Journal of Applied Analysis and Computation
Volume 11, Number 2, April 2021, 841–857
Website:http://www.jaac-online.com
DOI:10.11948/20200033

MULTIPLE POSITIVE SOLUTIONS OF THE DISCRETE DIRICHLET PROBLEM WITH ONE-DIMENSIONAL PRESCRIBED MEAN CURVATURE OPERATOR*

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Abstract We shall discuss the existence and multiplicity of positive solutions for the discrete Dirichlet problem with one-dimensional prescribed mean curvature operator. Based on the critical point theory, we shall show the existence of either one, or two, or three, or infinity many positive solutions depending on the asymptotic behavior of nonlinearity near zero.

Keywords Positive solutions, multiplicity, prescribed mean curvature operator, critical point theory.

MSC(2010) 34B15, 34B18.

1. Introduction

It's well known that the extrinsic mean curvature H_L of a graph $\Sigma_u = \{(x, u(x)) \,|\, x \in \Omega\}$ in flat Minkowski space $\mathbb{L}^{N+1}$ is given by

$$H_L = \frac{1}{N}\mathrm{div}\Big(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}\Big), \tag{1.1}$$

here $\mathbb{L}^{N+1} = \{(t, x) \in \mathbb{R} \times \mathbb{R}^N\}$ endowed with the Lorentzian metric $g_l = -dt^2 + \sum\limits_{i=1}^{N} dx_i^2$, the graph Σ_u is determined by a smooth function $u(x) : \Omega \subseteq \mathbb{R}^N \to \mathbb{R}$, which means the induced metric on it from the ambient space $\mathbb{L}^{N+1}$ is Riemannian, i.e. the norm of the gradient of u is less than 1. Calabi [8] proved that the only entire spacelike graphs(the case $\Omega = \mathbb{R}^N$) in $\mathbb{L}^{N+1}$, $N \leq 4$, with vanishing mean curvatures are necessarily the hyperplanes. Later, Cheng and Yau [10] extend Calabi's theorem to arbitrary dimension. Subsequently, this kind of problem has attracted the research of many scholars, such as Mawhin and Bereanu([3,5,6]), Torres([3]), Jebelean([11,15,16]), Obersnel and Omairi([12,13]) and so on.

Since the equation (1.1) generally admits the null solution, it may have some interest to study the existence of non-trivial (positive) solutions. Thus the existence

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*The authors were supported by National Natural Science Foundation of China (Nos. 11901464, 11801453, 11671322) and the Young Teachers' Scientific Research Capability Upgrading project of Northwest Normal University(NWNU-LKQN2020-20).

and multiplicity of (positive) solutions for the Dirichlet problem of prescribed mean curvature equations arising from (1.1) and the discrete analogue have been widely discussed by various methods, such as the method of lower and upper solutions([5]) and topological degree theory (see [2, 3, 6, 12, 20]) and critical point theory ([9, 12, 13, 26]), see [2, 5, 6, 9, 13, 15, 16, 21, 26] and the references therein.

Motivated it, we focus on the study of the existence and multiplicity of positive solutions for the discrete Dirichlet problem with one-dimensional prescribed mean curvature operator

$$\begin{aligned} & -\nabla\Big(\frac{\Delta u(k)}{\sqrt{1-\kappa(\Delta u(k))^2}}\Big) = f(k,u(k)), \quad k\in\{1,2,\cdots,N\} =: [1,N]_{\mathbb{Z}}, \\ & u(0)=u(N+1)=0, \end{aligned} \tag{1.2}$$

where $\kappa > 0$ is a constant, Δ is the forward difference operator with $\Delta u(k) = u(k+1)-u(k)$ and ∇ is the backward difference operator with $\nabla u(k) = u(k)-u(k-1)$, and $f \in C([1,N]_{\mathbb{Z}} \times [0,\infty), \mathbb{R})$. An interesting question is which techniques and theorems regarding the continuous differential equations can be adapted for difference equations. Many literatures discuss this question and give some common points and differences between the continuous case and its discrete analogue, see for example, see Kelly and Peterson [17], Agarwal and O'Regan [1], Bereanu and Mawhin [4], Elsayed et al. [14], Cabada et al. [7]).

It is worthy pointing out that when $\kappa = 0$, the problem (1.2) is degenerated to the classical second-order discrete boundary value problem

$$\begin{aligned} & -\nabla(\Delta u(k)) = f(k,u(k)), \quad k\in[1,N]_{\mathbb{Z}}, \\ & u(0)=u(N+1)=0. \end{aligned} \tag{1.3}$$

The equation (1.3) with other boundary value conditions has been discussed by Agarwal et al. [1], Bereanu and Mawhin [4], Cabada and Dimitrov [7], Zhang and Liu [25], Yu et al. [24], Luca [18, 19] and the references therein. Especially, by using Minimax principle and Mountain Pass Lemma in critical point theory, they proved the existence and multiplicity of positive solutions of (1.3) under the suitable conditions for the nonlinear term f, see [24, 25].

When $\kappa = -1$, Zhou and Ling [26] obtained infinitely many positive solutions for the second order nonlinear difference equation

$$\begin{aligned} & -\nabla\Big(\frac{\Delta u(k)}{\sqrt{1+(\Delta u(k))^2}}\Big) = \lambda f(k,u(k)), \quad k\in[1,T]_{\mathbb{Z}}, \\ & u(0)=u(T+1)=0. \end{aligned}$$

When $\kappa = 1$, Chen et al. [9] gave the existence and multiplicity of positive solutions for the following discrete boundary value problem

$$\begin{aligned} & -\nabla\Big(\frac{\Delta u(k)}{\sqrt{1-(\Delta u(k))^2}}\Big) = \lambda\mu(k)u^q(k), \quad k\in[1,N]_{\mathbb{Z}}, \\ & \Delta u(1)=u(N+1)=0, \end{aligned}$$

where $\lambda > 0$, $q > 1$, $N > 3$. They proved that there exists a constant Λ, such that this problem does not have a positive solution for $\lambda \in (0,\Lambda)$, has at least one positive solutions for $\lambda = \Lambda$ and has at least two positive solutions for $\lambda \in (\Lambda,\infty)$. Motivated by the above papers, we establish the existence and multiplicity of positive solutions of the problem (1.2) by variational methods, which develops the main results of [1, 4, 9, 16, 24, 25].

