Journal of Applied Analysis and Computation
Volume 11, Number 2, April 2021, 810–840
Website:http://www.jaac-online.com
DOI:10.11948/20200011

HOPF BIFURCATION IN A DELAYED PREDATOR-PREY SYSTEM WITH GENERAL GROUP DEFENCE FOR PREY*

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Abstract In this paper, a delayed predator-prey system with group defence for prey is investigated. Firstly, in the absence of spatial diffusion and time delay, the stability of positive equilibrium and existence of the Hopf bifurcation are investigated, as well as the direction of the Hopf bifurcation, which is determined by applying the first Lyapunov number. Then, the occurrence of the Hopf bifurcation in the diffusion-driven delayed system is further explored. By using the center manifold reduction and the normal form theorem, the conditions ensuring the occurrence of Hopf bifurcation, its direction and its stability are formulated in terms of different parameters. Finally, some numerical simulations are carried out to verify the theoretical results and the existence of the homogeneous periodic solution is exhibited by setting different values of parameters. Moreover, stable temporal periodic solutions and spatially inhomogeneous periodic solutions are identified from the numerical simulations. The obtained results are also explained and discussed from the practical point of view.

Keywords Stability, hopf bifurcation, time delay, group defense.

MSC(2010) 35B32, 34C23, 37G10.

1. Introduction

The predator-prey systems are the most important models to investigate interactions among species in real world. The study of dynamics of predator-prey models has become one of the most interesting themes in fields of ecology, biology and applied mathematics, since the pioneering work of Lotka [16] and Volterra [27] was published, where they proposed the classic predator-prey model to explain the relationship between the two species. Many researchers have carried out the research in this respect and acquired various excellent results. In ecological systems, a variety of strategies, such as refuging, group, herd behavior, etc are adopted by different species to search for resources and for defense purposes.

There are many factors affecting the dynamics of predator-prey population. The crucial one of them is adding functional response into these predator-prey systems, such as Holling type I-IV [1, 11], Beddington-DeAngelis type [4], ratio-dependent type [2]. In 2011, Ajraldi and Venturio [3] proposed a modified predator-prey model with a Holling-I type functional response of the predator to prey, in which the

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*The authors were supported by National Natural Science Foundation of China (No. 11971032, 11571016) and the research fund (No. 2019QDZ25).

hunting behavior of predators and defensive strategy of prey are explained in a different way. That is to say, it is based on the fact that when predators move freely in the nature to search for food for subsistence, the prey gather together to defend, instead of scattered escape. Hence, a new social behavior of prey has been introduced into interface, which is called group defence. Such proposed system is described as follows.

$$\begin{cases} \frac{du(t)}{dt} = r(1 - \frac{u(t)}{K})u(t) - a\sqrt{u(t)}v(t), \\ \frac{dv(t)}{dt} = ab\sqrt{u(t)}v(t) - cv(t), \end{cases} \tag{1.1}$$

where $u(t)$ and $v(t)$ represent respectively the prey and predator densities at time t. The parameter r denotes the logistic growth rate, K represents the carrying capacity of the environment and a is the searching efficiency of predators for prey. The parameters b and c denote the biomass conversion coefficient and the mortality rate of predator species in the absence of prey, respectively. All the parameters r, K, a, b, c in the model are assumed to be positive. In system (1.1), since the strongest and smartest prey will be situated on the boundary of the group and defend the weakest prey in the center of group, the predators will have difficulties to hunt prey and can not catch up with the inside group prey, so the interactions between predators and prey mainly fall on prey individuals situating on the boundary of group. Moreover, the social behavior could induce multistability in minimal competitive ecosystems, refer to [19]. Since population behaviors possess the feather of history memory, Yin and Wen in [34] introduced time fractional-order derivatives and spatial diffusion into system (1.1), and proved the existence and uniqueness of a global positive solution. The Hopf bifurcation and the global dynamics have been investigated by Lv et al. [17]. Xu and Song [32] studied the diffusion-driven Turing instability and derived the formulas determining the direction and stability of the Hopf bifurcation in the diffusive predator-prey system (1.1) with a herd behavior and quadratic mortality. For other results on such predator-prey system (1.1), one can refer to [23, 25, 29].

In [28], the model (1.1) was formulated with a generic exponent $\alpha \in (0, 1)$, which includes the special case $\alpha = \frac{1}{2}$ and allows to establish the sensitivity of a fundament parameter with respect to the exponent of u. Therefore, the new system could be represented as follows.

$$\begin{cases} \frac{du(t)}{dt} = r\left(1 - \frac{u(t)}{K}\right)u(t) - au^{\alpha}(t)v(t), \\ \frac{dv(t)}{dt} = abu^{\alpha}(t)v(t) - cv(t). \end{cases} \tag{1.2}$$

The first equation represents the population dynamics of the prey. Here, the exponent α is a kind of aggregation efficiency when the prey gather in a huge group to defend against the predators. The second term in first equation simulates the hunting process that the prey are subject to predators. The second equation describes the population dynamics of the predators, in which the same term is scaled by the biomass conversion coefficient b. Kumar and Kharbanda [14] investigated the dynamics of model (1.2) such as boundedness of the solutions, existence and stability conditions of the equilibrium points, saddle-node bifurcation, transcritical bifurcation and the Hopf bifurcation with non-linear harvesting in prey species. By establishing conditions of the nonexistence of periodic orbits, the existence and uniqueness of limit cycles, Xu, et al. [33] studied the global dynamics of the predator-prey model (1.2). Furthermore, Zhu and Zhang investigated the existence and nonexistence of constant steady state solutions in [35].

In the natural ecosystem, the prey and predators are always in movement. Such movement of predators and prey give rise to spatial diffusion. The spatial diffusion has been introduced in many works [6, 8, 13, 18, 20, 26, 30]. In addition, after Venturino and Petrovskii [28] extended system (1.1) to a spatiotemporal model, there are many special spatial distribution characteristics are discovered. For example, Zhou and Song investigated diffusion-driven Turing instability in [32] where the prey species exhibits group defence and the predator species has quadratic mortality. Now we will consider the temporal and spatial dispersion of the diffusive system with general group defence, described as follows.

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = d_1 \Delta u(x,t) + r\left(1 - \frac{u(x,t)}{K}\right) u(x,t) - a u^{\alpha}(x,t) v(x,t), \\ \frac{\partial v(x,t)}{\partial t} = d_2 \Delta v(x,t) + abu^{\alpha}(x,t) v(x,t) - cv(x,t), \end{cases} \tag{1.3}$$

where $d_1 > 0$ and $d_2 > 0$ are diffusion coefficients of prey and predator species. In order to simplify system (1.3), we take the following transformations

$$\bar{t} = rt,\ \bar{u} = \frac{u}{K},\ \bar{v} = \frac{aK^{\alpha-1}}{r}v,\ \bar{c} = \frac{c}{r},\ \beta = \frac{abK^{\alpha}}{r},\ \bar{d_1} = \frac{d_1}{r},\ \bar{d_2} = \frac{d_2}{r},$$

and we omit the bars of $\bar{t},\ \bar{u},\ \bar{v},\ \bar{c},\ \bar{d_1},\ \bar{d_2}$ for convenience of notations.

It is well known that the effects of population density change on growth are not instantaneous but with time delay and are associated with the past of the life. Since May [21] found that delay would destroy the stability of the positive equilibrium in the logistic model and lead to periodic oscillations, the effects of time delay on biological population are increasingly studied by many researchers, see [5, 7, 8, 10, 15, 29]. However, with respect to predator-prey model with group defence, there are few related results reported about the system (1.3) with time delay. Therefore, we will investigate the delayed predator-prey system with group defence in the presence of spatial diffusion basing on the model in [28], which is described by

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = d_1 \Delta u(x,t) + (1 - u(t-\tau))u(x,t) - u^{\alpha}(x,t)v(x,t), & x \in \Omega, \quad t > 0, \\ \frac{\partial v(x,t)}{\partial t} = d_2 \Delta v(x,t) + \beta u^{\alpha}(x,t)v(x,t) - cv(x,t), & x \in \Omega, \quad t > 0, \\ \frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta} = 0, & x \in \partial\Omega, \quad t \geq 0, \\ u(x,\theta) = u_0(x,\theta) > 0, \quad v(x,\theta) = v_0(x,\theta) > 0, & x \in \Omega,\ \theta \in [-\tau, 0], \end{cases} \tag{1.4}$$

where $\tau > 0$ is the time delay, $\Omega = [0, l\pi]$ is a bound domain in $\mathbb{R}^1$ with boundary $\partial\Omega$. $\Delta = \frac{\partial^2}{\partial x^2}$ represents the Laplacian operator, η is the outward unit normal vector to the boundary $\partial\Omega$. The Neumann boundary condition indicates that the prey and predator are in isolated pieces.

The remainder of this paper in organized as follows. In Section 2, we study the dynamics of the non-delayed system, including existence and stability of the Hopf bifurcation at the positive equilibrium, as well as applying the first Lyapunov number to determine the direction of the Hopf bifurcation. In Section 3, we explore the stability of the positive equilibrium and the existence of the Hopf bifurcation when the time delay is presented. Furthermore, the formulas determining the direction of the Hopf bifurcation are derived by using center manifold reduction and the normal form theorem. In Section 4, some numerical simulations are presented to verify the theoretical results. Some conclusions have been made in Section 5.

2. The non-delayed system

2.1. Local stability analysis of the model without diffusion

Without delay and diffusion, system (1.4) becomes

$$\begin{cases} \frac{du}{dt} = (1-u)u - u^\alpha v \triangleq f(u,v), \\ \frac{dv}{dt} = \beta u^\alpha v - cv \triangleq g(u,v). \end{cases} \tag{2.1}$$

The system has three equilibria $E_0 = (0,0), E_1 = (1,0)$ and $E_* = (u_*, v_*)$. If condition $0 < c < \beta$ holds, the system (2.1) has a unique positive equilibrium $E_* = (u_*, v_*)$ with $u_* = \left(\frac{c}{\beta}\right)^{\frac{1}{\alpha}}$, $v_* = (1-u_*)(u_*)^{1-\alpha}$. In view of the biological meaning of the model, we are only interested in the positive equilibrium $E_* = (u_*, v_*)$. For system (2.1), the Jacobian matrix at $E_* = (u_*, v_*)$ is

$$J_{E_*} = \begin{pmatrix} 1-\alpha+(\alpha-2)u_* & -\frac{c}{\beta} \\ \alpha\beta(1-u_*) & 0 \end{pmatrix}.$$

Thus the characteristic equation of J_{E_*} could be given by

$$\lambda^2 - T_0\lambda + D_0 = 0, \tag{2.2}$$

here

$$T_0 \triangleq Tr(J_{E_*}) = 1-\alpha+(\alpha-2)u_* = 1-\alpha+(\alpha-2)\left(\frac{c}{\beta}\right)^{\frac{1}{\alpha}},$$

$$D_0 \triangleq Det(J_{E_*}) = c\alpha(1-u_*) = c\alpha\left[1-\left(\frac{c}{\beta}\right)^{\frac{1}{\alpha}}\right] > 0.$$

The characteristic roots of equation (2.2) can be given as

$$\lambda = \frac{T_0 \pm \sqrt{T_0^2 - 4D_0}}{2}. \tag{2.3}$$

Therefore one could has the following result.

Theorem 2.1. *If $0 < \alpha < 1$ and $0 < c < \beta$, then the positive equilibrium E_* of system (2.1) is unstable when*

$$(H_1):\ 0 < c < c_H,$$

and is locally asymptotically stable when

$$(H_2):\ c_H < c < \beta.$$

Furthermore, system (2.1) undergoes a supercritical Hopf bifurcation when $l_1 < 0$, and undergoes a subcritical Hopf bifurcation when $l_1 > 0$ at critical value $c = c_H$. This implies that the periodic solutions bifurcated from the Hopf bifurcation are stable and unstable, respectively. Where $c_H = \beta(\frac{\alpha-1}{\alpha-2})^\alpha$ and l_1 can be found in (2.6).

Proof. Notice that $D_0 > 0$, so the stability of the positive equilibrium E_* can be determined by the sign of T_0. If (H_1) is satisfied, then $T_0 > 0$. This means all the eigenvalues of the characteristic equation (2.2) have positive real parts, i.e., E_* is unstable under condition (H_1). Similarly, one could prove that $T_0 < 0$ with condition (H_2). That is to say, the positive equilibrium E_* is locally asymptotically stable.

In what follows, it is found that if $c = c_H = \beta(\frac{\alpha-1}{\alpha-2})^\alpha$, one has $T_0 = 0$. So we take c as the Hopf bifurcation parameter. Then the characteristic equation (2.2) has a pair of purely imaginary roots due to $D_0 > 0$ when $c = c_H$. We therefore let $\lambda_{1,2} = \varphi(c) \pm i\omega(c)$ be the complex roots of characteristic equation (2.2), where

$$\varphi(c) = \frac{T_0}{2}, \quad \omega(c) = \frac{\sqrt{4D_0 - T_0^2}}{2},$$

with

$$\varphi(c_H) = 0, \quad \omega(c_H) = \sqrt{\frac{\alpha\beta}{2-\alpha}\left(\frac{\alpha-1}{\alpha-2}\right)^\alpha} \triangleq \omega_0 > 0.$$

In addition

$$\frac{dRe(\lambda)}{dc}\Big|_{c=c_H} = \frac{\alpha-2}{2\alpha\beta}\left(\frac{\alpha-1}{\alpha-2}\right)^{1-\alpha} < 0.$$

Thus, via the Poincaré-Andronov-Hopf bifurcation theorem, system (2.1) undergoes the Hopf bifurcation when $c = c_H = \beta(\frac{\alpha-1}{\alpha-2})^\alpha$.

