \int_0^T g(\xi) d\xi}$, equation (1.1) *has no positive T-periodic solution.*

Theorem 3.2. *Suppose that* (*A*0) *holds and $j_0 = j'_0 = j_\infty = j'_\infty = 0$.*
(1) If $f^0 B < f_\infty \sigma A$, when $\frac{1}{f_\infty \sigma A \int_0^T g(\xi) d\xi} < \lambda < \frac{1}{f^0 B \int_0^T g(\xi) d\xi}$, equation (1.1) *has at least a positive T-periodic solution.*
(2) If $f_0 \sigma A > f^\infty B$, when $\frac{1}{f_0 \sigma A \int_0^T g(\xi) d\xi} < \lambda < \frac{1}{f^\infty B \int_0^T g(\xi) d\xi}$, equation (1.1) *has at least a positive T-periodic solution.*

Proof. Still define

$$\Phi(t) = (\phi(t-\tau_0(t)), \phi(t-\tau_1(t)), ..., \phi(t-\tau_n(t)))$$

and $\overline{\Phi(t)} = \max_{0\le i\le n}\{\phi(t-\tau_i(t))\}$, for $\phi \in P\cap\partial\Omega_r$.

(1) Assume $f^0B<f_\infty\sigma A$, then $f^0<f_\infty$, when $\frac{1}{f_\infty\sigma A\int_0^T g(\xi)d\xi}<\lambda<\frac{1}{f^0B\int_0^T g(\xi)d\xi}$, then there exists $0<\varepsilon<f_\infty$, such that

$$\frac{1}{(f_\infty-\varepsilon)\sigma A\int_0^T g(\xi)d\xi}<\lambda<\frac{1}{(f^0+\varepsilon)B\int_0^T g(\xi)d\xi},$$

for the above ε, choose $r_1>0$, such that $\underline{f(t,x)} \le (f^0+\varepsilon)\overline{x}$ for $\overline{x}\in[0,r_1]$, $t\in[0,T]$. Thus, for all $\phi\in P\cap\partial\Omega_{r_1}$, we have $0\le\overline{\Phi(t)}\le r_1$, that is

$$f(t,\Phi(t))\le (f^0+\varepsilon)\overline{\Phi(t)}.$$

Thus, we have

$$\| Q_\lambda\phi \| \le \lambda(f^0+\varepsilon)B \| \phi \| \int_0^T g(\xi)d\xi < \| \phi \| , \tag{3.3}$$

for all $\phi\in P\cap\partial\Omega_{r_1}$.

On the other hand, there exists $H_1>0$, such that $f(t,x)\ge(f_\infty-\varepsilon)\overline{x}$ for $\overline{x}\ge H_1$ and $t\in[0,T]$. Moreover, select $r_2=\max\{2r_1,\frac{H_1}{\sigma}\}$, obviously, $\Omega_{r_1}\subset\Omega_{r_2}$.

Then $\overline{\Phi(t)}\ge\sigma \| \phi \| =\sigma r_2\ge H_1$, for all $\phi\in P\cap\partial\Omega_{r_2}$, $t\in[0,T]$. Thus

$$f(t,\Phi(t))\ge(f_\infty-\varepsilon)\overline{\Phi(t)}.$$

Then, for $\phi\in P\cap\partial\Omega_{r_2}$, we can obtain

$$\| Q_\lambda\phi \| \ge \lambda\sigma(f_\infty-\varepsilon)A \| \phi \| \int_0^T g(\xi)d\xi > \| \phi \| .$$

Thus, by Lemma 2.4(ii), the operator Q_λ has at least one fixed point in $P\cap(\overline{\Omega_{r_2}}\backslash\Omega_1)$, that is, (1.1) has at least a positive T-periodic solution for $\frac{1}{f_\infty\sigma A\int_0^T g(\xi)d\xi}<\lambda<\frac{1}{f^0B\int_0^T g(\xi)d\xi}$.