Clearly, (1.2) can be rewritten as

$$-\nabla(\phi(\Delta u(k))) = f(k, u(k)), \quad k \in [1, N]_{\mathbb{Z}}, \qquad u(0) = u(N+1) = 0, \tag{1.4}$$

where $\phi : (-\frac{1}{\sqrt{\kappa}}, \frac{1}{\sqrt{\kappa}}) \to \mathbb{R}$ defined by

$$\phi(s) = \frac{s}{\sqrt{1-\kappa s^2}}. \tag{1.5}$$

Clearly, $\phi^{-1}(y) = \frac{y}{\sqrt{1+\kappa y^2}}$ is globally Lipschitz in $\mathbb{R}$. Since ϕ is singular at $\pm\frac{1}{\sqrt{\kappa}}$, we can remove the singularity of ϕ by the equivalent transformation of problem (1.2) into the following problem

$$-\nabla(\psi(\Delta u(k))) = \tilde{f}(k, u(k)), \; k \in [1, N]_{\mathbb{Z}}, \qquad u(0) = u(N+1) = 0, \tag{1.6}$$

where ψ is an asymptotically linear increasing homeomorphism from $\mathbb{R}$ to $\mathbb{R}$ such that $\psi(s) = \frac{s}{\sqrt{1-\kappa s^2}}$ near $s = 0$, the function $\tilde{f}$ is bounded, actually vanishes outside the rectangle $[0, N+1]_{\mathbb{Z}} \times [-\frac{N+1}{2\sqrt{\kappa}}, \frac{N+1}{2\sqrt{\kappa}}]$, and $\tilde{f} = f$ in $[0, N+1]_{\mathbb{Z}} \times [0, \frac{N+1}{2\sqrt{\kappa}}]$. Hence, to obtain the existence of positive solutions of (1.2), we only need to establish the existence of positive solutions of the discrete Dirichlet problem (1.6).

The rest of the paper is organized as follows. In Section 2, we introduce some basic results on variational method for the functional defined on a real Banach space. In Section 3, we will first establish the variational framework of (1.2) and transfer the existence of positive solutions of (1.2) into the existence of critical points of the corresponding functional. Next we will prove the existence results of either one, or two, or three positives solutions. In Section 4, we prove the existence of infinity many positive solutions of (1.2) under suitable oscillatory assumptions at zero on the nonlinearity f via a variational principle by Ricceri [23, Theorem 2.5].

2. Preliminary results

For convenience, we list a few notations that will be used throughout this paper. Set $\mathbb{R}^+ = (0, \infty)$. Let $a, b \in \mathbb{Z}$ with $a < b$, denote $[a, b]_{\mathbb{Z}} = \{a, a+1, \cdots, b\}$, and $\sum\limits_{k=a}^{b} u(k) = 0$, $\prod\limits_{k=a}^{b} u(k) = 1$ with $b < a$. For $\mathbf{u} \in \mathbb{R}^N$, set

$$\|\mathbf{u}\|_\infty = \max_{k\in[1,N]_{\mathbb{Z}}} |u(k)|, \qquad \|\mathbf{u}\|_1 = \sum_{k=1}^{N} |u(k)|, \qquad \|\mathbf{u}\| = \Big(\sum_{k=1}^{N} u^2(k)\Big)^{\frac{1}{2}}.$$

It is easy to verify that the norms $\|\cdot\|_\infty$, $\|\cdot\|_1$, $\|\cdot\|$ are equivalent. Let $\|\Delta\mathbf{u}\|_\infty = \max\limits_{k\in[0,N]_{\mathbb{Z}}} |\Delta u(k)|$, $\|\Delta\mathbf{u}\|_1 = \sum_{k=0}^{N} |\Delta u(k)|$, $\|\Delta\mathbf{u}\| = \Big(\sum_{k=0}^{N} (\Delta u(k))^2\Big)^{\frac{1}{2}}$. Denote $\mathbf{u}^+ = \max\{\mathbf{u}, 0\}$, $\mathbf{u}^- = \min\{\mathbf{u}, 0\}$.

Let E be the class of function $u : [0, N+1]_{\mathbb{Z}} \to \mathbb{R}$ such that $u(0) = u(N+1) = 0$ with the usual inner product and the usual norm

$$(\mathbf{u}, \mathbf{v}) = \sum_{k=1}^{N} (u(k), v(k)), \qquad \|u\| = \Big(\sum_{k=1}^{N} u^2(k)\Big)^{\frac{1}{2}},$$

respectively. Then E is an N-dimensional Hilbert space. Furthermore, E is isomorphic to $\mathbb{R}^N$, its elements can be associated to the coordinates $(u(1), u(2), \cdots, u(N))$.

Lemma 2.1 (Discrete Poincaré inequality, see Lemma 3, [22])**.** *Let $a < b$ be two given integers, $w \in \mathbb{R}^{b-a+1}$ with $w(a) = 0$. Then*

$$\sum_{i=a}^{b} w^2(i) \leq \frac{(b-a)^2}{2} \sum_{j=a}^{b-1} (w(j+1) - w(j))^2.$$

Lemma 2.2 (Theorem 2.5, [23])**.** *Let X be a Hilbert space, $\Phi, \Psi : X \to \mathbb{R}$ two sequentially weakly lower semicontinuous, continuously Gâteaux differentiable functionals. Assume that Ψ is strongly continuous and coercive. For each $\rho > \inf_X \Psi$, set*

$$\varphi(\rho) := \inf_{\Psi^{-1}((-\infty,\rho))} \frac{\Phi(u) - \inf_{\overline{\Psi^{-1}((-\infty,\rho))}^w} \Phi}{\rho - \Psi(u)},$$

where $\Psi^{-1}((-\infty, \rho)) := \{u \in X : \Psi(u) < \rho\}$ and $\overline{\Psi^{-1}((-\infty, \rho))}^w$ is its closure in the weak topology of X. Furthermore, set

$$\delta := \liminf_{\rho \to (\inf_X \Psi)^+} \varphi(\rho).$$

If $\delta < +\infty$, then for every $\lambda > \delta$, either $\Phi + \lambda\Psi$ posseses a local minimum, which is also a global minimum of Ψ, or there is a sequence $\{u_k\}$ of pairwise distinct critical points of $\Phi + \lambda\Psi$, with $\lim_{k\to\infty} \Psi(u_k) = \inf_X \Psi$, weakly converging to a global minimum of Ψ.

3. Existence and multiplicity of positive solutions

We discuss the existence and multiplicity of positive solutions for problem (1.2). Assume that

(H1) $f : [1, N]_{\mathbb{Z}} \times \mathbb{R} \to \mathbb{R}$ is continuous.

(H2) $f(k, 0) \geq 0$ for $k \in [1, N]_{\mathbb{Z}}$.

(H3) there exist $a, b \in \mathbb{Z}$ with $1 \leq a < b \leq N$ such that $\liminf_{s\to 0+} \frac{F(k,s)}{s^2} > -\infty$ for $k \in [a, b]_{\mathbb{Z}}$, where $F(k, u) = \int_0^u f(k, s)ds$.

(H4) there exist $c, d \in \mathbb{Z}$ with $a < c < d < b$ such that $\limsup_{s\to 0+} \sum_{k=c}^{d} \frac{F(k,s)}{s^2} = +\infty$.