Next, in order to investigate the direction of the Hopf bifurcation to system (2.1), we need to calculate the first Lyapunov number l_1, see reference [22].

We first translate the positive equilibrium E_* into the origin by the translation $\tilde{u} = u - u_*$, $\tilde{v} = v - v_*$. As a result, system (2.1) can be rewritten as the form

$$\begin{cases} \frac{d\tilde{u}}{dt} = (1 - \tilde{u} - u_*)(\tilde{u} + u_*) - (\tilde{u} + u_*)^\alpha(\tilde{v} + v_*), \\ \frac{d\tilde{v}}{dt} = \beta(\tilde{u} + u_*)^\alpha(\tilde{v} + v_*) - c(\tilde{v} + v_*). \end{cases} \tag{2.4}$$

Then, we can write the Taylor expansion of the system (2.4) in a neighborhood of the origin as follows

$$\begin{cases} \dot{\tilde{u}} = \alpha_{10}\tilde{u} + \alpha_{01}\tilde{v} + \alpha_{11}\tilde{u}\tilde{v} + \alpha_{20}\tilde{u}^2 + \alpha_{02}\tilde{v}^2 + \alpha_{30}\tilde{u}^3 + \alpha_{21}\tilde{u}^2\tilde{v} + \alpha_{12}\tilde{u}\tilde{v}^2 \\ \qquad + \alpha_{03}\tilde{v}^3 + Q_1(\tilde{u}, \tilde{v}), \\ \dot{\tilde{v}} = \beta_{10}\tilde{u} + \beta_{01}\tilde{v} + \beta_{11}\tilde{u}\tilde{v} + \beta_{20}\tilde{u}^2 + \beta_{02}\tilde{v}^2 + \beta_{30}\tilde{u}^3 + \beta_{21}\tilde{u}^2\tilde{v} + \beta_{12}\tilde{u}\tilde{v}^2 \\ \qquad + \beta_{03}\tilde{v}^3 + Q_2(\tilde{u}, \tilde{v}), \end{cases} \tag{2.5}$$

where

$$\alpha_{10} = 1 - \alpha + (\alpha - 2)u_*, \quad \alpha_{01} = -u_*^\alpha, \quad \alpha_{11} = -\alpha u_*^{\alpha-1}, \quad \beta_{21} = -\alpha_{21},$$

$$\alpha_{02}=\alpha_{12}=\alpha_{03}=\beta_{01}=\beta_{02}=\beta_{12}=\beta_{03}=0,\quad \alpha_{21}=-\frac{1}{2}\alpha(\alpha-1)u_*^{\alpha-2},$$
$$\alpha_{20}=-1-\frac{1}{2}\alpha(\alpha-1)(u_*^{-1}-1),\quad \alpha_{30}=-\frac{1}{6}\alpha(\alpha-1)(\alpha-2)(1-u_*)u_*^{-2},$$
$$\beta_{10}=\alpha\beta(1-u_*),\quad \beta_{20}=-\beta(1+\alpha_{20}),\quad \beta_{11}=-\beta\alpha_{11},\quad \beta_{30}=-\beta\alpha_{30}.$$

It is noted that system (2.1) undergoes the Hopf bifurcation when $c=c_H$, we therefore obtain $\alpha_{10}+\beta_{01}=0$ and $D_0=\alpha_{10}\beta_{01}-\beta_{10}\beta_{01}>0$. $Q_1(\tilde{u},\tilde{v})$ and $Q_2(\tilde{u},\tilde{v})$ are high order terms $\mathcal{O}(\tilde{u}^i\tilde{v}^j)$, with terms $\tilde{u}^i\tilde{v}^j$ satisfying $i+j\geq 4$. Hence, the first Lyapunov number l_1 which determines the directions of the Hopf bifurcation is given by the following formula:

$$\begin{aligned}
l_1=&\frac{-3\pi}{2\alpha_{01}D_0^{\frac{3}{2}}}\{[\alpha_{10}\beta_{10}(\alpha_{11}^2+\alpha_{11}\beta_{02}+\alpha_{02}\beta_{11})-2\alpha_{10}\beta_{10}(\beta_{02}^2-\alpha_{20}\alpha_{02})\\
&-2\alpha_{10}\alpha_{01}(\alpha_{20}^2-\beta_{20}\beta_{02})+\alpha_{10}\alpha_{01}(\beta_{11}^2+\alpha_{20}\beta_{11}+\alpha_{11}\beta_{02})\\
&+(\alpha_{01}\beta_{10}-2\alpha_{10}^2)(\beta_{11}\beta_{02}-\alpha_{11}\alpha_{20})-\alpha_{01}^2(2\alpha_{20}\beta_{20}+\beta_{11}\beta_{20})\\
&+\beta_{10}^2(\alpha_{11}\alpha_{02}+2\alpha_{02}\beta_{02})]-(\alpha_{10}^2+\alpha_{01}\beta_{10})[3(\beta_{10}\beta_{03}-\alpha_{01}\alpha_{30})\\
&+2\alpha_{10}(\alpha_{21}+\beta_{12})+(\beta_{10}\alpha_{12}-\alpha_{01}\beta_{21})]\}\\
=&\frac{-3\pi}{2\alpha_{01}^2\beta_{10}}[\alpha_{10}\beta_{10}\alpha_{11}^2+\alpha_{10}\alpha_{01}\alpha_{11}\beta(\beta\alpha_{11}-\alpha_{20})+\alpha_{11}\alpha_{20}(2\alpha_{10}^2-\alpha_{01}\beta_{10})\\
&-\alpha_{10}^2\beta(1+\alpha_{20})(-2\alpha_{20}+\beta\alpha_{11})-2\alpha_{10}\alpha_{01}\alpha_{20}^2\\
&-(\alpha_{10}^2+\alpha_{01}\beta_{10})(-3\alpha_{01}\alpha_{30}+2\alpha_{10}\alpha_{21}+\alpha_{01}\alpha_{21})].
\end{aligned}\tag{2.6}$$

When $c=c_H$

$$\begin{aligned}
l_1(c_H)&=-(\alpha+1)(\alpha-2)^2\frac{3\pi}{4(\alpha-1)}+\frac{3\pi}{4}\alpha(\alpha-2)(\beta-1)(\frac{\alpha-1}{\alpha-2})^{\alpha-1}\\
&=\frac{3\pi(\alpha-2)^2}{4(1-\alpha)}\left[\alpha+1+\alpha(1-\beta)(\frac{\alpha-1}{\alpha-2})^{\alpha}\right].
\end{aligned}\tag{2.7}$$

From (2.7), we know $l_1<0$ if $\beta>1+\frac{\alpha+1}{\alpha}(\frac{\alpha-1}{\alpha-2})^{-\alpha}$, otherwise $l_1>0$. □

Remark 2.1. If $\alpha=\frac{2}{3},\ \beta=1.5,\ c=1$, we obtain $c_H=0.5953$. From Theorem 2.1, for $1=c>c_H$, the positive equilibrium $E_*=(0.5443,0.3721)$ is locally asymptotically stable. This result is shown in Figure 1. Moreover, to verify the existence of the Hopf bifurcation, we set parameters $\alpha=\frac{2}{3},\ \beta=10$, one gets $c_H=3.9685$ and the first Lyapunov number $l_1=-8.9779<0$. From Theorem 2.1, system (2.1) undergoes a supercritical Hopf bifurcation at critical value $c=c_H$, see Figure 2.

2.2. Hopf bifurcation for the diffusive system

In this subsection, we will investigate the Hopf bifurcation of system (1.4) without delay.

Firstly, we define a space:

$$\mathfrak{X}=\{(u,v)\in H^2([0,l\pi])\times H^2([0,l\pi]):u_x(0,t)=u_x(l\pi,t)=v_x(0,t)=v_x(l\pi,t)=0\},$$

where $H^2([0,l\pi])$ is the standard Sobolev space.

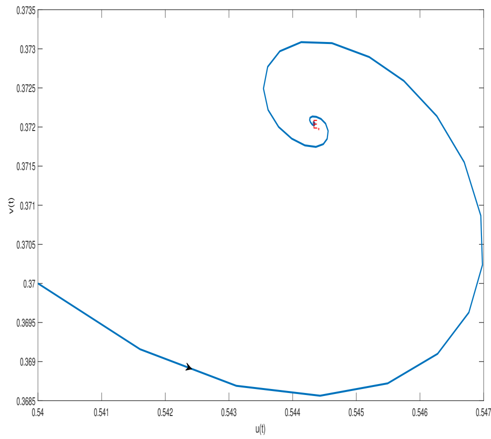

Figure 1. The equilibrium E_* is locally asymptotically stable.

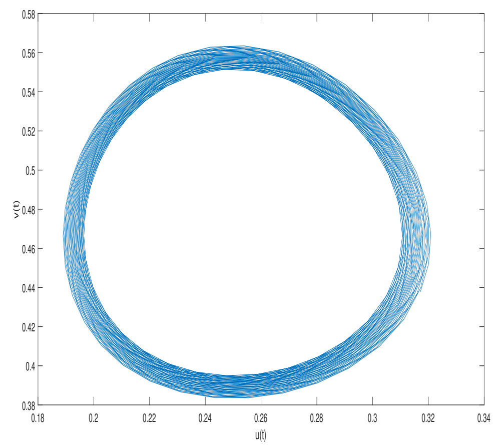

Figure 2. The periodic solution bifurcated from the Hopf bifurcation is stable.

Therefore, the linearized system of (1.4) without delay at the equilibrium E_* can be described as follows:

$$\begin{pmatrix} \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial t} \end{pmatrix} = \begin{pmatrix} 1-\alpha+(\alpha-2)u_*+d_1\Delta & -\frac{c}{\beta} \\ \alpha\beta(1-u_*) & d_2\Delta \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \triangleq L \begin{pmatrix} u \\ v \end{pmatrix}, \tag{2.8}$$

where L is a linear operator with domain $D_L = \mathfrak{X}_{\mathbb{C}} = \mathfrak{X} \bigoplus i\mathfrak{X} = \{a+ib : a, b \in \mathfrak{X}\}$. Hence, the formal solution can be written as

$$\begin{pmatrix} u \\ v \end{pmatrix} = \sum_{n=0}^{\infty} \begin{pmatrix} a_n \\ b_n \end{pmatrix} e^{\lambda_n t} \cos \frac{n}{l} x, \tag{2.9}$$

here a_n and b_n are coefficients, λ_n is the temporal spectrum, and n is the spatial spectrum. Then, substituting (2.8) into (2.9), we get

$$\sum_{n=0}^{\infty}\begin{pmatrix} a_n \\ b_n \end{pmatrix}\lambda_n e^{\lambda_n t}\cos\frac{n}{l}x = \sum_{n=0}^{\infty} J_n \begin{pmatrix} a_n \\ b_n \end{pmatrix} e^{\lambda_n t}\cos\frac{n}{l}x. \tag{2.10}$$

Equating equal powers of $e^{\lambda_n t}\cos\frac{n}{l}x$ on both sides, we obtain

$$(\lambda_n I - J_n)\begin{pmatrix} a_n \\ b_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where

$$J_n = \begin{pmatrix} 1-\alpha+(\alpha-2)u_* - d_1\frac{n^2}{l^2} & -\frac{c}{\beta} \\ \alpha\beta(1-u_*) & -d_2\frac{n^2}{l^2} \end{pmatrix}.$$

The characteristic equation of (2.9) is

$$\lambda_n^2 - T_n(c)\lambda + D_n(c) = 0, \tag{2.11}$$

here

$$\begin{aligned} T_n(c) &\triangleq Tr(J_n) = 1-\alpha+(\alpha-2)u_* - (d_1+d_2)\frac{n^2}{l^2}, \\ D_n(c) &\triangleq Det(J_n) = d_1 d_2\left(\frac{n}{l}\right)^4 - d_2[1-\alpha+(\alpha-2)u_*]\frac{n^2}{l^2} + c\alpha(1-u_*). \end{aligned} \tag{2.12}$$

The characteristic roots of the equation (2.11) can be shown as

$$\lambda_n = \frac{T_n(c) \pm \sqrt{T_n^2(c) - 4D_n(c)}}{2}. \tag{2.13}$$

Hence we can obtain the following theorem.

Theorem 2.2. *When time delay is absent, system (1.4) undergoes the Hopf bifurcation at* $c = c_{H_n}$, *if there exists some* $n \in N = \{0,1,2,\cdots\}$ *such that conditions* (H_3) *and* (H_4) *are satisfied.*

$$(H_3): \quad 0 < \alpha < 1 - (d_1+d_2)\frac{n^2}{l^2},$$

$$(H_4): \quad \beta > \frac{d_2^2(2-\alpha)\frac{n^4}{l^4}}{\alpha}\left[\frac{\alpha-1+(d_1+d_2)\frac{n^2}{l^2}}{\alpha-2}\right]^{-\alpha}.$$

Where

$$c_{H_n} = \beta\left[\frac{\alpha-1+(d_1+d_2)\frac{n^2}{l^2}}{\alpha-2}\right]^{\alpha}.$$

Proof. It is clear that the eigenvalues of the characteristic equation (2.11) are $\lambda_n = \frac{T_n(c)\pm\sqrt{T_n^2(c)-4D_n(c)}}{2}$, $n \in N = \{0,1,2\cdots\}$. It is coincident with the eigenvalues (2.3) of the corresponding ordinary differential equations (ODEs) of system (1.4) in the absence of time delay when $n = 0$. If $T_n(c) = 0$, we obtain $c = c_{H_n}$. As we know, the Hopf bifurcation may occur when the following conditions are satisfied

$$T_n(c_{H_n}) = 0, \quad D_n(c_{H_n}) > 0,$$

meanwhile

$$T_j(c_{H_n}) \neq 0, \quad D_j(c_{H_n}) \neq 0, \quad \text{for } n \neq j.$$

It is noted that when $c = c_{H_n}$,

$$\begin{aligned} D_n(c_{H_n}) &= -d_2^2\frac{n^4}{l^4} + \alpha\beta\frac{1+(d_1+d_2)\frac{n^2}{l^2}}{2-\alpha}\left[\frac{\alpha-1+(d_1+d_2)\frac{n^2}{l^2}}{\alpha-2}\right]^{\alpha} \\ &\geq -d_2^2\frac{n^4}{l^4} + \frac{\alpha\beta}{2-\alpha}\left[\frac{\alpha-1+(d_1+d_2)\frac{n^2}{l^2}}{\alpha-2}\right]^{\alpha}. \end{aligned}$$

Under conditions (H_3) and (H_4), we can get $D_n(c_{H_n}) > 0$.