(2) Assume $f_0\sigma A>f^\infty B$, then $f_0>f^\infty$, when $\frac{1}{f_0\sigma A\int_0^T g(\xi)d\xi}<\lambda<\frac{1}{f^\infty B\int_0^T g(\xi)d\xi}$, then there exists $0<\varepsilon<f_0$, such that

$$\frac{1}{(f_0-\varepsilon)\sigma A\int_0^T g(\xi)d\xi}<\lambda<\frac{1}{(f^\infty+\varepsilon)B\int_0^T g(\xi)d\xi},$$

for the above ε, choose $r_1>0$, such that $\underline{f(t,x)}\ge(f_0-\varepsilon)\overline{x}$ for $\overline{x}\in[0,r_1]$, $t\in[0,T]$. Thus, for all $\phi\in P\cap\partial\Omega_{r_1}$, we have $0\le\overline{\Phi(t)}\le r_1$, that is

$$f(t,\Phi(t))\ge(f_0-\varepsilon)\overline{\Phi(t)}.$$

Thus, we have

$$\| Q_\lambda\phi \| \ge \lambda\sigma(f_0-\varepsilon)A \| \phi \| \int_0^T g(\xi)d\xi > \| \phi \| , \tag{3.4}$$

for all $\phi \in P \cap \partial\Omega_{r_1}$.

On the other hand, there exists $H_2 > 0$, such that $f(t,x) \leq (f^{\infty}+\varepsilon)\overline{x}$ for $\overline{x} \geq H_2$ and $t \in [0,T]$. Moreover, select $r_2 = \max\{2r_1, \frac{H_2}{\sigma}\}$, obviously, $\Omega_{r_1} \subset \Omega_{r_2}$.

Then $\overline{\Phi(t)} \geq \sigma \parallel \phi \parallel = \sigma r_2 \geq H_2$, for all $\phi \in P \cap \partial\Omega_{r_2}$, $t \in [0,T]$. Thus

$$f(t,\Phi(t)) \leq (f^{\infty}+\varepsilon)\overline{\Phi(t)}.$$

Then, for $\phi \in P \cap \partial\Omega_{r_2}$, we can obtain

$$\parallel Q_{\lambda}\phi \parallel \leq \lambda(f^{\infty}+\varepsilon)B \parallel \phi \parallel \int_0^T g(\xi)d\xi < \parallel \phi \parallel .$$

Thus, by Lemma 2.4(i), the operator Q_λ has at least one fixed point in $P \cap (\overline{\Omega_{r_2}} \setminus \Omega_1)$, which is the positive T-periodic solution of (1.1) for $\frac{1}{f_0 \sigma A \int_0^T g(\xi)d\xi} < \lambda < \frac{1}{f^{\infty} B \int_0^T g(\xi)d\xi}$.

The proof is completed. □

Corollary 3.2. *Suppose $h(t) \equiv 0$, $a(t) \not\equiv 0$, then $(A0)$ holds if $\int_0^T a(\xi)d\xi \geq 0$ and $\int_0^T [a(\xi)]_+ d\xi \leq \frac{4}{T}$.*

4. Example

Example 4.1. Consider the following equations:

$$\phi'' + 2\phi' + \phi = \lambda(1+\sin 8t)\frac{2+\cos 8t}{2+\phi(t-\tau(t))^n}, \quad n > 0, \tag{4.1}$$

where $h(t) = 2$, $a(t) = 1$, $g(t) = 1+\sin 8t$, $f(t,x) = \frac{2+\cos 8t}{2+x^n}$, obviously, they are all $T = \frac{\pi}{4}$ periodic functions in t, moreover, $\tau(t)$ is an arbitrary $\frac{\pi}{4}$-periodic continuous function.