3.1. An equivalent formulation

Let us define $\tilde{f} : [1, N]_{\mathbb{Z}} \times \mathbb{R} \to \mathbb{R}$ as follows:

$$\tilde{f}(k, s) = \begin{cases} f(k, 0)(\frac{\sqrt{\kappa}}{N+1}s + 1), & \text{if } -\frac{N+1}{\sqrt{\kappa}} < s < 0, \\ f(k, s), & \text{if } 0 \leq s \leq \frac{N+1}{2\sqrt{\kappa}}, \\ f(k, \frac{N+1}{2\sqrt{\kappa}})(2 - \frac{2\sqrt{\kappa}}{N+1}s), & \text{if } \frac{N+1}{2\sqrt{\kappa}} < s < \frac{N+1}{\sqrt{\kappa}}, \\ 0, & \text{if } |s| \geq \frac{N+1}{\sqrt{\kappa}}. \end{cases} \tag{3.1}$$

Clearly, $\tilde{f}(k,\cdot)$ is continuous. Then looking for the positive solution of (1.2) is equivalent to looking for the positive solution of the same problem with f replaced by $\tilde{f}$. In fact, if u is a positive solution of (1.2), then $\|\Delta u\|_\infty < \frac{1}{\sqrt{\kappa}}$ and hence $\|u\|_\infty < \frac{N+1}{2\sqrt{\kappa}}$. In the following context, we shall replace f with $\tilde{f}$ and such a function satisfies all the assumptions (H1)-(H4). From (H1), for each $r>0$, there exists $\gamma \in \mathbb{R}^N$, such that $f(k,s) \le \gamma(k)$ for every $s \in [-r,r]$ and $k \in [1,N]_{\mathbb{Z}}$. Furthermore, by (H1), there exists $\gamma \in \mathbb{R}^N$ such that

$$|\tilde{f}(k,s)| \le \gamma(k), \qquad \text{for } k \in [1,N]_{\mathbb{Z}} \text{ and } s \in \mathbb{R}. \tag{3.2}$$

Set $\sigma = \phi'(\phi^{-1}(\|\gamma\|_1))$ and for $v \in \mathbb{R}$, define

$$\psi(v) = \begin{cases} \sigma(v+\phi^{-1}(\|\gamma\|_1)) - \|\gamma\|_1, & \text{if } v < -\phi^{-1}(\|\gamma\|_1), \\ \phi(v), & \text{if } |v| \le \phi^{-1}(\|\gamma\|_1), \\ \sigma(v-\phi^{-1}(\|\gamma\|_1)) + \|\gamma\|_1, & \text{if } v > \phi^{-1}(\|\gamma\|_1). \end{cases} \tag{3.3}$$

Set

$$\Psi(v) = \int_0^v \psi(s)ds, \tag{3.4}$$

and observe that

$$\frac{1}{2}v^2 \le \Psi(v) \le \frac{1}{2}\sigma v^2 \text{ for any } v \in \mathbb{R}. \tag{3.5}$$

We claim that *a function $u \in E$ is a positive solution of* (1.2) *if and only if it is a positive solution of the problem*

$$-\nabla\big(\psi(\Delta u(k))\big) = \tilde{f}(k,u(k)), \quad k \in [1,N]_{\mathbb{Z}}, \qquad u(0)=u(N+1)=0. \tag{3.6}$$

Suppose that u is a positive solution of (1.2). Then it follows from $u(0) = u(N+1) = 0$ and $u(k) > 0, k \in [1,N]_{\mathbb{Z}}$ that there at least exists $k_0 \in [1,N]_{\mathbb{Z}}$ such that $\Delta u(k_0-1) \ge 0 \ge \Delta u(k_0)$. If $k_0 < k \le N$, summing the equation in (1.2) between $i=k_0$ and k, we obtain that $-\phi(\Delta u(k)) = -\phi(\Delta u(k_0-1)) + \sum\limits_{i=k_0}^{k} \tilde{f}(i,u(i))$. If $1 \le k < k_0$, summing the equation in (1.2) between $i=k+1$ and k_0, we obtain that $\phi(\Delta u(k)) = \phi(\Delta u(k_0)) + \sum\limits_{i=k+1}^{k_0} \tilde{f}(i,u(i))$. Thus,

$$|\phi(\Delta u(k))| \le \sum_{i=1}^{N} |\tilde{f}(i,u(i))| \le \|\gamma\|_1, \ k \in [1,N]_{\mathbb{Z}},$$

and $|\Delta u(k)| \le \phi^{-1}(\|\gamma\|_1)$. Therefore $\phi(\Delta u(k)) = \psi(\Delta u(k))$ in $[1,N]_{\mathbb{Z}}$ and we conclude that u is a positive solution of (3.6).

Suppose that u is a positive solution of (3.6). Arguing as above we see that $|\Delta u(k)| \le \psi^{-1}(\|\gamma\|_1)$, from (3.3), $\psi(\Delta u(k)) = \phi(\Delta u(k))$ in $[1,N]_{\mathbb{Z}}$. In particular $\|\Delta u\|_\infty < \frac{1}{\sqrt{\kappa}}$ and we conclude that u is a positive solution of (1.2).

3.2. Existence of a positive solution

Theorem 3.1. *Suppose that (H1)-(H4) hold. Then problem* (1.2) *has at least one positive solution.*

Proof. It's easy to verify that a function $u \in E$ is a positive solution of (1.2) if and only if it is a positive solution of the problem (3.6). In the sequel of the proof we shall replace f with $\tilde{f}$. However, for the sake of simplicity in the notation, the modified function $\tilde{f}$ will still be denoted by f.

Define the functional $\mathcal{J}_1 : E \to \mathbb{R}$ as

$$\mathcal{J}_1(v) = \sum_{k=1}^{N+1} \Psi(\Delta v(k-1)) - \sum_{k=1}^{N} F(k, v). \tag{3.7}$$

Then $\mathcal{J}_1$ is $C^1(E, \mathbb{R})$ and weakly lower semicontinuous. For any $u \in E$, by using $u(0) = u(N+1) = 0$ and (3.7), we can compute the Fréchet derivative as

$$\begin{aligned}
&\frac{\partial \mathcal{J}_1(u)}{\partial u}\\
&= \lim_{t\to 0} \frac{\mathcal{J}_1(u+th) - \mathcal{J}_1(u)}{t}\\
&= \lim_{t\to 0} \frac{\sum\limits_{k=1}^{N+1}[\Psi(\Delta(u(k-1)+th(k-1))) - \Psi(\Delta u(k-1))] - \sum\limits_{k=1}^{N}[F(k, u(k+th(k)) - F(k, u(k))]}{t}\\
&= \lim_{t\to 0} \frac{\sum\limits_{k=1}^{N+1}\Psi'(\Delta(u(k-1)+\theta th(k-1)))\Delta h(k-1)t - \sum\limits_{k=1}^{N} F'(k, u(k+\theta th(k)))h(k)t}{t} \quad \big(\theta \in [0,1)\big)\\
&= \sum_{k=1}^{N} \psi(\Delta(u(k-1)))\Delta h(k-1) - f(k, u(k))h(k)\\
&= -\sum_{k=1}^{N} [\nabla\psi(\Delta(u(k))) + f(k, u(k))]h(k),
\end{aligned}$$

since $h \in E$. Thus u is a critical point of $\mathcal{J}_1$ on E (that is, $\mathcal{J}_1'(u) = 0$) if and only if $-\nabla\psi(\Delta u(k)) = f(k, u_k)$. By (3.2), there exists a constant $c_f > 0$ such that

$$\sum_{s=1}^{N} F(k, v) \le c_f \quad \text{for all } v \in E.$$

It follows from (3.5) that $\mathcal{J}_1$ is coercive and bounded from below.