Next we need to verify the transversality condition. To this end, we let $\lambda_n(c) = \varphi_n(c) \pm i\omega_n(c)$. Since $\varphi(c_{H_n}) = 0$, $\omega_n(c_{H_n}) = \sqrt{D_n(c_{H_n})} > 0$, note that condition (H_3) holds, it follows that

$$\begin{aligned} \frac{dRe(\lambda_n)}{dc}\Big|_{c=c_{H_n}} &= \frac{1}{2\alpha\beta}(\alpha-2)\left(\frac{c}{\beta}\right)^{\frac{1-\alpha}{\alpha}}\Big|_{c=c_{H_n}} \\ &= \frac{\alpha-2}{2\alpha\beta}\left(\frac{\alpha-1+(d_1+d_2)\frac{n^2}{l^2}}{\alpha-2}\right)^{1-\alpha} < 0. \end{aligned}$$

Hence the transversality condition for the Hopf bifurcation is satisfied. □

Remark 2.2. To verify that system (1.4) in the absence of time delay undergoes the Hopf bifurcation when c crosses c_{H_n} in Theorem 2.2, we set the following parameters

(i) $\alpha = 0.2,\ \beta = 2,\ d_1 = 0.8,\ d_2 = 1.7,\ n = 2,\ l = 10,$

(ii) $\alpha = 0.5,\ \beta = 0.85,\ d_1 = 0.4,\ d_2 = 0.6,\ n = 1,\ l = 10,$

(iii) $\alpha = 0.75,\ \beta = 0.9,\ d_1 = 0.3,\ d_2 = 0.5,\ n = 2,\ l = 10.$

From (i)-(iii), conditions (H_3) and (H_4) are always satisfied. Meanwhile, we successively obtain critical value $c_{H_n} = 1.69$, 0.4858, 0.2627, respectively. From Theorem 2.2, system (1.4) in the absence of time delay undergoes the Hopf bifurcation at $c = c_{H_n}$, see Figure 3 (from top to bottom). Our numerical simulation results indicate that the aggregation efficiency α may induce the complex dynamical behaviors, for example: the Hopf bifurcation, although $\alpha \neq \frac{1}{2}$, see Figure 3 for more details. It is an interesting finding for the case of $\alpha = \frac{1}{2}$.

Remark 2.3. If diffusion is present, the positive equilibrium E_* is always stable, so there is no the Turing instability for this positive equilibrium.

In fact, if we investigate the Turing instability for positive equilibrium E_* of system (1.4) when $\tau = 0$, we assume that the positive equilibrium E_* is stable for

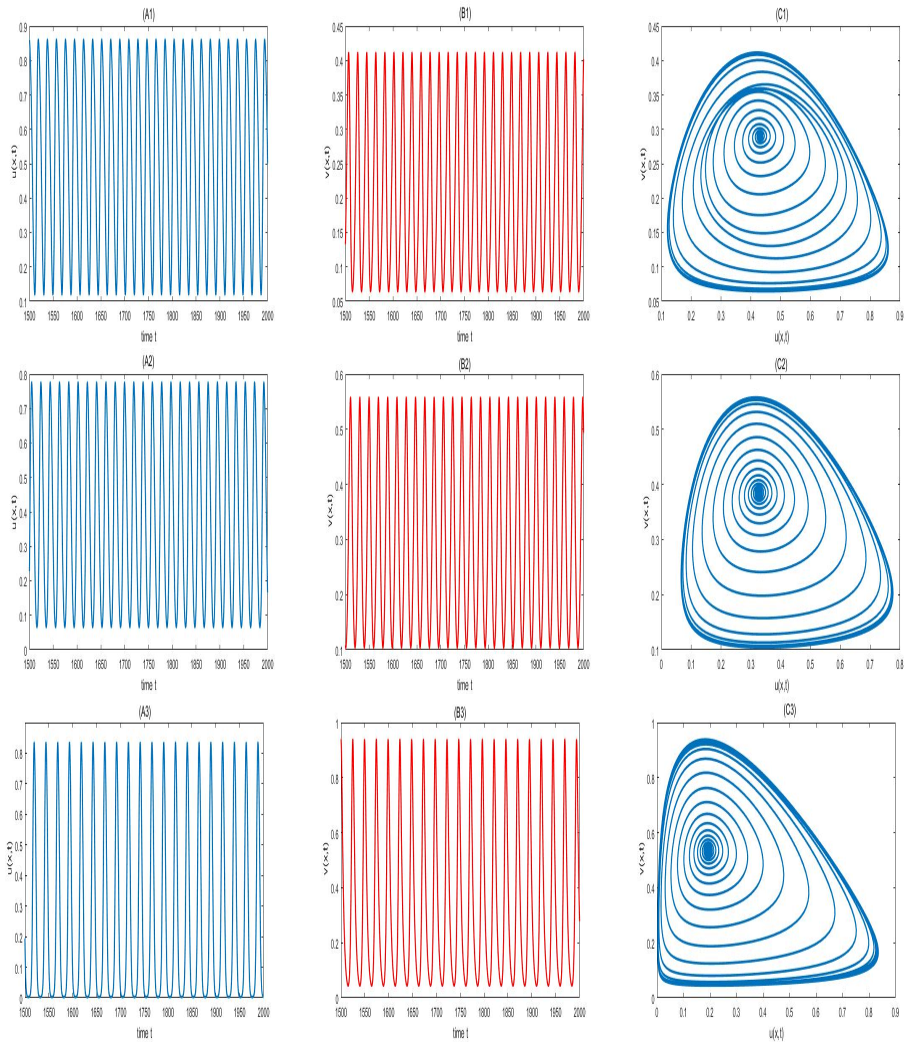

Figure 3. Solutions $u(x,t)$ and $v(x,t)$ with diffusion and different aggregation efficiency α. The Hopf bifurcation could exist in the following situations: (i) $\alpha = 0.2$ (top panel), (ii) $\alpha = 0.5$ (middle panel), (iii) $\alpha = 0.75$ (bottom panel), other parameters could refer to (i-iii) in Remark 2.2. Here choose the initial values $u(x,0) = u_* - 0.001\cos 2x,\ v(x,0) = u_* - 0.001\cos 2x$.

the corresponding ODEs, i.e., condition (H_2) is satisfied. According to (2.12), we know

$$T_n(c) \triangleq Tr(J_n) = T_0 - (d_1 + d_2)\frac{n^2}{l^2},$$

$$D_n(c) \triangleq Det(J_n) = D_0 + d_1 d_2 \left(\frac{n}{l}\right)^4 - a_1 d_2 \frac{n^2}{l^2},$$

with

$$a_1 = 1 - \alpha + (\alpha - 2)u_*.$$

When the condition (H_2) holds, $T_0 < 0$, then $T_n < 0$ for any $n = 0, 1, 2\cdots$. Therefore, the stability of the equilibrium E_* is determined by the sign of $D_n(c)$.

Under condition (H_2), we have $a_1 < 0$, then $D_n(c) > 0$ for any $n = 0, 1, 2\cdots$, which means that the positive equilibrium E_* is always stable, so, there is no Turning instability of E_*.

3. The delayed model

3.1. Stability analysis

In this section, we will investigate the stability for the positive equilibrium of system (1.4). Note that time delay does not change the positive equilibrium of a system, that is to say, the positive equilibrium of system (1.4) is still E_*.

Now we define

$$U = (u, v)^T,\ X = C([0, l\pi], \mathfrak{X}),\ \mathcal{A}_\tau = C([-\tau, 0], \mathfrak{X}).$$

The linearized system of (1.4) at the positive equilibrium E_* can be therefore expressed by

$$\dot{U} = D\Delta U + L(U_t), \tag{3.1}$$

with

$$D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix},\ Dom(\Delta) = \{(u, v)^T:\ u, v \in C^2([0, l\pi], \mathfrak{X}), u_x, v_x = 0,\ x = 0, l\pi\},$$

$U_t = (u_t, v_t) \in \mathcal{A}_\tau$ and operator $L : \mathbb{R}^2 \times \mathcal{A}_\tau \to \mathfrak{X}$ satisfying

$$L(\phi_t) = L_1\phi(0) + L_2\phi(-\tau), \tag{3.2}$$

where

$$\phi_t(\cdot) = \begin{pmatrix} \phi_1(t+\cdot) \\ \phi_2(t+\cdot) \end{pmatrix},\ L_1 = \begin{pmatrix} (1-\alpha)(1-u_*) & -\frac{c}{\beta} \\ \alpha\beta(1-u_*) & 0 \end{pmatrix},\ L_2 = \begin{pmatrix} -u_* & 0 \\ 0 & 0 \end{pmatrix}.$$

From [31], we can obtain the characteristic equation of the linear system (3.1)

$$\lambda y - d\Delta y - L(y) = 0, \quad y \neq 0. \tag{3.3}$$

It is well known that the eigenvalue problem under the Neumann boundary condition

$$\varphi'' = \rho\varphi, \quad x \in (0, l\pi), \quad \varphi'(0) = \varphi'(l\pi) = 0,$$

has eigenvalues $\rho_n = \frac{n^2}{l^2}$ $(n = 0, 1, 2\cdots)$ with corresponding eigenfunctions

$$\varphi_n = \cos\frac{n}{l}x, \quad n = 0, 1, 2\cdots.$$

So assuming that

$$y = \sum_{n=0}^{\infty} \begin{pmatrix} y_{1n} \\ y_{2n} \end{pmatrix} e^{\lambda t}\cos\frac{n}{l}x. \tag{3.4}$$

Substituting (3.4) into (3.1) and equating equal powers of $e^{\lambda_n t}\cos\frac{n}{l}x$ on both sides, then

$$\lambda\begin{pmatrix} y_{1n} \\ y_{2n} \end{pmatrix} = \begin{pmatrix} (1-\alpha)(1-u_*) - u_* e^{\lambda\tau} - d_1\frac{n^2}{l^2} & -\frac{c}{\beta} \\ \alpha\beta(1-u_*) & -d_2\frac{n^2}{l^2} \end{pmatrix}\begin{pmatrix} y_{1n} \\ y_{2n} \end{pmatrix}, \quad n = 0, 1\cdots.$$

Hence we obtain the characteristic equation

$$\triangle_n(\lambda,\tau) = \lambda^2 + A_n\lambda + B_n + u_*(\lambda + C_n)e^{-\lambda\tau} = 0, \tag{3.5}$$

in which

$$\begin{aligned} A_n &= (d_1+d_2)\frac{n^2}{l^2} - (1-\alpha)(1-u_*), \\ B_n &= d_1 d_2\frac{n^4}{l^4} - (1-\alpha)(1-u_*)d_2\frac{n^2}{l^2} + c\beta(1-u_*), \\ C_n &= d_2\frac{n^2}{l^2}. \end{aligned}$$

When $\tau = 0$, if $c > c_{H_n}$ and conditions $(H_3) - (H_4)$ in Theorem 2.2 hold, then all the characteristic roots of Eq.(3.5) have negative real parts for $n \in N$.

When $\tau > 0$, in order to derive some conditions for the Hopf bifurcation, we need to search for critical values τ such that there exists a pair purely imaginary eigenvalues. It is clear that $i\omega$ $(\omega > 0)$ is the root of characteristic equation (3.5) if and only if ω satisfies

$$-\omega^2 + iA_n\omega + B_n + u_*(i\omega + C_n)(\cos\omega\tau - i\sin\omega\tau) = 0.$$

Separating real and imaginary parts, we have

$$\begin{cases} -\omega^2 + B_n + u_*\omega\sin\omega\tau + u_*C_n\cos\omega\tau = 0, \\ A_n\omega + \omega u_*\cos\omega\tau - C_n u_*\sin\omega\tau = 0. \end{cases}$$

Square both sides of the equations above, then

$$\omega^4 + \omega^2(A_n^2 - 2B_n - u_*^2) + B_n^2 - C_n^2u_*^2 = 0. \tag{3.6}$$

Let $z = \omega^2$, then (3.6) can be rewritten as follows

$$z^2 - Pz + Q = 0, \tag{3.7}$$

with

$$\begin{aligned} P &= -A_n^2 + 2B_n + u_*^2, \\ Q &= B_n^2 - C_n^2u_*^2, \\ R &= (-A_n^2 + 2B_n + u_*^2)^2 - 4(B_n^2 - C_n^2u_*^2). \end{aligned}$$

Hence the roots of Eq.(3.7) are given by $z_\pm = \frac{P\pm\sqrt{R}}{2}$. Since $z = \omega^2 > 0$, so we shall seek the positive roots of Eq.(3.7). According to the sign of P, Q, R, there exist three cases.

Case I: (i) $R < 0$. (ii) $R > 0,\ P < 0,\ Q > 0$. (iii) $R = 0,\ P \leq 0$.
In **Case I**, Eq.(3.7) has no positive roots.

Case II: (i) $R > 0,\ Q < 0$. (ii)$R = 0,\ P > 0$.
In **Case II**, Eq.(3.7) has one positive root, therefore the characteristic equation (3.5) has a pair purely imaginary roots $\pm i\omega_n^+$ at $\tau = \tau_n^{j+},\ j = 0, 1, 2\cdots$.