Through some calculations, the conditions of Lemma 2.1 are satisfied,

$$A = \frac{\frac{\pi}{4}}{[\exp(\frac{\pi}{4})-1]^2}, \quad B = \frac{\frac{\pi}{4}\exp(\frac{\pi}{2})}{[\exp(\frac{\pi}{4})-1]^2}, \quad \sigma = \exp(-\frac{\pi}{2}),$$

and

$$\begin{aligned}
&\int_0^{\frac{\pi}{4}} g(\xi)d\xi = \int_0^{\frac{\pi}{4}} (1+\sin 8\xi)d\xi = \frac{\pi}{4},\\
&m(1) = \min\{f(t,x),\, 0 \leq t \leq \frac{\pi}{4},\, \exp(-\frac{\pi}{2}) \leq x \leq 1\}\\
&\qquad = \min\{\frac{2+\cos 8t}{2+x^n},\, 0 \leq t \leq \frac{\pi}{4},\, \exp(-\frac{\pi}{2}) \leq x \leq 1\} = \frac{1}{3},\\
&M(1) = \max\{f(t,x),\, 0 \leq t \leq \frac{\pi}{4},\, 0 \leq x \leq 1\}\\
&\qquad = \max\{\frac{2+\cos 8t}{2+x^n},\, 0 \leq t \leq \frac{\pi}{4},\, 0 \leq x \leq 1\} = \frac{3}{2}.
\end{aligned}$$

Moreover,

$$f^0 = \limsup_{x\to 0^+} \max_{t\in[0,\frac{\pi}{4}]} \frac{f(t,x)}{x} = \limsup_{x\to 0^+} \max_{t\in[0,\frac{\pi}{4}]} \frac{2+\cos 8t}{x(2+x^n)} = \infty,$$

$$f_\infty = \liminf_{x\to+\infty} \min_{t\in[0,\frac{\pi}{4}]} \frac{f(t,x)}{x} = \liminf_{x\to+\infty} \min_{t\in[0,\frac{\pi}{4}]} \frac{2+\cos 8t}{x(2+x^n)} = 0,$$

$$f_0 = \liminf_{x\to 0^+} \min_{t\in[0,\frac{\pi}{4}]} \frac{f(t,x)}{x} = \liminf_{x\to 0^+} \min_{t\in[0,\frac{\pi}{4}]} \frac{2+\cos 8t}{x(2+x^n)} = \infty,$$

$$f^\infty = \limsup_{x\to+\infty} \max_{t\in[0,\frac{\pi}{4}]} \frac{f(t,x)}{x} = \limsup_{x\to+\infty} \max_{t\in[0,\frac{\pi}{4}]} \frac{2+\cos 8t}{x(2+x^n)} = 0.$$

Thus, $j_0 = 1$, $j'_\infty = 1$, furthermore,

$$\lambda_{01} = \frac{1}{Am(1)\int_0^T g(\xi)d\xi} = \frac{48[\exp(\frac{\pi}{4})-1]^2}{\pi^2}, \quad \lambda_{02} = \frac{1}{BM(1)\int_0^T g(\xi)d\xi} = \frac{32[\exp(\frac{\pi}{4})-1]^2}{3\pi^2\exp(\frac{\pi}{2})}.$$

Therefore, by Theorem 3.1(1), Eq.(4.1) has at least a positive $\frac{\pi}{4}$-periodic solution for $\lambda > \lambda_{01} = \frac{48[\exp(\frac{\pi}{4})-1]^2}{\pi^2}$, and by Theorem 3.1(2), Eq.(4.1) has at least a positive $\frac{\pi}{4}$-periodic solution for $0 < \lambda < \lambda_{02} = \frac{32[\exp(\frac{\pi}{4})-1]^2}{3\pi^2\exp(\frac{\pi}{2})}$.

When $n = 5$, $\tau = 0.7$ and $\lambda = 10$, now $\lambda > \lambda_{01}$, Figure 1 is the numerical simulation of Example 4.1.