Therefore, there exists $u \in E$ such that

$$\mathcal{J}_1(u) = \min_{v\in E} \mathcal{J}_1(v).$$

It is easy to see that $u \in E$ and u is a solution of problem (3.6). To check that $u \ge 0$, we test the equation in (3.6) against u^-. It concludes from (H2) that

$$\sum_{k=1}^{N} \psi(\Delta u_k^-)\Delta u^-(k) \le 0,$$

which yields $u^- = 0$ by the monotonicity of ψ. Finally, we verify that $u \ne 0$. Let $\chi \in E$ with $0 \le \chi \le 1$, $\chi(k) = 0$ for $k \in [0, a-1]_{\mathbb{Z}} \cup [b+1, N+1]_{\mathbb{Z}}$ and $\chi(k) = 1$

for $k \in [a,b]_{\mathbb{Z}}$. From (H3) and (H4), there exist a constant $K > 0$ and a strictly decreasing sequence $\{x_n\}$ satisfying

$$\lim_{n\to+\infty} x_n = 0, \ \lim_{n\to+\infty} \sum_{k=c}^{d} \frac{F(k,x_n)}{x_n^2} = +\infty,$$
$$F(k, x_n\chi(k)) \geq -K x_n^2 \chi^2(k) \qquad \text{for } k \in [a,b]_{\mathbb{Z}} \text{ and all } n \geq 1. \tag{3.8}$$

By using (3.5), it is easy to compute that

$$\begin{aligned}\mathcal{J}_1(x_n\chi) &= \sum_{k=1}^{N+1} \Psi(x_n \Delta\chi(k-1)) - \sum_{k=1}^{N} F(k, x_n\chi)\\ &\leq x_n^2\Big(\frac{1}{2}\sigma \sum_{k=1}^{N+1} (\Delta\chi(k-1))^2 - \sum_{k=c}^{d} \frac{F(k,x_n)}{x_n^2} + K \sum_{k=1}^{N} \chi^2(k)\Big)\\ &= x_n^2\Big(\frac{1}{2}\sigma \|\Delta\chi\|^2 - \sum_{k=c}^{d} \frac{F(k,x_n)}{x_n^2} + K\|\chi\|^2\Big).\end{aligned}$$

Hence, we infer

$$\mathcal{J}_1(u) \leq \mathcal{J}_1(x_n\chi) < 0 \qquad \text{for all large enough } n,$$

implying $u \neq 0$.

Hence, u is a positive solution of (3.6), i.e. u is a positive solution of (1.2). □

Example 3.1. Let $m, n : [1,N]_{\mathbb{Z}} \to \mathbb{R}$ with $m^+ > 0$. Then from Theorem 3.1, the problem

$$-\nabla\Big(\frac{\Delta u(k)}{\sqrt{1-\kappa(\Delta u(k))^2}}\Big) = m(k)u^p(k) + n(k)u^q(k), \ k \in [1,N]_{\mathbb{Z}}, \ u(0) = u(N+1) = 0$$

has a positive solution, where $p \in (0,1)$ and $q \in (1,+\infty)$.

3.3. Existence of multiple positive solutions

Theorem 3.2. *Suppose that*

(H5) $g : [1,N]_{\mathbb{Z}} \times \mathbb{R} \to \mathbb{R}$ *is continuous and set* $G(k,s) = \int_0^s g(k,\xi)d\xi$*;*

(H6) there exists $\omega \in E$ *with* $\omega > 0$ *and* $\|\Delta\omega\|_\infty < \frac{1}{\sqrt{\kappa}}$ *such that* $\sum_{k=1}^{N} G(k,\omega) > 0$*;*

(H7) $\limsup_{s\to 0^+} \frac{G(k,s)}{s^2} \leq 0$ *for* $k \in [1,N]_{\mathbb{Z}}$*;*

(H8) $g(k,0) = 0$ *for* $k \in [1,N]_{\mathbb{Z}}$.

Then there exists $\mu^* > 0$ *such that the problem*

$$-\nabla\Big(\frac{\Delta u(k)}{\sqrt{1-\kappa(\Delta u(k))^2}}\Big) = \mu g(k,u(k)), \ k \in [1,N]_{\mathbb{Z}}, \quad u(0) = u(N+1) = 0 \tag{3.9}$$

has at least two positive solutions for all $\mu > \mu^*$.

Proof. By similar argument of the equivalent formulation of (1.2) in Section 3.1, we can replace g with the function $\tilde{g} : [1, N]_{\mathbb{Z}} \times \mathbb{R} \to \mathbb{R}$ defined by

$$\tilde{g}(k,s) = \begin{cases} g(k,s), & \text{if } 0 \le s \le \frac{N+1}{2\sqrt{\kappa}}, \\ g(k, \frac{N+1}{2\sqrt{\kappa}})(2 - \frac{2\sqrt{\kappa}}{N+1}s), & \text{if } \frac{N+1}{2\sqrt{\kappa}} < s < \frac{N+1}{\sqrt{\kappa}}, \\ 0, & \text{if } s \ge \frac{N+1}{\sqrt{\kappa}} \text{ or } s \le 0, \end{cases} \tag{3.10}$$

then $g(k, \cdot)$ is continuous and observe once more that, with the context of positive solutions of (3.9), to look for the solutions of problem (3.9) is equivalent to look for the solutions of the same problem with g replaced by $\tilde{g}$. In the sequel of the proof we shall replace g with $\tilde{g}$. However, for the sake of simplicity in the notation, the modified function $\tilde{g}$ will still be denoted by g. Moreover, such a function satisfies all the properties assumed in the statement of the theorem.

Let $\omega \in E$ be the function with the properties described in (H6) and let μ^* be such that

$$\sum_{k=1}^{N+1} \Phi(\Delta\omega(k-1)) - \mu^* \sum_{k=1}^{N} G(k, \omega) = 0, \tag{3.11}$$

where $\Phi(y) = \int_0^y \phi(\xi) d\xi$ for any $y \in (-\frac{1}{\sqrt{\kappa}}, \frac{1}{\sqrt{\kappa}})$ and ϕ defined by (1.5). Fixed $\mu > \mu^*$, it follows from (H5) that for each $r > 0$, there exists $\gamma \in \mathbb{R}^N$ such that $|g(k,s)| \le \gamma(k)$ for $k \in [1, N]_{\mathbb{Z}}$ and every $s \in [-r, r]$. Moreover, from (3.10), there exists $\gamma \in \mathbb{R}^N$ such that

$$\mu|g(k,s)| \le \gamma(k) \quad \text{for } k \in [1, N]_{\mathbb{Z}} \text{ and } s \in \mathbb{R}.$$

This together with (3.9),(3.11) and the argument of Section 3.1 implies that

$$\phi(\|\Delta\omega\|_\infty) < \|\gamma\|_1.$$

Define ψ as in (3.3), Ψ as in (3.4), and $\mathcal{J}_\mu : E \to \mathbb{R}$ by setting

$$\mathcal{J}_\mu(u) = \sum_{k=1}^{N+1} \Psi(\Delta u(k-1)) - \mu \sum_{k=1}^{N} G(k, u).$$

It's not difficult to verify that $\mathcal{J}_\mu$ is C^1 and weakly lower semicontinuous. Moreover, it is coercive and bounded from below.