Case III: (i) $R > 0,\ P > 0,\ Q > 0$.
In **Case III**, Eq.(3.7) has two positive roots, therefore the characteristic equation (3.5) has two pair purely imaginary roots $\pm i\omega_n^{\pm}$ at $\tau = \tau_n^{j\pm},\ j = 0, 1, 2\cdots$. Here

$$\omega_n^{\pm} = \sqrt{z_{\pm}}, \quad \tau_n^{j\pm} = \tau_n^{0\pm} + \frac{2j\pi}{\omega_n^{\pm}}, \quad j = 0, 1, 2\cdots,$$

with

$$\tau_n^{0\pm} = \begin{cases} \frac{1}{\omega_n^{\pm}} \arccos \frac{(\omega_n^{\pm})^2(C_n - A_n) - B_n C_n}{[(\omega_n^{\pm})^2 + C_n^2]u_*}, & \sin \omega_n^{\pm}\tau > 0, \\ \frac{-1}{\omega_n^{\pm}} \arccos \frac{(\omega_n^{\pm})^2(C_n - A_n) - B_n C_n}{[(\omega_n^{\pm})^2 + C_n^2]u_*}, & \sin \omega_n^{\pm}\tau < 0. \end{cases} \tag{3.8}$$

Fixing the parameters $\alpha, \beta, c, d_1, d_2, l$, we define

$$\mathfrak{D}_1 = \{\exists\ n \in N \mid \text{such that } \textbf{Case II} \text{ is satisfied}\},$$

$$\mathfrak{D}_2 = \{\exists\ n \in N \mid \text{such that } \textbf{Case III} \text{ is satisfied}\}.$$

Corollary 3.1. *Suppose* $0 < \alpha < 1$ *and* $0 < c < \beta$ *are satisfied:*

(i) If $R = 0$*, then* $Re(\frac{d\lambda}{d\tau})\Big|_{\tau=\tau_n^{j\pm}} = 0$*;*

(ii) If $R > 0$*, then* $Re(\frac{d\lambda}{d\tau})\Big|_{\tau=\tau_n^{j+}} > 0$*,* $Re(\frac{d\lambda}{d\tau})\Big|_{\tau=\tau_n^{j-}} < 0$ *for* $\tau \in \mathfrak{D}_1 \cup \mathfrak{D}_2$ *and* $j \in N$*.*

Proof. Differentiating two sides of (3.5) with respect to τ, one has

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{2\lambda + A_n + u_* e^{-\lambda\tau}}{\lambda u_*(\lambda + C_n)e^{-\lambda\tau}} - \frac{\tau}{\lambda},$$

then

$$\begin{aligned}
&\left(Re\left(\frac{d\lambda}{d\tau}\right)^{-1}\right)\Big|_{\tau=\tau_n^{j\pm}} \\
&= \left[Re\left(\frac{2i\omega + A_n + u_*(\cos\omega\tau - i\sin\omega\tau)}{i\omega u_*(i\omega + C_n)(\cos\omega\tau - i\sin\omega\tau)} - \frac{\tau}{i\omega}\right)\right]_{\tau=\tau_n^{j\pm}} \\
&= \left[Re\left(\frac{u_* + (2i\omega + A_n)(\cos\omega\tau + i\sin\omega\tau)}{-\omega^2 u_* + iC_n\omega u_*} - \frac{\tau}{i\omega}\right)\right]_{\tau=\tau_n^{j\pm}} \\
&= \left[\frac{-\omega u_*(u_* + A_n\cos\omega\tau - 2\omega\sin\omega\tau) + C_n\omega u_*(2\omega\cos\omega\tau + A_n\sin\omega\tau)}{\omega^4 u_*^2 + C_n^2\omega^2 u_*^2}\right]_{\tau=\tau_n^{j\pm}} \\
&= \left[\frac{\omega^2(2\omega^2 + A_n^2 - 2B_n - u_*^2)}{\omega^4 u_*^2 + C_n^2\omega^2 u_*^2}\right]_{\tau=\tau_n^{j\pm}} \\
&= \pm\frac{(\omega_n^{j\pm})^2\left[2(\omega_n^{j\pm})^2 + A_n^2 - 2B_n - u_*^2\right]}{(\omega_n^{j\pm})^4 u_*^2 + C_n^2(\omega_n^{j\pm})^2 u_*^2}
\end{aligned}$$

$$
\begin{aligned}
&= \pm \frac{(\omega_n^{j\pm})^2\sqrt{R}}{(\omega_n^{j\pm})^4 u_*^2 + C_n^2(\omega_n^{j\pm})^2 u_*^2} \\
&= \pm \frac{\sqrt{R}}{(\omega_n^{j\pm})^2 u_*^2 + C_n^2 u_*^2} \\
&= \begin{cases} \frac{\sqrt{R}}{(\omega_n^{j+})^2 u_*^2 + C_n^2 u_*^2} > 0, & \tau = \tau_n^{j+}, \\ -\frac{\sqrt{R}}{(\omega_n^{j-})^2 u_*^2 + C_n^2 u_*^2} < 0, & \tau = \tau_n^{j-}. \end{cases}
\end{aligned}
$$

Since

$$
\text{Sign}\left\{\mathbf{Re}\left(\frac{d\lambda}{d\tau}\right)\right\} = \text{Sign}\left\{\mathbf{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}\right\},
$$

therefore the result holds. □

From (3.8), we know $\tau_n^{0\pm} < \tau_n^{j\pm}$ $j \in N$, define the smallest τ so that the stability will change, $\tau = \widehat{\tau} \triangleq min\{\tau_n^{0+}\ or\ \tau_n^{0\pm},\ n \in \mathfrak{D}_1 \cup \mathfrak{D}_2\}$. According to the above analysis, we have the result below.

Theorem 3.1. *Suppose* $0 < c < \beta$ *and* $0 < \alpha < 1$ *are satisfied, then*

(i) in **Case I**, $\forall \tau \geq 0$ *all the characteristic roots of Eq.(3.5) have negative real parts, namely the positive equilibrium* E_* *of system (1.4) is locally asymptotically stable.*

(ii) in **Case II** *or* **Case III**, *the positive equilibrium* E_* *is locally asymptotically stable when* $\tau \in [0, \widehat{\tau})$, *and unstable when* $\tau \in (\widehat{\tau}, +\infty)$.

(iii) in **Case II** *or* **Case III**, *when* $\tau = \tau_n^{j+}$ $(\tau = \tau_n^{j\pm})$, *system* (1.4) *undergoes Hopf bifurcation and the periodic solutions bifurcated from the Hopf bifurcation are spatially inhomogeneous.*

Remark 3.1. (i) By using G.J. Butler's Lemma in Freedman and Rao [9], we can obtain that the real parts of the characteristic roots are all negative for $\tau \in [0, \widehat{\tau})$, namely E_* is locally stable for $\tau \in [0, \widehat{\tau})$, by using Proposition 6.5 in [24], we can obtain that E_* is unstable for $\tau \in (\widehat{\tau}, +\infty)$. Therefore, the result (ii) in Theorem 3.1 holds.

(ii) From this result, it is found that time delay could induce stability switches and successive Hopf bifurcations. When time delay is absent, dynamical behaviors in Cases II and III could not happen. These complex and interesting dynamics could be helpful in the biological strategies.

3.2. Direction of the Hopf bifurcation

In this part, we will investigate the direction of the Hopf bifurcation and stability of the bifurcating periodic solution using the center manifold reduction and the normal form theorem in [12,31]. In this section, we always set $\tau^* = \tau_n^{j+}$ and denote ω_n^+ by ω^* for simplicity.

To simplify calculation, we let $\bar{u} = u - u_*,\ \bar{v} = v - v_*$, and still denote the $\bar{u},\ \bar{v}$ by $u,\ v$. Thus we rewrite system (1.4) at positive equilibrium E_* in the following form

$$
\begin{cases}
\dot{u}(t) = d_1\Delta u + a_{11}u(t) + a_{12}v(t) + a_{13}u(t-\tau) + a_{21}u^2(t) + a_{22}u(t)v(t) \\
\qquad\qquad -u(t)u(t-\tau) + a_{31}u^3(t) + a_{32}u^2(t)v(t) + \mathcal{O}(4), \\
\dot{v}(t) = d_2\Delta v + b_{11}u(t) + b_{12}v(t) + b_{21}u^2(t) + b_{22}u(t)v(t) + b_{31}u^3(t) \\
\qquad\qquad +b_{32}u^2(t)v(t) + \mathcal{O}(4),
\end{cases}
\tag{3.9}
$$

where

$$
\begin{aligned}
&a_{11} = (1-\alpha)(1-u_*),\ a_{12} = -u_*^{\alpha},\ a_{13} = -u_*,\ a_{21} = -\frac{1}{2}\alpha(\alpha-1)(u_*^{-1}-1), \\
&a_{22} = -\alpha u_*^{\alpha-1},\ a_{31} = -\frac{1}{6}\alpha(\alpha-1)(\alpha-2)(1-u_*)u_*^{\alpha-2},\ a_{32} = -\frac{1}{2}\alpha(\alpha-1)u_*^{\alpha-2}, \\
&b_{11} = \alpha\beta(1-u_*),\ b_{12} = 0,\ b_{21} = -\beta a_{21},\ b_{22} = -\frac{1}{2}a_{22},\ b_{31} = -\beta a_{31},\ b_{32} = -a_{32}.
\end{aligned}
$$

We denote the solution of (3.9) as $\mathbf{X} = (u(t), v(t))^T$ and let $\mathbf{X}_t(\theta) = \mathbf{X}(t+\theta)$, $\theta \in [-\tau, 0]$, where $\mathbf{X}_t \in C([-\tau, 0], \mathbb{R}^2) \triangleq \mathcal{C}$. Let $\tau = \tau^* + \mu$, it is clear that system (3.9) undergoes a Hopf bifurcation when $\mu = 0$ with corresponding eigenvalues $\pm i\omega^*$. Then we rewrite (3.9) as the following functional differential equation,

$$
\dot{\mathbf{X}}_t = L_\mu(\mathbf{X}_t) + R_\mu(\mathbf{X}_t), \tag{3.10}
$$

where $L_\mu :\ \mathcal{C} \to \mathbb{R}^2$ and $R_\mu :\ \mathcal{C} \to \mathbb{R}^2$ are defined by

$$
L_\mu(\phi) = M_0 \begin{pmatrix} \phi_1(0) \\ \phi_2(0) \end{pmatrix} + M_1 \begin{pmatrix} \phi_1(-\tau) \\ \phi_2(-\tau) \end{pmatrix}, \tag{3.11}
$$

here

$$
M_0 = \begin{pmatrix} a_{11} - d_1\frac{n^2}{l^2} & a_{12} \\ b_{11} & -d_2\frac{n^2}{l^2} \end{pmatrix},\ M_1 = \begin{pmatrix} a_{13} & 0 \\ 0 & 0 \end{pmatrix},
$$

$$
R_\mu(\phi) = \begin{pmatrix} a_{21}\phi_1^2(0) + a_{22}\phi_1(0)\phi_2(0) - \phi_1(0)\phi_1(-\tau) + a_{31}\phi_1^3(0) + a_{32}\phi_1^2(0)\phi_2(0) + O(4) \\ b_{21}\phi_1^2(0) + b_{22}\phi_1(0)\phi_2(0) + b_{31}\phi_1^3(0) + b_{32}\phi_1^2(0)\phi_2(0) + O(4) \end{pmatrix}. \tag{3.12}
$$

By using Riesz representation theorem, we obtain

$$
L_\mu(\phi) = \int_{-\tau}^{0} \phi(\theta)d\eta(\theta,\mu) \quad for\ \phi \in \mathcal{C}, \tag{3.13}
$$

in which we choose

$$
\eta(\theta,\mu) = M_0\delta(\theta) + M_1\delta(\theta+\tau), \tag{3.14}
$$

where $\delta(\theta)$ is the Dirac function and $d\eta(\theta,\mu) = M_0\delta(\theta)d(\theta) + M_1\delta(\theta+\tau)d(\theta)$.

Next, we define two operators A_μ and F_μ on $C^1([-\tau,0],\mathbb{R}^2) \triangleq \mathcal{C}^1$ in the following form

$$(A_\mu\phi)(\theta) = \begin{cases} \frac{d\phi}{d\theta}, & \theta \in [-\tau, 0), \\ \int_{-\tau}^{0} \phi(\xi) d\eta(\xi,\mu), & \theta = 0, \end{cases}$$

$$(F_\mu\phi)(\theta) = \begin{cases} 0, & \theta \in [-\tau, 0), \\ R_\mu(\phi), & \theta = 0. \end{cases} \tag{3.15}$$

Then (3.10) is transformed into

$$\dot{\mathbf{X}}_t = A_\mu(\mathbf{X}_t) + F_\mu(\mathbf{X}_t). \tag{3.16}$$

Furthermore, the adjoint operator A_μ^* of A_μ is defined as

$$(A_\mu^*\psi)(\theta^*) = \begin{cases} -\frac{d\psi}{d\theta^*}, & \theta^* \in (0, \tau], \\ \int_{-\tau}^{0} \psi(-\xi) d\eta^T(\xi,\mu), & \theta^* = 0, \end{cases} \tag{3.17}$$

in which $\psi \in C^1([0,\tau],\mathbb{R}^2)$. Note that the solution space of (3.16) complex space is $\mathbb{C}^2$ instead of $\mathbb{R}^2$. To calculate the coordinate of center manifold, we need to define bilinear inner product

$$\langle \psi(\theta^*), \phi(\theta) \rangle = \bar{\psi}^T(0)\phi(0) - \int_{\theta=-\tau}^{0} \int_{\xi=0}^{\theta} \bar{\psi}^T(\xi-\theta) d\eta(\theta)\phi(\xi) d\xi, \tag{3.18}$$

where $\phi \in C([-\tau,0],\mathbb{C}^2)$, $\psi \in C^1([0,\tau],\mathbb{C}^2)$ and $\eta(\theta) = \eta(\theta,0)$.