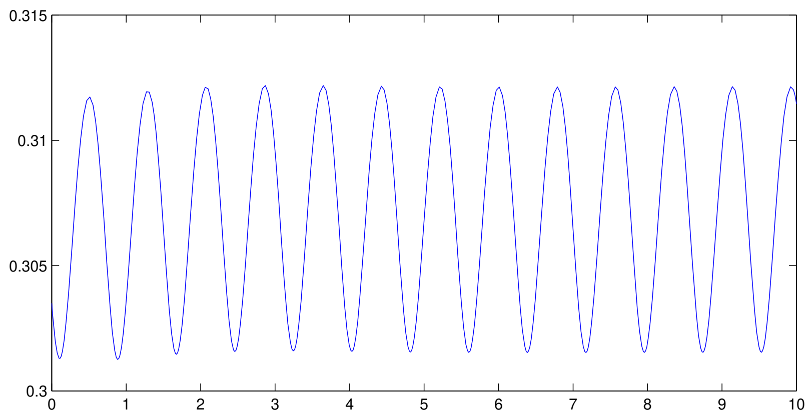

Figure 1. The numerical simulation of Example 4.1.

Example 4.2. Now consider the following equations:

$$\phi'' + 2\phi' + \phi = \lambda(1+\sin 8t)\frac{\phi(t-\tau(t))^2(2+\cos 8t)}{2+\phi(t-\tau(t))^6}, \tag{4.2}$$

note that $f(t,x) = \frac{x^2(2+\cos 8t)}{2+x^6}$, moreover, $\tau(t)$ is still an arbitrary $\frac{\pi}{4}$-periodic continuous function.

Now the conditions of Lemma 2.1 are still satisfied,

$$A = \frac{\frac{\pi}{4}}{[\exp(\frac{\pi}{4})-1]^2}, \quad B = \frac{\frac{\pi}{4}\exp(\frac{\pi}{2})}{[\exp(\frac{\pi}{4})-1]^2}, \quad \sigma = \exp(-\frac{\pi}{2}),$$

and

$$\int_0^{\frac{\pi}{4}} g(\xi)d\xi = \int_0^{\frac{\pi}{4}} (1+\sin 8\xi)d\xi = \frac{\pi}{4},$$

$$\begin{aligned} m(1) &= \min\{f(t,x),\ 0 \le t \le \frac{\pi}{4},\ \exp(-\frac{\pi}{2}) \le x \le 1\} \\ &= \min\{\frac{x^2(2+\cos 8t)}{2+x^6},\ 0 \le t \le \frac{\pi}{4},\ \exp(-\frac{\pi}{2}) \le x \le 1\} \\ &= \frac{1}{2\exp(\pi)+\exp(-2\pi)}, \end{aligned}$$

moreover,

$$\begin{aligned} f^0 &= \limsup_{x\to 0^+} \max_{t\in[0,\frac{\pi}{4}]} \frac{f(t,x)}{x} = \limsup_{x\to 0^+} \max_{t\in[0,\frac{\pi}{4}]} \frac{x^2(2+\cos 8t)}{x(2+x^6)} \\ &= \limsup_{x\to 0^+} \max_{t\in[0,\frac{\pi}{4}]} \frac{x(2+\cos 8t)}{2+x^6} = 0, \\ f^\infty &= \limsup_{x\to +\infty} \max_{t\in[0,\frac{\pi}{4}]} \frac{f(t,x)}{x} = \limsup_{x\to +\infty} \max_{t\in[0,\frac{\pi}{4}]} \frac{x^2(2+\cos 8t)}{x(2+x^6)} \\ &= \limsup_{x\to +\infty} \max_{t\in[0,\frac{\pi}{4}]} \frac{x(2+\cos 8t)}{2+x^6} = 0. \end{aligned}$$

Thus, $j_0 = 2$,

$$\lambda_{01} = \frac{1}{Am(1)\int_0^T g(\xi)d\xi} = \frac{16[\exp(\frac{\pi}{4})-1]^2[2\exp(\pi)+\exp(-2\pi)]}{\pi^2}.$$

Therefore, by Theorem 3.1(1), Eq.(4.2) has at least two positive $\frac{\pi}{4}$-periodic solutions for $\lambda > \lambda_{01} = \frac{16[\exp(\frac{\pi}{4})-1]^2[2\exp(\pi)+\exp(-2\pi)]}{\pi^2}$.

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