Let $u_1 \in E$ be such that

$$\mathcal{J}_\mu(u_1) = \min_{u \in E} \mathcal{J}_\mu(u)$$

and by (3.11), it concludes that

$$\mathcal{J}_\mu(u_1) < 0. \tag{3.12}$$

Notice that $u_1 \in E$ is a nontrivial solution of the problem

$$-\nabla\big(\psi(\Delta u(k))\big) = \mu g(k, u(k)), \ k \in [1, N]_{\mathbb{Z}}, \quad u(0) = u(N+1) = 0. \tag{3.13}$$

By using the condition $g(k,s) \le 0$ for $s \le 0$, $k \in [1, N]_{\mathbb{Z}}$, and arguing as in the proof of Theorem 3.1, we see that any solution u of (3.13) satisfies $u \ge 0$. In particular

u_1 is a positive solution of (3.13). A second solution u_2 can be found using the mountain pass theorem (see e.g. [27]). Note that the coercivity of $\mathcal{J}_\mu$ implies that the Palais-Smale condition holds. Let us check that the functional has a mountain pass geometry near the origin. Take $\varepsilon > 0$ such that

$$\frac{1}{N^2} - \mu\varepsilon > 0.$$

From (H7), there exists r such that $0 < r < \|u_1\|$ and

$$G(k, s) \le \varepsilon s^2 \quad \text{for } k \in [1, N]_{\mathbb{Z}},\ s \in [0, r].$$

For any $v \in E$, by Cauchy-Schwarz inequality, we get that

$$\|v\|_\infty \le \sqrt{N}\|v\|. \tag{3.14}$$

Therefore, for all $v \in E$ with $0 < \|v\| \le \frac{r}{\sqrt{N}}$, by using (3.5) and Lemma 2.1, we have that

$$\begin{aligned} \mathcal{J}_\mu(v) &= \sum_{k=1}^{N+1} \Psi(\Delta v(k-1)) - \mu \sum_{k=1}^{N} G(k, v) \\ &\ge \frac{1}{2}\sum_{k=1}^{N+1} (\Delta v(k-1))^2 - \mu\varepsilon \sum_{k=1}^{N} v^2(k) \ge \|v\|^2 (\frac{1}{N^2} - \mu\varepsilon) > 0. \end{aligned}$$

Since (3.12) also holds, it yields that the functional $\mathcal{J}_\mu$ has a critical point u_2, with $\mathcal{J}_\mu(u_2) > 0$. Thus, u_2 is a positive solution of (3.13), which is different from u_1. By the claim in Section 3.1, we conclude that u_1 and u_2 are actually solutions of problem (3.9). □

Example 3.2. Let $q \in (1, +\infty)$, and $n : [1, N]_{\mathbb{Z}} \to \mathbb{R}$ with $n^+ > 0$. Then by using Theorem3.2, we get that there exists $\mu^* > 0$, such that for any $\mu > \mu^*$, the problem

$$-\nabla\Big(\frac{\Delta u(k)}{\sqrt{1 - \kappa(\Delta u(k))^2}}\Big) = \mu n(k) u^q(k),\ k \in [1, N]_{\mathbb{Z}}, \quad u(0) = u(N+1) = 0$$

has at least two positive solutions.

Now we give a result about the existence of at least three positive solutions of the following two-parameter problem

$$\begin{aligned} &-\nabla\Big(\frac{\Delta u(k)}{\sqrt{1 - \kappa(\Delta u(k))^2}}\Big) = \lambda f(k, u(k)) + \mu g(k, u(k)), \quad k \in [1, N]_{\mathbb{Z}}, \\ &u(0) = u(N+1) = 0. \end{aligned} \tag{3.15}$$

Theorem 3.3. *Let (H1-H8) and*

(H9) $\liminf_{s \to 0+} \frac{G(k,s)}{s^2} > -\infty$ *for* $k \in [a, b]_{\mathbb{Z}}$ *with* a, b *defined in (H2).*

Then there exists $\mu^* > 0$ *and a function* $\lambda : (\mu^*, +\infty) \to \mathbb{R}^+ \cup \{+\infty\}$ *such that the problem* (3.15) *has at least three positive solutions for all* $\mu > \mu^*$ *and for all* $\lambda \in (0, \lambda(\mu))$.

Proof. Like in the proofs of Theorem 3.1 and Theorem 3.2, we replace f and g with functions $\tilde{f}$ and $\tilde{g}$, we will still denote by f and g, which satisfy all assumptions of the theorem, agree with the original functions in $[1,N]_{\mathbb{Z}}\times[0,\frac{N+1}{2\sqrt{\kappa}})$ and such that $f(k,s)\geq 0$ and $g(k,s)=0$ for any $k\in[1,N]_{\mathbb{Z}}$ and $s\leq 0$. Thus, there exist constants $c_f, c_g>0$ such that

$$\sum_{s=1}^{N}|F(s,v)|\leq c_f \qquad \text{and} \qquad \sum_{s=1}^{N}|G(s,v)|\leq c_g \ \text{ for all } \ v\in E.$$

The proof will follow closely the lines of the proof of Theorem 3.2. The properties of g yield that a first solution as a global minimizer and a second solution as a mountain pass critical point for large μ. Next, for small λ, the properties of f produce an additional local minimum point close to the origin.

As in the proof of Theorem 3.2, let $\omega\in E$ be the function with the properties described in (H6) and let μ^* be such that

$$\sum_{k=1}^{N+1}\Phi(\Delta\omega(k-1))-\mu^*\sum_{k=1}^{N}G(k,\omega)+2c_f=0. \tag{3.16}$$

Fixed $\mu>\mu^*$, it follows from (H1) and (H5) that there exists $\gamma\in\mathbb{R}^N$, such that

$$|f(k,s)|+\mu|g(k,s)|\leq\gamma(k) \qquad \text{for any } k\in[1,N]_{\mathbb{Z}} \text{ and } s\in\mathbb{R}.$$

From (3.15),(3.16) and the argument of Section 3.1, we have that $\phi(\|\Delta\omega\|_\infty)<\|\gamma\|_1$. We define ψ as in (3.3), Ψ as in (3.4) and for all $\lambda>0$, the functional $\mathcal{J}_{\lambda,\mu}:E\to\mathbb{R}$ by setting

$$\mathcal{J}_{\lambda,\mu}(v)=\sum_{k=1}^{N+1}\Psi(\Delta v(k-1))-\lambda\sum_{k=1}^{N}F(k,v)-\mu\sum_{k=1}^{N}G(k,v).$$