It is clear that $\pm i\omega^*$ are eigenvalues of characteristic equation (3.5) when $\mu = 0$. So, according to definition of A_μ and A_μ^*, $\pm i\omega^*$ are eigenvalues of A_0 and A_0^* respectively. We assume that the eigenvectors of A_0 and A_0^* corresponding to $i\omega^*$ and $-i\omega^*$ are $q(\theta)$ and $q^*(\theta^*)$ respectively, i.e.,

$$A_0 q(\theta) = i\omega^* q(\theta), A_0^* q^*(\theta^*) = -i\omega^* q^*(\theta^*), \tag{3.19}$$

which need to satisfy the normalized condition $\langle q^*, q \rangle = 1$ and orthogonal condition $\langle q^*, \bar{q} \rangle = 0$. To simplify calculations, we set

$$q(\theta) = q(0)e^{i\omega^*\theta} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} e^{i\omega^*\theta}, \quad q^*(\theta^*) = q^*(0)e^{i\omega^*\theta^*} = \begin{pmatrix} q_1^* \\ q_2^* \end{pmatrix} e^{i\omega^*\theta^*}. \tag{3.20}$$

When $\theta = 0$, by using the definition of A_μ, we can obtain

$$\begin{aligned} A_0 q(0) &= \int_{-\tau}^{0} d\eta(\theta) q(\theta) \\ &= M_0 \int_{-\tau}^{0} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} e^{i\omega^*\theta} \delta(\theta) d(\theta) + M_1 \int_{-\tau}^{0} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} e^{i\omega^*\theta} \delta(\theta+\tau) d(\theta) \end{aligned}$$

$$= M_0 \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} + M_1 \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} e^{-i\omega^*\tau^*}$$
$$= i\omega^* \begin{pmatrix} q_1 \\ q_2 \end{pmatrix},$$

then

$$(M_0 + M_1 e^{-i\omega^*\tau^*} - i\omega^* I) \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

since $(q_1, q_2)^T$ is the nonzero vector, then

$$q(\theta) = \begin{pmatrix} 1 \\ \frac{b_{11}}{i\omega^* + d_2 \frac{n^2}{l^2}} \end{pmatrix} e^{i\omega^*\theta}. \tag{3.21}$$

Similarly, by using the definition of A_μ^*, we can obtain

$$q^*(\theta^*) = \sigma \begin{pmatrix} \frac{-i\omega^* + d_2 \frac{n^2}{l^2}}{b_{11}} \\ 1 \end{pmatrix} e^{i\omega^*\theta^*}. \tag{3.22}$$

Notice that $\langle q^*, q\rangle = 1$, we need to determine the value σ. From (3.18), one has

$$\begin{aligned}
\langle q^*, q\rangle &= \bar{q}^{*T}(0)q(0) - \int_{\theta=-\tau}^{0} \int_{0}^{\xi=\theta} \bar{q}^{*T}(\xi - \theta) d\eta(\theta) q(\xi) d(\xi) \\
&= \bar{\sigma}(\bar{q}_1^*, \bar{q}_2^*)(q_1, q_2)^T - \int_{\theta=-\tau}^{0} \int_{0}^{\xi=\theta} \bar{\sigma}(\bar{q}_1^*, \bar{q}_2^*) e^{-i\omega^*(\xi-\theta)} d\eta(\theta)(q_1, q_2)^T e^{i\omega^*\xi} d\xi \\
&= \bar{\sigma}(\bar{q}_1^* q_1 + \bar{q}_2^* q_2) - \bar{\sigma}(\bar{q}_1^*, \bar{q}_2^*) M_1 (q_1, q_2)^T (-\tau^* e^{-i\omega^*\tau^*}) \\
&= \bar{\sigma}(\bar{q}_1^* + q_2 + a_{13}\bar{q}_1^* \tau^* e^{-i\omega^*\tau^*}).
\end{aligned}$$

Thus we can choose σ as

$$\sigma = 1/(q_1^* + \bar{q}_2 + a_{13} q_1^* \tau^* e^{i\omega^*\tau^*}). \tag{3.23}$$

After computing q and q^*, one can decompose the whole solution space. Assuming the whole solution space is $\mathbb{X}$, we denote $x_t = x_t(\theta)$ as a solution of (3.16) when $\mu = 0$ and define

$$z(t) = \langle q^*, x_t\rangle. \tag{3.24}$$

Decompose $\mathbb{X} = \mathbb{X}^C + \mathbb{X}^S$, where $\mathbb{X}^C = \{zq + \bar{z}\bar{q} \mid z \in \mathbb{C}\}$ represents the center manifold C_0 at $\mu = 0$ with the method of Hassard, Kazarinoff, and Wan [12], $\mathbb{X}^S = \{u \in \mathbb{X} \mid \langle q^*, u\rangle = 0\}$ is the stable manifold.

Hence there exists $z \in \mathbb{C}$ and $W = (W_1, W_2)^T \in \mathbb{X}^S$ so that $x_t = zq + \bar{z}\bar{q} + W$, then

$$W(t, \theta) \triangleq x_t(\theta) - zq - \bar{z}\bar{q}$$

$$= x_t(\theta) - 2Re\{z(t)q(\theta)\}. \tag{3.25}$$

In terms of the center manifold reduction, one has $W(t,\theta) = W(z(t), \bar{z}(t), \theta)$ on $\mathbb{X}^C$. According to the center eigenspace at the equilibrium, one further has

$$W(t,\theta) = W(z(t), \bar{z}(t), \theta) = W_{20}(\theta)\frac{z^2}{2} + W_{02}(\theta)\frac{\bar{z}^2}{2} + W_{11}(\theta)z\bar{z} + W_{30}(\theta)\frac{z^3}{6} + \cdots, \tag{3.26}$$

where $W_{20}(\theta)$, $W_{02}(\theta)$, $W_{11}(\theta)$ and $W_{30}(\theta)$ are undetermined.

Since $\dot{\mathbb{X}}_t = A_0(\mathbb{X}_t) + F_0(\mathbb{X}_t)$, it holds that

$$\frac{dx_t}{dt} = \frac{d(zq + \bar{z}\bar{q} + W)}{dt} = A_0(zq + \bar{z}\bar{q} + W) + F_0(zq + \bar{z}\bar{q} + W). \tag{3.27}$$

Combining (3.24) with (3.27), one gets

$$\begin{aligned}\frac{d\langle q^*, x_t\rangle}{dt} = \frac{dz(t)}{dt} &= A_0(z\langle q^*, q\rangle + \bar{z}\langle q^*, \bar{q}\rangle + \langle q^*, W\rangle) + \langle q^*, F_0\rangle,\\ &= A_0 z + \langle q^*, F_0\rangle,\end{aligned}$$

that is to say

$$\frac{dz}{dt} = i\omega^* z + \langle q^*, F_0\rangle. \tag{3.28}$$

According to the definition of F_0

$$\begin{aligned}\langle q^*(\theta), F_0\rangle &= \bar{q}^{*T}(0)F_0(z,\bar{z},0) - \int_{\theta=-\tau}^{0}\int_{\xi=0}^{\theta} \bar{q}^{*T}(\xi-\theta)d\eta(\theta)F(\xi)d(\xi)\\ &= \bar{q}^{*T}(0)F_0(z,\bar{z},0),\end{aligned} \tag{3.29}$$

then

$$\frac{dz}{dt} = i\omega^* z + g(z,\bar{z}), \tag{3.30}$$

where

$$\begin{aligned}g(z,\bar{z}) =& \bar{q}^{*T}(0)F_0(z,\bar{z},0) \triangleq g_{20}\frac{z^2}{2} + g_{02}\frac{\bar{z}^2}{2} + g_{11}z\bar{z} + g_{30}\frac{z^3}{6} + g_{21}\frac{z^2\bar{z}}{2}\\ &+ g_{12}\frac{z\bar{z}^2}{2} + g_{03}\frac{\bar{z}^3}{6} + \cdots\\ =& \bar{\sigma}(\bar{q}_1^*, 1)\cdot\\ &\begin{pmatrix} a_{21}\phi_1^2(0) + a_{22}\phi_1(0)\phi_2(0) - \phi_1(0)\phi_1(-\tau) + a_{31}\phi_1^3(0) + a_{32}\phi_1^2(0)\phi_2(0)\\ b_{21}\phi_1^2(0) + b_{22}\phi_1(0)\phi_2(0) + b_{31}\phi_1^3(0) + b_{32}\phi_1^2(0)\phi_2(0)\end{pmatrix}.\end{aligned} \tag{3.31}$$

From (3.25) and (3.26), we have

$$x_t(\theta) = W(z,\bar{z},\theta) + zq + \bar{z}\bar{q}$$

$$=W_{20}(\theta)\frac{z^2}{2}+W_{02}(\theta)\frac{\bar{z}^2}{2}+W_{11}(\theta)z\bar{z}+W_{30}(\theta)\frac{z^3}{6}+\begin{pmatrix}1\\q_2\end{pmatrix}e^{i\omega^*\theta}z$$
$$+\begin{pmatrix}1\\\bar{q}_2\end{pmatrix}e^{-i\omega^*\theta}\bar{z}+\cdots. \tag{3.32}$$

Since the corresponding coefficients are equal, then g_{ij} can be explicitly expressed and its expressions are given in Appendix for the convenience.

It is clear that g_{21} depends on $W_{20}(\theta)$ and $W_{11}(\theta)$, thus it is necessary to find the values of $W_{20}(\theta)$ and $W_{11}(\theta)$. Substituting (3.28) into (3.27), we have

$$\frac{dW}{dt}=A_0W-\bar{q}^{*T}(0)F_0(z,\bar{z})q-q^{*T}(0)\bar{F}_0(z,\bar{z})\bar{q}+F_0,$$

from (3.25), it follows that

$$\begin{aligned}\dot{W}&=\dot{x}_t(\theta)-\dot{z}q-\dot{\bar{z}}\bar{q}\\&=\begin{cases}A_0W-2Re\{\bar{q}^{*T}(0)F_0(z,\bar{z})q(\theta)\}, & \theta\in[-\tau,0),\\A_0W-2Re\{\bar{q}^{*T}(0)F_0(z,\bar{z})q(\theta)\}+F_0, & \theta=0,\end{cases}\\&\triangleq A_0W+H(z,\bar{z},\theta),\end{aligned} \tag{3.33}$$

where

$$H(z,\bar{z},\theta)=H_{20}\frac{z^2}{2}+H_{11}z\bar{z}+H_{02}\frac{\bar{z}^2}{2}+\cdots. \tag{3.34}$$

From (3.26), we know

$$\begin{aligned}W(t,\theta)&=W(z(t),\bar{z}(t),\theta)\\&\triangleq W_{20}(\theta)\frac{z^2}{2}+W_{02}(\theta)\frac{\bar{z}^2}{2}+W_{11}(\theta)z\bar{z}+W_{30}(\theta)\frac{z^3}{6}+\cdots.\end{aligned}$$

Differentiating two sides above with respect to t, one has

$$\begin{aligned}\dot{W}&=W_z\cdot\dot{z}(t)+W_{\bar{z}}\cdot\dot{\bar{z}}(t)\\&=(W_{20}(\theta)z+W_{11}(\theta)\bar{z}+\cdots)(i\omega^*z+g(z,\bar{z}))\\&\quad+(W_{02}(\theta)\bar{z}+W_{11}(\theta)z+\cdots)(-i\omega^*\bar{z}+\bar{g}(z,\bar{z}))\\&=i\omega^*W_{20}(\theta)z^2-i\omega^*W_{02}(\theta)\bar{z}^2+\cdots,\end{aligned} \tag{3.35}$$

from (3.33) and (3.34), we have

$$\begin{aligned}\dot{W}&\triangleq A_0W+H(z,\bar{z},\theta)\\&=(\frac{1}{2}A_0W_{20}(\theta)+\frac{1}{2}H_{20})z^2+(A_0W_{11}(\theta)+H_{11})z\bar{z}+(\frac{1}{2}A_0W_{02}(\theta)+\frac{1}{2}H_{02})\bar{z}^2.\end{aligned} \tag{3.36}$$

Comparing the coefficients of z^2 and $z\bar{z}$ from (3.35) and (3.36), we arrive at

$$(A_0-2i\omega^*I)W_{20}(\theta)=-H_{20}(z,\bar{z},\theta),$$

$$A_0W_{11}(\theta) = -H_{11}(z, \bar{z}, \theta). \tag{3.37}$$

When $-\tau \le \theta < 0$, from (3.33) and (3.34), we have

$$\begin{aligned}
H(z, \bar{z}, \theta) =& - 2Re\{\bar{q}^{*T}(0)F_0(z, \bar{z})q(\theta)\} \\
=& - \bar{q}^{*T}(0)F_0(z, \bar{z})q - q^{*T}(0)\bar{F}_0(z, \bar{z})\bar{q} \\
=& - g(z, \bar{z})q(\theta) - \bar{g}(z, \bar{z})\bar{q}(\theta) \\
=&(-g_{20}q(\theta) - \bar{g}_{02}\bar{q}(\theta))\frac{z^2}{2} + (-g_{11}q(\theta) - \bar{g}_{11}\bar{q}(\theta))z\bar{z} \\
&+ (-g_{02}q(\theta) - \bar{g}_{20}\bar{q}(\theta))\frac{\bar{z}^2}{2} + \cdots \\
=&H_{20}\frac{z^2}{2} + H_{11}z\bar{z} + H_{02}\frac{\bar{z}^2}{2} + \cdots ,
\end{aligned} \tag{3.38}$$

then we can obtain

$$\begin{aligned}
H_{20}(\theta) &= -g_{20}q(\theta) - \bar{g}_{02}\bar{q}(\theta), \\
H_{11}(\theta) &= -g_{11}q(\theta) - \bar{g}_{11}\bar{q}(\theta).
\end{aligned} \tag{3.39}$$

Substituting (3.39) into (3.37) yields that

$$\begin{aligned}
A_0W_{20}(\theta) &= 2i\omega^*W_{20}(\theta) + g_{20}q(\theta) + \bar{g}_{02}\bar{q}(\theta), \\
A_0W_{11}(\theta) &= g_{11}q(\theta) + \bar{g}_{11}\bar{q}(\theta).
\end{aligned} \tag{3.40}$$

By applying the definition of A_μ when $-\tau \le \theta < 0$, we have

$$\begin{aligned}
\dot{W}_{20}(\theta) &= 2i\omega^*W_{20}(\theta) + g_{20}q(\theta) + \bar{g}_{02}\bar{q}(\theta), \\
&= 2i\omega^*W_{20}(\theta) + g_{20}q(0)e^{i\omega^*\theta} + \bar{g}_{02}\bar{q}(0)e^{-i\omega^*\theta},
\end{aligned} \tag{3.41}$$

which is similar to the linear ordinary differential equation of the first order

$$\frac{dW_{20}(\theta)}{d\theta} = P(\theta)W_{20}(\theta) + Q(\theta),$$

where

$$P(\theta) = 2i\omega^*, \quad Q(\theta) = g_{20}q(0)e^{i\omega^*\theta} + \bar{g}_{02}\bar{q}(0)e^{-i\omega^*\theta}.$$

By applying the method of variation of constants, we can obtain

$$W_{20}(\theta) = \frac{ig_{20}q(0)}{\omega^*}e^{i\omega^*\theta} + \frac{i\bar{g}_{02}\bar{q}(0)}{3\omega^*}e^{-i\omega^*\theta} + \tilde{C}_1e^{2i\omega^*\theta}, \tag{3.42}$$

here $\tilde{C}_1 = (\tilde{C}_1^1, \tilde{C}_1^2)^T$ is a constant vector. Similarly, we can obtain

$$W_{11}(\theta) = \frac{-ig_{11}q(0)}{\omega^*}e^{i\omega^*\theta} + \frac{i\bar{g}_{11}\bar{q}(0)}{\omega^*}e^{-i\omega^*\theta} + \tilde{C}_2, \tag{3.43}$$

where $\tilde{C}_2 = (\tilde{C}_2^1, \tilde{C}_2^2)^T$ is a constant vector. Next, we need to calculate the values of $\tilde{C}_1$ and $\tilde{C}_2$.

When $\theta = 0$, according to the definition of A_0 and (3.27), we have

$$H_{20}(0) =2i\omega^*W_{20}(0) - A_0W_{20}(0)$$

$$
\begin{aligned}
&=2i\omega^* W_{20}(0) - \int_{-\tau}^{0} [d\eta(\theta)] W_{20}(\theta) \\
&= -\, 2g_{20}q(0) - \frac{2}{3}\bar{g}_{02}\bar{q}(0) + 2i\omega^*\tilde{C}_1 - \frac{ig_{20}}{\omega^*}\int_{-\tau}^{0} e^{i\omega^*\theta} d\eta(\theta) q(0) \\
&\quad - \frac{i\bar{g}_{02}}{3\omega^*}\int_{-\tau}^{0} e^{-i\omega^*\theta} d\eta(\theta)\bar{q}(0) - \tilde{C}_1 \int_{-\tau}^{0} e^{2i\omega^*\theta} d\eta(\theta),
\end{aligned} \tag{3.44}
$$

and

$$
\begin{aligned}
H_{11}(0) &= -\, A_0 W_{11}(0) = -\int_{-\tau}^{0} [d\eta(\theta)] W_{11}(\theta) \\
&= \frac{ig_{11}}{\omega^*}\int_{-\tau}^{0} e^{i\omega^*\theta} d\eta(\theta) q(0) - \frac{i\bar{g}_{11}}{\omega^*}\int_{-\tau}^{0} e^{-i\omega^*\theta} d\eta(\theta)\bar{q}(0) - \int_{-\tau}^{0} \tilde{C}_2 d\eta(\theta).
\end{aligned} \tag{3.45}
$$

Note that $i\omega^*$ is eigenvalue of A_0 with corresponding eigenvector $q(0)$, i.e., $A_0q(0) = i\omega^* q(0)$, from (3.17), one has

$$
i\omega^* q(0) = \int_{-\tau}^{0} e^{i\omega^*\theta} d\eta(\theta) q(0). \tag{3.46}
$$

Similarly, according to $A_0^*\bar{q}(0) = -i\omega^*\bar{q}(0)$, we have

$$
-i\omega^*\bar{q}(0) = \int_{-\tau}^{0} e^{-i\omega^*\theta} d\eta(\theta)\bar{q}(0). \tag{3.47}
$$

Substituting (3.46) and (3.47) into (3.44) and (3.45) respectively, then

$$
H_{20}(0) = -g_{20}q(0) - \bar{g}_{02}\bar{q}(0) + 2i\omega^*\tilde{C}_1 - \tilde{C}_1\int_{-\tau}^{0} e^{2i\omega^*\theta} d\eta(\theta), \tag{3.48}
$$

and

$$
H_{11}(0) = -g_{11}q(0) - \bar{g}_{11}\bar{q}(0) - \int_{-\tau}^{0} \tilde{C}_2 d\eta(\theta). \tag{3.49}
$$

From (3.33) and (3.34), we infer that when $\theta = 0$,

$$
\begin{aligned}
H(z,\bar{z},0) &= -\, 2Re\{\bar{q}^{*T}(0) F_0(z,\bar{z},0) q(0)\} + F_0(z,\bar{z},0) \\
&= -\, g(z,\bar{z}) q(0) - \bar{g}(z,\bar{z})\bar{q}(0) + F_0(z,\bar{z},0) \\
&= (-g_{20}q(0) - \bar{g}_{02}\bar{q}(0))\frac{z^2}{2} + (-g_{11}q(0) - \bar{g}_{11}\bar{q}(0)) z\bar{z} \\
&\quad + (-g_{02}q(0) - \bar{g}_{20}\bar{q}(0))\frac{\bar{z}^2}{2} + F_0(z,\bar{z},0) \\
&= H_{20}(0)\frac{z^2}{2} + H_{11}(0) z\bar{z} + H_{02}(0)\frac{\bar{z}^2}{2} + \cdots .
\end{aligned} \tag{3.50}
$$

Notice that there exist the terms of z^2 and $z\bar{z}$ in F_0, then compare with the coefficient of z^2 and $z\bar{z}$, we get

$$
H_{20}(0) = -g_{20}q(0) - \bar{g}_{02}\bar{q}(0) + \begin{pmatrix} 2a_{21} + 2a_{22}q_2 - 2e^{-i\omega^*\tau^*} \\ 2b_{21} + 2b_{22}q_2 \end{pmatrix}, \tag{3.51}
$$

and

$$H_{11}(0) = -g_{11}q(0) - \bar{g}_{11}\bar{q}(0) + \begin{pmatrix} 2a_{21} + 2a_{22}Re(q_2) - 2Re(e^{-i\omega^*\tau^*}) \\ 2b_{21} + 2b_{22}Re(q_2) \end{pmatrix}. \quad (3.52)$$

Substituting (3.51) into (3.48), then

$$G_1\tilde{C}_1 = \begin{pmatrix} 2a_{21} + 2a_{22}q_2 - 2e^{-i\omega^*\tau^*} \\ 2b_{21} + 2b_{22}q_2 \end{pmatrix},$$

where

$$\begin{aligned} G_1 &= 2i\omega^* I - \int_{-\tau}^{0} e^{2i\omega^*\theta} d\eta(\theta) = 2i\omega^* I - M_0 - M_1 e^{-2i\omega^*\tau^*} \\ &= \begin{pmatrix} 2i\omega^* - a_{11} + d_1\frac{n^2}{l^2} - a_{13}e^{-2i\omega^*\tau^*} & -a_{12} \\ -b_{11} & 2i\omega^* + d_2\frac{n^2}{l^2} \end{pmatrix}. \end{aligned}$$

According Cramer's Rule, we have

$$\tilde{C}_1^1 = \frac{\begin{vmatrix} 2a_{21} + 2a_{22}q_2 - 2e^{-i\omega^*\tau^*} & -a_{12} \\ 2b_{21} + 2b_{22}q_2 & 2i\omega^* + d_2\frac{n^2}{l^2} \end{vmatrix}}{\begin{vmatrix} 2i\omega^* - a_{11} + d_1\frac{n^2}{l^2} - a_{13}e^{-2i\omega^*\tau^*} & -a_{12} \\ -b_{11} & 2i\omega^* + d_2\frac{n^2}{l^2} \end{vmatrix}},$$

and

$$\tilde{C}_1^2 = \frac{\begin{vmatrix} 2i\omega^* - a_{11} + d_1\frac{n^2}{l^2} - a_{13}e^{-2i\omega^*\tau^*} & 2a_{21} + 2a_{22}q_2 - 2e^{-i\omega^*\tau^*} \\ -b_{11} & 2b_{21} + 2b_{22}q_2 \end{vmatrix}}{\begin{vmatrix} 2i\omega^* - a_{11} + d_1\frac{n^2}{l^2} - a_{13}e^{-2i\omega^*\tau^*} & -a_{12} \\ -b_{11} & 2i\omega^* + d_2\frac{n^2}{l^2} \end{vmatrix}}.$$

Similarly, substituting (3.52) into (3.49) yields that

$$G_2\tilde{C}_2 = \begin{pmatrix} 2a_{21} + 2a_{22}Re(q_2) - 2Re(e^{-i\omega^*\tau^*}) \\ 2b_{21} + 2b_{22}Re(q_2) \end{pmatrix},$$

where

$$G_2 = -\int_{-\tau}^{0} d\eta(\theta) = -M_0 - M_1 = \begin{pmatrix} -a_{11} - a_{13} + d_1\frac{n^2}{l^2} & -a_{12} \\ -b_{11} & d_2\frac{n^2}{l^2} \end{pmatrix},$$

thus

$$\tilde{C}_2^1 = \frac{\begin{vmatrix} 2a_{21} + 2a_{22}q_2 - 2Re(e^{-i\omega^*\tau^*}) & -a_{12} \\ 2b_{21} + 2b_{22}Re(q_2) & d_2\frac{n^2}{l^2} \end{vmatrix}}{\begin{vmatrix} -a_{11} - a_{13} + d_1\frac{n^2}{l^2} & -a_{12} \\ -b_{11} & d_2\frac{n^2}{l^2} \end{vmatrix}},$$

and

$$\tilde{C}_2^2 = \frac{\begin{vmatrix} -a_{11} - a_{13} + d_1\frac{n^2}{l^2} & 2a_{21} + 2a_{22}q_2 - 2Re(e^{-i\omega^*\tau^*}) \\ -b_{11} & 2b_{21} + 2b_{22}Re(q_2) \end{vmatrix}}{\begin{vmatrix} -a_{11} - a_{13} + d_1\frac{n^2}{l^2} & -a_{12} \\ -b_{11} & d_2\frac{n^2}{l^2} \end{vmatrix}}.$$

According to the analysis above, we can determine the values of $W_{20}(\theta)$ and $W_{11}(\theta)$ from (3.42) and (3.43).

Therefore, the direction of the Hopf bifurcation and the stability of the periodic solutions can be determined by the sign of formulas below:

$$c_1(0) = \frac{i}{2\omega^*\tau^*}\left(g_{20}g_{11} - 2\mid g_{11}\mid^2 - \frac{1}{3}\mid g_{02}\mid^2\right) + \frac{1}{2}g_{21}, \quad \mu_2 = -\frac{Re(c_1(0))}{Re(\lambda'(\tau^*))},$$
$$T_2 = -\frac{1}{\omega^*\tau^*}\left[Im(c_1(0)) + \mu_2 Im(\lambda'(\tau^*))\right], \quad \beta_2 = 2Re(c_1(0)).$$

Theorem 3.2. *If $Re(c_1(0)) \neq 0$, then*

(i) μ_2 determines the direction of the Hopf bifurcation: if $\mu_2 > 0$ (resp. $\mu_2 < 0$), then the Hopf bifurcation is supercritical (resp. subcritical) and the bifurcating periodic solutions exist for $\mu > 0$ (resp. $\mu < 0$).

(ii) β_2 determines the stability of the bifurcating periodic solutions£º if $\beta_2 < 0$ (resp. $\beta_2 > 0$), then the bifurcating periodic solutions are stable (resp. unstable).

(iii) T_2 determines the period of bifurcating periodic solutions: if $T_2 > 0$ (resp. $T_2 < 0$), then the period increases (resp. decreases).