Then $\mathcal{J}_{\lambda,\mu}$ is C^1 and weakly lower semicontinuous and coercive. In particular, $\mathcal{J}_{\lambda,\mu}$ satisfies the Palais-Smale condition. Consequently, for each $\lambda>0$, there exists $u_1\in E$ such that

$$\mathcal{J}_{\lambda,\mu}(u_1)=\min_{v\in E}\mathcal{J}_{\lambda,\mu}(v).$$

Clearly, $u_1\in E$ is a solution of the problem

$$\begin{aligned}&-\nabla\big(\psi(\Delta u(k))\big)=\lambda f(k,u(k))+\mu g(k,u(k)), \quad k\in[1,N]_{\mathbb{Z}},\\ &u(0)=u(N+1)=0.\end{aligned} \tag{3.17}$$

Meanwhile, by (3.16), if $\lambda\in(0,1)$, then

$$\mathcal{J}_{\lambda,\mu}(u_1)\leq\mathcal{J}_{\lambda,\mu}(\omega)<-c_f<0. \tag{3.18}$$

By using the fact that $\lambda f(k,s)+\mu g(k,s)\geq 0$ for $k\in[1,N]_{\mathbb{Z}}$, $s\leq 0$, and arguing as in the proof of Theorem 3.1, we see that any solution u of (3.17) satisfies $u\geq 0$. Therefore, u_1 is a positive solution of (3.17).

By a similar way of the proof of Theorem 3.2, the second solution will be found by using the mountain pass theorem. Take $\varepsilon>0$, such that

$$\frac{1}{N^2}-\mu\varepsilon>0.$$

From (H7), there exists r such that $0 < r < \|\omega\|$ and

$$G(k,s) \le \varepsilon s^2 \qquad \text{for any } k \in [1,N]_{\mathbb{Z}} \text{ and } s \in [0,r].$$

Take $v \in E$ with $0 < \|v\| \le \frac{r}{\sqrt{N}}$. Then it follows from (3.5) and Lemma 2.1 yields that

$$\begin{aligned}\sum_{k=1}^{N+1} \Psi(\Delta v(k-1)) - \mu \sum_{k=1}^{N} G(k,v) &\ge \frac{1}{2}\sum_{k=1}^{N+1}(\Delta v(k-1))^2 - \mu\varepsilon \sum_{k=1}^{N} v^2(k) \\ &\ge \|v\|^2\Big(\frac{1}{N^2} - \mu\varepsilon\Big) > 0. \qquad (3.19)\end{aligned}$$

Take a constant $\lambda(\mu) \in (0,1)$ such that

$$\frac{r^2}{N}\Big(\frac{1}{N^2} - \mu\varepsilon\Big) - \lambda(\mu)c_f > 0$$

and choose any $\lambda \in (0, \lambda(\mu))$. By (3.19), we have

$$\mathcal{J}_{\lambda,\mu}(v) > 0 \qquad \text{for all } v \in E \text{ with } \|v\| = \frac{r}{\sqrt{N}}.$$

Since (3.18) holds, by the mountain pass theorem we conclude that the functional $\mathcal{J}_{\lambda,\mu}$ has a critical point u_2 with $\mathcal{J}_{\lambda,\mu}(u_2) > 0$. Therefore u_2 is a positive solution of (3.17). Since $\mathcal{J}_{\lambda,\mu}(u_1) < 0$, it concludes that $u_1 \ne u_2$.

Finally, we prove that there exists a local minimum point u_3 of $\mathcal{J}_{\lambda,\mu}$ with $\|u_3\| < \frac{r}{\sqrt{N}}$. To verify that $u_3 \ne 0$, we argue as in the proof of Theorem 3.1. Consider a function $\chi \in E$, a constant $K > 0$ and a strictly decreasing sequence $\{c_n\}$ as in (3.8), with the further property, which follows from (H9) that

$$G(k, c_n\chi(k)) \ge -Kc_n^2\chi^2(k) \qquad \text{for any } k \in [1,N]_{\mathbb{Z}} \text{ and all } n.$$

Then by using (3.5) and (3.8), we compute that

$$\begin{aligned}\mathcal{J}_{\lambda,\mu}(c_n\chi) &= \sum_{k=1}^{N+1} \Psi(c_n\Delta\chi(k-1)) - \lambda\sum_{k=1}^{N} F(k, c_n\chi) - \mu \sum_{k=1}^{N} G(k, c_n\chi) \\ &\le c_n^2\Big(\frac{1}{2}\sigma\sum_{k=1}^{N+1}(\Delta\chi(k-1))^2 - \lambda\sum_{k=c}^{d}\frac{F(k,c_n)}{c_n^2} + (\lambda+\mu)K\sum_{k=1}^{N}\chi^2(k)\Big) \\ &= c_n^2\Big(\frac{1}{2}\sigma\|\Delta\chi\|^2 - \lambda\sum_{k=c}^{d}\frac{F(k,c_n)}{c_n^2} + (\lambda+\mu)K\|\chi\|^2\Big).\end{aligned}$$

Hence, we conclude that $\mathcal{J}_{\lambda,\mu}(u_3) \le \mathcal{J}_{\lambda,\mu}(c_n\chi) < 0$ for all large enough n and in particular, $u_3 \ne 0$. Obviously, (3.19) yields $\mathcal{J}_{\lambda,\mu}(u_3) > -c_f$. Since $\mathcal{J}_{\lambda,\mu}(u_1) < -c_f$ by (3.18), it concludes that $u_1 \ne u_3$. Therefore u_1, u_2 and u_3 are positive solutions of (3.17) and by the claim in Section 3.1, u_1, u_2 and u_3 are positive solutions of (3.15). □

Example 3.3. Let $p \in (0,1)$, $q \in (1,+\infty)$ and $m, n : [1,N]_{\mathbb{Z}} \to \mathbb{R}$ with $m^+ > 0$ and $n^+ > 0$. Then Theorem 3.3 implies that there exist a constant $\mu^* > 0$ and of a function $\lambda : (\mu^*, +\infty) \to \mathbb{R}^+ \cup \{\infty\}$ such that the problem

$$-\nabla\Big(\frac{\Delta u(k)}{\sqrt{1-\kappa(\Delta u(k))^2}}\Big) = \lambda m(k)u^p(k) + \mu n(k)u^q(k), \ k \in [1,N]_{\mathbb{Z}},$$

$$u(0) = u(N+1) = 0$$

has at least three positive solutions for all $\mu > \mu^*$ and all $\lambda \in (0, \lambda(\mu))$.

4. Infinitely many positive solutions

Theorem 4.1. *Assume that (H1)-(H4) and*

(H10) $f(k,u) = g(k)h(u)$ with $g(k) > 0$ for all $k \in [1,N]_{\mathbb{Z}}$;

(H11) there exist two strictly decreasing sequences $\{a_i\}$ and $\{b_i\}$ with $b_{i+1} < a_i < b_i$ and $b_1 < \frac{N+1}{2\sqrt{\kappa}}$ such that $\lim_{i\to\infty} b_i = 0$, $\lim_{i\to\infty} \frac{a_i}{b_i} = 0$ and $h(s) \leq 0$ for every $s \in [a_i, b_i]$.

Then problem (1.2) *has a sequence of positive solutions $\{u_i\} \subset E$ which satisfies*

$$\lim_{i\to\infty} \|u_i\|_\infty = 0.$$

Proof. Like in the proof of Theorem 3.1, we replace h with the auxiliary function $\hat{h}$, which satisfies all the assumptions of the theorem, and $\hat{h}(u) = h(u)$ on $[0, \frac{N+1}{2\sqrt{\kappa}}]$. We split the proof in several steps.

Step 1. A modified problem. Let us define $\hat{h} : \mathbb{R} \to \mathbb{R}$ as follows:

$$\hat{h}(s) = \begin{cases} 0, & \text{if } s < 0, \\ h(s), & \text{if } 0 \leq s < b_1, \\ h(b_1), & \text{if } s \geq b_1. \end{cases} \tag{4.1}$$

Obviously, $\hat{h}$ is continuous and $\hat{H}$ denote its primitive, that is $\hat{H}(s) = \int_0^s \hat{h}(\xi)d\xi$. It is easy to see that $\hat{H}$ and $\hat{h}$ have the same properties with H and h, respectively. Therefore, we argue as the claim of Theorem 3.1, a function $u \in E$ is a solution of (1.2) if and only if it is a solution of the problem

$$-\nabla\big(\psi(\Delta u(k))\big) = g(k)\hat{h}(u(k)), \quad k \in [1,N]_{\mathbb{Z}}, \qquad u(0) = u(N+1) = 0. \tag{4.2}$$

Let $\mathcal{J}, \mathcal{H} : E \to \mathbb{R}$ be the functionals defined by

$$\mathcal{J}(u) = \sum_{k=1}^{N+1} \Psi(\Delta u(k-1)), \quad \mathcal{H}(u) = -\sum_{k=1}^{N} g(k)\hat{H}(u(k)), \quad u \in E.$$

Due to (3.5), $\mathcal{J}$ is well defined on E, continuous, coercive and weakly lower semicontinuous and $\mathcal{H}$ is also well defined and sequentially weakly continuous. Moreover, $\mathcal{J}$ and $\mathcal{H}$ are continuously Gâteaux differentiable with derivative given by

$$\frac{\partial \mathcal{J}}{\partial u}(v) = \sum_{k=1}^{N+1} \psi(\Delta u(k-1))\Delta v(k-1), \quad \frac{\partial \mathcal{H}}{\partial u}(v) = -\sum_{k=1}^{N} g(k)\hat{h}(u(k))v(k), \; \forall\, u,\, v \in E.$$

With these assumption, the function φ from Lemma 2.2 reads as follows

$$\varphi(\rho) = \inf_{\mathcal{J}^{-1}((-\infty,\rho))} \frac{\mathcal{H}(u) - \inf_{\overline{\mathcal{J}^{-1}((-\infty,\rho))}} \mathcal{H}}{\rho - \mathcal{J}(u)},$$

where $\mathcal{J}^{-1}((-\infty,\rho)) := \{u \in E : \mathcal{J}(u) < \rho\}$.

Step 2. We claim that $\delta < 1$.

Recall that $\delta = \liminf_{\rho\to 0^+} \varphi(\rho)$ and clearly $\delta \geq 0$. It follows from $\hat{H}(s) = 0,\ s \leq 0$ that

$$\max_{[-b_i,b_i]} \hat{H} = \max_{[0,b_i]} \hat{H} = \max_{[0,a_i]} \hat{H}.$$

Let $\bar{s}_i \in [0, a_i]$ such that $\hat{H}(\bar{s}_i) = \max_{[-b_i,b_i]} \hat{H}$ and denote $s_i = \frac{1}{N^2\sigma} b_i^2$, here σ is defined by (3.3). Then it follows that

$$\mathcal{J}^{-1}((-\infty, s_i]) \subseteq \{v \in E \,|\, \|v\|_\infty \leq b_i\}.$$

In fact, if $v \in E$ is such that $\mathcal{J} \leq s_i$, then by (3.5) and Lemma 2.1, we have that

$$\frac{1}{N^2}\|v\|^2 \leq \frac{1}{2}\|\Delta v\|^2 \leq \mathcal{J}(v) \leq s_i,$$

and clearly, $\|v\|_\infty^2 \leq b_i^2$. Hence

$$\sup_{\mathcal{J}^{-1}((-\infty,\rho))} (-\mathcal{H}(v)) \leq \max_{v\in[-b_i,b_i]} \hat{H}(v) \cdot \sum_{k=1}^{N} g(k) = \hat{H}(\bar{s}_i)\|g\|_1. \tag{4.3}$$

By (H3) $\liminf_{s\to 0^+} \frac{\hat{H}(s)}{s^2} > -\infty$, there exist $M_1 > 0$ and $\tau \in (0, b_1)$ such that

$$\hat{H}(s) > -M_1 s^2 \quad \text{for every} \quad s \in (0, \tau). \tag{4.4}$$

Set $\eta = \frac{\sigma[N+1-(b-a)]}{2a(N+1-b)}$ and $\theta = \frac{(2a+1)(a+1)}{a} + \frac{[2(N-b)+1](N-b)}{N+1-b}$. Choose a suitable constant l, such that

$$\limsup_{i\to\infty} \frac{\max_{[0,a_i]} \hat{H}}{b_i^2} < l < \frac{1}{\|g\|_1}.$$

For i large enough, it follows from (H11) that

$$\frac{\max_{[0,a_i]} \hat{H}}{b_i^2}\|g\|_1 + \Big(\frac{M_1\theta}{6}\|g\|_\infty + \eta l\|g\|_1\Big)\frac{a_i^2}{b_i^2} < l\|g\|_1,$$

which implies that for any $\bar{s}_i \leq a_i$, we get that

$$\frac{\hat{H}(\bar{s}_i)}{s_i}\|g\|_1 + \Big(\frac{M_1\theta}{6}\|g\|_\infty + \eta l\|g\|_1\Big)\frac{\bar{s}_i^2}{s_i} < l\|g\|_1. \tag{4.5}$$

Define

$$w_{\bar{s}_i}(k) = \begin{cases} \frac{\bar{s}_i}{a}k, & \text{if } k \in [0,a]_{\mathbb{Z}}, \\ \bar{s}_i, & \text{if } k \in [a,b]_{\mathbb{Z}}, \\ \frac{\bar{s}_i}{N+1-b}(N+1-k), & \text{if } k \in [b,N+1]_{\mathbb{Z}}. \end{cases}$$

Clearly, $w_{\bar{s}_i} \in E$ and

$$\mathcal{J}(w_{\bar{s}_i}) \leq \frac{1}{2}\sigma\|\Delta w_{\bar{s}_i}\|^2 \leq \frac{\sigma}{2}(\frac{1}{a} + \frac{1}{N+1-b})\bar{s}_i^2 = \eta\bar{s}_i^2. \tag{4.6}$$