4. Numerical simulation

In this section, we carry out some numerical simulations to illustrate theoretical analysis given in Section 3. For system (1.4), set the following parameters:

$$\alpha = 0.2,\ \beta = 2,\ c = 1.8,\ l = 10,\ d_1 = 0.4,\ d_2 = 0.8, \tag{4.1}$$
$$\alpha = 0.5,\ \beta = 2.5,\ c = 1.7,\ l = 1,\ d_1 = 2.5,\ d_2 = 1.7, \tag{4.2}$$
$$\alpha = 0.7,\ \beta = 1.6,\ c = 1.2,\ l = 10,\ d_1 = 1.3,\ d_2 = 0.7. \tag{4.3}$$

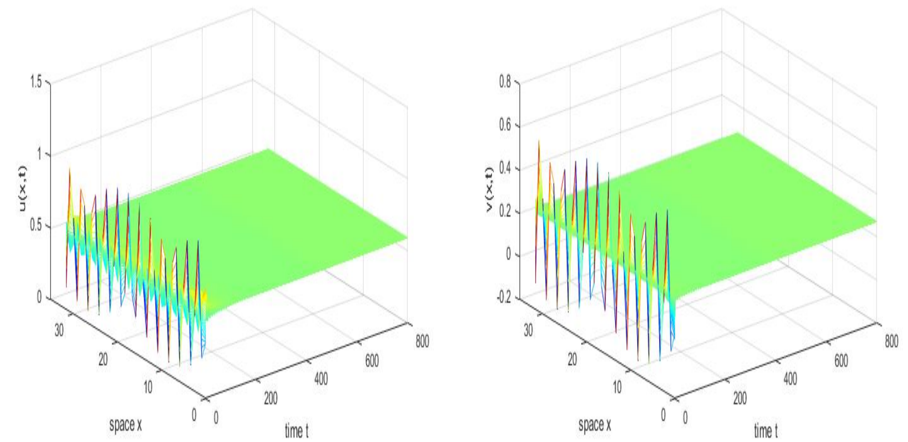

Figure 4. $E_* = (0.5905, 0.2687)$ is locally asymptotically stable, where $\tau = 1.34$ and initial condition is $(u(x,0), v(x,0)) = (0.5905 + 0.5\cos(2x), 0.2687 + 0.4\cos(2x))$.

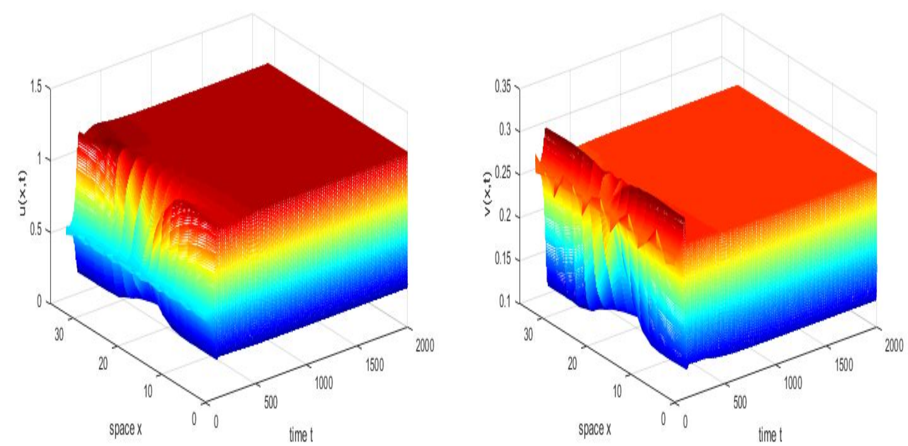

Figure 5. When $\alpha = 0.2$, the bifurcating periodic solutions are stable, where $\tau = 1.48$ and initial condition is $(u(x,0), v(x,0)) = (0.5905 + 0.02\cos(2x), 0.2687 + 0.02\cos(2x))$.

Choosing the parameters as in (4.1), by direct computation, we have $\tau^* = \tau_0^0 = 1.4544$. From Theorem 3.1, one obtains that if $\tau \in [0, \hat{\tau})$, then the positive equilibrium E_* is locally asymptotically stable. This result is shown in Figure 4.

From Theorem 3.1, we know that system (1.4) undergoes the Hopf bifurcation at the positive equilibrium E_* when $\tau = \hat{\tau}$. According to Theorem 3.2, it is easy to check that the condition $Re(c_1(0)) \neq 0$ is always satisfied by using mathematical software MATLAB, and we have

$$\mu_2 \approx 0.1931 > 0, \ \beta_2 \approx -0.6119 < 0, \ T_2 \approx -0.4826 < 0.$$

Moreover, from (4.2)-(4.3), the condition $Re(c_1(0)) \neq 0$ holds. Accordingly, from

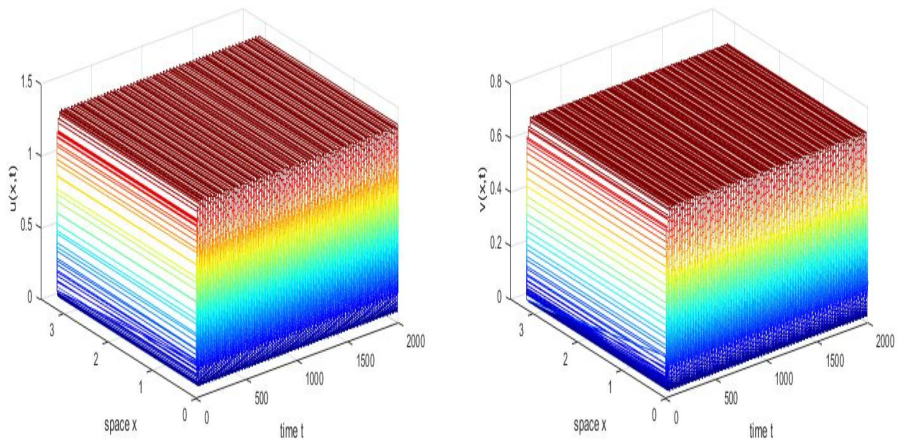

Figure 6. When $\alpha = 0.5$, the bifurcating periodic solutions are stable, where $\tau = 1.29$ and initial condition is $(u(x, 0), v(x, 0)) = (0.4624 + 0.5\cos(2x), 0.3656 + 0.5\cos(2x))$.

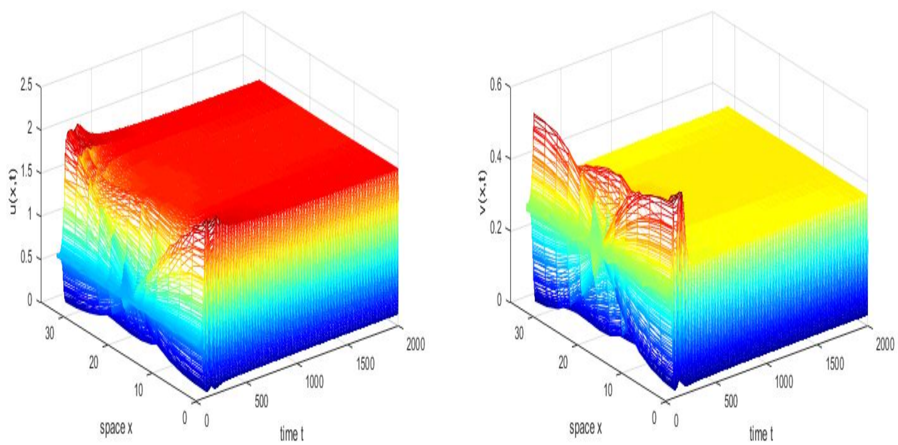

Figure 7. When $\alpha = 0.7$, the bifurcating periodic solutions are stable, where $\tau = 1.5$ and initial condition is $(u(x, 0), v(x, 0)) = (0.663 - 0.02\cos(2x), 0.2979 - 0.02\cos(2x))$.

(4.2), we can obtain

$$\mu_2 \approx 0.7496 > 0,\ \beta_2 \approx -2.7443 < 0,\ T_2 \approx -0.3466 < 0,$$

similarly, we can obtain the following results from (4.3),

$$\mu_2 \approx 0.0594 > 0,\ \beta_2 \approx -0.215 < 0,\ T_2 \approx -0.2822 < 0.$$

By setting parameters in (4.1), (4.2) and (4.3), we find that the delayed reaction-diffusion system (1.4) possesses a supercritical Hopf bifurcation with different values of parameter α. That is, when choosing $\alpha = 0.2$, $\alpha = 0.5$ and $\alpha = 0.7$ in conditions (4.1), (4.2) and (4.3), respectively, the stable spatially homogeneous periodic solutions will emerge with time increasing, see Figures 5, 6 and 7.

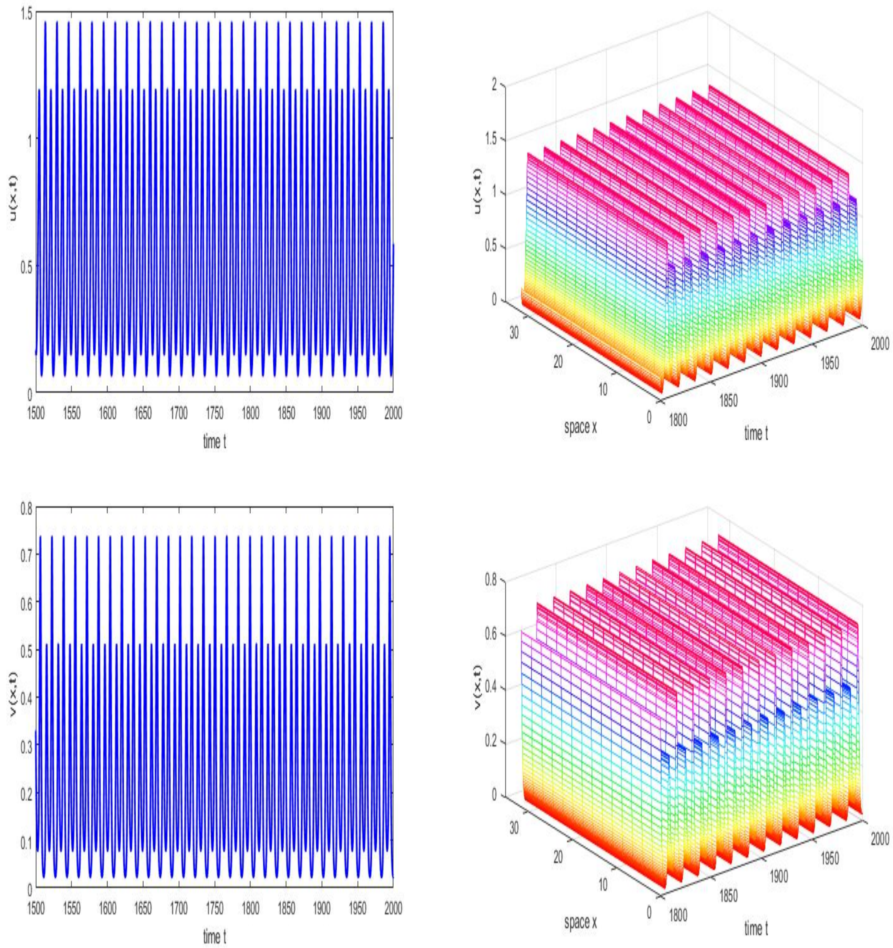

Figure 8. For system (1.4), there exist quasi periodic solution, where $\alpha = 0.2$, $\tau = 1.2942$ and initial condition is $(u(x,0), v(x,0)) = (0.4624 + 0.005\cos(5x), 0.3656 + 0.05\cos(5x))$.

In particular, when we let $n = 2$ and set the other parameters as in (4.2), then we obtain $\tau^* = \tau_2^{0+} = 1.2942$, our numerical simulation results indicate that there exist double periodic solutions induced by time delay in system (1.4), see Figure 8. In addition, when we choose

$$\alpha = 0.93,\ \beta = 0.6,\ c = 0.35,\ n = 2,\ l = 10,\ d_1 = 0.3,\ d_2 = 4,$$

from Theorem 3.2, we have $\tau^* = \tau_2^{0+} = 2.067$, and

$$\mu_2 \approx 0.094 > 0,\ \beta_2 \approx -0.3340 < 0,\ T_2 \approx 0.441 > 0.$$

As is shown in Figure 9, the delayed predator-prey system (1.4) undergoes a supercritical Hopf bifurcation and admits stable spatially inhomogeneous periodic solutions with time evolving.

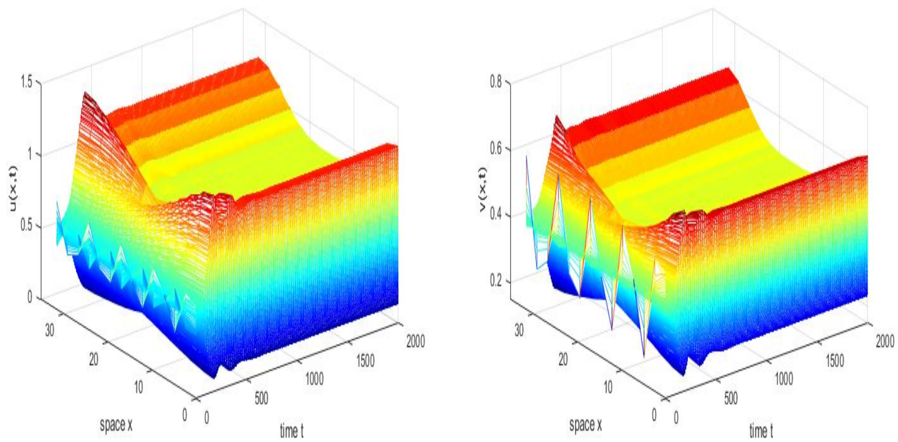

Figure 9. For system (1.4), the spatially inhomogeneous bifurcating periodic solution are stable, where $\alpha = 0.93,\ \tau = 2.08$ and initial condition is $(u(x,0), v(x,0)) = (0.5601 - 0.2\cos(2x), 0.4224 - 0.2\cos(2x))$.

5. Conclusion

In this paper, the delayed reaction-diffusion predator-prey system with general group defence mechanism for the prey species is considered. The stability of the positive equilibrium, the existence of the Hopf bifurcation are researched and its direction is investigated by the method of the first Lyapunov number when the spatial diffusion and time delay are absent. It is found that the supercritical Hopf bifurcation will occur in the ODEs. Then, the conditions for the occurrence of the Hopf bifurcation in the diffusion-driven delayed system are derived. The formulas that depend on different parameters and determine the direction and stability of the Hopf bifurcation are derived by using the center manifold reduction and the normal form theorem.