So, it follows from the definition of $w_{\bar{s}_i}$ that

$$
\begin{aligned}
&-\mathcal{H}(w_{\bar{s}_i})\\
=&\sum_{k=1}^{N} g(k)\hat{H}(w_{\bar{s}_i}(k))\\
=&\sum_{k=1}^{a} g(k)\hat{H}(w_{\bar{s}_i}(k))+\sum_{k=a+1}^{b} g(k)\hat{H}(w_{\bar{s}_i}(k))+\sum_{k=b+1}^{N} g(k)\hat{H}(w_{\bar{s}_i}(k))\\
>&-\frac{M_1(2a+1)(a+1)}{6a}\|g\|_\infty \bar{s}_i^2+\hat{H}(\bar{s}_i)\min_{[a+1,b]_{\mathbb{Z}}} g-\frac{M_1(2(N-b)+1)(N-b)}{6(N+1-b)}\|g\|_\infty \bar{s}_i^2\\
>&-\frac{M_1\bar{s}_i^2}{6}\Big[\frac{(2a+1)(a+1)}{a}+\frac{(2(N-b)+1)(N-b)}{(N+1-b)}\Big]\|g\|_\infty=-\frac{M_1\theta}{6}\|g\|_\infty \bar{s}_i^2.
\end{aligned}
\tag{4.7}
$$

This together with (4.3) and (4.5)-(4.7) deduces that

$$
\begin{aligned}
\sup_{J^{-1}((-\infty,s_k])}(-\mathcal{H}(v)+\mathcal{H}(w_{\bar{s}_i})&\le \|g\|_1\hat{H}(\bar{s}_i)+\frac{M_1\theta}{6}\|g\|_\infty \bar{s}_i^2\\
&\le l\|g\|_1(s_i-\eta\bar{s}_i^2)\le l\|g\|_1(s_i-\mathcal{J}(w_{\bar{s}_i})).
\end{aligned}
$$

Since $s_i\to 0$ as $i\to\infty$, we get that

$$
\delta\le\liminf_i \frac{\mathcal{H}(w_{\bar{s}_i})-\inf_{\mathcal{J}^{-1}((-\infty,s_i])}\mathcal{H}(v)}{s_i-\mathcal{J}(w_{\bar{s}_i})}\le l\|g\|_1<1.
$$

Step 3. 0 is not a local minimum of $\mathcal{J}+\mathcal{H}$.

We will construct a sequence of functions in E tending in norm to zero where the energy attains negatives value. By the assumption (H4), it follows that $\limsup\limits_{s\to 0^+}\frac{\hat{H}(s)}{s^2}=+\infty$, so there exists $M_2>0$ such that

$$
M_2>\frac{6\eta+M_1\theta\|g\|_\infty}{6\min\limits_{[a+1,b]_{\mathbb{Z}}} g},
\tag{4.8}
$$

then there exists a sequence $\{\tilde{s}_i\}\subset(0,\tau)$ converging to zero such that

$$
\hat{H}(\tilde{s}_i)>M_2\tilde{s}_i^2.
\tag{4.9}
$$

Let $w_{\tilde{s}_i}(k)$ be defined by

$$
w_{\tilde{s}_i}(k)=\begin{cases}\frac{\tilde{s}_i}{a}k, & \text{if } k\in[0,a]_{\mathbb{Z}},\\ \tilde{s}_i, & \text{if } k\in[a,b]_{\mathbb{Z}},\\ \frac{\tilde{s}_i}{N+1-b}(N+1-k), & \text{if } k\in[b,N+1]_{\mathbb{Z}}.\end{cases}
$$

It is clear that $\|w_{\tilde{s}_i}\|\to 0$ as $k\to\infty$. We claim that

$$
\mathcal{J}(w_{\tilde{s}_i})+\mathcal{H}(w_{\tilde{s}_i})<0.
$$

Indeed, it follows from (4.4) and (4.8) that

$$\begin{aligned}
&\mathcal{J}(w_{\tilde{s}_i})+\mathcal{H}(w_{\tilde{s}_i})\\
\le&\eta\tilde{s}_i^2-\sum_{k=1}^{a}g(k)\hat{H}(w_{\tilde{s}_i}(k))-\sum_{k=a+1}^{b}g(k)\hat{H}(w_{\tilde{s}_i}(k))-\sum_{k=b+1}^{N}g(k)\hat{H}(w_{\tilde{s}_i}(k))\\
\le&\eta\tilde{s}_i^2+\frac{M_1(2a+1)(a+1)}{6a}\|g\|_\infty\tilde{s}_i^2-\hat{H}(\tilde{s}_i)\min_{[a+1,b]_{\mathbb{Z}}}g+\frac{M_1(2(N-b)+1)(N-b)}{6(N+1-b)}\|g\|_\infty\tilde{s}_i^2\\
\le&\tilde{s}_i^2\Big(\eta+\frac{M_1\theta}{6}\|g\|_\infty-M_2\min_{[a+1,b]_{\mathbb{Z}}}g\Big)\\
<&0=\mathcal{J}(0)+\mathcal{H}(0).
\end{aligned}$$

Hence, the claim is achieved.

Step 4. Existence of a sequence of critical points for $\mathcal{J}+\mathcal{H}$.

We apply Lemma2.2 to the functionals $\mathcal{J}$ and $\mathcal{H}$ with $\lambda=1$. One has that 0 is the global minimum of $\mathcal{J}$ and by Step 3, it is not a local minimum of $\mathcal{J}+\mathcal{H}$, hence there exists a sequence $\{u_i\}$ of pairwise distinct critical points of the energy such that $\lim_{i\to\infty}\mathcal{J}(u_i)=0$. In particular,

$$\lim_{i\to\infty}\|u_i\|_\infty=0. \tag{4.10}$$

Now, we shall prove that the critical points of the energy functional are positive. Assume that u is a critical point of $\mathcal{J}+\mathcal{H}$ and the set $\{k\,|\,u(k)<0\}\neq\emptyset$, then by using the fact $\hat{h}(u)=0$ for $u\le 0$, we have

$$\begin{aligned}
0=\frac{\partial\mathcal{J}}{\partial u}(u)u^-+\frac{\partial\mathcal{H}}{\partial u}(u)u^-&=\sum_{k=1}^{N}\psi(\Delta u(k))\Delta u^-(k-1)-\sum_{k=1}^{N}\hat{h}(u(k))u^-(k)\\
&=\sum_{k=1}^{N}\psi(\Delta u^-(k))\Delta u^-(k-1),
\end{aligned}$$

which implies $u^-=0$, a contradiction. By the similar argument of the proof of Theorem 3.1, we can verify that $u_i\neq 0$. Hence, by (4.10), for i large enough, $0\le u_i(k)\le b_1$ for every $k\in[0,N+1]_{\mathbb{Z}}$, and (1.2) has a sequence positive solution $\{u_i\}$ satisfies (4.10). □

Acknowledgements

We are very grateful to the reviewers for their valuable suggestions.

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