By setting different values of parameter α, our numerical results indicate that not only in non-delayed system but also in diffusion-driven delayed model, the aggregation efficiency α could induce instability, bifurcation and nonhomogeneous solutions. For the non-delayed model, the Hopf bifurcation not only happens in case of $\alpha = \frac{1}{2}$, but also it could still occur when taking $\alpha \neq \frac{1}{2}$, which is an interesting finding. Moreover, for the delayed model with spatial diffusion, when different values of parameter α are set, such as $\alpha = 0.2,\ 0.5,\ 0.7$, respectively, and other parameters as in (4.1)-(4.3), it is found that the delayed reaction-diffusion system (1.4) possesses a supercritical Hopf bifurcation and the stable spatially homogeneous periodic solutions will emerge with time increasing. Further, from the numerical simulations it is noted that there may exist double periodic solutions and stable spatially inhomogeneous periodic solution when $\tau^* = \tau_n^{0\pm}$ for $n \in N$, see Figures 8 and 9.

The aggregation efficiency α of prey has an impact on the population size of prey and predators. A small increase or decrease of the aggregation efficiency of prey does not affect coexistence of prey and predators, only causes the small shift of coexistence equilibrium point. From the critical values c_H and $\tau_n^{j\pm}$, we see that the aggregation efficiency could induce the Hopf bifurcations in the system. When time delay is present, switches of stability will also happen. The resulting periodic solutions correspond to the coexistence of prey and predators, that implies the balance between them. That could be beneficial for prey and predators. However, from the analysis, when the aggregation efficiency of prey $\alpha = 1$, no Hopf bifurcations will occur for the system (1.3). If the predator death rate c tends increasingly to β, the coexistence equilibrium point E_* will coincide with boundary equilibrium point $(1, 0)$, then the predators will tend to be extinct. Some more interesting and complex dynamical behaviors about this model will be further investigated.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (No.11971032, 11571016) and the research fund (No.2019QDZ25).

Appendix

The explicit expressions of g_{ij} as follows

$$
\begin{aligned}
g_{20} =& 2\bar{\sigma}\Big(\bar{q}_1^* a_{21} + \bar{q}_1^* q_2 a_{22} - \bar{q}_1^* e^{-i\omega^*\tau^*} + b_{21} + b_{22} q_2\Big),\\
g_{11} =& 2\bar{\sigma}\left(\bar{q}_1^* a_{21} + \bar{q}_1^* a_{22} Re(q_2) - \bar{q}_1^* Re(e^{i\omega^*\tau^*}) + b_{21} + b_{22} Re(q_2)\right),\\
g_{02} =& 2\bar{\sigma}\Big(\bar{q}_1^* a_{21} + \bar{q}_1^* \bar{q}_2 a_{22} - \bar{q}_1^* e^{i\omega^*\tau^*} + b_{21} + b_{22} \bar{q}_2\Big),\\
g_{30} =& 6\bar{\sigma}\Big[\bar{q}_1^* a_{21} W_{20}^{(1)}(0) + \bar{q}_1^* a_{22}\left(\frac{1}{2}W_{20}^{(1)}(0) q_2 + \frac{1}{2}W_{20}^{(2)}(0)\right)\\
& - \bar{q}_1^*\left(\frac{1}{2}W_{20}^{(1)}(0)e^{-i\omega^*\tau^*} + \frac{1}{2}W_{20}^{(1)}(-\tau)\right) + \bar{q}_1^* a_{31} + \bar{q}_1^* q_2 a_{32}\\
& + b_{21} W_{20}^{(1)}(0) + b_{22}\left(\frac{1}{2}W_{20}^{(1)}(0) q_2 + \frac{1}{2}W_{20}^{(2)}(0)\right) + b_{31} + b_{32} q_2\Big],\\
g_{21} =& 2\bar{\sigma}\Big[\bar{q}_1^* a_{21}(W_{20}^{(1)}(0) + 2W_{11}^{(1)}(0)) + 3b_{31} + b_{32}(\bar{q}_2 + 2q_2)\\
& + \bar{q}_1^* a_{22}\left(\frac{1}{2}W_{20}^{(1)}(0)\bar{q}_2 + W_{11}^{(1)}(0) q_2 + W_{11}^{(2)}(0) + \frac{1}{2}W_{20}^{(2)}(0)\right)\\
& - \bar{q}_1^*\left(\frac{1}{2}W_{20}^{(1)}(0)e^{i\omega^*\tau^*} + W_{11}^{(1)}(0)e^{-i\omega^*\tau^*} + W_{11}^{(1)}(-\tau) + \frac{1}{2}W_{20}^{(1)}(-\tau)\right)\\
& + 3\bar{q}_1^* a_{31} + \bar{q}_1^* a_{32}(\bar{q}_2 + 2q_2) + b_{21}\Big(W_{20}^{(1)}(0) + 2W_{11}^{(1)}(0)\Big)\\
& + b_{22}\left(\frac{1}{2}W_{20}^{(1)}(0)\bar{q}_2 + W_{11}^{(1)}(0) q_2 + W_{11}^{(2)}(0) + \frac{1}{2}W_{20}^{(2)}(0)\right)\Big]\\
g_{12} =& 2\bar{\sigma}\Big[\bar{q}_1^* a_{21}(W_{02}^{(1)}(0) + 2W_{11}^{(1)}(0)) + 3b_{31} + b_{32}(q_2 + 2\bar{q}_2)\\
& + \bar{q}_1^* a_{22}\left(\frac{1}{2}W_{02}^{(1)}(0) q_2 + W_{11}^{(1)}(0)\bar{q}_2 + W_{11}^{(2)}(0) + \frac{1}{2}W_{02}^{(2)}(0)\right)\\
& - \bar{q}_1^*\left(\frac{1}{2}W_{02}^{(1)}(0)e^{-i\omega^*\tau^*} + W_{11}^{(1)}(0)e^{i\omega^*\tau^*} + W_{11}^{(1)}(-\tau) + \frac{1}{2}W_{02}^{(1)}(-\tau)\right)\\
& + 3\bar{q}_1^* a_{31} + \bar{q}_1^* a_{32}(q_2 + 2\bar{q}_2) + b_{21}\Big(W_{02}^{(1)}(0) + 2W_{11}^{(1)}(0)\Big)
\end{aligned}
$$

$$+ b_{22}\left(\frac{1}{2}W_{02}^{(1)}(0)q_2 + W_{11}^{(1)}(0)\bar{q}_2 + W_{11}^{(2)}(0) + \frac{1}{2}W_{02}^{(2)}(0)\right)\Big]$$

$$\begin{aligned} g_{03} =& 6\bar{\sigma}\Big[\bar{q}_1^* a_{21} W_{02}^{(1)}(0) + \bar{q}_1^* a_{22}\left(\frac{1}{2}W_{02}^{(1)}(0)\bar{q}_2 + \frac{1}{2}W_{02}^{(2)}(0)\right) \\ &- \bar{q}_1^*\left(\frac{1}{2}W_{02}^{(1)}(0)e^{i\omega^*\tau^*} + \frac{1}{2}W_{02}^{(1)}(-\tau)\right) + \bar{q}_1^* a_{31} + \bar{q}_1^*\bar{q}_2 a_{32} \\ &+ b_{21}W_{02}^{(1)}(0) + b_{22}\left(\frac{1}{2}W_{02}^{(1)}(0)\bar{q}_2 + \frac{1}{2}W_{02}^{(2)}(0)\right) + b_{31} + b_{32}\bar{q}_2\Big]. \end{aligned}$$

References

[1] J. F. Andrews, *A mathematical model for the continuous culture of microorganisms utilizing inhibitory substrates*, Biotechnol. Bioeng., 1968, 10, 707–723.

[2] R. Arditi and L. R. Ginzburg, *Coupling in predator-prey dynamics: ratio-dependence*, J. Theor. Biol., 1989, 139, 311–326.

[3] V. Ajraldi, M. Pittavino and E. Venturino, *Modeling herd behavior in population systems*, Nonlinear Anal.: RWA, 2011, 12, 2319–2338.

[4] J. R. Beddington, *Mutual interference between parasites or predators and its effect on searching efficiency*, J. Anim. Ecol., 1975, 44, 331–340.

[5] J. Cao and R. Yuan, *Bifurcation analysis in a modified Lesile-Gower model with Holling type II functional response and delay*, Nonlinear Dyn., 2016, 84, 1341–1352.

[6] B. Chakraborty and N. Bairagi, *Complexity in a prey-predator model with prey refuge and diffusion*, Ecol. Complex., 2019, 37, 11–23.

[7] M. Chen, R. Wu, B. Liu and L. Chen, *Spatiotemporal dynamics in a ratio-dependent predator-prey model with time delay near the Turning-Hopf bifurcation point*, Commun. Nonlinear Sci. Numer. Simulat., 2019, 77, 141–167.

[8] S. Djilali, *Impact of prey herd shape on the predator-prey interaction*, Chaos Soliton Fract., 2019, 120, 139–148.

[9] H. I. Freefman and V. S. H. Rao, *The trade-off between mutual interference and time lag in predator-prey-systems*, Bull. Math. Biol., 1983, 45, 991–1004.

[10] W. Gao and B. Dai, *Dynamics of a predator-prey model with delay and fear effect*, J. Nonlinear Model. Anal., 2019, 1, 57–72.

[11] C. S. Holling, *The functional response of predators to prey density and its role in mimicry and population regulation*, Mem. Entomol. Soc. Can., 1965, 97, 1–60.

[12] B. D. Hassard, N. D. Kazarinoff and Y. Wan, *Theory and Applications of Hopf bifurcation*, Cambridge University Press, Cambridge-New York, 1981.

[13] R. Han and B. Dai, *Cross-diffusion induced Turing instability and amplitude equation for a toxic-phytoplankton-zooplankton model with nonmonotonic functional response*, Int. J. Bifurcat. Chaos, 2017, 27, 1750088.

[14] S. Kumar and H. Kharbanda, *Chaotic behavior of predator-prey model with group defense and non-linear harvesting in prey*, Chaos Soliton Fract., 2019, 119, 19–28.

[15] S. Kayan and H. Merdan, *An algorithm for Hopf bifurcation analysis of a delayed reaction-diffusion model*, Nonlinear Dyn., 2017, 89, 345–366.

[16] A. J. Lotka, *Elements of Physical Biology*, Williams and Wilkins, New York, 1925.

[17] Y. Lv, Y. Pei and R. Yuan, *Hopf bifurcation and global stability of a diffusive Gause-type predator-prey models*, Comput. Math. Appl., 2016, 72, 2620–2635.

[18] B. Liu, R. Wu and L. Chen, *Turing-Hopf bifurcation analysis in a superdiffusive predator-prey model*, Chaos, 2018, 28, 113118.

[19] D. Melchionda, E. Pastacaldi, C. Perria and M. Banerjee, *Social behavior-induced multistability in minimal competitive ecosystems*, J. Theor. Biol., 2018, 439, 24–38.

[20] Z. Ma, J. Liu and J. Yue, *Spatiotemporal dynamics induced by delay and diffusion in a predator-prey model with mutual interference among the predator*, Comput. Math. Appl., 2018, 75, 3488–3507.

[21] R. M. May, *Time-delay versus stability in population models with two and three trophic levels*, Ecology, 1973, 54, 315–325.

[22] L. Perko, *Differential Equations and Dynamical Systems. third edition*, Springer-Verlag, New York, 2001.

[23] S. Sun, C. Guo and C. Qin, *Dynamic behaviors of a modified predator-prey model with state-dependent impulsive effects*, Adv. Differ. Equ., 2016, 50.

[24] H. L. Smith, *An Introduction to Delay Differential Equations with Sciences Applications to the Life*, Springer, New York, 2010.

[25] X. Tang, H. Jiang and Z. Deng, *Delay induced subcritical Hopf bifurcation in a diffusive predator-prey model with herd behavior and hyperbolic mortality*, J. Appl. Anal. Comput., 2017, 7, 1385–1401.

[26] C. Tian and S. Ruan, *Pattern formation and synchronism in an allelopathic plankton model with delay in a network*, SIAM. J. Appl. Dyn. Syst., 2019, 18, 531–557.

[27] V. Volterra, *Fluctuations in the abundance of a species considered mathematically*, Nature, 1926, 118, 558–560.

[28] E. Venturino and S. Petrovskii, *Spatiotemporal behavior of a prey-predator system with a group defense for prey*, Ecol. Complex., 2013, 14, 37–47.

[29] F. Wei, C. Wang and S. Yuan, *Spatial dynamics of a diffusive predator-prey modek with Leslie-Gower functional response and strong Allee effect*, J. Nonlinear Model. Anal., 2020, 2, 267–285.

[30] R. Wu, M. Chen and B. Liu, *Hopf bifurcation and Turing instability in a predator-prey model with Michaelis-Menten functional response*, Nonlinear Dyn., 2018, 91, 2033–2047.

[31] J. Wu, *Theory and Applications of Partial Functional Differential Equations*, Springer Berlin, 1996.

[32] Z. Xu and Y. Song, *Bifurcation analysis of a diffusive predator-prey system with a herd behavior and quadratic mortality*, Math. Methods Appl. Sci., 2015, 38, 2994–3006.

[33] C. Xu, S. Yuan and T. Zhang, *Global dynamics of a predator-prey model with defense mechanism for prey*, Appl. Math. Lett., 2016, 62, 42–8.

[34] H. Yin and X. Wen, *Hopf bifurcation of a diffusive Gause-type predator-prey model induced by time fractional-order derivatives*, Math. Methods Appl. Sci., 2018, 41, 5178–5189.

[35] H. Zhu and X. Zhang, *Dynamics and patterns of a diffusive prey-predator system with a group defense for prey*, Discrete Dyn. Nat. Soc., 2018, 6519696.